A Implementation of a Exact Finite Reduction Scheme for Nonlinear PDE's

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Abstract

In this paper we study a semilinear Dirichlet problem applying a non local version of the Lyapunov-Schmidt reduction introduced in [1] for fields theory. We devise an algorithm based on the reduction and apply it to a sample model, we test the feasibility of the procedure and check the theoretical estimates on the convergence. We obtain two non trivial solution for the problem taken as a model.

Introduction

A global finite parameters reduction in fields theory was introduced in [1], as an application of the fundamental works by Amann, Conley and Zehnder in Hamiltonian systems. We consider a semilinear Dirichlet problem, and construct an algorithm based on the finite parameters reduction. Applying the algorithm to a sample model, we test the feasibility of the technique and we check the theoretical estimates on the convergence. Applying Peano-Picard and Newton-Raphson procedures we obtain two non trivial solutions.

1 Analytical setting

Our investigation takes place in $H := H_0^1(\Omega, \mathbb{R}^k)$, where we consider a non linear perturbation F of an elliptic operator L. We are trying to solve the following simple Dirichlet boundary value problem

$$\begin{cases} -Lu = F(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Assume the nonlinear operator $F : H \to H$ be a Nemitski operator, *i.e.*$F(u) := f \circ u$, where $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz,

$$|f(s_1) - f(s_2)| \leqslant C\,|s_1 - s_2|\,.$$

The core of the method consists in the spectral decomposition of H w.r.t. the eigenspaces of $-L$, and in the exploitation of the Green operator $g = (-L)^{-1}$, *i.e.*$g : H \to H$, $g \circ (-L) = -L \circ g = id_H$. The problem is translated through g and successively decomposed into a finite and an infinite part by means of a suitable cut-off.

Here is an outline of these steps. The spectral decomposition and the Green operator of (1) are given by,

$$-L\hat{u}_j = \lambda_j \hat{u}_j, \qquad \langle \hat{u}_i, \hat{u}_j \rangle = \delta_{ij}, \qquad 0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \dots. \tag{2}$$

$$g(v) = g\left(\sum_{j=1}^{+\infty} v_j \hat{u}_j\right) = \sum_{j=1}^{+\infty} v_j \frac{1}{\lambda_j} \hat{u}_j, \tag{3}$$

thus the cut-off of the space H is written as:

$$v = \sum_{j=1}^{+\infty} v_j \hat{u}_j = \sum_{j=1}^{m} v_j \hat{u}_j + \sum_{j=m+1}^{+\infty} v_j \hat{u}_j \in H, \tag{4}$$

$$v = \mathbb{P}_m v + \mathbb{Q}_m v = \mu + \eta, \qquad H = \mathbb{P}_m H \oplus \mathbb{Q}_m H. \tag{5}$$

Here the crucial starting point: we are going to search solutions of (1) represented by the form: $u = g(v)$, for suitable $v \in H$,

$$\begin{array}{lcl} -Lu & = & F(u), \\ -L\left(g\left(v\right)\right) & = & F\left(g\left(v\right)\right), \quad v = \mu + \eta, \\ \mu + \eta & = & F(g(\mu + \eta)), \end{array} \tag{6}$$

so the problem is splitted into

$$\begin{array}{lcl} \eta & = & \mathbb{Q}_m F(g(\mu + \eta)) \quad \text{(infinite part)} \\ \\ \mu & = & \mathbb{P}_m F(g(\mu + \eta)) \quad \text{(finite part)} \end{array} \tag{7}$$

The infinite part of the equation, for suitable fixed cut-off m, is uniquely solved, for every fixed finite part $\mu \in \mathbb{P}_m H$. Indeed the map

$$\begin{array}{lcl} \mathbb{Q}_m H & \longrightarrow & \mathbb{Q}_m H \\ \eta & \longmapsto & \mathbb{Q}_m F(g(\mu + \eta)), \end{array} \tag{CTR}$$

is contractive, provided m being suitably large. Using the Lipschitz constant C of F and recalling the monotone character of the spectral sequence $\{\lambda_j\}$, we obtain:

$$\begin{aligned}\|\mathbb{Q}_m F(g(\mu+\eta_1)) - \mathbb{Q}_m F(g(\mu+\eta_2))\| \leqslant \\ \leqslant C\,\|g(\mu+\eta_1) - g(\mu+\eta_2)\| \leqslant C\frac{1}{\lambda_{m+1}}\,\|\eta_1-\eta_2\|\,.\end{aligned}$$

Thus, we can choose $m \in \mathbb{N}$ large enough to achieve $\frac{C}{\lambda_{m+1}} < 1$, so that the *unique fixed point* $\tilde{\eta}(\mu)$ of this contraction solves the (infinite part) of the equation (2). It can be easily proved that the fixed point $\tilde{\eta}(\mu)$ inherits the regularity of F, being expressible as the implicit function of an equation involving F:

$$\mathcal{F}(\mu,\eta) = 0, \qquad \mathcal{F}(\mu,\eta) := \mu - \mathbb{Q}_m F(g(\mu+\eta)).$$

By substituting $\tilde{\eta}(\mu)$ into the finite equation (finite part), we get a *finite dimensional problem*:

$$\mu = \mathbb{P}_m F\left(g\left(\mu + \tilde{\eta}\left(\mu\right)\right)\right), \qquad \mu \in \mathbb{R}^m. \tag{8}$$

Although it is finite, in general we have not an *a priori* control about existence and uniqueness; more precisely, we could find no solutions, or many solutions, and possible bifurcation phenomena could happen for increasing (Lipschitz constant C of) F. Every solution μ^* of (8) gives rise to a solution of the original nonlinear Dirichlet problem:

$$u = g\left(\mu^* + \tilde{\eta}\left(\mu^*\right)\right).$$

2 Numerical implementation

The previously outlined procedure can be implemented in a numerical framework by substituting the differential operator of the PDE with its discretized version. Using a finite element approach, denoting by T_h a generic discretization of Ω, formed by n nodes and N subdivisions with characteristic mesh size h, the discrete elliptic operator reduces to a symmetric positive-definite matrix $-L_h$. The numerical solution vector $u_h \in H_h := \mathbb{R}^n$, is given by the solution of the system of the nonlinear algebraic equations:

$$-L_h(u_h) = F_h(u_h), \tag{9}$$

where F_h is the discretization of the nonlinear function operator F.

The symmetric eigenproblem of the corresponding linear system can be written as:

$$-L_h u = \lambda u, \tag{10}$$

where

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n,$$
$$u_1, u_2, \ldots, u_n,$$

are the real positive eigenvalues and the corresponding eigenvectors. Note that the eigenvalues and eigenvectors thus defined converge to the eigenvalues and eigenfunctions of the continuous problem (1) in the limit when $h \to 0$ and $n \to +\infty$ [2].

Remark 1. The first eigenvalues of the finite element problem (10) converge to fixed values when the nodal spacing of the discretization is made arbitrarily small, *i.e.*, when the dimension of the linear system (10) goes to infinity. Hence the leftmost eigenpairs of the eigenproblems converge to the same leftmost eigenspectrum.

Remark 2. The largest eigenvalues cannot converge and grow as n^2 when $h \to 0$, consistently with theory. This suggests that the highest frequencies are in essence inversely related to the magnitude of the discretization error, while the lowest frequencies represent the fundamental natural modes of the physical system described by the PDE (1).

In analogy to the continuous case (2), the discrete Green operator g_h of $-L_h$ can be easily written, w.r.t. the basis of H given by the eigenvectors of $-L_h$:

$$g_h(u_k) = \left(-L_h^{-1}\right)(u_k) := \frac{1}{\lambda_k} u_k \qquad k = 1, \ldots, n.$$

Since $-L_h$ is s.p.d., any vector $v \in H$ can be expressed as:

$$v = a_1 u_1 + \cdots + a_n u_n = Ua,$$

where U denotes the matrix whose columns are the eigenvectors u_k of $-L_h$:

$$U := [u_1, \ldots, u_n].$$

The Green operator applied to v gives:

$$g_h(v) = \frac{a_1}{\lambda_1} u_1 + \cdots + \frac{a_n}{\lambda_n} u_n = U\Lambda^{-1} a,$$

where Λ is the diagonal matrix of the eigenvalues (ordered accordingly to the corresponding eigenvectors).

The algorithm described in the previous section applies directly to the discretized problem, and proceeds as follows. For a given m, the vector space H is split into two subspaces $P_m H$ and $Q_m H$, where $P_m H \subseteq H$ is generated by the first m eigenvectors $u_1, \ldots, u_m$, while $Q_m H \subseteq H$ is generated by $u_{m+1}, \ldots, u_n$. Consequently, the projectors P_m and Q_m, which are the discrete counterparts of $\mathbb{P}_m$ and $\mathbb{Q}_m$ in (5), can be explicitly written by means of the two matrices V_1 and V_2:

$$V_1 := [u_1, \ldots, u_m], \qquad V_2 := [u_{m+1}, \ldots, u_n], \qquad [V_1, V_2] = U.$$

For every $v = \hat{\mu} + \hat{\eta} \in H$, we have:

$$\hat{\mu} := P_m v = V_1 V_1^T v = V_1 a', \qquad a' \in \mathbb{R}^m, \tag{11}$$

$$\hat{\eta} := Q_m v = V_2 V_2^T v = V_2 a'', \qquad a'' \in \mathbb{R}^{n-m}. \tag{12}$$

The discrete version of (6) becomes then:

$$-L_h u = F_h(u), \tag{13}$$

$$-L_h\left(g_h(v)\right) = F_h\left(g_h(v)\right), \tag{14}$$

$$\hat{\mu} \oplus \hat{\eta} = F_h\left(g_h(\hat{\mu} + \hat{\eta})\right). \tag{15}$$

The numerical algorithm is thus formed by two finite dimensional fixed point iterations:

$$\hat{\eta} = Q_m F_h\left(g_h(\hat{\mu} + \hat{\eta})\right), \qquad \in \mathbb{R}^{n-m}, \tag{16}$$

$$\hat{\mu} = P_m F_h\left(g_h(\hat{\mu} + \hat{\eta})\right), \qquad \in \mathbb{R}^m, \tag{17}$$

with (16) satisfying:

$$\left\| Q_m F_h\left(g_h(\hat{\mu} + \hat{\eta}_1)\right) - Q_m F_h\left(g_h(\hat{\mu} + \hat{\eta}_2)\right) \right\| \leqslant \frac{C}{\lambda_{m+1}} \left\| \hat{\eta}_1 - \hat{\eta}_2 \right\|.$$

To prove the last assertion, we first note that F_h is Lipschitz whenever f is, *i.e.*, for every $u, \bar{u} \in H_h$,

$$\left\| F_h(u) - F_h(\bar{u}) \right\| = \left\| \begin{pmatrix} f(u_1) - f(\bar{u}_1) \\ \vdots \\ f(u_n) - f(\bar{u}_n) \end{pmatrix} \right\| =$$

$$= \left\| \begin{pmatrix} c_1(u_1 - \bar{u}_1) \\ \vdots \\ c_n(u_n - \bar{u}_n) \end{pmatrix} \right\| \leqslant C \left\| u - \bar{u} \right\|,$$

where C is the Lipschitz constant of f, while $c_1, \ldots, c_n$ are suitable positive numbers such that $c_j < C, \forall j$. The proof is completed by considering that:

$$\|g_h(\hat{\mu} + \hat{\eta}_1) - g_h(\hat{\mu} + \hat{\eta}_2)\| = \|g_h(\hat{\eta}_1 - \hat{\eta}_2)\| \leqslant \frac{1}{\lambda_{m+1}} \|\hat{\eta}_1 - \hat{\eta}_2\| .$$

The value of m is chosen so that the Lipschitz constant $\frac{C}{\lambda_{m+1}}$ becomes sufficiently smaller than 1 and guarantees a fast convergence of the fixed point iteration (16). Teh fixed point problem (17) can be solved either by Peano-Picard iterations or by a Newton-Raphson technique, but we do not have estimates on contractivity.

Numerical performance does not allow for the full solution of the eigenproblem (10), which has been assumed so far. However, the ellipticity of $-L_h$ suggests that the discrete Green function tends to zero rapidly, according to the continuous case, but we are not aware of any useful theoretical estimate of this behavior. Thus in the following section we will try to evaluate experimentally, applying the proposed scheme to a sample problem, the minimum size of the eigensolution of (10) that gives a predefined accuracy.

2.1 Implementation and sample test

Consider the one-dimensional Laplacian operator on the unit interval;

$$L := \Delta = \frac{\partial^2}{\partial x^2} \qquad \Omega := [0,1] \subseteq \mathbb{R}, \qquad \mathcal{H} := H_0^1([0,1], \mathbb{R}), \tag{18}$$

which, after the discretization with characteristic length $h = \frac{1}{n+1}$, and elimination from $-L_h$ of the two Dirichlet boundary conditions, will be represented by the well known tridiagonal matrix:

$$-L_h = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}. \tag{19}$$

operating on the finite dimensional space $H := \mathbb{R}^n$. By very classical computations, the eigenpairs of (10) are found to be:

$$\lambda_k = 4(n+1)^2 \sin^2\left(\frac{k\pi}{2(n+1)}\right), \qquad k = 1, \ldots, n, \tag{20}$$

$$u_k = \{u_{k,i}\} = \left\{ \sqrt{\frac{2}{n+1}} \sin\left(\frac{k\pi}{n+1} i\right) \right\}, \qquad k = 1, \ldots, n. \tag{21}$$

Since $\sin t \approx t$ as $t \to 0$, the leftmost eigenvalues converge to the quantities

$$\lambda_k(-L_h) \longrightarrow (\pi k)^2, \qquad k = 1, \ldots, n, \tag{22}$$

while the eigenvectors behave as:

$$u_k \longrightarrow \sin(k\pi x), \qquad k = 1, \ldots, n, \quad 0 \leqslant x \leqslant 1. \tag{23}$$

The largest eigenvalue is provided by (20) with $k = n$:

$$\lambda_n(-L_h) = \frac{4}{h^2} \sin^2 \left(\frac{\pi}{2} \frac{n}{n+1} \right) \approx \frac{4}{h^2}. \tag{24}$$

Note that the eigenpairs satisfy Remarks 1 and 2.

Our sample test considers the Nemitski operator F associated to the function

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = \alpha \left(1 - e^{-\frac{x^2}{2}} \right).$$

Note that F is Lipschitz, with Lipschitz constant equal to

$$C = \sup |f'| = \alpha / \sqrt{e} \approx 18.1959 (\alpha = 30).$$

Discretizing with $n = 80$ nodes, we can look for a suitable eigenvalue to perform the cutoff. The first candidates for contractive factors M are found to be

$$M = \frac{|f'|}{\lambda_1} \approx 1.84364, \qquad M = \frac{|f'|}{\lambda_2} \approx 0.460912, \qquad M = \frac{|f'|}{\lambda_3} \approx 0.204852, \qquad \ldots$$

We set $m = 2$, so that the cotractive factor for the fixed point iteration given in (16) is 0.204852. The splitting is organized as follows:

$$H = \mathbb{R}^{80} = P_2 H \oplus Q_2 H,$$
$$v = \hat{\mu} + \hat{\eta} = a'_1 u_1 + a'_2 u_2 + a''_1 u_3 + \cdots + a''_{n-2} u_n,$$

To show the convergence characteristics of map (16), we perform several iterations by applying the Peano-Picard procedure. All the eigenpairs of (19) are employed in the calculations. The effects of using a partial eigenspectrum ($k \ll n$) will be addressed in the next section.

We fix as a first guess $\hat{\mu}^{(0)} = 500u_1 + 500u_2$ (u_1 and u_2 are the first two eigenvectors), and randomly generate $\hat{\eta}^{(0)}$. Denoting by $\hat{\eta}^{(j)}$ the j-th iterate of the contraction map, we expect the following estimate to be fulfilled,

$$\left\| \hat{\eta}^{(j+1)} - \hat{\eta}^{(j)} \right\| \leqslant 0.21 \left\| \hat{\eta}^{(j)} - \hat{\eta}^{(j-1)} \right\|,$$

j	$\varepsilon_\eta^{(j)}$	M	j	$\varepsilon_\eta^{(j)}$	M
1	8.21310×10^{1}	$\star$	11	6.33852×10^{-7}	0.169058
2	5.56597×10^{0}	0.067770	12	1.07158×10^{-7}	0.169058
3	9.43390×10^{-1}	0.169492	13	1.81159×10^{-8}	0.169059
4	1.60178×10^{-1}	0.169790	14	3.06254×10^{-9}	0.169053
5	2.71269×10^{-2}	0.169355	15	5.17750×10^{-10}	0.169059
6	4.58876×10^{-3}	0.169159	16	8.75862×10^{-11}	0.169167
7	7.75914×10^{-4}	0.169090	17	1.47106×10^{-11}	0.167956
8	1.31182×10^{-4}	0.169067	18	2.57688×10^{-12}	0.175171
9	2.21776×10^{-5}	0.169060	19	4.18135×10^{-13}	0.162264
10	3.74931×10^{-6}	0.169058	20	1.28513×10^{-13}	0.307348

Table 1: Convergence of the Peano-Picard iterations applied to (16) with $m = 2$, $\mu_0 = 500u_1 + 500u_2$ and using all the eigenspectrum of (19), $(n = 640, l = 640)$.

The results are reported in Table 1. After 18 iterations, the L^2 norm of the difference between two successive iterations, $\varepsilon_\eta^{(j)} = \left\|\hat{\eta}^{(j)} - \hat{\eta}^{(j-1)}\right\|_{L^2}$, becomes smaller than 10^{-11} with a contractive factor of approximately 0.17. Note that M is always smaller than the theoretical predictions and seems to stabilize after the 3^{rd} iteration. After 16 iterations, small oscillations appear due to round-off errors. Similar behavior is found when changing the initial guess $\mu^{(0)}$ or increasing the number of nodes $n = 160, 320, 640$. The number of iterations changes slightly, while M always converges to about 0.17.

The solution of the full problem is obtained by solving

$$\hat{\mu} = P_m F_h \left(g_h \left(\hat{\mu} + \hat{\eta}(\hat{\mu}) \right) \right). \tag{25}$$

Here we iterate by means again of the Peano-Picard procedure, though we do not possess any contraction result. At each of the iterations of the $\hat{\mu}$–map we have to solve the $\hat{\eta}$–map. To ensure convergence of the latter we perform a fixed number of iterations equal to 20. This allows $\varepsilon_\eta^{(20)}$ to become always smaller than 10^{-12}. Table 2 reports the fixed point μ for the complete equation. Note that numerically calculated contractive factor of this small scale $(m = 2)$ problem is rather small, achieving a value of about 0.22 (see table 2, 4^{th} column).

As apparent from table 2, the map converges to

$$\hat{\mu} = 626.853u_1 - (7.67118 \times 10^{-14})u_2,$$

in 20 iterations. By means of the contraction map we can build the approximate solution of the discretized problem,

$$\bar{u} = g_h(\hat{\mu} + \hat{\eta}(\hat{\mu})).$$

j	$\mu^{(j)}$	$\varepsilon_\mu^{(j)}$	M
1	$(230.589, -40.3486)$	6.7037×10^1	$\star$
2	$(283.236, -16.8716)$	5.7644×10^1	0.859881
3	$(369.349, -6.96820)$	8.6680×10^1	1.503710
4	$(482.325, -2.46623)$	1.1307×10^2	1.304410
5	$(573.516, -0.605729)$	9.1209×10^1	0.806688
6	$(612.605, -0.103476)$	3.9092×10^1	0.428595
7	$(623.499, -0.0150424)$	1.0895×10^1	0.278693
$\vdots$	$\vdots$	$\vdots$	$\vdots$
16	$(626.853, -2.61551 \times 10^{-10})$	1.7451×10^{-5}	0.225253
17	$(626.853, -3.56424 \times 10^{-11})$	3.9308×10^{-6}	0.225253
18	$(626.853, -4.93335 \times 10^{-12})$	8.8543×10^{-7}	0.225253
19	$(626.853, -5.30742 \times 10^{-13})$	1.9945×10^{-7}	0.225253
20	$(626.853, -7.67118 \times 10^{-14})$	4.4925×10^{-8}	0.225252

Table 2: Convergence behavior of the complete map (25) starting with $m = 2$, $\mu^{(0)} = 100u_1 - 100u_2$ and using all the eigenspectrum of (19), $(n = 640, l = 640)$.

By interpolation, we construct a candidate solution for the analytical problem (1),

$$\mathcal{H} \ni \tilde{u}(x) := \text{interpolation}(\bar{u}).$$

interpoliamo con un polinomio di grado 3!

Now we give an estimate of the goodness of the solution evaluating the residual function $E(x) := -\frac{\partial^2}{\partial x^2}\tilde{u}(x) - F(\tilde{u}(x))$. The theoretical accuracy of the finite difference method does not degenerate as the discretization scale $n = 10, 20, \ldots, 320, 640$ varies. Theoretically, the norm of the error function should decrease proportionally to the square of the number of subdivisions. Doubling the subdivisions, the error asymptotically decreases by a factor $\frac{1}{4}$, as expected by theoretical predictions.

2.1.1 Using a partial eigenspectrum

As claimed in the previous section, in general the complete eigensolution of the elliptic operator $-L_h$ is computationally too demanding and is seldom . Nevertheless, theoretical considerations suggest that a not so big number of eigenvectors could suffice to evaluate a good approximate solution of the b.v.p. We consider the solution $\bar{u}$ so far determined employing $l = n = 640$ eigenvectors as the "exact" solution, and we try to approximate it progressively reducing the number $l \ll n$ of eigenvectors involved to generate the solution.

L^1–norm of the residue			L^2–norm of the residue		
n	n^{th}–residue	$\frac{n^{th}\text{–residue}}{2n^{th}\text{–residue}}$	n	n^{th}–residue	$\frac{n^{th}\text{–residue}}{2n^{th}\text{–residue}}$
10	9.07257×10^{-1}	$\star$	10	1.53448	$\star$
20	2.81267×10^{-1}	3.22561	20	7.778759×10^{-1}	1.97042
40	7.59899×10^{-2}	3.701369	40	2.47547×10^{-1}	3.14590
80	1.55879×10^{-2}	4.874926	80	5.12011×10^{-2}	4.83480
160	3.30348×10^{-3}	4.718633	160	9.88305×10^{-3}	5.18070
320	7.46062×10^{-4}	4.427889	320	1.94486×10^{-3}	5.08160
640	1.76359×10^{-4}	4.230356	640	4.02237×10^{-4}	4.83512

Table 3: Behavior of the residue of the approximated solution when the number of subdivisions n increases ($m = 2, l = n$).

First we test the contractiveness of the generator of the queue $\hat{\eta}$. We will calculate and follow the behavior of $\varepsilon_\eta^{(j)} := \left\|\hat{\eta}^{(j+1)} - \hat{\eta}^{(j)}\right\|$ and the contractive factor $M_j = \frac{\varepsilon_\eta^{(j)}}{\varepsilon_\eta^{(j-1)}}$. Recall that theoretically the contractive factor should be $M \leqslant 0.20485$. See the table 4 Next we try the Peano-Picard procedure to obtain successive approximate solutions, increasing the number $l < 640$ of eigenvectors involved, and check the eventual rate of convergence to the more accurate, but more expensive, solution $\tilde{u}$, obtained employing all eigenvectors ($l = n = 640$).

See tables 5.

2.2 Newton-Raphson procedure

The exact finite reduction could also be more deeply exploited in order to perform more refined strategies for searching the solutions. In fact, the solutions $\hat{\mu}$ of the reduced equation (25)

$$\hat{\mu} = P_m F_h(g_h(\hat{\mu} + \tilde{\eta}(\hat{\mu}))),$$

can be considered as the fixed points of the map

$$\mu \mapsto PP(\mu) := P_m F_h(g_h(\hat{\mu} + \tilde{\eta}(\hat{\mu}))).$$

The Peano-Picard procedure consists in the iterated application of $PP(\cdot)$ from a tentative starting point μ_0.

$$\mu_1 = PP(\mu_0), \ldots, \mu_i = PP^i(\mu_0), \ldots$$

$l = 160$		
j	$\varepsilon_\eta^{(j)}$	M
1	7.84170×10^{1}	$\star$
2	3.99984×10^{0}	0.051007
3	2.89345×10^{-1}	0.072339
4	2.34318×10^{-2}	0.080982
5	2.18111×10^{-3}	0.093083
6	2.22913×10^{-4}	0.102202
7	2.37485×10^{-5}	0.106537
8	2.56871×10^{-6}	0.108163
9	2.79277×10^{-7}	0.108722
10	3.04167×10^{-8}	0.108912
11	3.31465×10^{-9}	0.108975
12	3.61567×10^{-10}	0.109081
13	3.94282×10^{-11}	0.109048
14	4.31287×10^{-12}	0.109385
15	3.79512×10^{-13}	0.111259

$l = 40$		
j	$\varepsilon_\eta^{(j)}$	M
1	6.99369×10^{1}	$\star$
2	3.90643×10^{0}	0.055857
3	2.76129×10^{-1}	0.070686
4	2.17559×10^{-2}	0.078789
5	1.98066×10^{-3}	0.091040
6	2.00151×10^{-4}	0.101052
7	2.12306×10^{-5}	0.106073
8	2.29291×10^{-6}	0.108000
9	2.49164×10^{-7}	0.108667
10	2.71326×10^{-8}	0.108894
11	2.95671×10^{-9}	0.108973
12	3.22190×10^{-10}	0.108969
13	3.51193×10^{-11}	0.109002
14	3.89529×10^{-12}	0.110916
15	4.26181×10^{-13}	0.109409

$l = 10$		
j	$\varepsilon_\eta^{(j)}$	M
1	6.63682×10^{1}	$\star$
2	4.00190×10^{0}	0.060298
3	2.79898×10^{-1}	0.069941
4	2.15908×10^{-2}	0.077138
5	1.93161×10^{-3}	0.089464
6	1.93531×10^{-4}	0.100192
7	2.04669×10^{-5}	0.105755
8	2.20834×10^{-6}	0.107898
9	2.39886×10^{-7}	0.108628
10	2.61154×10^{-8}	0.108866
11	2.84542×10^{-9}	0.108955
12	3.09856×10^{-10}	0.108896
13	3.37722×10^{-11}	0.108993
14	3.72282×10^{-12}	0.110233
15	3.92485×10^{-13}	0.105427

$l = 4$		
j	$\varepsilon_\eta^{(j)}$	M
1	5.69242×10^{1}	$\star$
2	3.07766×10^{0}	0.054066
3	1.81176×10^{-1}	0.058868
4	1.29883×10^{-2}	0.071689
5	1.13693×10^{-3}	0.087535
6	1.09579×10^{-4}	0.096382
7	1.08595×10^{-5}	0.099102
8	1.08213×10^{-6}	0.099648
9	1.07857×10^{-7}	0.099671
10	1.07438×10^{-8}	0.099611
11	1.06966×10^{-9}	0.099561
12	1.06517×10^{-10}	0.099580
13	1.06174×10^{-11}	0.099678
14	1.05226×10^{-12}	0.099107
15	1.35749×10^{-13}	0.129007

Table 4: Convergence of Peano-Picard iterations applied to (16) starting with $m = 2$, $\mu^{(0)} = 500u_1 + 500u_2$ and $(n = 640, l = 160, 40, 10, 4)$.

$\boldsymbol{l = 160}$			
j	$\mu^{(j)}$	$\mu^{(j)} - \bar{\mu}$	M_j
1	$(557.101, 115.658)$	135.063	135.060
2	$(605.908, 21.1875)$	29.7917	0.220576
3	$(621.783, 3.16681)$	5.97733	0.200637
4	$(625.692, 0.443249)$	1.24227	0.207830
5	$(626.590, 0.0610481)$	0.269357	0.216826
6	$(626.793, 0.00837702)$	0.0597325	0.221759
7	$(626.839, 0.00114853)$	0.0133737	0.223893
8	$(626.849, 0.000157440)$	0.00300559	0.224738
9	$(626.851, 0.0000215808)$	0.000676438	0.225059
10	$(626.852, 2.95813454 \times 10^{-6})$	0.000152321	0.225180

$\boldsymbol{l = 40}$			
j	$\mu^{(j)}$	$\mu^{(j)} - \bar{\mu}$	M_j
1	$(557.101, 115.658)$	135.063	135.063
2	$(605.908, 21.1875)$	29.7917	0.2205767
3	$(621.783, 3.16681)$	5.97733	0.2006370
4	$(625.692, 0.4432489)$	1.24227	0.2078307
5	$(626.590, 0.06104813)$	0.2693578	0.2168263
6	$(626.793, 0.008377023)$	0.05973271	0.2217596
7	$(626.839, 0.001148533)$	0.01337393	0.2238963
8	$(626.849, 0.0001574401)$	0.003005773	0.2247486
9	$(626.851, 0.00002158085)$	0.0006766204	0.2251069
10	$(626.852, 2.95813 \times 10^{-6})$	0.0001525030	0.2253893

$\boldsymbol{l = 4}$			
j	$\mu^{(j)}$	$\mu^{(j)} - \bar{\mu}$	M_j
1	$(557.082, 115.788)$	135.184	135.18476
2	$(605.986, 21.2413)$	29.7753	0.220257
3	$(621.865, 3.17323)$	5.91074	0.198510
4	$(625.757, 0.443697)$	1.18196	0.199969
5	$(626.646, 0.0610455)$	0.215283	0.182140
6	$(626.846, 0.00836796)$	0.0105572	0.0490387
7	$(626.890, 0.00114610)$	0.0384218	3.63938
8	$(626.901, 0.000156945)$	0.0484506	1.26101
9	$(626.903, 0.0000214909)$	0.0507005	1.04643
10	$(626.903, 2.94278 \times 10^{-6})$	0.0512045	1.00994

Table 5: Convergence of the Peano-Picard iterations applied to (25) with $m = 2$, $\mu_0 = 600u_1 + 0u_2$ and with $n = 640$ nodes and employing $l = 160, 40, 4$ eigenvectors.

If a limit is reached then a solution of the original problem is found. Alternatively, the solutions of (25) can also be considered as the zeros of the map:

$$\begin{aligned} NR : \mathbb{R}^m &\to \mathbb{R}^m, \\ \mu &\mapsto NR(\mu) := \mu - P_m F_h(g_h(\hat{\mu} + \tilde{\eta}(\hat{\mu}))), \end{aligned}$$

which could be sought by means of the Newton-Raphson procedure. Namely, we search for a limit in the sequence defined as follows:

$$\begin{cases} J_{NR}(\mu)s &= -NR(\mu), \\ \mu &\mapsto \mu + s. \end{cases}$$

At this stage, the exactly finitely reduced equation (25) allows us to determine the jacobian of the non linear system:

$$J_{NR}(\mu) = \left(\frac{\partial NR_i}{\partial \mu_j}(\mu) \right)_{i,j=1,\dots,m},$$

$$\begin{aligned} \frac{\partial NR_i}{\partial \mu_j}(\mu) = \delta_{ij} - \frac{\partial (F_h)_i}{\partial u_r} \cdot \left(\frac{1}{\lambda_r} \frac{\partial}{\partial \mu_j}(\mu + \tilde{\eta}(\mu)) \right) = \\ = \delta_{ij} - \frac{\partial (F_h)_i}{\partial u_j} \frac{1}{\lambda_j} - \sum_{r=3}^{k} \frac{\partial (F_h)_i}{\partial u_r} \cdot \frac{1}{\lambda_r} \frac{\partial \tilde{\eta}_r(\mu)}{\partial \mu_j}. \end{aligned} \quad (26)$$

First we calculate the Jacobian of $F_h(u)$ with respect to the eigenvectors coordinates, *i.e.*considering $u[u_1, \dots, u_k](x) = u_1\hat{u}_1(x) + \cdots + u_k\hat{u}_k(x)$. Thus,

$$\begin{aligned} \frac{\partial (F_h)_i}{\partial u_r}(u) &= \frac{\partial}{\partial u_r} \left((F_h)_i (u_1\hat{u}_1 + \cdots + u_k\hat{u}_k) \right) = \\ &= \frac{\partial}{\partial u_r} \left(\langle F_h(u_1\hat{u}_1 + \cdots + u_k\hat{u}_k), \hat{u}_i \rangle \right) = \\ &= \frac{\partial}{\partial u_r} \sum_{l=1}^{n} f(u[u_1, \dots, u_k](x_l)) \hat{u}_i(x_l) = \\ &= \sum_{l=1}^{n} \frac{df}{du}(u(x_l)) \frac{\partial u}{\partial u_r}(x_l) \hat{u}_i(x_l) = \\ &= \sum_{l=1}^{n} f'(u(x_l)) \hat{u}_r(x_l) \hat{u}_i(x_l). \end{aligned} \quad (27)$$

$\boldsymbol{\mu_0 = (100, 0)}$			
j	μ_j	ε_μ^j	$\frac{\varepsilon_\mu^{j-1}}{(\varepsilon_\mu^j)^2}$
1	$(205.949, 1.66223 \times 10^{-15})$	105.95	105.95
2	$(160.633, -1.08118 \times 10^{-15})$	45.316	0.00403693
3	$(159.585, -5.48825 \times 10^{-16})$	1.04871	0.000510684
4	$(159.582, 6.32719 \times 10^{-16})$	0.00283031	0.0025735
5	$(159.582, 2.17691 \times 10^{-15})$	2.10223×10^{-8}	0.00262429
6	$(159.582, 3.53473 \times 10^{-15})$	1.70535×10^{-13}	385.881
L^2-Residue = 0.00103257			

$\boldsymbol{\mu_0 = (600, 0)}$			
j	μ_j	ε_μ^j	$\frac{\varepsilon_\mu^{j-1}}{(\varepsilon_\mu^j)^2}$
1	$(627.622, 1.59563 \times 10^{-15})$	27.6225	27.6225
2	$(626.853, -4.35704 \times 10^{-17})$	0.769375	0.00100835
3	$(626.852, 1.88864 \times 10^{-16})$	0.000522536	0.000882756
4	$(626.852, 1.73901 \times 10^{-15})$	2.41811×10^{-10}	0.000885613
5	$(626.852, 2.70311 \times 10^{-15})$	9.64099×10^{-16}	16487.9
L^2-Residue = 0.0205632			

Table 6: Solutions found starting from $\mu_0 = (100, 0)$ and $\mu_0 = (600, 0)$, with $n = 640$ subdivisions, employing $k = 32$ eigenvectors

Next, in order to determine the derivative of $\tilde{\eta}(\mu)$, we differentiate the defining equation,

$$\tilde{\eta}(\mu) = Q_m F_h(g_h(\mu + \tilde{\eta}(\mu)))$$

$$\frac{\partial \tilde{\eta}_s}{\partial \mu_j}(\mu) = \frac{\partial (F_h)_r}{\partial u_j}\frac{1}{\lambda_j} + \sum_{s=3}^{k} \frac{\partial (F_h)_r}{\partial u_s}\frac{1}{\lambda_r}\frac{\partial \tilde{\eta}_r}{\partial \mu_j}(\mu),$$

and solve the linear system:

$$\sum_{s=3}^{k}\left(\delta_{rs} - \frac{\partial (F_h)_r}{\partial u_s}\frac{1}{\lambda_r}\right)\frac{\partial \tilde{\eta}_r}{\partial \mu_j} = \frac{1}{\lambda_j}\frac{\partial (F_h)_r}{\partial u_j}.$$

2.2.1 Application of Newton-Raphson

Implementing the previous algorithm we obtained the same solution found by Peano-Picard, and also a new solution. As apparent from the tables 6, the ratio $\frac{\varepsilon_\mu^j}{(\varepsilon_\mu^{j+1})^2}$ converges, in agree with the theoretical prediction of Newton-Raphson algorithm.

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