Loss of Gevrey Regularity for Asymptotic Optics

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Abstract

In this paper we will investigate some aspects of the asymptotic behavior of oscillatory integrals from the Gevrey point of view. We will give formal asymptotic expansions and study the Gevrey character of oscillatory integrals, in comparison with the Gevrey character of their amplitudes. We will deduce a formula for the loss of Gevrey regularity both for phase functions in the Morse class and for degenerate phase function.

Contents

1 Oscillatory integrals and Gevrey character

Our aim is to give an asymptotic analysis from the Gevrey point of view of oscillatory integrals of the type

$$I_a^{\varphi}(q,\lambda) := \int_{\mathbb{R}^n} e^{i\lambda\varphi(x,q)} a(x,q,\lambda)\, dx \tag{1}$$

for $\lambda \to +\infty$, where $q \in \mathbb{R}^\ell$, the phase function φ is a real–valued smooth function in $\Omega \subset \mathbb{R}^n \times \mathbb{R}^\ell$, while the amplitude a is a complex–valued smooth function in $\mathbb{R}^n \times [1,+\infty)$, and also a symbol of order m (see the definitions below).

We will concentrate on the simpler case

$$I_p^{\varphi}(\lambda) := \int_{\mathbb{R}^n} e^{i\lambda\varphi(x)} p(x,\lambda)\, dx \tag{2}$$

where no external parameters q appear, and first consider Morse non degeneracy for φ, *i.e.* we require that every critical point x_0, $d\varphi(x_0) = 0$, to be non degenerate, *i.e.* $\det d^2\varphi(x_0) \neq 0$.

The first motivation for this investigation is to extend the results of [4] to a first situation where critical points occur. In fact, the problems in optics require a rather precise analysis of integrals of the type of (2). Indeed, the Huygens principle in wave optics, states that the illumination in a point $q \in \mathbb{R}^3$ suscitated by a light wave which is completely determined, say, on a surface Σ, can be computed as a superposition of elementary contributions coming from every infinitesimal portion of Σ. This superposition of infinitesimal waves can be expressed precisely by means of an oscillatory integral.

Moreover, for $q \in \mathbb{R}^3$ fairly distant from the caustic and λ sufficiently large, the short wave approximation holds, *i.e.* the oscillatory integral can be substituted by a finite sum of contributions which can be plainly interpreted as rays. This fact is proved by means of the stationary phase method, and in a single time makes descend geometrical optics from wave optics and offers a support for the Fermat principle.

We set ourselves in the Gevrey class of functions in order to obtain finer estimates than which obtainable in the C^∞ class and also to allow ourselves to employ compactly supported amplitudes, which cannot be done in the analytical class.

We will also make use of refined scales of anisotropic Gevrey spaces splitting (separating) the Gevrey regularity with respect to the variables x and q, and the large parameter λ.

Given $\rho, \sigma, \theta \geqslant 1$ we define the following spaces of formal symbols:

1. Let $F_m^\theta[1,+\infty)$ be the set of all formal series

$$\sum_{j=0}^{\infty} \kappa_{m-j}\lambda^{m-j} \in F_m[1,+\infty) \tag{3}$$

such that there exists a constant C such that

$$|\kappa_{m-j}| \leqslant C^{j+1} (j!)^{\theta}, \qquad \forall j \in \mathbb{N} \tag{4}$$

2. Let $F_m^{\sigma,\theta}(\mathbb{R}^n \times [1,+\infty))$ be the set of all

$$\sum_{j=0}^{\infty} p_{m-j}(x)\lambda^{m-j} \in F_m(\mathbb{R}^n \times [1,+\infty)), \tag{5}$$

such that, for every compact subset $K \subset \mathbb{R}^n$, there exists a constant C_K, such that

$$|\partial^{\alpha} p_{m-j}(x)| \leqslant C_K^{|\alpha|+j+1} (\alpha!)^{\sigma} (j!)^{\theta}, \qquad \forall \alpha \in \mathbb{N}^n, \forall x \in K. \tag{6}$$

3. Let $F_m^{\rho,\sigma,\theta}(\mathbb{R}^n \times [1,+\infty))$ be the set of all

$$\sum_{j=0}^{\infty} p_{m-j}(x,q)\lambda^{m-j} \in F_m(\mathbb{R}^n \times [1,+\infty)), \tag{7}$$

such that, for every compact subset $K \subset \mathbb{R}^n$, there exists a constant C_K, such that

$$\left|\partial_x^{\alpha} \partial_q^{\beta} p_{m-j}(x,q)\right| \leqslant C_K^{|\alpha|+|\beta|+j+1} (\alpha!)^{\sigma} (\beta!)^{\rho} (j!)^{\theta}, \qquad \forall \alpha \in \mathbb{N}^n, \forall x \in K. \tag{8}$$

We observe that setting $\varepsilon = \lambda^{-1}$ we obtain that θ coincides with the Gevrey index for the formal Gevrey power series.

We emphasize that there are essentially two major issues related to the study of the asymptotics of oscillatory integrals $I(\lambda)$, $\lambda \gg 1$ as above. The first goal is to derive a formal asymptotic expansion

$$I(\lambda) \sim \sum_{j=0}^{\infty} I_j \lambda^{\mu - j\nu}, \qquad I_j \in \mathbb{C},\ \lambda \gg 1 \tag{9}$$

for some $\mu \in \mathbb{R}$, $\nu > 0$, and secondly, to study the Gevrey character of the formal series above.

Secondly, one deals with the remainder, namely we investigate, roughly speaking, the rate of decay for $\lambda \to +\infty$ of

$$R_N(\lambda) := I(\lambda) - \sum_{j=0}^{N} I_j \lambda^{\mu - j\nu}, \qquad N \in \mathbb{N},\ \lambda \gg 1 \tag{10}$$

Again the Gevrey classes are a natural framework for estimating the type of decay of $R_N(\lambda)$.

The main result we will prove is the following universal formula for the remainder:

$$\left| I_g^k(\lambda) - \sum_{m=0}^{\infty} I_m \lambda^{-\frac{1}{k}-\frac{m}{k}} \right| \leqslant A e^{-a\lambda^{-\frac{1}{k(\sigma-1)+1}}}, \tag{11}$$

where σ is the Gevrey class of the amplitude g and φ has the first k derivatives equal to zero.

2 Morse critical points for the phase

2.1 Asymptotic expansions and remainder

The main result of this chapter is on the loss

$$p \in F^{\sigma,\theta} \qquad \Rightarrow \qquad I \in F^{\max\{2\sigma-1,\theta\}}$$

of formal Gevrey regularity for the stationary phase method (cf. [6]) when the phase function is nonanalytic.

Note that when p is non analytic, *i.e.* $\sigma > 1$, then $2\sigma - 1 > \sigma$, so the Gevrey character of I is strictly greater (*worse*). Otherwise, when p is analytic, there is no degeneracy, because $2\sigma - 1 = 2 \cdot 1 - 1 = 1 = \sigma$.

We decouple the influence of the formal Gevrey character on the well known formal series appearing in SPM.

THEOREM 1. *Let* $\varphi(x) \in G^{\sigma}(\mathbb{R}^n, \mathbb{R})$ *and* $p(x,\lambda) \sim \sum p_j(x)\lambda^{-j} \in F_m^{\sigma,\theta}(\mathbb{R}^n, \mathbb{R})$. *Suppose there is a unique critical point* x_0 *for* φ *and suppose Morse non degeneracy,* i.e. $\exists! x_0, \nabla\varphi(x_0) = 0$, *and* $\nabla^2\varphi(x_0) \neq 0$. *Then*

$$I(\lambda) = \int e^{i\lambda\varphi(x)} p(x,\lambda)dx = e^{i\lambda\varphi(x_0)} q(\lambda), \tag{12}$$

$$\textit{where} \quad q(\lambda) \sim \sum_{k=0}^{\infty} q_{-\frac{n}{2}+m-k} \lambda^{-\frac{n}{2}+m-k} \in F_{m-n/2}^{\max\{2\sigma-1,\theta\}}([1,+\infty)) \tag{13}$$

where, as it is well known from the complete asymptotic expansion for the stationary phase method

$$q_{-\frac{n}{2}+m-k} = \frac{e^{i(\frac{\pi}{4})\operatorname{sgn} Q}}{\det Q} \sum_{j+s=k} \frac{(2i)^{-s}}{s!} \left\langle Q^{-1}\frac{\partial}{\partial y}, \frac{\partial}{\partial y} \right\rangle^{s} \times$$

$$\times \left(p_{m-j}(\kappa(y),\lambda) \left|\det \kappa'(y)\right| \right) \Big|_{y=0} \tag{14}$$

with $x = \kappa(y)$ *being the change of the variables transforming the phase function* $\varphi(x)$ *into* $\langle Qy, y\rangle$

Proof. By a generalization of the Morse lemma in G^{σ} classes (see Appendix 3.2 for the proof) there exists an appropriate G^{σ} change of variables $x = \kappa(y) \in G^{\sigma}$, with respect to which the phase function $\kappa^{*}\varphi(y) = \varphi(\kappa(y))$ becomes a quadratic form $\langle Qy, y\rangle$.

The new amplitude $\tilde{p}(y, \lambda)$ is defined by

$$\tilde{p}(y,\lambda) = p(\kappa(y),\lambda)\left|\det \kappa'(y)\right|.$$

Thus we can write,

$$q(\lambda) = \int e^{i\lambda \frac{\langle Qy,y\rangle}{2}} \tilde{p}(y,\lambda)dx,$$

where $\tilde{p}$ is still a formal $G^{\sigma,\theta}$ symbol.

According to the well known formula,

$$q_{-\frac{n}{2}+m-k} = \frac{e^{i(\frac{\pi}{4})\operatorname{sgn} Q}}{\det Q} \sum_{j+s=k} \frac{(2i)^{-s}}{s!} \left\langle Q^{-1}\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right\rangle^{s} \tilde{p}_{m-j}(y)\Big|_{y=0}.$$

The degree of derivation to which $\tilde{p}_{m-j}(y)$ is subject is $2s$, thus, by the inequality (4),

$$\begin{aligned}\left|q_{m-\frac{n}{2}-k}\right| &\leqslant A \sum_{j+s=k} \frac{1}{2^{s}}\frac{1}{s!}\left|\partial^{2s}\tilde{p}_{m-j}(0)\right| \leqslant \\ &\leqslant A \sum_{j+s=k} \frac{1}{2^{s}}\frac{1}{s!} C^{2s+j+1}\left((2s)!\right)^{\sigma}(j!)^{\theta} \approx \\ &\approx Ak \max_{j+s=k} \frac{1}{2^{s}}\frac{1}{s!} C^{2s+j+1}(s)!^{2\sigma}(j!)^{\theta} \approx \\ &\approx \left(\tilde{C}\right)^{2k+1}(k!)^{\max\{2\sigma-1,\theta\}}, \quad (15)\end{aligned}$$

and the expression above yields the end of the proof. □

Osservazione 1. One can introduce Banach spaces of formal $G^{\sigma,\theta}$ symbols with norms of the type

$$\|p\|_{\sigma,\theta;T} := \sum_{\alpha\in\mathbb{Z}^{n}_{+}, j\in\mathbb{Z}_{+}} \frac{T^{|\alpha|+j}}{(\alpha!)^{\sigma}(j!)^{\theta}} \sup_{x\in K}\left|\partial^{\alpha}p_{m-j}(x)\right| \quad (16)$$

and, after precise combinatorial estimates via the Stirling formula, consider SPM with fixed phase as a linear operator acting between two Banach spaces with formal symbols. Such results, a part of the theoretical value in itself might be useful in future investigations. Motivations for such approach are based on results for singular PDE in Gevrey spaces, divergent Gevrey formal power series in Dynamical Systems.

We point out that if $\sigma = \theta$ we recover the loss of $\sigma - 1$ Gevrey regularity studied in [6, 2].

2.2 Gevrey anisotropy for the amplitude

Let Q be a symmetric non–degenerate real matrix in $\mathbb{R}^{2n}$ with signature type (n, n). The next theorem shows that we may reduce the loss of Gevrey smoothness if we impose additional regularity in anisotropic Gevrey spaces $G^{\sigma,\rho}$. Let $\sigma, \rho \geqslant 1$. We define $G^{\sigma,\rho;Q}(\mathbb{R}^{2n})$ as the set of all $g \in C^\infty(\mathbb{R}^{2n})$ such that there exist $S \in GL(2n; \mathbb{R})$ satisfying

$$S \circ Q \circ S^{-1} = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \tag{17}$$

and the function $S^*g(z) = g(Sz)$ satisfies the following σ, ρ anisotropic Gevrey estimates: for every $K \subset\subset \mathbb{R}^{2n}$ there exists $C > 0$ such that

$$\sup_{(x,y)\in K} |\partial_x^\alpha \partial_y^\beta S^*g(x, y)| \leqslant C^{|\alpha|+|\beta|+1}(\alpha!)^\sigma(\beta!)^\rho, \qquad \alpha, \beta \in \mathbb{Z}_+^n \tag{18}$$

We note that $G^{\sigma,\rho;Q}(\mathbb{R}^{2n}) \subset G^{\max\{\sigma,\rho\}}(\mathbb{R}^{2n})$.

Setting $f(x, y) := S^*g(x, y)$, let us estimate its multi–derivative with respect to $\gamma = (\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n$,

$$\begin{aligned} \left|\left(\frac{\partial}{\partial(x,y)}\right)^\gamma f\right| = \left|\left(\frac{\partial}{\partial x}\right)^\alpha \left(\frac{\partial}{\partial y}\right)^\beta f\right| &\leqslant C^{|\alpha|+|\beta|+1}(\alpha!)^\sigma(\beta!)^\rho \leqslant \\ &\leqslant C^{|\alpha|+|\beta|+1}(\alpha!\beta!)^{\max\{\sigma,\rho\}} = C^{|\gamma|+1}(\gamma!)^{\max\{\sigma,\rho\}}, \end{aligned}$$

because, plainly, $\gamma! = \alpha!\beta!$ and $|\gamma| = |\alpha|+|\beta|$. So f is in $G^{\max\{\rho,\sigma\}}$, and as a result also g is, being S an analytical (linear!) diffeomorphism.

The next result shows that the $G^{\sigma,\rho;Q}(\mathbb{R}^{2n})$ anisotropic regularity of the amplitude leads to an improvement of the Gevrey index of the asymptotic expansion.

THEOREM 2. *Let Q be a $2n \times 2n$ non–degenerate real symmetric matrix with signature type (n, n). Let*

$$a \in G_0^{\sigma,\rho;Q}(\mathbb{R}^{2n}). \tag{19}$$

Then

$$I(\lambda) = \int e^{i\lambda\langle Qz,z\rangle} a(z)dz \sim \sum_{k=0}^\infty a_{-n-k}\lambda^{-n-k} \in F_{-n}^{\sigma+\rho-1}(\mathbb{R}^{2n}). \tag{20}$$

Proof. The proof follows from the fact that

$$\langle Qz, z\rangle = xy, \qquad z = (x, y) \in \mathbb{R}^{2n},$$

the explicit formulas for the asymptotic expansion and the assumption $a \in G_0^{\sigma,\rho;Q}(\mathbb{R}^{2n})$. Indeed, the form of Q implies that

$$\left\langle Q^{-1}\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right\rangle = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j \partial y_j}. \tag{21}$$

Next, in view of (19), we get

$$a_{-n-k} = \frac{e^{i(\frac{\pi}{4})n}}{(-1)^n}\frac{(2i)^{-k}}{k!}\left(\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j \partial y_j}\right)^k S^* a(x, y)\Big|_{(x,y)=(0,0)}.$$

which allows us to estimate in the following way

$$\begin{aligned} |a_{-n-k}| \leqslant C_0 \frac{1}{2^k}\frac{1}{k!}\left|\partial_x^k \partial_y^k S^* a(x, y)\right|\Big|_{(x,y)=(0,0)} \leqslant \\ \leqslant C_0 \frac{1}{2^k}\frac{1}{k!} C^{2k+1} (k!)^\sigma (k!)^\rho \approx \\ \approx C_0 \tilde{C}^{k+1} (k)!^{\sigma+\rho-1}, \end{aligned} \tag{22}$$

as desired.

□

3 The case of degenerate phase

Our aim is to give a rather complete asuymptotic analysis from the Gevrey point of view of oscillatory integrals of the type

$$I_g^k(\lambda) := \int_{\mathbb{R}} \exp\left(it^k \lambda\right) g(t) dt, \qquad g \in G_0^\sigma(\mathbb{R}), \tag{23}$$

where $k = 1, 2, \dots,$ and $\lambda \in \mathbb{R}$ is considered as a large parameter $\lambda \to \infty$.

As a result, we will be able to analyze more general oscillatory integrals with arbitrary G^ρ phase,

$$I_g^\varphi(\lambda) := \int_{\mathbb{R}} \exp\left(i\varphi(t)\lambda\right) g(t) dt, \qquad \varphi \in G^\rho(\mathbb{R}), \qquad g \in G_0^\sigma(\mathbb{R}), \tag{24}$$

applying partitions of unity, changes of coordinates and other results on preservation of Gevrey classes. In fact, integrals as (23) locally represent every possible situation which can occur with oscillatory integrals: lack of critical points (k=1), Morse critical points (k=2) and degenerate critical points ($k \geqslant 3$). In the optics interpretation, these situations are represented respectively by darkness, rays and caustics.

We will give a formal asymptotic expansion of (23) and study its Gevrey character with respect to which of g. We will deduce a universal formula for the loss of Gevrey regularity. Next we will study the decay of the remainder and prove a general exponential trend which was already found for the Morse case (k=2).

Consider first the G^σ Mac Laurin expansion of $g(t)$,

$$g(t) \sim \sum_{\alpha=0}^{\infty} \frac{g^{(\alpha)}(0)}{\alpha!} t^\alpha, \qquad \left| g^{(\alpha)}(0) \right| \leqslant C^{\alpha+1} (\alpha!)^\sigma.$$

Substituting into (23), we, formally, get

$$I_g^k(\lambda) = \int_{\mathbb{R}} e^{it^k\lambda} g(t) dt \sim \sum_{\alpha=0}^{\infty} \frac{g^{(\alpha)}(0)}{\alpha!} \int_{\mathbb{R}} e^{it^k\lambda} t^\alpha dt. \tag{25}$$

The integrals of the form $\int e^{it^k\lambda} t^\alpha dt$, are singular for $\lambda = 0$, and does not exists in a classical sense. Nevertheless, they can be defined as homogeneous distributions in Hörmander sense,

$$I^{k,\alpha}(\lambda) = \int e^{it^k\lambda} t^\alpha dt := \lim_{\delta \to 0} \int e^{it^k\lambda} t^\alpha \psi(\delta t) dt,$$

where ψ is a suitable compact supported function $\psi \equiv 1$ in a neighborhood of 0. As distributions, they coincide, for $\lambda \neq 0$, with a homogeneous function of λ.

For $k = 1$, we recall a well known homogeneity result.

Lemma 1. *If $u \in \mathcal{D}'(\mathbb{R})$ is a homogeneous distribution of order μ, then its Inverse Fourier Transform*

$$\hat{u}(\lambda) = \int_{\mathbb{R}} \exp(ix\lambda)\, u(x) dx$$

is homogeneous of degree $-1-\mu$.

Proof. Indeed, this is evident by the change of the variable

$$\begin{aligned} \hat{u}(s\lambda) &= \int_{\mathbb{R}} \exp(ixs\lambda)\, u(x) dx \\ &= s^{-1} \int_{\mathbb{R}} \exp(ix\lambda)\, u(xs^{-1}) dx \\ &= s^{-1-\mu} \int_{\mathbb{R}} \exp(ix\lambda)\, u(x) dx. \end{aligned} \tag{26}$$

for every $s > 0$. □

The cases in which $k > 1$ can be reconduced to this by means of suitable changes of variable.

Proposition 1. *Let k odd, then*

$$I^{k,\alpha}(\lambda) = \begin{cases} \frac{1}{k} C^{k,\alpha}_{odd} \lambda^{-\frac{\alpha+1}{k}}, & \alpha = mk + p - 1, \quad p = 1, \dots, k-1, \\ 0, & \alpha = mk - 1 \quad m \in \mathbb{Z}_+. \end{cases}$$

Proof. Being k odd, we can perform the following Holder-continuous change of coordinates: $t^k = r, t = r^{\frac{1}{k}}, dt = \frac{1}{k} r^{\frac{1}{k}-1} dr$, so

$$\int_{\mathbb{R}} e^{it^k \lambda} t^\alpha dt = \frac{1}{k} \int_{\mathbb{R}} e^{ir\lambda} r^{\frac{\alpha+1}{k}-1} dr = \frac{1}{k} C \lambda^{-\frac{\alpha+1}{k}}.$$

Recall that this integral is zero whenever $\frac{\alpha+1}{k} \in \mathbb{Z}_+$, because it coincides with one of the derivatives of the Dirac delta distribution. (which are all homogeneous distributions, being derivatives of the Heaviside distribution, which is 0-degree homogeneous, and are all zero for $\lambda \neq 0$.) □

Proposition 2. *Let k even,* i.e. *$k = 2l$, then*

$$I^{2l,2\beta}(\lambda) = \frac{1}{l} C^{l,\beta}_{even} \lambda^{-\frac{\beta}{l}-\frac{1}{2l}}, \qquad \forall \beta \in \mathbb{Z}_+$$

$$I^{2l,2\beta+1}(\lambda) = 0.$$

Proof. We cannot perform a global change of variable here, nevertheless,

$$\int_{\mathbb{R}} e^{it^{2l}\lambda} t^\alpha dt = \int_{-\infty}^0 + \int_0^{+\infty} = \begin{cases} 2\int_0^\infty, & \alpha \text{ even} \\ 0 & \alpha \text{ odd} \end{cases}.$$

So the nontrivial case is $\alpha = 2\beta$.

$$I^{2l,2\beta} = 2 \int_0^\infty e^{it^{2l}\lambda} t^{2\beta} dt = 2 \frac{1}{2l} \int_0^\infty e^{ir\lambda} r^{\frac{\beta}{l}+\frac{1}{2l}-1} dt = \\ = \frac{1}{l} \int_{\mathbb{R}} e^{ir\lambda} H(r) r^{\frac{\beta}{l}+\frac{1}{2l}-1} dt,$$

where $H(r)$ is the Heaviside function. So we are still considering the inverse Fourier transform of a homogeneous distribution, which is a Schwartz distribution again coinciding with an homogeneous function of λ, when $\lambda \neq 0$. □

We give now an estimate on the coefficients found above: $C^{k,\alpha}_{odd}$ and $C^{l,\beta}_{even}$, when k, l are considered as fixed, while α, β are spanning all $\mathbb{Z}_+$.

LEMMA 2. *Let k odd, $p = 1, \dots, k-1$. Let $C^{k,\alpha}_{odd}$ as in Proposition 1. Then there exist $A_1, A_2 > 0$ such that*

$$A_1 < \left| \frac{C^{k,mk+p-1}_{odd}}{m!} \right|^{\frac{1}{m}} < A_2. \tag{27}$$

Proof. For $\lambda \neq 0$ we can write,

$$H^{k,p-1}(\lambda) := \int_{\mathbb{R}} e^{it^k\lambda} t^{p-1} dt = \frac{1}{k} C^{k,p-1} \lambda^{-\frac{p}{k}}.$$

Differentiating the left side, we get

$$\frac{\partial}{\partial\lambda} \int_{\mathbb{R}} e^{it^k\lambda} t^{p-1} dt = i \int_{\mathbb{R}} e^{it^k\lambda} t^{k+p-1} dt,$$

so, differentiating m times,

$$\left(\frac{\partial}{\partial\lambda}\right)^m \int_{\mathbb{R}} e^{it^k\lambda} t^{p-1} dt = i^m \int_{\mathbb{R}} e^{it^k\lambda} t^{mk+p-1} dt) = i^m H^{k,mk+p-1}(\lambda).$$

On the other hand, differentiating the right side m times, we get

$$\left(\frac{\partial}{\partial\lambda}\right)^m \frac{1}{k} C^{k,p-1} \lambda^{-\frac{p}{k}} =$$
$$= \left(-\frac{p}{k}\right)\left(-1-\frac{p}{k}\right)\dots\left(-(m-1)-\frac{p}{k}\right)\frac{1}{k} C^{k,p-1} \lambda^{-m-\frac{p}{k}}.$$

Equating both sides,

$$i^m \frac{1}{k} C^{k,mk+p-1} \lambda^{-\frac{mk+p}{k}} =$$
$$= (-1)^m \left(\frac{p}{k}\right)\left(1+\frac{p}{k}\right)\dots\left((m-1)+\frac{p}{k}\right)\frac{1}{k} C^{k,p-1} \lambda^{-m-\frac{p}{k}}.$$

As a result,

$$\left|C^{k,mk+p-1}\right| = \left|C^{k,p-1}\right| \left((m-1)+\frac{p}{k}\right)!$$

thus,

$$\inf_p \left|C^{k,p-1}\right| (m-1)! < \left|C^{k,mk+p-1}\right| < \sup_p \left|C^{k,p-1}\right| m!$$

Dividing by $m!$ and extracting the m-th root the thesis follows. $\square$

LEMMA 3. *Let $k = 2l$, $l \in \mathbb{Z}_+$. Let $C^{l,\beta}_{even}$ as in Proposition 2. Then there exist $B_1, B_2 > 0$ such that*

$$B_1 < \left| \frac{C^{l,\beta}_{even}}{\beta!} \right|^{\frac{l}{\beta}} < B_2. \tag{28}$$

Proof. Setting

$$K^{l,\beta}(\lambda) := \int_{\mathbb{R}} e^{it^{2l}\lambda} t^{2\beta} dt = \frac{1}{l} C^{l,\beta} \lambda^{-\frac{\beta}{l} - \frac{1}{2l}},$$

the same argument of Lemma 2 works with minor changes. □

COROLLARY 1. *There exists $\bar{C}_1 > 0$ s.t.*

$$\left| C^{k,\alpha}_{odd} \right| \leqslant \bar{C}_1^{\alpha} (\alpha!)^{\frac{1}{k}}, \qquad \forall \alpha \in \mathbb{Z}_+,$$

and the estimate is sharp.

COROLLARY 2. *There exists $\bar{C}_2 > 0$ s.t.*

$$\left| C^{l,\beta}_{even} \right| \leqslant \bar{C}_2^{\beta} (\beta!)^{\frac{1}{l}}, \qquad \forall \alpha \in \mathbb{Z}_+,$$

and the estimate is sharp.

3.1 Asymptotic expansions

THEOREM 3. *Let k odd, $g \in G^{\sigma}_0(\mathbb{R})$. Then*

$$I^k_g(\lambda) \asymp \frac{1}{k} \sum_{p=1}^{k-1} \lambda^{-\frac{p}{k}} \sum_{m=0}^{\infty} \frac{g^{(mk+p-1)}(0)}{(mk+p-1)!} C^{k,mk+p-1}_{odd} \lambda^{-m} \qquad \in F^{k(\sigma-1)+1}_{-\frac{1}{k}}[1, +\infty).$$

Proof. Considering the Mac Laurin expansion of g and recalling Proposition 1,

$$\begin{aligned} I^k_g(\lambda) &\asymp \sum_{\alpha=0}^{\infty} \frac{g^{(\alpha)}(0)}{\alpha!} \int_{\mathbb{R}} e^{it^k\lambda} t^{\alpha} dt = \sum_{\alpha=0}^{\infty} \frac{g^{(\alpha)}}{\alpha!} \frac{1}{k} C^{k,\alpha}_{odd} \lambda^{-\frac{\alpha+1}{k}} = \\ &= \frac{1}{k} \sum_{p=1}^{k-1} \sum_{m=0}^{\infty} \frac{g^{(mk+p-1)}(0)}{(mk+p-1)!} C^{k,mk+p-1}_{odd} \lambda^{-\frac{mk+p}{k}}. \end{aligned}$$

The expansion can be considered as the sum of $k-1$ series with dominant term $\lambda^{-\frac{p}{k}}$. Let us estimate the coefficients. Being $g \in G^\sigma$ and recalling Lemma 2,

$$\left|\frac{g^{(mk+p-1)}(0)}{(mk+p-1)!} C_{odd}^{k,mk+p-1}\right| \leqslant$$
$$\leqslant \frac{(mk+p-1)!^\sigma}{(mk+p-1)!} C^m (m!) \asymp C^m (m!)^{k(\sigma-1)+1},$$

as claimed. ☐

THEOREM 4. *Let $k = 2l$, $l \in \mathbb{Z}_+$. Then*

$$I_g^{2l}(\lambda) \asymp \frac{1}{l} \sum_{q=0}^{l-1} \lambda^{-\frac{q}{l}-\frac{1}{2l}} \sum_{m=0}^{\infty} \frac{g^{2(ml+q))}(0)}{(2(ml+q))!} C_{even}^{l,ml+q} \lambda^{-m} \qquad \in F_{-\frac{1}{2l}}.$$

Proof. Recalling Proposition 2 we write

$$I_g^{2l}(\lambda) \asymp \frac{1}{l} \sum_{\beta=0}^{\infty} \frac{g^{(2\beta)}(0)}{\beta} C_{even}^{l,\beta} \lambda^{-\frac{\beta}{l}-\frac{1}{2l}} =$$
$$= \frac{1}{l} \sum_{q=0}^{l-1} \sum_{m=0}^{\infty} \frac{g^{(2(ml+q))}(0)}{(2(ml+q))!} C_{even}^{l,ml+q} \lambda^{-m-\frac{q}{l}-\frac{1}{2l}} =$$
$$= \frac{1}{l} \sum_{q=0}^{l-1} \lambda^{-\frac{q}{l}-\frac{1}{2l}} \sum_{m=0}^{\infty} \frac{g^{(2(ml+q))}(0)}{(2(ml+q))!} C_{even}^{l,ml+q} \lambda^{-m}.$$

The expansion can be considered as the sum of l series with dominant term $\lambda^{-\frac{q}{l}-\frac{1}{2l}}$. Let us estimate the coefficients. Recalling Lemma 3,

$$\left|\frac{g^{(2(ml+q))}(0)}{(2(ml+q))!} C_{even}^{l,ml+q}\right| \leqslant$$
$$\leqslant (2(ml+q))!^{\sigma-1} C^m ((ml+q)!^{\frac{1}{l}} \asymp C^m (m!)^{2l(\sigma-1)+1},$$

as claimed, and exactly as in Theorem 3, being $k = 2l$. ☐

Remark 3.1. Note that the loss of Gevrey regularity from σ of g to $k(\sigma-1)+1$ of I_g^k, amounts to $(k-1)(\sigma-1)$. For $k=1$ the loss is 0, while for $k=2$, the Stationary Phase case, equals $\sigma-1$, as already observed in [6].

3.2 The exponential estimate for the remainder

In this section we will prove a universal exponential decay result for the remainder of the expansion series of $I_g^k(\lambda)$:

$$\left| I_g^k(\lambda) - \sum_{m=0}^{\infty} I_m \lambda^{-\frac{1}{k}-\frac{m}{k}} \right| \leqslant A e^{-a\lambda^{-\frac{1}{k(\sigma-1)+1}}}. \tag{29}$$

Let us face first some preliminaires.

THEOREM 5. *There exist* $C_0, C_1 > 0$*, such that*

$$\left| I_N^R(\lambda) \right| := \left| I_g^k(\lambda) - \sum_{m=0}^{N-1} I_m \lambda^{-\frac{1}{k}-\frac{m}{k}} \right| \leqslant C_0 C_1^N (N!)^{k(\sigma-1)+1} \lambda^{-N}, \tag{30}$$

LEMMA 4. *If*

$$I \leqslant c^n (n!)^\theta \lambda^{-n}, \tag{31}$$

then there exists $A, a > 0$ *such that*

$$I \leqslant A e^{-a\lambda^{\frac{1}{\theta}}}.$$

Proof. Let us divide both sides of the hypothesis by $N!^{2\theta}$,

$$\frac{I}{N!^{2\theta}} \leqslant \frac{c^N \lambda^{-N}}{N!^\theta},$$

elevate to the power $\frac{1}{\theta}$

$$\frac{I^{\frac{1}{\theta}}}{N!^2} \leqslant \frac{c^{\frac{N}{\theta}} \lambda^{-\frac{N}{\theta}}}{N!},$$

and sum from 0 to ∞

$$\sum_{N=0}^{\infty} \frac{I^{\frac{1}{\theta}}}{N!^2} \leqslant \sum_{N=0}^{\infty} \frac{1}{N!} \left(\frac{\lambda}{c}\right)^{-\frac{N}{\theta}},$$

which gives, elevating finally to θ,

$$I \leqslant A e^{-a\lambda^{\frac{1}{\theta}}},$$

where $A = \frac{1}{J_B(0,2)^\theta}$, $a = \theta c^{-\frac{1}{\theta}}$ and J_B is the Bessel J function. □

THEOREM 6. *Let $g \in G_0^\sigma(\mathbb{R})$ be flat in 0. Then*

$$\left| I_g^k(\lambda) \right| \leqslant A e^{-a\lambda^{\frac{1}{k(\sigma-1)+1}}}. \tag{32}$$

Proof. Easy corollary of the Theorem 5 and of the Lemma 4. $\square$

Remark 3.2. The thesis of Theorem 6 still holds for non flat g, but also when

[$k = 1$] It suffices $g \in G_0^\sigma(\mathbb{R})$.

[k **even**] $g^{(2\alpha)}(0) = 0, \qquad \forall \alpha \in \mathbb{Z}_+$.

[k **odd**] $g^{(km+p-1)}(0) = 0, \qquad \forall k \in \mathbb{Z}_+, \quad p = 1, \dots, k-1$..

Proof. It is a direct consequence of Propositions 1 and 2. $\square$

Sketch of the proof of Theorem 5. The integral can be split in two parts:

$$I_g^k(\lambda) = \sum_{\alpha=0}^{N} \frac{g^{(\alpha)}(0)}{\alpha!} \int_{\mathbb{R}} e^{it^k\lambda} t^\alpha dt + \int_{\mathbb{R}} e^{it^k\lambda} \frac{g^{(N+1)}(\theta t)}{(N+1)!} t^{N+1} dt = I_N^0 + I_N^R$$

The estimate of the remainder gives:

$$\begin{aligned}
I_N^R(\lambda) &= \int_{\mathbb{R}} e^{it^k\lambda} \frac{g^{(N+1)}(\theta t)}{(N+1)!} t^{N+1} dt = \frac{1}{k} \int_{\mathbb{R}} e^{ir\lambda} \frac{g^{(N+1)}(\theta r^{\frac{1}{k}})}{(N+1)!} r^{\frac{N-k}{k}} dr = \\
&\overset{IPP}{=} \frac{1}{k} \frac{1}{(N+1)!} \left(-\frac{1}{i\lambda} \right) \int_{\mathbb{R}} e^{ir\lambda} \frac{d}{dr} \left(r^{\frac{N-k}{k}} g^{(N+1)}(\theta r^{\frac{1}{k}}) \right) dr = \cdots = \\
&\overset{IPP}{=} \frac{1}{k} \frac{1}{(N+1)!} \left(-\frac{1}{i\lambda} \right)^m \int_{\mathbb{R}} e^{ir\lambda} \left(\frac{d}{dr} \right)^m \left(r^{\frac{N-k}{k}} g^{(N+1)}(\theta r^{\frac{1}{k}}) \right) dr \asymp \\
&\asymp N!^{k(\sigma-1)+1} \lambda^{-N}.
\end{aligned}$$

$\square$

Appendix: The Morse lemma for Gevrey functions

Lemma 5 (Morse lemma). *Let $f(x) : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a C^∞ function such that*

$$\begin{cases} f(0) = 0, \\ \nabla f(0) = 0, \\ \nabla^2 f(0) \equiv A \in M(n \times n, \mathbb{R}), \qquad \det A \neq 0. \end{cases} \tag{33}$$

Then there exists local coordinates $y = y(x)$ in a neighborhood of 0 *with respect to which*

$$f(x(y)) = \frac{1}{2} \langle Ay, y \rangle .$$

Proof. We can rewrite f as a quadratic form with non constant coefficients applying twice the Hadamard's lemma:

$$\varphi(x) = \frac{1}{2} \langle B(x)x, x \rangle = \sum_{j,k=1,\dots,n} \frac{1}{2} b_{jk}(x) x_j x_k, \tag{34}$$

where

$$b_{jk} = 2 \int_0^1 (1-t) \frac{\partial^2 f}{\partial x_j \partial x_k}(tx) dt. \tag{35}$$

Let us consider the function

$$\begin{aligned} B : \mathbb{R}^n &\longrightarrow M(n \times n, \mathbb{R}), \\ x &\longmapsto (b_{jk})(x), \end{aligned}$$

for which

$$B(0) = A = \left(\frac{\partial^2 f}{\partial x_j \partial x_k} \right).$$

We are looking for a change of coordinates $y = y(x) = R(x)x$ with respect to which the matrix $\langle B(x(y))x(y), x(y) \rangle \equiv \langle Ay, y \rangle$ for all y in a neighborhood of 0, *i.e.*

$$\begin{aligned} y = R(x)x, \qquad & R(x)^T A R(x) = B(x), \\ R(0) = \mathbb{I}, \qquad & R(x) \in M(n \times n, \mathbb{R}). \end{aligned}$$

Indeed, with respect such coordinates,

$$\begin{aligned} f(x) = f(x) = \langle B(x)x, x \rangle = \left\langle R(x)^T A R(x) x, x \right\rangle = \\ = \langle AR(x)x, R(x)x \rangle = \langle Ay, y \rangle . \end{aligned}$$

Now, the existence of such an $R(x)$ is assured if the map

$$\begin{aligned} \mathrm{Sym}(n \times n, \mathbb{R}) &\longrightarrow \mathrm{Sym}(n \times n, \mathbb{R}), && (36) \\ R &\longmapsto R^T A R, && (37) \end{aligned}$$

is an isomorphism, and we will prove its invertibility by the surjectivity of its differential in $x = 0, R = \mathbb{I}$. Differentiating (36) then, we obtain

$$d[R^T A R](S)\Big|_{\substack{x=0 \\ R=\mathbb{I}}} = \\ = \frac{d}{d\lambda}\left((R + \lambda S)^T A (R + \lambda S)\right)\Big|_{\substack{\lambda=0 \\ x=0 \\ R=\mathbb{I}}} = S^T A + A S, \quad (38)$$

which clearly is surjective, for every symmetric matrix C is image of $S = \frac{A^{-1}C}{2}$. □

Osservazione 2. Moreover, the invertible map (36) is polynomial in the entries of R, thus its inverse, say $F(B(x)) = F(R^T(x)AR(x)) = R(x)$ is analytic in the entries of B. We obtain then $y(x) = R(x)x$ is a G^σ change of coordinates whenever the Morse function $f(x)$, and by consequence $B(x)$, as apparent from (35), is G^σ.

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