Phragmèn-Lindelöf Principles for Nonlinear Elliptic Equations

Abstract

This paper contains Phragmèn-Lindelöf type results for viscosity solutions of fully nonlinear second-order uniformly elliptic equations with superlinear gradient term in a wide class of unbounded domains. Under suitable assumptions on the coefficients, as classically, we show that the Maximum Principle holds in a generalized version of cylindrical and conical domains, resp., for subsolutions with exponential and polynomial growth at infinity.

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1 Introduction and main results

A classical result concerning the Laplace operator is following maximum principle (MP) in a two-dimensional cone Σ, based on the Phragmèn-Lindelöf theory, see [26]. Let Γ be an open arc of length θ, possibly $\theta = 2\pi$, on the unit circumference S^1 and $\Sigma = \{\rho s \mid \rho > 0,\ s \in \Gamma\}$: if $\alpha = \frac{\pi}{\theta}$, then

$$\Delta u \geq 0 \text{ in } \Sigma,\ u \leq 0 \text{ on } \partial\Sigma,\ u = o(|x|^\alpha) \ \Rightarrow\ u \leq 0 \text{ in } \Sigma. \tag{1.1}$$

This result has been generalized to higher dimensions by Berestycki-Caffarelli-Nirenberg [3]. Let Γ be a sufficiently regular open subset of the unit spherical surface S^{n-1} such that $\overline{\Gamma} \neq S^{n-1}$, Σ be defined as before and $\lambda > 0$ the principal eigenvalue of the Laplace-Beltrami operator Δ_S on Γ with Dirichlet boundary conditions

$$\begin{cases} -\Delta_S \varphi = \lambda\varphi,\ \varphi > 0 & \text{in } \Gamma \\ \qquad \varphi = 0 & \text{on } \partial\Gamma \end{cases},$$

then (1.1) holds for $\alpha > 0$ such that $\alpha(n + \alpha - 2) = \lambda$.

An explicit formula for the admissible growth exponent α was also established by Oddson [25] and Miller [24], respectively, in dimension $n = 2$ and $n \geq 3$, when more general second-order uniformly elliptic operator in nondivergent form $Lu = a_{ij}D_{ij}u$ is considered instead of Δu. It is also worth to mention that $\alpha = 1$ in the case of the half-plane in dimension $n = 2$ and of the half-space in dimension $n \geq 3$, as shown by Gilbarg and Hopf, respectively, in their pioneeristic works [16] and [18]. For a general survey on the topics we refer to [26], [21] and [23].

Here we will consider more general unbounded domains Ω, called **wG**-domains, which are characterized by a measure-geometric property depending on a real positive parameter $\sigma < 1$: for all $y \in \Omega$ there exists a ball B such that

$$y \in B, \quad |B \backslash \Omega_y| \geq \sigma |B|, \tag{1.2}$$

where Ω_y is the (connected) component of $B \backslash \partial\Omega$ containing y.

This is an extension of the **G** condition of Cabrè [7], which is based on an idea of Berestycki-Nirenberg-Varadhan [5].

Thus we can consider domains Ω in $\mathbb{R}^2$ between the graphs of two continuous functions with sub-linear growth, defined on a half-line, and similar domains in higher dimension $n > 2$, defined on a $2^{-(n-1)}$-hyperspace lying between two hypersurfaces. But, as we will see, we can also consider larger domains, as the complement of only one of such graphs, see Section 2 below.

Since no regularity is required to the domains, no optimal barrier functions are available in general. Therefore in this case we do not look for an explicit formula for α, but only for the existence of the admissible growth exponent $\alpha > 0$, i.e. a qualitative Phragmèn-Lindelöf principle.

We will be concerned with viscosity subsolutions of fully nonlinear equations

$$F(x, u, Du, D^2u) = 0 \tag{$F = 0$}$$

in Ω, where $F(x, t, \xi, X) = P(x, t, \xi, X) + H(x, t, \xi)$ satisfies the structure conditions

$$P(x, t, \xi, X) \leq \mathcal{P}^+_{\lambda,\Lambda}(X), \quad H(x, t, \xi) \leq b_1(x)|\xi| + b_2(x)|\xi|^q + c(x)t, \tag{1.3}$$

for $(x, t, \xi, X) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^n \times \mathcal{S}^n$. Here $\mathcal{S}^n$ is the set of the real symmetric $n \times n$ matrices endowed with the partial ordering induced by the positive semidefiniteness, $\mathcal{P}^+_{\lambda,\Lambda}$ is the maximal Pucci operator, $1 \leq q \leq 2$, while $b_1(x)$, $b_2(x)$ and $c(x)$ are non-negative continuous functions. We outline that if P is linear, then the first of structure conditions (1.3) implies that P is uniformly elliptic, with positive ellipticity constants $\lambda \leq \Lambda$, i.e.

$$\lambda \mathrm{Tr}(Y) \leq P(x, t, \xi, X + Y) - P(x, t, \xi, X) \leq \Lambda \mathrm{Tr}(Y), \quad \text{for } Y \geq 0, \tag{1.4}$$

where Tr denotes the trace operator, but this is no more true in general when P is not linear, see e.g. [12] or [1]. Conversely, we recall that each unifomly elliptic operator is controlled from above and from below, resp., by two extremal uniformly elliptic operators, the Pucci minimal and maximal operators

$$\mathcal{P}^+_{\lambda,\Lambda}(X) = \Lambda \mathrm{Tr}(X^+) - \lambda \mathrm{Tr}(X^-) = \sup_{\lambda I \le A \le \Lambda I} \mathrm{Tr}(AX), \tag{1.5}$$

$$\mathcal{P}^-_{\lambda,\Lambda}(X) = \lambda \mathrm{Tr}(X^+) - \Lambda \mathrm{Tr}(X^-) = \inf_{\lambda I \le A \le \Lambda I} \mathrm{Tr}(AX), \tag{1.6}$$

where $X^\pm$ are the positive and negative part of X, which can be decomposed in an unique way as $X = X^+ - X^-$ with $X^\pm \ge 0$ and $X^+X^- = 0$. Other examples of fully nonlinear uniformly elliptic operators can be found in [17], [15], [10].
Here is our first main result.

Theorem 1.1 *Let $0 < \sigma < 1$, $R_0 \ge 0$, $\beta > 0$, $0 \le \gamma \le 1$, $0 < \lambda \le \Lambda$, $1 \le q \le 2$, $b_1' \ge 0$, $b_2' \ge 0$, $\overline{\alpha} > 0$ and N be real numbers. Assume that Ω is an unbounded* **wG***-domain such that for all y the ball B provided by (1.2) has radius*

$$R_y \le R_0 + \beta |y|^\gamma. \tag{1.7}$$

Suppose that F satisfy the structure conditions (1.3) with $b_1(x)$, $b_2(x)$ and $c(x)$ non-negative bounded continuous functions. If $u \in USC(\Omega)$ is a viscosity subsolution of the equation $F = 0$ in Ω, then there exists a positive constant c_0, depending on the dimension n and the above constants, such that, provided $0 \le \alpha \le \overline{\alpha}$ and $\alpha + c_1 \le c_0$:

- *for $\gamma > 0$ (parabolic shaped and conical domains), if*

$$b_1(x) \le \frac{b_1'}{1+|x|^\gamma}, \quad b_2(x) \le \frac{b_2'}{1+|x|^{\gamma(2-q)+\overline{\alpha}(q-1)}}, \quad c(x) \le \frac{c_1}{1+|x|^{2\gamma}}, \tag{1.8}$$

  we have

$$\limsup_{x \to \partial\Omega} u(x) \le 0, \quad \sup_{x\in\Omega} \frac{u(x)}{1+|x|^\alpha} \le N \quad \Rightarrow \quad u(x) \le 0 \ \text{ in } \Omega.$$

- *for $\gamma = 0$ (cylindrical domains), if*

$$b_1(x) \le b_1', \quad b_2(x) \le b_2' e^{-\overline{\alpha}(q-1)|x|}, \quad c(x) \le c_1, \tag{1.9}$$

  we have

$$\limsup_{x \to \partial\Omega} u(x) \le 0, \quad \sup_{x\in\Omega} \frac{u(x)}{e^{\alpha|x|}} \le N \quad \Rightarrow \quad u(x) \le 0 \ \text{ in } \Omega.$$

This says that the Maximum Principle holds in parabolic shaped and conical domains for viscosity solutions with polynomial growth and in cylindrical domains for viscosity solutions with exponential growth. In the case of cylindrical domains ($\gamma = 0$), similar results are stated for strong solutions of linear equations in [2] and [6], as well as in [28] in the case of conical domains ($\gamma = 1$). In [13], [14] analogous results have been obtained for viscosity solutions of fully nonlinear equations with linear and quadratic growth in the gradient ($q = 1$ and $q = 2$). Here also parabolic shaped domains ($0 < \gamma < 1$) are considered. But more general domains, which we call conical domains by components, can also be treated.

Theorem 1.2 *Let $0 < \sigma < 1$, $R_0 \geq 0$, $\beta > 0$, $0 < \lambda \leq \Lambda$, $1 \leq q \leq 2$, $b_1' \geq 0$, $b_2' \geq 0$, $\overline{\alpha} > 0$ and N be real numbers. Assume that Ω is an unbounded conical domain by components with parameter σ, R_0 and β, see Definition 2.8 below. Suppose that F satisfy the structure conditions (1.3) with $b_1(x)$, $b_2(x)$ and $c(x)$ non-negative bounded continuous functions such that*

$$b_1(x) \leq \frac{b_1'}{1+|x|}, \quad b_2(x) \leq \frac{b_2'}{1+|x|^{(2-q)+\overline{\alpha}(q-1)}}, \quad c(x) \leq \frac{c_1}{1+|x|^2}. \tag{1.10}$$

If $u \in USC(\Omega)$ is a viscosity subsolution of the equation $F = 0$ in Ω, then there exists a positive constant c_0, depending on the dimension n and the above constants, such that, for $\alpha \in [0, \overline{\alpha}]$ such that $\alpha + c_1 \leq c_0$, we have

$$\limsup_{x \to \partial\Omega} u(x) \leq 0, \quad \sup_{x \in \Omega} \frac{u(x)}{1+|x|^\alpha} \leq N \quad \Rightarrow \quad u(x) \leq 0 \ \text{ in } \Omega.$$

This yields indeed Phragmèn-Lindelöf results in a wider class of domains than **wG**, for instance the cut plane and the complement of continuous semi-infinite graphs in $\mathbb{R}^2$ and similar hypersurfaces in $\mathbb{R}^n$. Different examples will be given in the next Sections.

The basic tool is a variant of ABP estimate, see Lemma 2.4 below, which holds in **wG**-domains. It is based on the boundary weak Harnack inequality, see Krylov-Safonov [22] and Trudinger [27], for viscosity solutions of fully nonlinear equations. Our version is provided by our earlier result, see [1], but some basic ideas are already contained in [9], [10] and other arguments in [20] and [19].

We also notice that in the above Theorems, according to the general Phragmèn-Lindelöf theory, see [26], the growth control can be prescribed on an increasing sequence of spherical shells. More precisely, in Theorems 1.1 and 1.2 we suppose

$$\sup_{x \in \Omega} \frac{u(x)}{\varphi(|x|)} \leq N,$$

where $\varphi(r) = 1 + r^\alpha$ or $\varphi(r) = e^{\alpha r}$. What we observe is that this condition can be weakened by requiring the existence of an increasing sequence of radii r_k such that $\lim_{k\to\infty} r_k = +\infty$ and

$$\lim_{k\to\infty} \sup_{x\in\Omega\cap S_k} \frac{u(x)}{\varphi(|x|)} \le N,$$

where $S_k = \partial B_{r_k}(0)$.
The paper is organized as follows: in Section 2 we fix the notations and rearrange in a suitable form the fundamental tools that we need; in Section 3 we deduce Maximum Principles when zero order coefficients are allowed to change sign; in Section 4 we prove our main results, i.e. the qualitative Phragmèn-Lindelöf Theorems 1.1 and 1.2.

2 Preliminaries and auxiliary results

Throughout this paper we will denote by $USC(\Omega)$ and $LSC(\Omega)$, resp., the sets of the upper and lower semicontinuous functions in Ω.

Definition 2.1 *A function $u \in USC(\Omega)$ is a viscosity subsolution of the equation $F = f$ (equivalently, a solution of $F \ge f$ in the viscosity sense) if*

$$F(x, u(x), D\varphi(x), D^2\varphi(x)) \ge f(x)$$

at any point $x \in \Omega$ and for all $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local maximum in x. In the above definition we may also assume that $\varphi(x) = u(x)$, i.e. the test function φ "touches" u from above.
Similarly, a function $u \in LSC(\Omega)$ is a viscosity supersolution of $F = f$ (equivalently, a solution of $F \le f$ in the viscosity sense) if

$$F(x, u(x), D\varphi(x), D^2\varphi(x)) \le f(x)$$

at any point $x \in \Omega$ and for all $\varphi \in C^2(\Omega)$ touching u from below.
Finally, a continuous function f is a viscosity solution (equivalently, a solution in the viscosity sense) if it is both a viscosity sub- and supersolution.

For a general account and useful readings on viscosity solutions of fully nonlinear second-order elliptic equations we refer to [15], [10] and [11].

Definition 2.2 *Let $0 < \sigma < 1$. We say that $y \in \mathbb{R}^n$ satisfies condition $\mathbf{G}_\sigma$ in Ω, a domain of $\mathbb{R}^n$, if there exists a ball B such that*

$$y \in B, \quad |B \backslash \Omega_y| \ge \sigma|B|, \tag{2.1}$$

where Ω_y is the (connected) component of $B\backslash\partial\Omega$ containing y.
We will call Ω a **wG***-domain (with parameter σ) if there exists a positive number $\sigma < 1$ such that all points $y \in \Omega$ satisfy condition $\mathbf{G}_\sigma$ in Ω.*

Remark 2.3 In the sequel we use the fact that, if Ω is a **wG**-domain and R_y is the radius of the ball provided by condition (2.1) at $y \in \Omega$, then there exists a ball B containing y which realizes the equality in (2.1), see [29]. We denote by r_y the radius of this ball, noticing that $r_y \leq R_y$. □

Now, we quote from [1] our basic tool, a variant of the Alexandroff-Bakelman-Pucci estimate, see Theorem 4.2 there. It is based on **wG**-condition. Regarding that, here below we will use the following notation. Let $B_{r_y} = B_{r_y}(z_y)$. For any $\rho, t > 0$ we will set $B_{tr_y} = B_{tr_y}(z_y)$ and $B_{\rho,tr_y} = B_{tr_y}(z_y)\backslash\overline{B}_\rho(0)$.

Lemma 2.4 *Let $0 < \sigma < 1$, $N > 0$ and $1 \leq q \leq 2$ be real numbers. Let Ω be a* **wG** *domain (with parameter σ). Suppose that $w \in USC(\bar{\Omega})$ is a viscosity solution of $F(x, w, Dw, D^2w) \geq f$ [1] , under the structure condition (1.3), with $c = 0$ and $b_1, b_2, f \in C(\bar{\Omega})$ such that*

$$\overline{b}_i := \sup_{y\in\Omega;\,|y|\geq\varepsilon r_y} r_y^{2-q_i}\|b_i\|_{L^\infty(\Omega\cap B_{\tau\varepsilon r_y,\tau' r_y})} < +\infty, \tag{2.2}$$

where $q_1 = 1$ and $q_2 = q$, for all $\varepsilon > 0$ small enough, all $\tau < 1$ sufficiently close to 1 and some $\tau' > 1$.
If $w \leq N$ in Ω and $w \leq 0$ on $\partial\Omega$, then

$$\sup_\Omega w \leq C \sup_{y\in\Omega;\,|y|\geq\varepsilon r_y} r_y\|f^-\|_{L^n(\Omega\cap B_{\tau\varepsilon r_y,\tau' r_y})} + \sup_{y\in\Omega;\,|y|\leq\varepsilon r_y} C_y\, r_y\|f^-\|_{L^n(\Omega\cap B_{\varepsilon r_y})} \tag{2.3}$$

for possibly smaller $\varepsilon > 0$ and larger $\tau < 1$, depending on n and σ.
Here C and C_y are positive constants depending on n, q, λ, Λ, $\overline{b}_i N^{q_i-1}$, σ, ε, τ, τ', while C_y also depends on $N^{q_i-1}r_y^{2-q_i}\|b_i\|_{L^\infty(\Omega\cap B_{\varepsilon r_y})}$.

Remark 2.5 In Theorem 4.2 of [1] the above Lemma is stated with one only nonzero gradient term, but it is easy to verify that it still holds by adding any finite number of gradient term, with at least linear and at most quadratic growth, provided the corresponding condition (2.2) is satisfied. □

[1]"viscosity subsolution of $F(x, w, Dw, D^2w) = f$", oppure dire "satisfies $F(x, w, Dw, D^2w) \geq f$ in the viscosity sense" e aggiungere che significa nella Definition 2.1. Stessa cosa negli enunciati successivi, che ho segnato con "@"

Remark 2.6 Let $0 \le \gamma \le 1$. In Lemma 2.4 suppose the first order coefficients have the following decay:

$$b_i(x) \le \frac{b_{0,i}}{1+|x|^{\gamma(2-q_i)}}, \quad i=1,2, \tag{2.4}$$

for constants $b_{0,1}, b_{0,2} \ge 0$, and the **wG**-condition holds with $R_y \le R_0 + \beta|y|^\gamma$, as in Theorem 1.1.
Case $0 \le \gamma < 1$.
Let $R_1 = R_1(R_0, \beta, \gamma, \tau') \ge 1$ be large enough in order that $|y| - 2\tau'(R_0 + \beta|y|^\gamma) \ge \frac{1}{2}|y|$ for $|y| \ge R_1$. Then, using Remark 2.3, we get

$$r_y^{2-q_i}\|b_i\|_{L^\infty(\Omega\cap B_{\tau\varepsilon r_y,\tau' r_y})} \le \frac{b_{0,i}\, r_y^{2-q_i}}{1+(|y|-2\tau' r_y)^{\gamma(2-q_i)}} \le \frac{b_{0,i}\,(R_0+\beta|y|^\gamma)^{2-q_i}}{1+(\frac{|y|}{2})^{\gamma(2-q_i)}}.$$

Supposing $\beta \ge R_0\, 2^{-\gamma\,(2-q_i)}$, as we can, we have therefore

$$r_y^{2-q_i}\|b_i\|_{L^\infty(\Omega\cap B_{\tau\varepsilon r_y,\tau' r_y})} \le b_{0,i}\,\beta^{2-q_i} 2^{\gamma\,(2-q_i)}$$

for $|y| \ge R_1$, and simply

$$r_y^{2-q_i}\|b_i\|_{L^\infty(\Omega\cap B_{\tau\varepsilon r_y,\tau' r_y})} \le b_{0,i}\,(R_0+\beta\, R_1^\gamma)^{2-q_i}$$

if $|y| \le R_1$. Thus condition (2.2) holds true for all $\varepsilon > 0$, $0 < \tau < 1$ and $\tau' > 1$, with a bound independent of ε.
Case $\gamma = 1$.
In this case we have

$$r_y^{2-q_i}\|b_i\|_{L^\infty(\Omega\cap B_{\tau\varepsilon r_y,\tau' r_y})} \le \frac{b_{0,i}\, r_y^{2-q_i}}{(1+\varepsilon\tau r_y)^{2-q_i}} \le \frac{b_{0,i}}{(\varepsilon\tau)^{2-q_i}}$$

and come to the same conclusion as above, but with a bound depending on ε. □

In the same way as Theorem 4.2 of [1] leads to the ABP type inequality of Theorem 1.3 there, the above Lemma 2.4, by virtue of Remark 2.6, can be used to obtain the following result.

Lemma 2.7 (ABP) *Let σ, R_0, β, γ, λ, Λ, q, $b_{0,1}$, $b_{0,2}$, Ω and F be as in Lemma 2.4 and Remark 2.6. Suppose the b_i's satisfy (2.4) and $c = 0$.*
If $w \in USC(\Omega)$ is a viscosity solution of $F(x, w, Dw, D^2w) \ge f$ @ such that

$$\limsup_{x\to\partial\Omega} w(x) \le 0, \quad \sup_\Omega w \le N,$$

for a positive constant N, then

$$\sup_{\Omega} w \leq C \lim_{\varepsilon\to 0^+} \sup_{y\in\Omega;\,|y|\geq\varepsilon r_y} r_y \,\|f^-\|_{L^n(\Omega\cap B_{\varepsilon r_y,2r_y})}, \tag{2.5}$$

where C is a positive constant depending only on n, λ, Λ, q, $b_{0,1}$, $b_{0,2}N^{q-1}$, σ, R_0, β and γ.

The above Lemma will be used for cylindrical ($\gamma = 0$) and parabolic shaped domains ($0 < \gamma < 1$). For conical domains ($\gamma = 1$) we will refer directly to Lemma 2.4, as well as for larger domains, obtained by extension of the **wG** condition as follows.

Definition 2.8 (**wG** by components). *Let Ω be a domain of $\mathbb{R}^n$. Suppose that there exists a positive real parameter $\sigma < 1$ such that Ω is a* **wG***-domain with parameter σ, otherwise there exists a closed subset H of $\mathbb{R}^n$ such that:*
(i) all components Ω' of $\Omega\backslash H$ are **wG***-domains with parameter σ, i.e. all points $y \in \Omega'$ satisfy condition $\mathbf{G}_\sigma$ in Ω';*
(ii) all points $y \in \Omega \cap H$ satisfy condition $\mathbf{G}_\sigma$ in Ω.
In this case we refer to Ω as to a **wG***-domain by components. If the radii R_y provided by condition $\mathbf{G}_\sigma$ grow at most as $R_y \leq R_0 + \beta|y|$, for positive constants R_0 and β, then we say that Ω is a conical domain (by components) with parameters σ, R_0 and β.*

Example 2.9 If $f : [0, +\infty) \to \mathbb{R}$ is a continuous function such that $|f(x)| \leq k|x|$ for some $k > 0$, and G is its graph, then $\Omega = \mathbb{R}^2\backslash G$ is a **wG**-domain by components. In fact, we can take $H = \{(x, y) \in \mathbb{R}^2 \mid y = \pm kx,\ x \geq 0\}$. Similarly we can reason, in higher dimensions, for $\Omega = \mathbb{R}^n\backslash G$, where G is the graph of a continuous function $f : [0, +\infty)^{n-1} \to \mathbb{R}$ such that $|f(x)| \leq k|x|$. □

Since the proof of Lemma 2.4 relies on pointwise inequalities depending only on the local geometric condition $\mathbf{G}_\sigma$, see [1], the ABP type estimate (2.3) continues to hold if we consider **wG**-domains by components and specifies as follows for conical domains by components.

Lemma 2.10 (ABP) *Let $0 < \sigma < 1$, $R_0 > 0$, $\beta > 0$, $0 < \gamma < 1$, $0 < \lambda \leq \Lambda$, $1 \leq q \leq 2$, $b_{0,1} \geq 0$, $b_{0,2} \geq 0$. Suppose Ω to be a conical domain by components with parameters σ, R_0 and β. Assume that F satisfy the structure condition (1.3) with $b_1, b_2, f \in C(\overline{\Omega})$ and*

$$b_1(x) \leq \frac{b_{0,1}}{1 + |x|}, \quad b_2(x) \leq \frac{b_{0,2}}{1 + |x|^{2-q}}, \quad c(x) = 0. \tag{2.6}$$

If $w \in USC(\Omega)$ is a viscosity solution of $F(x,w,Dw,D^2w) \geq f$ such that

$$\limsup_{x\to\partial\Omega} w(x) \leq 0, \quad \sup_\Omega w \leq N,$$

for a positive constant N, then, for a sufficently small $\varepsilon < 1$, depending on n, q, λ, Λ, $b_{0,1}$, $b_{0,2}N^{q-1}$, σ,

$$\sup_\Omega w \leq C \sup_{y\in\Omega;\,|y|\geq\varepsilon r_y} r_y \|f^-\|_{L^n(\Omega\cap B_{\varepsilon r_y,2r_y})} + \sup_{y\in\Omega;\,|y|\leq\varepsilon r_y} C_y\, r_y \|f^-\|_{L^n(\Omega\cap B_{\varepsilon r_y})}. \tag{2.7}$$

where C and C_y are positive constants depending on the above quantities, while C_y also depends on $N^{q_i-1} r_y^{2-q_i} \|b_i\|_{L^\infty(\Omega\cap B_{\varepsilon r_y})}$, $i = 1,2$, where $q_1 = 1$ and $q_2 = q$.

Remark 2.11 By our assumptions on Ω in the previous result, every point $y \in \Omega$ satisfies the local condition $\mathbf{G}_\sigma$ in Ω or in some subdomain of Ω. Since what we need on the components is just the ABP type estimate (2.3), then Lemma 2.10 can be extended to the more general domains Ω satisfying Def.2.8 (i) with components which are in turn **wG**-domains by components, instead that simply **wG**-domains. □

Example 2.12 The iterative procedure on **wG**-domains by components considered in Remark 2.10 is motivated by the following example, which could not be explicitely covered by Lemma 2.10. For instance, let $K = \cup_{k\in\mathbb{N}} \overline{B}_{\frac{1}{4}}((k,0))$ and $\Omega = \mathbb{R}^2 \setminus K$. If $H = \{(x,0) \in \mathbb{R}^2 \mid x \geq 0\}$, then $\Omega\setminus H$ is not a **wG**-domain. But by Example 2.9 it is a **wG**-domain by components. Thus, Ω satisfy the conditions of the above mentioned results. □

3 Maximum principles.

Here we show a consequence of Lemma 2.7 for operators F with slightly positive zero order coefficients.

Proposition 3.1 (MP) *Let $0 < \sigma < 1$, $R_0 \geq 0$, $\beta \geq 0$, $0 \leq \gamma \leq 1$, $0 < \lambda \leq \Lambda$, $1 \leq q \leq 2$, $b_{0,1} > 0$, $b_{0,2} > 0$, $N > 0$, Ω and F be as in the above Lemma 2.7, but*

$$c(x) \leq \frac{c_0}{1+|x|^{2\gamma}}. \tag{3.1}$$

If c_0 is a positive constant small enough, depending on the same parameters of Lemma 2.7, and $w \in USC(\Omega)$ is a solution of $F(x,w,Dw,D^2w) \geq 0$ in the viscosity sense, then

$$\limsup_{x\to\partial\Omega} w(x) \leq 0, \quad \sup_\Omega w \leq N \quad \Rightarrow \quad w \leq 0 \ \text{ in } \Omega.$$

Proof. Suppose that w is continuous, otherwise we argue by approximation from above. Passing to $w^+ = \max(w, 0)$ we can consider non-negative supersolutions. By structure conditions, we put the equation $F \geq 0$ in the form

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2 w^+) + b_1(x)|Dw^+| + b_2(x)|Dw^+|^q \geq -c(x)w^+. \tag{3.2}$$

Case $0 \leq \gamma < 1$.
We apply Lemma 2.7 with $f(x) = -c(x)w^+(x)$ to the above equation to get

$$\sup_\Omega w^+ \leq C \lim_{\varepsilon\to 0^+} \sup_{y\in\Omega;\,|y|\geq\varepsilon r_y} r_y \|c^+ w^+\|_{L^n(\Omega\cap B_{\varepsilon r_y, 2r_y})} \tag{3.3}$$

Let R_1 a positive number be such that $|y| - 4(R_0 + \beta|y|^\gamma) \geq \frac{|y|}{2}$ for $|y| \geq R_1$. If $|y| \leq R_1$, then

$$C r_y \|c^+\|_{L^n(\Omega\cap B_{\varepsilon r_y, 2r_y})} \leq 2\, c_0\, C(R_0 + \beta\, R_1^\gamma)^2 |B_1|^{1/n} \leq c_0 K_1.$$

On the other hand, if $|y| > R_1$, using the decay assumption (3.1), then for $x \in B_{2r_y}$ we have

$$c^+(x) \leq \frac{c_0}{1 + |x|^{2\gamma}} \leq \frac{c_0}{1 + (|y| - 4r_y)^{2\gamma}} \leq \frac{c_0}{1 + (\frac{|y|}{2})^{2\gamma}},$$

whence

$$C r_y \|c^+\|_{L^n(\Omega\cap B_{\varepsilon r_y, 2r_y})} \leq 2\, c_0\, C|B_1|^{1/n} \frac{(R_0 + \beta|y|^\gamma)^2}{1 + (\frac{|y|}{2})^{2\gamma}} \leq c_0 K_2,$$

for positive constants K_1 and K_2. Let $K = \max(K_1, K_2)$, then, inserting in (3.3), we get

$$\sup_\Omega w^+ \leq c_0 K \sup_\Omega w^+, \tag{3.4}$$

from which the assert follows at once, choosing a sufficiently small c_0, in this case.
Case $\gamma = 1$.
In this case we apply Lemma 2.10, obtaining

$$\begin{aligned} \sup_\Omega w^+ \leq\; & C \sup_{y\in\Omega;\,|y|\geq\varepsilon r_y} r_y \|c^+ w^+\|_{L^n(\Omega\cap B_{\varepsilon r_y, 2r_y})} \\ & + \sup_{y\in\Omega;\,|y|\leq\varepsilon r_y} C_y\, r_y \|c^+ w^+\|_{L^n(\Omega\cap B_{\varepsilon r_y})}. \end{aligned} \tag{3.5}$$

Let $\varepsilon = \overline{\varepsilon} \in (0, \frac{1}{2})$ be small enough in order that $\overline{\varepsilon}\beta < \frac{1}{2}$, then $|y| \leq \overline{\varepsilon} r_y$ implies $|y| \leq R_0$, and this yields

$$C_y\, r_y \|c^+\|_{L^n(\Omega\cap B_{\overline{\varepsilon} r_y})} \leq c_0\, C_1(R_0 + \beta\, R_0^\gamma)^2 |B_1|^{1/n} \leq c_0 K_1.$$

On the other side, for $|y|\geq \bar{\varepsilon} r_y$ we have

$$Cr_y\|c^+\|_{L^n(\Omega\cap B_{\bar{\varepsilon}r_y,2r_y})}\leq 2\,c_0\,C|B_1|^{1/n}\,\frac{r_y^2}{1+(\bar{\varepsilon}r_y)^2}\leq c_0K_2,$$

where K_1 and K_2 are positive constants. Finally, setting $K=K_1+K_2$, we get (3.4) and conclude as in the previous case. □

We also need the following consequence of the above Proposition in the case of positive value on the boundary.

Corollary 3.2 (extended MP) *Let σ, R_0, β, γ, λ, Λ, q, $b_{0,1}$, $b_{0,2}$, N e Ω as in Prop.3.1, and $\rho>0$ such that $B_\rho\cap\Omega\neq\emptyset$ and $\Omega\backslash B_\rho\neq\emptyset$. Suppose that F and w are as in Prop.3.1, but restricted to $\Omega\backslash\overline{B}_\rho$, then for a sufficiently small c_0 we have*

$$\sup_{\Omega\backslash\overline{B}_\rho} w\leq M_\rho:=\limsup_{x\to\Omega\cap\partial B_\rho} w^+$$

Proof. Set $w_\rho=w^+-M_\rho$. Since $\partial(\Omega\backslash\overline{B}_\rho)\subset\partial\Omega\cup(\Omega\cap\partial B_\rho)$ and $\limsup_{x\to\partial\Omega} w\leq 0$, then

$$\limsup_{x\to\partial(\Omega\backslash\overline{B}_\rho)} w_\rho\leq 0.$$

Firstly, we put the equation $F\geq 0$ in $\Omega\backslash\overline{B}_\rho$ in the form

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2w^+)+b_1(x)|Dw^+|+b_2(x)|Dw^+|^q\geq -c^+w^+,$$

from which

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2w_\rho)+b_1(x)|Dw_\rho|+b_2(x)|Dw_\rho|^q\geq -c^+(w_\rho+M_\rho)\geq -2\,c^+\sup_{\Omega\backslash\overline{B}_\rho} w_\rho^+.$$

Arguing as above, in the proof of Prop.3.1, we get, for c_0 small enough,

$$w_\rho\leq 0\quad\text{in }\Omega\backslash\overline{B}_\rho,$$

as claimed. □

The same consequences can be drawn in the general case of conical domains by components, using Lemma 2.10 as for conical domains.

Proposition 3.3 (MP) *Let σ, R_0, β, λ, Λ, q, $b_{0,1}$, $b_{0,2}$, N, Ω and F be as in Lemma 2.10, but with*

$$c(x)\leq\frac{c_0}{1+|x|^2}. \tag{3.6}$$

If c_0 is a positive constant small enough, depending on the same parameters of Lemma 2.10, and $w\in USC(\Omega)$ is a solution of $F(x,w,Dw,D^2w)\geq 0$ in the viscosity sense, then

$$\limsup_{x\to\partial\Omega} w(x)\leq 0,\quad \sup_\Omega w\leq N\quad\Rightarrow\quad w\leq 0\ \text{ in }\Omega.$$

Corollary 3.4 (extended MP) *Let σ, R_0, β, γ, λ, Λ, q, $b_{0,1}$, $b_{0,2}$, N e Ω as in Prop.3.3, and $\rho > 0$ such that $B_\rho \cap \Omega \neq \emptyset$ and $\Omega \backslash B_\rho \neq \emptyset$. Suppose that F and w are as in Prop.3.3, but restricted to $\Omega \backslash \overline{B}_\rho$, then for a sufficiently small c_0 we have*

$$\sup_{\Omega \backslash \overline{B}_\rho} w \leq M_\rho := \limsup_{x \to \Omega \cap \partial B_\rho} w^+$$

4 Proof of the Phragmèn-Lindelöf principles.

Throughout this section we assume the structure conditions (1.3).
Suppose that $F(x, u, Du, D^2 u) \geq 0$. Let $\varphi : [0, +\infty) \to \mathbb{R}^+$ be a concave non-decreasing function, $\Phi(x) = \varphi(|x|)$ and $u^+ \leq N\Phi$ in Ω for a positive constant N. Supposing u continuous, we can restrict the argument to $\Omega^+ := \mathrm{int}\{x \in \Omega | u(x) \geq 0\}$, where $u = u^+$, observing that Ω^+ maintains the same geometric conditions of Ω. If this is not the case, we directly deal with u^+ in the viscosity sense, observing that, for a subsolution u of equation $F(x, u, Du, D^2 u) = 0$ the function u^+ is in turn a subsolution of the equation

$$\mathcal{P}^+(D^2 u^+) + b_1(x)|Du^+| + b_2(x)|Du^+|^q + c(x)u^+ = 0.$$

If we define $w(x) = \frac{u^+(x)}{\Phi(x)}$, then

$$\sup_\Omega w \leq N, \tag{4.1}$$

as required by Prop.3.1 and Prop.3.3. Reasoning as for a smooth $w \geq 0$, then

$$\mathcal{P}^+(D^2(w\,\Phi) + b_1(x)|D(w\,\Phi)| + b_2(x)|D(w\,\Phi)|^q + c^+(x)(w\,\Phi)$$
$$= \mathcal{P}^+(\Phi\, D^2 w + Dw \otimes D\Phi + D\Phi \otimes Dw(x) + wD^2\Phi)$$
$$+ b_1(x)\,|\Phi Dw + wD\Phi| + b_2(x)\,|\Phi Dw + wD\Phi|^q + c^+(x)\,\Phi\, w$$
$$\leq \Phi\, \mathcal{P}^+(D^2 w) + 2\Lambda\,|D\Phi|\,|Dw| + b_1(x)\,\Phi|Dw| + 2^q b_2(x)\,\Phi^q\,|Dw|^q$$
$$+ \mathcal{P}^+(\,D^2\Phi)\,w\, + b_1(x)\,|D\Phi|w + \,2^q b_2(x)\,|D\Phi|^q\,w^q + c^+(x)\,\Phi\, w.$$

Since $\varphi(r)$ is a positive non-decreasing function of r and $\Phi(x) = \varphi(|x|)$, then

$$\mathcal{P}^+(D^2\Phi) = \frac{(n-1)\,\Lambda}{|x|}\,\varphi' + \lambda\,\varphi''$$

and therefore

$$\mathcal{P}^+(D^2u^+) + b_1(x)|Du^+| + b_2(x)|Du^+|^q + c(x)u^+$$

$$\leq \varphi\,\mathcal{P}^+(D^2w) + 2\Lambda\,\varphi'\,|Dw| + b_1(x)\,\varphi|Dw| + 2^q b_2(x)\,\varphi^q\,|Dw|^q$$

$$+\left\{\lambda\varphi'' + \Lambda\,\frac{n-1}{|x|}\,\varphi' + b_1(x)\,\varphi' + 2^q b_2(x)\,N^{q-1}(\varphi')^q + c(x)\varphi\right\}\,w,$$

from which for a subsolution u of the equation $F = 0$ we get, in the viscosity sense,

$$\mathcal{P}^+(D^2w)\,+\,2\Lambda\,\frac{\varphi'}{\varphi}\,|Dw| + b_1(x)\,|Dw| +\,2^q b_2(x)\,\varphi^{q-1}\,|Dw|^q$$

$$+\left\{\lambda\,\frac{\varphi''}{\varphi} + \Lambda\,\frac{n-1}{|x|}\,\frac{\varphi'}{\varphi} + b_1(x)\,\frac{\varphi'}{\varphi} + 2^q b_2(x)\,N^{q-1}\,\frac{(\varphi')^q}{\varphi} + c(x)\right\}\,w \geq 0 \tag{4.2}$$

Proof of Theorem 1.1 (parabolic shaped and conical domains). Here $0 < \gamma \leq 1$. Let u be a subsolution of $F = 0$ in Ω such that $u^+ \leq N\,(1 + |x|^\alpha)$ in Ω, with $\alpha \in (0, \overline{\alpha}]$ to be chosen in the sequel. Taking $\varphi(r) = 1 + r^\alpha$ in (4.2), by the assumptions on the decay of the coefficients we have

$$\mathcal{P}^+(D^2w)\,+\,\left(\frac{\alpha}{|x|}\,+\frac{b_1'}{|x|^\gamma}\right)|Dw| + \frac{K_\rho\,b_2'}{|x|^{\gamma(2-q)}}\,|Dw|^q 2\Lambda$$

$$+\left(\frac{\lambda\alpha^2 + \Lambda(n-1)\,\alpha}{|x|^2} + \frac{b_1'\alpha}{|x|^{1+\gamma}} + \frac{K_\rho\,b_2'\,N^{q-1}\alpha^q}{|x|^{2\gamma+q(1-\gamma)}} + \frac{c_1}{|x|^{2\gamma}}\right)w \geq 0 \tag{4.3}$$

outside a ball $B_\rho = B_\rho(0)$, where K_ρ is a constant independent of α.
We can assume, up to a translation, that $0 \notin \Omega$.
Next, as a first case, suppose that there exists a ball $B_\rho = B_\rho(0)$ such that $\overline{B}_\rho \cap \overline{\Omega} = \emptyset$. In this case we have $1 + |x| \leq \frac{1+\rho}{\rho}\,|x|$ for $x \in \Omega$ and

$$\mathcal{P}^+(D^2w)\,+\,\frac{b_{1,\rho}}{1+|x|^\gamma}\,|Dw|\,+\,\frac{b_{2,\rho}}{1+|x|^{\gamma(2-q)}}\,|Dw|^q\,+\,\frac{c_{2,\rho}(\alpha)+c_1}{1+|x|^{2\gamma}}\,w \geq 0, \tag{4.4}$$

for positive constants $b_{i,\rho}$, $i = 1, 2$, and a polynomial $c_{2,\rho}(\alpha)$ of degree 2 such that $c_{2,\rho}(0) = 0$. Thus by Proposition 3.1, choosing $c_{2,\rho}(\alpha) + c_1 \leq c_0$, we infer that $w \leq 0$, therefore $u \leq 0$ in Ω and the proof is done.
If the above is not the case, then

$$\limsup_{x\to 0} w(x) \leq 0.$$

In this case, suppose, by contradiction, that $\sup_\Omega w > 0$. Taking $0 < \varepsilon < \sup_\Omega w$ and $\rho > 0$ such that $w < \varepsilon$ in $\Omega \cap \overline{B}_\rho$, then equation (4.4) is satisfied in $\Omega \backslash \overline{B}_\rho$ and therefore by Corollary 3.2 we get, again for $c_{2,\rho}(\alpha) + c_1 \le c_0$,

$$\varepsilon < \sup_\Omega w = \sup_{\Omega\backslash\overline{B}_\rho} w \le \limsup_{x\to\Omega\cap\partial B_\rho} w \le \varepsilon,$$

i.e. a contradiction. Hence again $\sup_\Omega w \le 0$ and the proof is complete. □

Proof of Theorem 1.1 (cylindrical domains). Here $\gamma = 0$. Let u be a subsolution of $F = 0$ in Ω such that $u^+ \le N\, e^{\alpha|x|}$ in Ω, with $\alpha \in (0, \overline{\alpha}]$ to be chosen in the sequel. In this case we use $\varphi(r) = e^{\alpha r}$ in (4.2), in order that $w(x) = \frac{u^+(x)}{\varphi(|x|)} \le N$. Hence from (4.2), using the decay of the coefficients, we get

$$\begin{aligned} &\mathcal{P}^+(D^2 w) \; + \; (2\Lambda\,\alpha + b_1')\,|Dw| + \; 2^q b_2' |Dw|^q \\ &+ \left(\lambda\alpha^2 + \Lambda(n-1)\frac{\alpha}{|x|} + b_1'\,\alpha + 2^q b_2'\, N^{q-1}\alpha^q + c_1\right) w \ge 0, \end{aligned} \tag{4.5}$$

if $|x| \ne 0$. As before, we suppose $0 \notin \Omega$, as we can, and observe that

$$\mathcal{P}^+(D^2 w) \; + (2\Lambda\,\alpha + b_1')\,|Dw| + \; 2^q b_2' |Dw|^q + (c_{2,\rho}\,(\alpha) + c_1) w \ge 0$$

outside $B_\rho = B_\rho(0)$. Proceeding along the same lines as above and choosing a suitable $\rho > 0$, by Proposition 3.1 and Corollary 3.2 we finish the proof. □

Proof of Theorem 1.2. We can proceed in a similar manner as in the Proof of Theorem 1.1 (parabolic shaped and conical domains) with $\gamma = 1$, using Proposition 3.3 and Corollary 3.4 instead of Proposition 3.1 and Corollary 3.2, resp.

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