REACTION-DIFFUSION EQUATIONS FOR POPULATION DYNAMICS WITH FORCED SPEED
I - THE CASE OF THE WHOLE SPACE

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Abstract. This paper is concerned with time-dependent reaction-diffusion equations of the following type:

$$\partial_t u = \triangle u + f(x - cte, u), \qquad t > 0, \quad x \in \mathbb{R}^N.$$

These kind of equations have been introduced in [1] in the case $N = 1$ for studying the impact of a climate shift on the dynamics of a biological species.

In the present paper, we first extend the results of [1] to arbitrary dimension N and to a greater generality in the assumptions on f. We establish a necessary and sufficient condition for the existence of travelling wave solutions, that is, solutions of the type $u(t, x) = U(x - cte)$. This is expressed in terms of the sign of the generalized principal eigenvalue λ_1 of an associated linear elliptic operator in $\mathbb{R}^N$. With this criterion, we then completely describe the large time dynamics for this equation. In particular, we characterize situations in which there is either extinction or persistence.

Moreover, we consider the problem obtained by adding a term $g(x, u)$ periodic in x in the direction e:

$$\partial_t u = \triangle u + f(x - cte, u) + g(x, u), \qquad t > 0, \quad x \in \mathbb{R}^N.$$

Here, g can be viewed as representing geographical characteristics of the territory which are not subject to shift. We derive analogous results as before, with λ_1 replaced by the generalized principal eigenvalue of the parabolic operator obtained by linearization about $u \equiv 0$ in the whole space. In this framework, travelling waves are replaced by pulsating travelling waves, which are solutions of the form $U(t, x - cte)$, with $U(t, x)$ periodic in t. These results still hold if the term g is also subject to the shift, but on a different time scale, that is, if $g(x, u)$ is replaced by $g(x - c'te, u)$, with $c' \in \mathbb{R}$.

1. Introduction and main results

1.1. **Introduction.** In a recent paper [1], a model to study the impact of climate shift (global warming) on the dynamics of a species facing it was proposed. This model involves the following reaction-diffusion equation on the real line

$$\partial_t u = \partial_{xx} u + f(x - ct, u), \qquad t > 0, \quad x \in \mathbb{R}.$$

The first part of this paper is dedicated to the mathematical study of higher dimensional versions of this problem, that we call the **pure shift case**:

$$\partial_t u = \triangle u + f(x - cte, u), \qquad t > 0, \quad x \in \mathbb{R}^N, \tag{1}$$

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with $f : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$, $c > 0$ and $e \in S^{N-1}$ given. In the one dimensional ecological model, the variable x is thought of as the latitude, say in the northern hemisphere. The solution $u(t, x)$ represents the population density of the species at time t and point x. The net effect of reproduction and mortality - which depends on the population density - is represented by the reaction term f. The x-dependence of f embodies the fact that climate conditions are not uniform in space, and there are regions more favourable than others for the species. Since the space dependence of f is affected by the time under the action $G_t(x) = x - cte$, the zones with favourable climate shift in the direction e with speed c. Heuristically, for the population to persist it is necessary for a portion of the population large enough to "migrate" in the direction e, and indeed this is what happens if the speed c is not too large. This is proved in [1] when the dimension N is equal to 1. Our aim is to extend the results of [1] to higher dimensions.

A crucial point to establish the large time behavior of solutions of (1) consists in the study of the existence and uniqueness of travelling wave solutions tracking the imposed shift. That is, positive bounded solutions of the type $u(t, x) = U(x - cte)$. Such solutions are obtained from the following elliptic problem in U:

$$(2) \qquad \begin{cases} \triangle U + ce \cdot \nabla U + f(x, U) = 0 \quad \text{a. e. in } \mathbb{R}^N \\ U > 0 \quad \text{in } \mathbb{R}^N \\ U \quad \text{is bounded.} \end{cases}$$

First, we establish a necessary and sufficient condition for the existence and uniqueness of travelling wave solutions. Then, we show that when such a travelling wave solution exists it is stable, in the sense that it attracts the orbits of solutions of the evolution problem with nontrivial initial conditions $u_0 \geq 0$. Otherwise, the solutions of (1) converge to zero as t goes to infinity.

In the second part of the paper, we consider problem (1) with the addition of a nonlinear term which does not depend on t:

$$(3) \qquad \partial_t u = \triangle u + f(x - cte, u) + g(x, u), \qquad t > 0, \quad x \in \mathbb{R}^N.$$

If one looks for travelling wave solutions $u(t, x) = U(x - cte)$, one is led to the equation

$$\triangle U + ce \cdot \nabla U + f(x, U) + g(x + cte, U) = 0 \quad \text{for a. e. } x \in \mathbb{R}^N,$$

for any $t > 0$. Clearly, this problem does not admit a solution unless $g(\cdot, s)$ is constant in the direction e (in which case it can be incorporated into f). In other words, the function $U(t, x) = u(t, x + cte)$ cannot be constant in t, that is, no travelling wave solutions exist. However, if the function $x \mapsto g(x, s)$ is periodic in the direction e, with period l, then $g(x+cte, s)$ is periodic in t, with period l/c. This suggests that we look for solutions u such that $U(t, x) := u(t, x + cte)$ is periodic in t, with period l/c. Then, our problem becomes

$$(4) \qquad \begin{cases} \partial_t U = \triangle U + ce \cdot \nabla U + f(x, U) + g(x + cte, U), \quad (t, x) \in \mathbb{R}^{N+1} \\ U > 0 \quad \text{in } \mathbb{R}^{N+1} \\ U \quad \text{is bounded} \\ U \quad \text{is } l/c\text{-periodic in } t, \end{cases}$$

where we have extended U by periodicity for $t < 0$. A function solving (4) is called a *pulsating travelling wave*, and we refer to this framework as the **mixed periodic/shift case**. As for the pure shift case, we first establish an existence and uniqueness result for pulsating travelling waves, then we prove that as $t \to \infty$ solutions of (3) with positive bounded initial datum converge either to the pulsating travelling wave, when it exists, or to 0 otherwise.

The arguments in the proofs of the mixed periodic/shift case also work for the **two-speeds case**, that is, for a problem of the type:

$$\partial_t u = \triangle u + f(x - cte, u) + g(x - c'te, u), \qquad t > 0, \quad x \in \mathbb{R}^N, \tag{5}$$

with arbitrary $c' \in \mathbb{R}$. In the ecological context, the term $g(x-c'te, u)$ may be viewed as representing the influence of some environmental factors - such as, for instance, the presence of vegetation - which are affected by the climate shift but in a different time scale. We point out that this does not include the case of two cooperating or competing species living in the same environment. For that kind of problem one has to consider a system of equations (see in particular [8] and the references therein) for which eigenvalue problems and maximum principles are more delicate to handle.

In the forthcoming paper [7], we treat the case where f is periodic in some space directions, orthogonal to the direction of the shift e, as well as the case where the problem is set in a straight infinite cylinder, with Neumann boundary conditions, or in sets which have an asymptotic cylindrical shape.

1.2. **Main results in the pure shift case.** Let us now describe precisely the assumptions on f. Even in the one dimensional case, they yield a slightly more general framework than in [1]. Throughout the paper, we will assume that the nonlinearity $f(x, s) : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ is a Carathéodory function such that

$$\begin{cases} s \mapsto f(x, s) \text{ is locally Lipschitz continuous, uniformly for a. e. } x \in \mathbb{R}^N, \\ \exists\, \delta > 0 \text{ such that } s \mapsto f(x, s) \in C^1([0, \delta]), \text{ uniformly for a. e. } x \in \mathbb{R}^N. \end{cases} \tag{6}$$

In some statements, we will require the following assumptions:

$$f(x, 0) = 0 \quad \text{for a. e. } x \in \mathbb{R}^N, \tag{7}$$

$$\exists\, S > 0 \text{ such that } f(x, s) \leq 0 \quad \text{for } s \geq S \text{ and for a. e. } x \in \mathbb{R}^N. \tag{8}$$

Condition (8) is usual in population dynamics and is related to a maximum carrying capacity of the environment. Another assumption needed is

$$\begin{cases} s \mapsto \dfrac{f(x, s)}{s} \text{ is nonincreasing for a. e. } x \in \mathbb{R}^N, \\ \text{and it is strictly decreasing for a. e. } x \in D \subset \mathbb{R}^N, \text{ with } |D| > 0. \end{cases} \tag{9}$$

Lastly, the condition of boundedness of the favourable zone is expressed by

$$\limsup_{|x|\to\infty} f_s(x, 0) < 0. \tag{10}$$

Such a condition is weaker than that in [1], where $f_s(x, 0)$ is assumed to have a negative limit as $|x| \to \infty$.

A typical example of f satisfying (7)-(10) is

$$f(x, s) = s(\zeta(x) - \eta(x)s),$$

with $\zeta, \eta \in L^\infty(\mathbb{R}^N)$ such that $\eta \geq 0$ a. e. in $\mathbb{R}^N$, $\eta > 0$ in $D \subset \mathbb{R}^N$, with $|D| > 0$,

$$\inf_{\{x\in\mathbb{R}^N \,:\, \zeta(x)>0\}} \eta(x) > 0, \qquad \limsup_{|x|\to\infty} \zeta(x) < 0.$$

If f satisfies (7) then the linearized operator about $w \equiv 0$ associated with the elliptic equation in (2) is

$$\mathcal{L}w = \triangle w + ce \cdot \nabla w + f_s(x, 0)w.$$

The main results here are that the existence and uniqueness of travelling wave solutions of (1) - as well as the large time behavior of solutions with nonnegative initial datum - depend on the stability of the solution $w \equiv 0$ for the equation $\mathcal{L}w = 0$, that is, on the sign of the *generalized principal eigenvalue* $\lambda_1(-\mathcal{L}, \mathbb{R}^N)$. The generalized principal eigenvalue of a linear elliptic operator $-L$ in a domain $\Omega \subset \mathbb{R}^N$ is defined by

(11)
$$\lambda_1(-L, \Omega) := \sup\{\lambda \in \mathbb{R} : \ \exists\, \phi \in W^{2,N}_{loc}(\Omega), \ \phi > 0 \text{ and } (L + \lambda)\phi \leq 0 \text{ a. e. in } \Omega\}.$$

In the sequel, we will set $\lambda_1 := \lambda_1(-\mathcal{L}, \mathbb{R}^N)$.

Theorem 1.1. *Assume that* (7)-(10) *hold. Then problem* (2) *admits solution if and only if* $\lambda_1 < 0$. *Moreover, the solution is unique when it exists.*

Theorem 1.2. *Let* $u(t, x)$ *be the solution of* (1) *with an initial condition* $u(0, x) = u_0(x) \in L^\infty(\mathbb{R}^N)$ *which is nonnegative and not identically equal to zero. Under assumptions* (7)-(10) *the following properties hold:*
(i) if $\lambda_1 \geq 0$ *then*

$$\lim_{t\to\infty} u(t, x) = 0,$$

uniformly with respect to $x \in \mathbb{R}^N$*;*
(ii) if $\lambda_1 < 0$ *then*

$$\lim_{t\to\infty} (u(t, x) - U(x - cte)) = 0,$$

uniformly with respect to $x \in \mathbb{R}^N$*, where* U *is the unique solution of* (2).

We will see that there exists a critical speed c_0, depending only on $f_s(x, 0)$ and not on e, such that $\lambda_1 < 0$ if and only if $c < c_0$, provided $\lambda_1 < 0$ when $c = 0$ (cf. Definition 2.1 and Proposition 3 in Section 2.1). Hence, the biological interpretation of Theorem 1.2 is that the population manages to persist by migrating if the speed of the climate shift is not too large, otherwise there is extinction.

We further consider the problem

$$\partial_t u = a\triangle u + \gamma f(x - cte, u), \qquad t > 0, \quad x \in \mathbb{R}^N, \tag{12}$$

and we examine the dependence of the critical speed c_0 with respect to the positive parameters a and γ (see Section 2.6).

1.3. **Main results in the mixed periodic/shift case.** It will always be assumed that $g(x, s) : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ is a Carathéodory function satisfying the same regularity assumptions (6) as f. Moreover, the function $x \mapsto g(x, s)$ is periodic in the direction e, with period $l > 0$, that is

$$g(x + le, s) = g(x, s) \quad \text{for } s \geq 0 \text{ and for a. e. } x \in \mathbb{R}^N.$$

Henceforth, we set

$$h(t, x, s) := f(x, s) + g(x + cte, s).$$

Then, the function h is l/c-periodic in t. The assumptions on the dependence of h with respect to x and s are the same as those on f in the pure shift case. More precisely, we assume that

$$h(t, x, 0) = 0 \quad \text{for a. e. } (t, x) \in \mathbb{R}^{N+1}, \tag{13}$$

$$\exists\, S>0 \;\text{ such that }\; h(t,x,s)\le 0 \quad \text{for } s\ge S \text{ and for a. e. } (t,x)\in\mathbb{R}^{N+1}, \tag{14}$$

$$\begin{cases} s\mapsto \dfrac{h(t,x,s)}{s} \;\text{ is nonincreasing for a. e. } (t,x)\in\mathbb{R}^{N+1}, \\ \text{and it is strictly decreasing for a. e. } (t,x)\in D\subset\mathbb{R}^{N+1}, \text{ with } |D|>0. \end{cases} \tag{15}$$

The analogue of condition (10) is required uniformly in $t\in\mathbb{R}$, that is,

$$\lim_{R\to\infty} \sup_{\substack{t\in\mathbb{R}\\ |x|>R}} h_s(t,x,0)<0. \tag{16}$$

A sufficient condition for (16) to hold is

$$\limsup_{|x|\to\infty} f_s(x,0) < -\sup_{x\in\mathbb{R}^N} g_s(x,0).$$

If (13) holds then the linearized operator about $w\equiv 0$ associated with the parabolic equation in (4) is

$$\mathcal{P}w=\partial_t w-\triangle w-ce\cdot\nabla w-h_s(t,x,0)w.$$

Note that the coefficients of $\mathcal{P}$ are l/c-periodic in t. We now introduce a notion of generalized principal eigenvalue associated with a general parabolic operator P with T-periodic coefficients with respect to t in a domain $\mathbb{R}\times\Omega$, where $T>0$ and Ω is a domain in $\mathbb{R}^N$. The generalized T-periodic (with respect to t) principal eigenvalue of P in $\mathbb{R}\times\Omega$ is defined as

$$\begin{aligned} \mu_1(P,\mathbb{R}\times\Omega) := \sup\{\mu\in\mathbb{R} \ :\ \exists\,\phi\in W^{1,2}_{N+1,loc}(\mathbb{R}\times\Omega) \ \ T\text{-periodic in } t,\\ \phi>0 \text{ and } (P-\mu)\phi\ge 0 \ \text{ a. e. in } \mathbb{R}\times\Omega\}. \end{aligned} \tag{17}$$

The test functions ϕ in the above definition belong to $W^{1,2}_{N+1,loc}(\mathbb{R}\times\Omega)$ in order to satisfy the maximum principle (see e. g. [14]). Unless otherwise specified, we will set $T:=l/c$ and we will denote by μ_1 the T-periodic generalized principal eigenvalue $\mu_1(\mathcal{P},\mathbb{R}^{N+1})$.

Theorem 1.3. *Assume that* (13)-(16) *hold. Then problem* (4) *admits solution if and only if $\mu_1<0$. Moreover, the solution is unique when it exists.*

Theorem 1.4. *Let $u(t,x)$ be the solution of* (3) *with an initial condition $u(0,x)=u_0(x)\in L^\infty(\mathbb{R}^N)$ which is nonnegative and not identically equal to zero. Under assumptions* (13)-(16) *the following properties hold:*
(i) if $\mu_1\ge 0$ then

$$\lim_{t\to\infty} u(t,x)=0,$$

uniformly with respect to $x\in\mathbb{R}^N$;
(ii) if $\mu_1<0$ then

$$\lim_{t\to\infty}(u(t,x)-U(t,x-cte))=0,$$

uniformly with respect to $x\in\mathbb{R}^N$, where U is the unique solution of (4)*.*

A natural question arises as to whether there exists a critical speed c_0 such that if $c<c_0$ then $\mu_1<0$ and if $c\ge c_0$ then $\mu_1\ge 0$. This would be the analogue here of the pure shift case. Such a result however is an open question. It is not clear whether such a property holds for the mixed case. What we can prove is that there exist a subcritical speed $\underline{c}$ and a supercritical speed $\overline{c}$ such that $\mu_1<0$ if $c<\underline{c}$ and $\mu_1\ge 0$ if $c\ge\overline{c}$ (cf. Definition 3.1 and Proposition 8 in Section 3.1).

1.4. **Main results in the two-speeds case.** Actually, the results in the mixed periodic/shift case hold good for the more general two-speeds problem (5) as well, for any two speeds $c' \neq c > 0$. If u solves (5) then the function $\tilde{u}(t,x) := u(t, x+cte)$ satisfies

$$\partial_t \tilde{u} = \triangle \tilde{u} + ce \cdot \nabla \tilde{u} + \tilde{h}(t,x,\tilde{u}) = 0 \quad \text{for a. e. } x \in \mathbb{R}^N,$$

with $\tilde{h}(t,x,s) := f(x,s) + g(x+(c-c')te, s)$. Therefore, under the same periodicity assumption $g(x+le,s) = g(x,s)$ as before, we see that $\tilde{h}$ is $l/(c-c')$-periodic in t. This suggests that we look for pulsating travelling wave solutions U of (5) with time period $l/(c-c')$, that is,

$$(18) \qquad \begin{cases} \partial_t U = \triangle U + ce \cdot \nabla U + f(x,U) + g(x+(c-c')te, U), & (t,x) \in \mathbb{R}^{N+1} \\ U > 0 \quad \text{in } \mathbb{R}^{N+1} \\ U \quad \text{is bounded} \\ U \quad \text{is } l/(c-c')\text{-periodic in } t. \end{cases}$$

As before, we denote by $\mathcal{P}$ the linearized operator about $w \equiv 0$:

$$\mathcal{P}w = \partial_t w - \triangle w - ce \cdot \nabla w - \tilde{h}_s(t,x,0)w,$$

and with μ_1 the generalized $l/(c-c')$-periodic (with respect to t) principal eigenvalue $\mu_1(\mathcal{P}, \mathbb{R}^{N+1})$ given by (17) with $T = l/(c-c')$.

Theorem 1.5. *The results of Theorems 1.3 and 1.4 hold true with h,* (3) *and* (4) *replaced respectively by $\tilde{h}$,* (5) *and* (18).

Subcritical and supercritical speeds $\underline{c}$, $\overline{c}$ exist even in this case, and are defined exactly as in the mixed periodic/shift case.

1.5. **Plan of the paper and strategy of the proofs.** The paper is divided into two parts. The first one (Section 2) deals with the pure shift case and the second one (Section 3) is concerned with the mixed periodic/shift case.

In Section 2.1, we first recall some properties of the generalized principal eigenvalue λ_1 that will be used in the sequel. Next, we make a change of unknown in order to transform problem (2) into a problem with an elliptic equation having self-adjoint linear part. This allows one to define the critical speed c_0 and to establish its relations with the sign of λ_1. Another consequence of the new formulation of the problem is that, owing to (10), we are able to construct some rotationally invariant supersolutions that are then used to derive exponential decay of travelling wave solutions. This is done in Section 2.2. In the following one, we make use of the self-adjoint structure of the equation and the exponential decay to prove a comparison principle for travelling waves, stated in Theorem 2.3 below. The necessary condition for the existence result as well as the uniqueness result are consequences of this comparison principle, as it is shown in Section 2.4. We wish to emphasize that this approach differs from that of [1], where the case $\lambda_1 > 0$ is handled by establishing a lower bound for the decay of the generalized principal eigenfunction. This is possible in [1] only because $f_s(x,0)$ is assumed to have a negative limit as $|x| \to \infty$, which is not the case in general here. The proof of the sufficient condition is essentially the same as in [1], based on the method of sub and supersolutions and a characterization of the generalized principal eigenvalue. In Section 2.5, we derive the large time behavior of any solution of (1) with nonnegative initial datum by comparison with some sub and supersolutions monotone in t, as was done in [2]. Some extra work is required to prove that the convergence is uniform in x. The dependence of the critical speed c_0 with respect to the amplitude of the reaction and diffusion terms is discussed in Section 2.6.

At the beginning of Section 3.1, we present some results concerning the generalized time-periodic principal eigenvalue μ_1 of the evolution operator. Then, we reduce (4) to an equivalent problem via the same change of function as in Section 2.1. As a consequence of the new formulation, we define sub and supercritical speeds $\underline{c}$, $\overline{c}$ and, in Section 3.2, we derive exponential decay of pulsating travelling wave solutions. In contrast with the pure shift case, the linear part of the new operator being parabolic is not self-adjoint. Thus, the method used in Section 2.3 to prove the comparison principle for travelling wave solutions is not applicable. Instead, a method based on the maximum principle is presented in Section 3.3. This is possible because, by (15) and (16), the function $h(t,x,s)$ has the right monotonicity in s for $|x|$ large. Section 3.4 is concerned with the large time behavior of solutions of (3). As a consequence, the sufficient condition for the existence of pulsating travelling wave solutions is obtained. Since they are of independent interest, we state such results for more general time periodic parabolic equations (for which neither (15) nor (16) are required). We prove Theorem 1.3 and Theorem 1.4 in Section 3.5, by putting together all the previous tools. Lastly, at the end of the paper we show that the arguments in the mixed periodic/shift case also apply to the two speeds problem (5).

1.6. **Notation.** We denote by $B_R(x)$ the ball of $\mathbb{R}^N$ with radius $R>0$ and centre $x\in\mathbb{R}^N$, and $B_R=B_R(0)$. The symbol ∇ stands for the vector $(\partial_1,\cdots,\partial_N)$ of partial derivatives with respect to the space variables $x_1,\cdots,x_N$, and $\triangle=\sum_{i=1}^N\partial_{ii}$.

We require the weak notion of solutions for parabolic equations such as (1) or (3), because it only assumes the initial datum to be in L^2_{loc}. Let us briefly recall the definition. We say that $u(t,x)$ is a solution (resp. sub, supersolution) of (1) with initial condition $u(0,\cdot)=u_0\in L^2_{loc}(\mathbb{R}^N)$ if, for all $t,r>0$, the functions $u,\nabla u$ belong to $L^2((0,t)\times B_r)$ and $u(t,\cdot)\in L^2(B_r)$, and

$$\int_{B_r}(u(\tau,\cdot)\phi(\tau,\cdot)-u_0\phi(0,\cdot))+\int_{(0,\tau)\times B_r}(-u\partial_t\phi+\nabla u\cdot\nabla\phi)$$
$$=\int_{(0,\tau)\times B_r}f(x-cte,u)\phi,$$

(resp. $\le$, $\ge$) for a. e. $\tau>0$ and any test function $\phi\in C^1([0,\tau]\times\overline{B}_r)$ such that $\phi=0$ on $[0,\tau]\times\partial B_r$. We will also make use of theory of strong solutions for parabolic equations. A strong solution of a parabolic equation in an open set $\mathcal{Q}\subset\mathbb{R}^{N+1}$ is a function $u\in W^{1,2}_{1,loc}(\mathcal{Q})$ which satisfies the equation a. e. in $\mathcal{Q}$. Here, for $p\ge1$, $W^{1,2}_p(\mathcal{Q})$ stands for the space of functions $\phi\in L^p(\mathcal{Q})$ with weak derivatives $\partial_t\phi$, $\partial_i\phi$ and $\partial_{ij}\phi$ in $L^p(\mathcal{Q})$, equipped with the norm

$$\|\phi\|_{p,\mathcal{Q}}:=\|\phi\|_{L^p(\mathcal{Q})}+\|\partial_t\phi\|_{L^p(\mathcal{Q})}+\sum_{i=1}^N\|\partial_i\phi\|_{L^p(\mathcal{Q})}+\sum_{i,j=1}^N\|\partial_{ij}\phi\|_{L^p(\mathcal{Q})}.$$

For elliptic equations such as that in (2) it is understood that we refer to strong solutions. In particular, it follows that if U is a solution of (2) then $U(x-cte)$ is a (travelling wave) solution of (1) with initial condition $U(x)$.

The regularity assumptions (6) on the function f are understood in the following sense:

(1) for any $\xi > 0$ there exists a positive constant k_ξ such that

$$\forall\, s_1, s_2 \in [0,\xi], \quad \|f(\cdot, s_1) - f(\cdot, s_2)\|_{L^\infty(\mathbb{R}^N)} \leq k_\xi |s_1 - s_2|;$$

(2) for a. e. $x \in \mathbb{R}^N$ the function $s \mapsto f(x,s)$ belongs to $C^1([0,\delta])$ and its derivative f_s satisfies

$$\forall\, s \in [0,\delta], \qquad \lim_{\substack{h\to 0\\ s+h\in[0,\delta]}} \|f_s(\cdot, s+h) - f_s(\cdot, s)\|_{L^\infty(\mathbb{R}^N)} = 0.$$

Finally, we denote for brief by “inf” and “sup” respectively the essential infimum and supremum of a measurable function.

2. Pure shift case

Recall that in Section 1.2 we introduced the following notation:

$$\mathcal{L}w := \triangle w + ce \cdot \nabla w + f_s(x,0)w,$$

$$\lambda_1 := \lambda_1(-\mathcal{L}, \mathbb{R}^N).$$

Throughout this section, we set

$$\zeta := -\limsup_{|x|\to\infty} f_s(x,0). \tag{19}$$

Note that $\zeta \in \mathbb{R}$ because $f_s(x,0) \in L^\infty(\mathbb{R}^N)$ thanks to the Lipschitz continuity of f. If (10) holds then $\zeta > 0$.

2.1. **The generalized principal eigenvalue and definition of the critical speed.**

The generalized principal eigenvalue $\lambda_1(-L,\Omega)$ defined by (11) has been introduced in [4]. Its properties have been widely investigated in our previous paper [5] and in the one in collaboration with F. Hamel [2]. We refer to our work in progress [6] for a comprehensive treatment of the subject, with rather general assumptions on the coefficients.

A basic result of [4] is that $\lambda_1(-L,\Omega)$ is always a well defined real number, which coincides with the Dirichlet principal eigenvalue of $-L$ in Ω if Ω is bounded and smooth. We recall that the Dirichlet principal eigenvalue of $-L$ in Ω is the unique real number λ such that the problem

$$\begin{cases} -L\varphi = \lambda\varphi & \text{a. e. in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \end{cases}$$

admits a positive solution φ (called Dirichlet principal eigenfunction, which is unique up to multiplication). Henceforth, we denote by $\lambda(R)$ the Dirichlet principal eigenvalue of $-\mathcal{L}$ in B_R and φ_R the associated principal eigenfunction such that $\varphi_R(0) = 1$. Another fundamental result for our purpose is

Proposition 1. ([4] and Proposition 4.2 in [2]) *The function $\lambda(R) : \mathbb{R}^+ \to \mathbb{R}$ decreases and satisfies*

$$\lim_{R\to\infty} \lambda(R) = \lambda_1.$$

Furthermore, there exists a generalized principal eigenfunction of $-\mathcal{L}$ in $\mathbb{R}^N$, that is, a positive function $\varphi \in W^{2,p}_{loc}(\mathbb{R}^N)$, for any $1 \leq p < \infty$, such that

$$-\mathcal{L}\varphi = \lambda_1\varphi \quad \textit{a. e. in } \mathbb{R}^N. \tag{20}$$

It is classical in this framework to make a Liouville transformation of the unknown U. The function $U(x)$ is a solution of (2) if and only if $v(x) := U(x)e^{\frac{c}{2}x\cdot e}$ is a solution of

$$\begin{cases} \triangle v + f(x, v(x)e^{-\frac{c}{2}x\cdot e})e^{\frac{c}{2}x\cdot e} - \dfrac{c^2}{4}v = 0 & \text{a. e. in } \mathbb{R}^N \\ v > 0 \quad \text{in } \mathbb{R}^N \\ v(x)e^{-\frac{c}{2}x\cdot e} \quad \text{is bounded.} \end{cases} \tag{21}$$

Under assumption (7), the linearized operator about $w \equiv 0$ associated with the equation in (21) is

$$\tilde{\mathcal{L}}w := \triangle w + (f_s(x,0) - c^2/4)w.$$

Proposition 2. *For any domain Ω in $\mathbb{R}^N$ the following identity holds:*

$$\lambda_1(-\tilde{\mathcal{L}}, \Omega) = \lambda_1(-\mathcal{L}, \Omega).$$

Proof. It follows immediately from definition (11) and the fact that

$$\tilde{\mathcal{L}}\phi = (\mathcal{L}(\phi e^{-\frac{c}{2}x\cdot e}))e^{\frac{c}{2}x\cdot e}$$

for any function $\phi \in W^{2,1}_{loc}(\Omega)$. □

We can now define the critical speed c_0. Let us introduce the operator

$$\mathcal{L}_0 w := \triangle w + f_s(x,0)w$$

and set $\lambda_0 := \lambda_1(-\mathcal{L}_0, \mathbb{R}^N)$.

Definition 2.1. (Critical speed) We define

$$c_0 := \begin{cases} 2\sqrt{-\lambda_0} & \text{if } \lambda_0 < 0 \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 2 we infer

$$\lambda_1 = \lambda_1(-\mathcal{L}_0 + \frac{c^2}{4}, \mathbb{R}^N) = \lambda_0 + \frac{c^2}{4}.$$

Hence, the following equivalence holds

Proposition 3. $\lambda_1 < 0$ *iff* $c < c_0$.

Note that, owing to Theorem 1.1 and Proposition 3, our condition $c_0 = 0$ if $\lambda_0 \geq 0$ implies that in such a case, there does not exist a stationary solution even without climate change, that is $c = 0$.

2.2. **Exponential decay.** The next result will be useful to derive the behaviour of solutions of (21) far from the origin. An analogous property is proved in [1] in dimension $N = 1$.

Lemma 2.2. *Let $v \in W^{2,N}_{loc}(\mathbb{R}^N)$ be a positive function satisfying*

$$\forall\, x \in \mathbb{R}^N, \quad v(x) \leq Ce^{\sqrt{\gamma}|x|}, \qquad \liminf_{|x|\to\infty} \frac{\triangle v(x)}{v(x)} > \gamma,$$

for some positive constants C and γ. Then,

$$\lim_{|x|\to\infty} v(x)e^{\sqrt{\gamma}\,|x|} = 0.$$

Proof. From the Sobolev embedding theorem we know that $v \in C^0(\mathbb{R}^N)$. By the assumptions on v there exist $\varepsilon, R > 0$ such that $\triangle v > (\gamma + 2\varepsilon)v$ a. e. in $\mathbb{R}^N \setminus B_R$. For $\rho \geq R$ and $a > 0$ let $\vartheta_{\rho,a} : [\rho, \rho + a] \to \mathbb{R}$ be the solution of

$$\begin{cases} \vartheta'' = (\gamma + \varepsilon)\vartheta & \text{in } (\rho, \rho + a) \\ \vartheta(\rho) = Ce^{\sqrt{\gamma}\rho} \\ \vartheta(\rho + a) = Ce^{\sqrt{\gamma}(\rho+a)}. \end{cases}$$

That is, $\vartheta_{\rho,a}(r) = A_{\rho,a}e^{-\sqrt{\gamma+\varepsilon}\,r} + B_{\rho,a}e^{\sqrt{\gamma+\varepsilon}\,r}$, with

$$A_{\rho,a} = Ce^{(\sqrt{\gamma}+\sqrt{\gamma+\varepsilon})\rho}\left(1 - \frac{e^{\sqrt{\gamma}a} - e^{-\sqrt{\gamma+\varepsilon}a}}{e^{\sqrt{\gamma+\varepsilon}a} - e^{-\sqrt{\gamma+\varepsilon}a}}\right),$$

$$B_{\rho,a} = Ce^{(\sqrt{\gamma}-\sqrt{\gamma+\varepsilon})\rho}\frac{e^{\sqrt{\gamma}a} - e^{-\sqrt{\gamma+\varepsilon}a}}{e^{\sqrt{\gamma+\varepsilon}a} - e^{-\sqrt{\gamma+\varepsilon}a}}.$$

The function $\theta_{\rho,a}(x) := \vartheta_{\rho,a}(|x|)$ satisfies

$$\triangle\theta_{\rho,a}(x) = (\gamma + \varepsilon)\vartheta_{\rho,a}(|x|) + \frac{N-1}{|x|}\vartheta'_{\rho,a}(|x|), \qquad x \in B_{\rho+a} \setminus B_\rho.$$

Since $\vartheta'_{\rho,a}(|x|) \leq \sqrt{\gamma+\varepsilon}\vartheta_{\rho,a}(|x|)$, there exists $\tilde\rho \geq R$ independent of a such that $\triangle\theta_{\tilde\rho,a} - (\gamma+2\varepsilon)\theta_{\tilde\rho,a} \leq 0$ in $B_{\tilde\rho+a} \setminus B_{\tilde\rho}$. Hence, v and $\theta_{\tilde\rho,a}$ are respectively a sub and a supersolution of $-\triangle + (\gamma + 2\varepsilon) = 0$ in $B_{\tilde\rho+a} \setminus B_{\tilde\rho}$ and $v \leq \theta_{\tilde\rho,a}$ on $\partial(B_{\tilde\rho+a} \setminus B_{\tilde\rho})$. Consequently, the weak maximum principle yields $v \leq \theta_{\tilde\rho,a}$ in $B_{\tilde\rho+a} \setminus B_{\tilde\rho}$, for any $a > 0$. Therefore, for $|x| > \tilde\rho$ we get

$$v(x) \leq \lim_{a\to\infty}\theta_{\tilde\rho,a}(x) = Ce^{(\sqrt{\gamma}+\sqrt{\gamma+\varepsilon})\tilde\rho}e^{-\sqrt{\gamma+\varepsilon}\,|x|},$$

which concludes the proof. ∎

The assumptions in Lemma 2.2 are sharp. Indeed, the function $v(x) = e^{(\sqrt{\gamma}+\varepsilon)x}$, $\varepsilon > 0$ and $x \in \mathbb{R}$, shows that the exponential factor $\sqrt{\gamma}$ in the first assumption is optimal. On the other hand, owing to the next example from [1], the strict inequality in the second assumption is needed: let $v \in C^2(\mathbb{R})$ be a function such that $v(x) = (1 + |x|)e^{-\sqrt{\gamma}|x|}$ for $|x|$ grater than some positive constant R. Then,

$$\lim_{|x|\to\infty} v(x)e^{-\sqrt{\gamma}|x|} = 0, \qquad \lim_{|x|\to\infty}\frac{v''(x)}{v(x)} = \gamma,$$

but the conclusion of Lemma 2.2 does not hold.

Proposition 4. *Let v be a solution of* (21) *and assume that* (7), (9), (10) *hold. Then, for any $0 < \gamma < \zeta + c^2/4$ there exists a positive constant $k = k_\gamma$ such that*

$$\forall\, x \in \mathbb{R}^N, \quad v(x) + |\nabla v(x)| \leq ke^{-\sqrt{\gamma}\,|x|}.$$

Proof. It is sufficient to prove the statement for $c^2/4 < \gamma < \zeta + c^2/4$. Set

$$z(x) := \frac{f(x, v(x)e^{-\frac{c}{2}x\cdot e})}{v(x)e^{-\frac{c}{2}x\cdot e}}.$$

The function z belongs to $L^\infty(\mathbb{R}^N)$ because f is locally Lipschitz continuous and (7) holds. By (7) and (9) we get

$$\frac{\triangle v(x)}{v(x)} = -z(x) + \frac{c^2}{4} \geq -f_s(x,0) + \frac{c^2}{4} \quad \text{for a. e. } x \in \mathbb{R}^N. \tag{22}$$

Hence, by (19),

$$\liminf_{|x|\to\infty} \frac{\triangle v(x)}{v(x)} \geq \zeta + \frac{c^2}{4} > \gamma.$$

The function $v(x)e^{-\sqrt{\gamma}|x|}$ is less than $v(x)e^{-\frac{c}{2}x\cdot e}$, which is bounded. Lemma 2.2 then yields $v(x) \leq C_0 e^{-\sqrt{\gamma}\,|x|}$ for some constant $C_0 > 0$. Now that we have derived the exponential decay of v, the estimate on ∇v follows from (22) by standard arguments. Indeed, by the L^p estimates, for every $1 \leq p < \infty$ there exists $\tilde{C}_1 > 0$ such that

$$\forall\, x \in \mathbb{R}^N, \quad \|v\|_{W^{2,p}(B_1(x))} \leq \tilde{C}_1 \|v\|_{L^\infty(B_2(x))},$$

where $\tilde{C}_1$ depends on $\|z\|_{L^\infty(\mathbb{R}^N)}$ and not on x. Hence, using the injection of $W^{2,p}$ in C^1 for $p > N$, we can find another constant $C_1 > 0$ such that $\|\nabla v\|_{L^\infty(B_1(x))} \leq C_1 e^{-\sqrt{\gamma}\,|x|}$. □

2.3. **Comparison principle.** In this section, we derive a comparison result between solutions and supersolutions of (21). In particular this yields the uniqueness result. The method here consists in assuming, by way of contradiction, that the set $\mathcal{O}$ where the supersolution lies below the solution is nonempty. Then, thanks to the self-adjoint structure of the linear part of the operator and the exponential decay of the solution, one gets a contradiction by using the Stokes theorem and Hopf's lemma. We mention that another approach can be followed. Based on the maximum principle, it applies also to operators with non-self-adjoint linear part and hence will come handy in the mixed periodic/shift case (see Section 3.3). We have chosen to present the first method in the pure shift framework, because it is of independent interest and also because it is the natural extension of the method used in [1] to prove nonexistence of solutions when $\lambda_1 = 0$. The main difficulty with respect to the one dimensional case treated in [1] is that $\mathcal{O}$ could be non-smooth and then neither the Stokes theorem nor Hopf's lemma do apply directly.

Theorem 2.3. *Assume that* (7)*,* (9)*,* (10) *hold. Let v be a solution of* (21) *and $w \in W^{2,N}_{loc}(\mathbb{R}^N)$ be a positive supersolution of the elliptic equation in* (21)*, that is,*

$$-\triangle w + \frac{c^2}{4} w \geq f(x, w(x)e^{-\frac{c}{2}x\cdot e})e^{\frac{c}{2}x\cdot e} \quad \textit{a. e. in } \mathbb{R}^N.$$

In addition, assume that there exist $k_0, k_1 > 0$ such that

$$w(x) \leq k_0 \;\Rightarrow\; |\nabla w(x)| \leq k_1. \tag{23}$$

Then $v \leq w$.

Proof. Set

$$\mathcal{O} := \{x \in \mathbb{R}^N \;:\; v(x) > w(x)\}$$

and assume, by way of contradiction, that $\mathcal{O} \neq \emptyset$. We distinguish different cases.

Case 1: $\mathcal{O} = \mathbb{R}^N$.

Multiplying the equation for v by w and that for w by v and integrating the difference over B_R we obtain, for any $R > 0$,

$$\int_{B_R} (-w\triangle v + v\triangle w) \leq \int_{B_R} [wf(x, ve^{-\frac{c}{2}x\cdot e}) - vf(x, we^{-\frac{c}{2}x\cdot e})]e^{\frac{c}{2}x\cdot e}.$$

The Stokes theorem yields

$$\int_{\partial B_R} (v\nabla w - w\nabla v)\cdot\nu \leq \int_{B_R} \left[\frac{f(x, ve^{-\frac{c}{2}x\cdot e})}{ve^{-\frac{c}{2}x\cdot e}} - \frac{f(x, we^{-\frac{c}{2}x\cdot e})}{we^{-\frac{c}{2}x\cdot e}}\right] vw. \tag{24}$$

By Proposition 4, there exists $R_0 > 0$ such that $v(x) \le k_0$ for $|x| \ge R_0$. Hence, by assumption (23), for $|x| \ge R_0$ $w(x) < k_0$ and $|\nabla w(x)| \le k_1$. Using again Proposition 4, we find that the left-hand side of (24) goes to zero as $R \to \infty$. This is a contradiction because, by (9), the right-hand side is nonincreasing in R and it is negative if $|D \cap B_R| > 0$, where D is the set in (9).

Case 2: $\mathcal{O} \neq \mathbb{R}^N$.

We would like to proceed as before, applying the Stokes theorem in $\mathcal{O}$, and deriving (24) with B_R replaced by $\mathcal{O}$. The problem is that $\mathcal{O}$ could be unbounded and it is not necessarily smooth. To deal with this, we introduce a family of cut-off functions. Let $\beta \in C^\infty(\mathbb{R})$ be such that

$$\beta = 0 \text{ in } (-\infty, 1/2], \qquad 0 < \beta' < 4 \text{ in } (1/2, 1), \qquad \beta = 1 \text{ in } [1, +\infty).$$

Then, we define $\beta_\varepsilon(s) := \beta(s/\varepsilon)$. Let us set $\sigma := v - w$. For $\varepsilon > 0$ we get

$$\int_{\mathbb{R}^N} (-w\triangle v + v\triangle w)\beta_\varepsilon(\sigma) \le \int_{\mathbb{R}^N} [wf(x, ve^{-\frac{c}{2}x\cdot e}) - vf(x, we^{-\frac{c}{2}x\cdot e})]e^{\frac{c}{2}x\cdot e}\beta_\varepsilon(\sigma). \tag{25}$$

Note that the function $\beta_\varepsilon(\sigma)$ is compactly supported because β_ε vanishes on $(-\infty, \varepsilon/2]$ and $\sigma(x) < \varepsilon/2$ for $|x|$ large enough by Proposition 4. The Stokes theorem yields

$$\begin{aligned}\int_{\mathbb{R}^N} (-w\triangle v + v\triangle w)\beta_\varepsilon(\sigma) &= \int_{\mathbb{R}^N} \beta'_\varepsilon(\sigma)\nabla\sigma \cdot (w\nabla v - v\nabla w)\\ &= \int_{\mathcal{O}} \beta'_\varepsilon(\sigma) w|\nabla\sigma|^2 - \int_{\mathcal{O}} \beta'_\varepsilon(\sigma)\sigma\nabla\sigma\cdot\nabla w.\end{aligned}$$

The function $\beta'_\varepsilon(\sigma)\sigma\nabla\sigma\cdot\nabla w$ converges pointwise to 0 as $\varepsilon \to 0^+$ and it satisfies

$$|\beta'_\varepsilon(\sigma)\sigma\nabla\sigma\cdot\nabla w| \le |\sigma||\nabla\sigma||\nabla w|\frac{4}{\varepsilon}\chi_{\{\sigma<\varepsilon\}} \le 4|\nabla\sigma||\nabla w|,$$

where $\chi_{\{\sigma<\varepsilon\}}$ stands for the characteristic function of the set $\{x : \sigma(x) < \varepsilon\}$. Since $4|\nabla\sigma||\nabla w| \in L^1(\mathcal{O})$, it follows from Lebesgue's dominated convergence theorem that the left-hand side of (25) satisfies

$$\limsup_{\varepsilon\to 0^+} \int_{\mathbb{R}^N} (-w\triangle v + v\triangle w)\beta_\varepsilon(\sigma) = \limsup_{\varepsilon\to 0^+} \int_{\mathcal{O}} w|\nabla\sigma|^2\beta'_\varepsilon(\sigma) \ge 0. \tag{26}$$

We argue differently according to whether w is a solution of the elliptic equation in (21) or not. Let us denote by $\mathcal{U}$ the set where w is a strict supersolution, that is,

$$\mathcal{U} := \left\{ x \in \mathbb{R}^N : \ -\triangle w(x) + \frac{c^2}{4}w(x) > f(x, w(x)e^{-\frac{c}{2}x\cdot e})e^{\frac{c}{2}x\cdot e} \right\}.$$

Case 2a: $|\mathcal{U} \cap \mathcal{O}| > 0$.

By (26) and (9) we get

$$\begin{aligned} 0 &\le \limsup_{\varepsilon\to 0^+} \int_{\mathbb{R}^N} \left(f(x, v(x)e^{-\frac{c}{2}x\cdot e})e^{\frac{c}{2}x\cdot e}w - \frac{c^2}{4}vw + v\triangle w \right)\beta_\varepsilon(\sigma)\\ &\le \limsup_{\varepsilon\to 0^+} \int_{\mathbb{R}^N} \left(f(x, w(x)e^{-\frac{c}{2}x\cdot e})e^{\frac{c}{2}x\cdot e} - \frac{c^2}{4}w + \triangle w \right) v\beta_\varepsilon(\sigma)\\ &= \int_{\mathcal{O}} \left(f(x, w(x)e^{-\frac{c}{2}x\cdot e})e^{\frac{c}{2}x\cdot e} - \frac{c^2}{4}w + \triangle w \right) v.\end{aligned}$$

This is a contradiction because the last term is strictly negative (possibly equal to $-\infty$).

Case 2b: $|\mathcal{U} \cap \mathcal{O}| = 0$.

By (9), the right-hand side of (25) is nonpositive. Hence, by (26),

$$\lim_{\varepsilon\to 0^+}\int_{\mathcal{O}} w|\nabla\sigma|^2\beta'_\varepsilon(\sigma)=0. \tag{27}$$

We will find a contradiction by using the fact that σ is a positive solution of a linear elliptic equation in $\mathcal{O}$, vanishing on the boundary, and applying the Hopf lemma in a smooth region of $\partial\mathcal{O}$. To this end, we will define such a suitable region. Let $x_0\in\mathcal{O}$ and $d:=\operatorname{dist}(x_0,\partial\mathcal{O})$. Thus, $B_d(x_0)\subset\mathcal{O}$ and there exists a point $x_1\in\partial\mathcal{O}\cap\partial B_d(x_0)$. Since the function σ is a positive solution of a linear elliptic equation in $B_d(x_0)$ (because $f(x,\cdot)$ is Lipschitz continuous) and vanishes on $x_1\in\partial B_d(x_0)$, the Hopf lemma implies that $\nabla\sigma(x_1)\cdot(x_1-x_0)<0$. Let us consider a coordinate system for $\mathbb{R}^N$ such that $\nabla\sigma(x_1)\in\{0\}^{N-1}\times\mathbb{R}^+$ and denote the generic point in $\mathbb{R}^N$ by $(y,z),\ y\in\mathbb{R}^{N-1},\ z\in\mathbb{R}$. From the implicit function theorem it follows that, in a suitable neighborhood of x_1, the set $\partial\mathcal{O}$ is given by $\{(y,z)\in\mathbb{R}^N\ :\ y\in A,\ z=F(y)\}$, where A is a domain in $\mathbb{R}^{N-1}$ and F is a function in $C^1(A)$, and in this neighborhood $\mathcal{O}\subset\{z>F(y)\}$. Furthermore, there exist $A'\subset\subset A$, with $|A'|>0$, and $k,\gamma>0$ such that the bounded set

$$\mathcal{O}':=\{(y,z)\in\mathbb{R}^N\ :\ y\in A',\ F(y)<z<F(y)+k\}$$

is contained in $\mathcal{O}$ and $\partial_z\sigma\geq\gamma$ holds in $\mathcal{O}'$. Therefore,

$$\begin{aligned}\int_{\mathcal{O}} w|\nabla\sigma|^2\beta'_\varepsilon(\sigma)\geq\int_{\mathcal{O}'} w|\nabla\sigma|^2\beta'_\varepsilon(\sigma)&\geq\gamma\left(\inf_{\mathcal{O}'}w\right)\int_{A'}dy\int_{F(y)}^{F(y)+k}\beta'_\varepsilon(\sigma)\partial_z\sigma\,dz\\&=\gamma\left(\inf_{\mathcal{O}'}w\right)\int_{A'}\beta_\varepsilon(\sigma(y,F(y)+k))\,dy\geq\gamma\left(\inf_{\mathcal{O}'}w\right)|A'|\beta_\varepsilon(\gamma k).\end{aligned}$$

Since $\inf_{\mathcal{O}'}w>0$ and $\beta_\varepsilon(\gamma k)\to 1$ as $\varepsilon\to 0^+$, this yields a contradiction with (27). □

2.4. **Existence and uniqueness of travelling waves.**

Proof of Theorem 1.1. Since problems (2) and (21) are equivalent, the uniqueness result for (2) immediately follows from the comparison principle in Theorem 2.3. Recall that any solution of (21) belongs to $W^{1,\infty}(\mathbb{R}^N)$ by Proposition 4 and then satisfies (23). Let us now prove the criterion for existence in Theorem 1.1.

Case 1: $\lambda_1<0$.

We proceed exactly as in [2]. By Proposition 1 there exists $R>0$ such that $\lambda(R)<0$. Define the function

$$\underline{U}(x):=\begin{cases}\kappa\varphi_R(x) & x\in B_R\\ 0 & \text{otherwise,}\end{cases}$$

with $\kappa>0$ to be chosen. Since

$$-\triangle(\kappa\varphi_R)-ce\cdot\nabla(\kappa\varphi_R)=(f_s(x,0)+\lambda(R))\kappa\varphi_R\quad\text{a. e. in } B_R,$$

$f(x,0)=0$ by (7) and $f(x,\cdot)\in C^1([0,\delta])$, it follows that, for κ small enough, $\underline{U}$ satisfies $-\triangle\underline{U}-ce\cdot\nabla\underline{U}\leq f(x,\underline{U})$ a. e. in B_R. As was shown in [3], $\underline{U}$ is a weak subsolution of the equation in (2). On the other hand, the function $\overline{U}(x)\equiv S$ (where S is the constant in (8)) is a supersolution of the equation in (2). Also, choosing a smaller κ if need be, we get $\underline{U}\leq\overline{U}$. Consequently, (see e. g. [3]) we find a function U such that

$$\begin{cases}\triangle U+ce\cdot\nabla U+f(x,U)=0 & \text{a. e. in } \mathbb{R}^N\\ \underline{U}\leq U\leq\overline{U} & \text{in } \mathbb{R}^N.\end{cases}$$

The strong maximum principle implies that U is strictly positive and then satisfies (2).

Case 2: $\lambda_1 \geq 0$.

From Proposition 2 it follows that $\lambda_1(-\tilde{\mathcal{L}}, \mathbb{R}^N) = \lambda_1 \geq 0$. Assume, by way of contradiction, that (2) admits a solution U, that is, (21) admits the solution $v(x) = U(x)e^{\frac{c}{2}x\cdot e}$. Let φ be a generalized principal eigenfunction of $-\mathcal{L}$ in $\mathbb{R}^N$ (cf. Proposition 1). Then, the function $\tilde{\varphi}(x) := \varphi(x)e^{\frac{c}{2}x\cdot e}$ is a positive solution of $-\tilde{\mathcal{L}}\tilde{\varphi} = \lambda_1\tilde{\varphi}$ a. e. in $\mathbb{R}^N$. Normalize $\tilde{\varphi}$ in such a way that $\tilde{\varphi}(0) < v(0)$. By (9) we see that

$$-\triangle\tilde{\varphi} + \frac{c^2}{4}\tilde{\varphi} = (f_s(x,0) + \lambda_1)\tilde{\varphi} \geq f(x, \tilde{\varphi}(x)e^{-\frac{c}{2}x\cdot e})e^{\frac{c}{2}x\cdot e} \quad \text{a. e. in } \mathbb{R}^N,$$

that is, $\tilde{\varphi}$ is a supersolution of the equation in (21). Let us show that $\tilde{\varphi}$ satisfies (23), for some $k_0, k_1 > 0$. By interior elliptic estimates and Harnack inequality, for $1 \leq p < \infty$ we get

$$\forall\, x \in \mathbb{R}^N, \quad \|\tilde{\varphi}\|_{W^{2,p}(B_1(x))} \leq C_1\|\tilde{\varphi}\|_{L^\infty(B_2(x))} \leq C_2\tilde{\varphi}(x),$$

with C_1, C_2 positive constants independent of x. As a consequence, the embedding theorem yields $|\nabla\tilde{\varphi}| \leq C_3\tilde{\varphi}$ in $\mathbb{R}^N$, for some $C_3 > 0$, and then (23) holds for any $k_0 > 0$ with $k_1 = C_3k_0$. Therefore, Theorem 2.3 implies $v \leq \tilde{\varphi}$, that is a contradiction. □

2.5. **Large time behavior.** In this section, we consider the initial value problem (1).

Proof of Theorem 1.2. Let S be the positive constant in (8) and set $S' := \max\{S, \|u_0\|_{L^\infty(\mathbb{R}^N)}\}$. The functions $u_1 \equiv 0$ and $u_2 \equiv S'$ are a sub and a supersolution respectively of (28), because $f(x,0) = 0$ and $f(x,S') \leq 0$ for a. e. $x \in \mathbb{R}^N$. Therefore, thanks to the parabolic weak maximum principle, the existence of a unique solution u of (1) with initial condition $u(0,x) = u_0(x)$ follows from standard parabolic theory of weak solutions. In addition, $0 \leq u \leq S'$. The function $\tilde{u}(t,x) := u(t, x + cte)$ is a solution of

$$\partial_t\tilde{u} = \triangle\tilde{u} + ce\cdot\nabla\tilde{u} + f(x,\tilde{u}), \qquad t > 0, \quad x \in \mathbb{R}^N, \tag{28}$$

with initial condition $\tilde{u}(0,x) = u_0(x)$. Let w be the solution of (28) with initial datum $w(0,x) = S'$. By comparison, $\tilde{u} \leq w \leq S'$ in $\mathbb{R}^+ \times \mathbb{R}^N$. Furthermore, using once again the maximum principle, we infer that $t \mapsto w(t,x)$ is nonincreasing and then, as $t \to \infty$, it converges to a function $W(x)$ satisfying

$$\forall\, x \in \mathbb{R}^N, \qquad \limsup_{t\to\infty}\tilde{u}(t,x) \leq \lim_{t\to\infty} w(t,x) = W(x) \leq S'. \tag{29}$$

The L^p regularity theory up to the boundary (see e. g. [13] or Chapter VII in [14]) yields that, for any $\rho > 0$, there exists $C_\rho > 0$ such that

$$\forall\, t_0 \geq 0,\ \forall\, x_0 \in \mathbb{R}^N, \quad \|w\|_{N+1,(t_0,t_0+\rho)\times B_\rho(x_0)} \leq C_\rho. \tag{30}$$

Then, as $t \to \infty$, $w(t,x)$ converges to $W(x)$ locally uniformly in $\mathbb{R}^N$ and, as $n \to \infty$, $\partial_i w(t+n, x)$ and $\partial_{ij} w(t+n, x)$ converge respectively to $\partial_i W(x)$ and $\partial_{ij} W(x)$ weakly locally in $L^{N+1}(\mathbb{R}^+ \times \mathbb{R}^N)$. Clearly, $0 \leq W \leq S'$ and $\triangle W + ce\cdot\nabla W + f(x,W) = 0$ a. e. in $\mathbb{R}^N$. Furthermore, we claim that for any sequences $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+$ and $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$,

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty}|x_n| = +\infty \quad \Rightarrow \quad \lim_{n\to\infty} w(t_n, x_n) = 0. \tag{31}$$

Let us postpone the proof of (31) and conclude the proof. We consider two different cases.

Case 1: $\lambda_1 \geq 0$.

Owing to Theorem 1.1, the function W cannot be strictly positive and then, by the elliptic strong maximum principle, $W \equiv 0$. This shows that $w(t,x)$ converges locally uniformly to zero as $t \to \infty$. If this convergence is not uniform then there exists $\varepsilon > 0$, $(t_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ such that $t_n, |x_n| \to \infty$ as $n \to \infty$ and $w(t_n, x_n) \geq \varepsilon$ for every $n \in \mathbb{N}$. This is in contradiction with our claim (31). Therefore, $w(t,x)$ converges to zero as $t \to \infty$, uniformly with respect to $x \in \mathbb{R}^N$, and by (29) the same is true for $\tilde{u}$. Statement (i) then follows, because $u(t,x) = \tilde{u}(t, x - cte)$.

Case 2: $\lambda_1 < 0$.

By Proposition 1 there exists $R > 0$ such that $\lambda(R) < 0$. Consider the Dirichlet principal eigenfunction φ_R of $-\mathcal{L}$ in B_R. As we have seen in the case 1 of the proof of Theorem 1.1, for $\kappa > 0$ small enough the function

$$\underline{U}(x) := \begin{cases} \kappa\varphi_R(x) & x \in B_R \\ 0 & \text{otherwise} \end{cases}$$

is a weak subsolution of the elliptic equation in (2). Moreover, even if it means decreasing κ, we can assume that $\underline{U}(x) \leq \tilde{u}(1,x)$. Indeed, $\tilde{u}(1,x) > 0$ by the parabolic strong maximum principle. Thus, the comparison principle yields that the solution v of (28) with initial condition $v(0,x) = \underline{U}(x)$ is nondecreasing in t and satisfies

$$\forall\, t > 0,\ \forall\, x \in \mathbb{R}^N, \quad \underline{U}(x) \leq v(t,x) \leq \tilde{u}(t+1,x) \leq w(t+1,x). \tag{32}$$

Arguing as before one finds that, as t goes to infinity, $v(t,x)$ converges locally uniformly to a function V satisfying $\underline{U} \leq V \leq W \leq S'$ and $\triangle V + ce\cdot\nabla V + f(x,V) = 0$ a. e. in $\mathbb{R}^N$. Therefore, by the strong maximum principle $V > 0$ and then Theorem 1.1 yields $V \equiv W \equiv U$, where U is the unique solution of (2). We have shown that v and w converge locally uniformly to U as t goes to infinity and then, owing to (32), the same is true for $\tilde{u}$. Assume by contradiction that the convergence of $\tilde{u}$ to U is not globally uniform in x. Hence, there exist $\varepsilon > 0$, $(t_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} |x_n| = \infty$ and $|\tilde{u}(t_n,x_n) - U(x_n)| \geq \varepsilon$ for every $n \in \mathbb{N}$. Recall that $U(x)e^{\frac{c}{2}\cdot ex}$ is the unique solution of problem (21). Thus, Proposition 4 yields $\lim_{|x|\to\infty} U(x) = 0$. Consequently, for n big enough we have that

$$\varepsilon \leq \tilde{u}(t_n,x_n) - U(x_n) \leq \tilde{w}(t_n,x_n) - U(x_n).$$

Therefore, $\limsup_{n\to\infty} w(t_n,x_n) \geq \varepsilon$, which is impossible by (31). This means that $\lim_{t\to\infty} u(t, x + cte) = U(x)$, uniformly in $x \in \mathbb{R}^N$, and then statement (ii) holds.

It only remains to prove the claim (31). Assume that (31) does not hold. Then, there exist $\varepsilon > 0$, $(t_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ such that $t_n, |x_n| \to \infty$ as n goes to infinity and $w(t_n,x_n) \geq \varepsilon$ for every $n \in \mathbb{N}$. Using estimate (30) and the compact injection we find that, as n goes to infinity and up to subsequences, $w(t, x + x_n)$ converges to a function $\tilde{w}(t,x)$ uniformly in $[0,\rho] \times \overline{B}_\rho$, for any $\rho > 0$. Moreover, by (9) and (19), the function $\tilde{w}$ satisfies

$$\partial_t \tilde{w} \leq \triangle \tilde{w} + ce \cdot \nabla \tilde{w} - \zeta \tilde{w} \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^N,$$

with initial condition $\tilde{w}(0,x) = S'$. Define the function $\theta(t,x) := S'e^{-\zeta t}$. We see that $\partial_t \theta = -\zeta\theta$ and $\theta(0,x) = \tilde{w}(0,x)$. Therefore, the parabolic maximum principle yields $\tilde{w}(t,x) \leq \theta(t,x)$ for $t > 0$, $x \in \mathbb{R}^N$ and then $\lim_{t\to\infty} \tilde{w}(t,x) = 0$. This is impossible, because

$$\forall\, t > 0, \quad \tilde{w}(t,0) = \lim_{n\to\infty} w(t,x_n) \geq \limsup_{n\to\infty} w(t_n,x_n) \geq \varepsilon.$$

The proof of Theorem 1.2 is thereby complete. □

2.6. **Influence of the parameters on the critical speed c_0.** We examine now the dependence of the critical speed c_0 with respect to the amplitude of the reaction and diffusion terms. Consider the problem (12) with a and γ positive constants. Travelling wave solutions $u(t,x) = U(x - cte)$ of (12) satisfy

$$\begin{cases} a\triangle U + ce\cdot\nabla U + \gamma f(x,U) = 0 \quad \text{a. e. in } \mathbb{R}^N \\ U > 0 \quad \text{in } \mathbb{R}^N \\ U \quad \text{is bounded.} \end{cases} \tag{33}$$

The associated linearized operator about $U \equiv 0$ is given by

$$\mathcal{L}_{a,c,\gamma} w := a\triangle w + ce\cdot\nabla w + \gamma f_s(x,0)w.$$

For any $a, c, \gamma > 0$ set $\lambda_1(a,c,\gamma) := \lambda_1(-\mathcal{L}_{a,c,\gamma}, \mathbb{R}^N)$.

Theorems 1.1 and 1.2 hold true with (1), (2) and λ_1 replaced respectively by (12), (33) and $\lambda_1(a,c,\gamma)$. This is readily seen through the time change $t \to t/\gamma$, which reduces problem (12) to

$$\partial_t u = \triangle u + \frac{\gamma}{a} f(x - \frac{c}{a}te, u), \qquad t > 0, \quad x \in \mathbb{R}^N. \tag{34}$$

Indeed, if f satisfies (7)-(10) then the same is true for $a^{-1}\gamma f$ and we can apply Theorems 1.1 and 1.2 with

$$c = \frac{c}{a}, \qquad f = \frac{\gamma}{a} f, \qquad \lambda_1 = \lambda_1(1, c/a, \gamma/a) = \frac{1}{a}\lambda_1(a,c,\gamma).$$

We show that the critical speed for the problem (12) is given by

$$c_0(a,\gamma) := \begin{cases} 2\sqrt{-\lambda_1(a/\gamma,0,1)\, a\gamma} & \text{if } \lambda_1(a/\gamma,0,1) < 0 \\ 0 & \text{otherwise.} \end{cases}$$

That is,

Proposition 5. $\lambda_1(a,c,\gamma) < 0$ *iff* $c < c_0(a,\gamma)$.

The following result describes the behavior of the function $(a,\gamma) \mapsto c_0(a,\gamma)$.

Theorem 2.4. *Assume that* (10) *holds. If* $f_s(x,0) \leq 0$ *for a. e.* $x \in \mathbb{R}^N$ *then* $c_0 \equiv 0$. *Otherwise,* $c_0 \in C^0(\mathbb{R}^+ \times \mathbb{R}^+)$ *and there exists* $\sigma > 0$ *such that*

$$c_0(a,\gamma) > 0 \;\Leftrightarrow\; \frac{a}{\gamma} < \sigma,$$

$$c_0(a,\gamma) \leq 2\sqrt{\sup_{x\in\mathbb{R}^N} f_s(x,0)\, a\gamma},$$

$\gamma \mapsto c_0(a,\gamma)$ *is nondecreasing, and it is strictly increasing for* $\gamma > \frac{a}{\sigma}$,

$$\lim_{\gamma\to\infty} \frac{c_0(a,\gamma)}{\sqrt{a\gamma}} = 2\sqrt{\sup_{x\in\mathbb{R}^N} f_s(x,0)} \quad \textit{uniformly in } a \in (0,R), \textit{ for any } R > 0.$$

Owing to Proposition 5, Theorem 2.4 has the following biological interpretation:

(1) if the population diffusion is rather low then a slow climate change is sufficient for extinction;

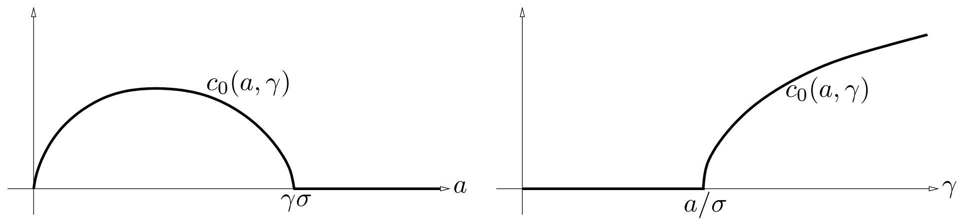

FIGURE 1. graphs of $a \mapsto c_0(a,\gamma)$ and $\gamma \mapsto c_0(a,\gamma)$

(2) if the diffusion is too high then extinction occurs even if the climate conditions do not change at all;
(3) the larger is the amplitude of the reaction term, the higher are the chances of persistence of the species.

Proof of Proposition 5. Applying Proposition 3 with $c = a^{-1}c$ and $f = a^{-1}\gamma f$ we derive

$$\lambda_1(1, c/a, \gamma/a) < 0 \;\Leftrightarrow\; \frac{c}{a} < \tilde{c}_0,$$

where

$$\tilde{c}_0 = \begin{cases} 2\sqrt{-\lambda_1(1,0,\gamma/a)} & \text{if } \lambda_1(1,0,\gamma/a) < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, since $\lambda_1(a,c,\gamma) = a\lambda_1(1,c/a,\gamma/a)$ and $a^2\lambda_1(1,0,\gamma/a) = \lambda_1(a/\gamma,0,1)\,a\gamma$,

$$\lambda_1(a,c,\gamma) < 0 \;\Leftrightarrow\; c < a\tilde{c}_0 = c_0(a,\gamma).$$

□

To prove Theorem 2.4 we will make use of a result for elliptic operators in self-adjoint form which we quote from [6].

Lemma 2.5. *The function $\rho \mapsto \lambda_1(\rho,0,1) : \mathbb{R}^+ \to \mathbb{R}$ is continuous, nondecreasing and satisfies*

$$\lim_{\rho\to 0^+} \lambda_1(\rho,0,1) = -\sup_{x\in\mathbb{R}^N} f_s(x,0),$$

$$-\limsup_{|x|\to\infty} f_s(x,0) \le \lim_{\rho\to\infty} \lambda_1(\rho,0,1) \le -\liminf_{|x|\to\infty} f_s(x,0).$$

Proof of Theorem 2.4. If $\sup_{x\in\mathbb{R}^N} f_s(x,0) > 0$ then Lemma 2.5 and condition (10) imply that there exists a unique positive number σ such that $\lambda_1(\sigma,0,1) = 0$. The result then follows from Lemma 2.5. □

3. Mixed periodic/shift case

In Section 1.3 we introduced the following notation:

$$T := \frac{l}{c},$$

$$h(t,x,s) := f(x,s) + g(x+cte,s),$$

$$\mathcal{P}w := \partial_t w - \triangle w - ce\cdot\nabla w - h_s(t,x,0)w,$$

$$\mu_1 := \mu_1(\mathcal{P}, \mathbb{R}^{N+1}).$$

Henceforth, we set

$$\zeta := -\lim_{R\to\infty} \sup_{\substack{t\in\mathbb{R}\\ |x|>R}} h_s(t,x,0). \tag{35}$$

Note that $\zeta \in \mathbb{R}$ because $h_s(t,x,0) \in L^\infty(\mathbb{R}^N)$ thanks to the Lipschitz continuity of f and g. If (16) holds, then $\zeta > 0$.

3.1. **Generalized time-periodic principal eigenvalue and definitions of $\underline{c}$, $\overline{c}$.** The definition (17) of generalized principal eigenvalue for time-periodic parabolic operators has been introduced in [15], in the case of operators in divergence form with Hölder continuous coefficients. Here, we will use some results about $\mu_1(\mathcal{P}, \mathbb{R}\times\Omega)$ which we quote form [6], regarding operators with only bounded zero-order coefficient such as $\mathcal{P}$.

The first property - which motivates the name of "generalized principal eigenvalue" - is that if Ω is a bounded smooth domain in $\mathbb{R}^N$ then $\mu_1(\mathcal{P}, \mathbb{R}\times\Omega)$ coincides with the T-periodic Dirichlet principal eigenvalue of $\mathcal{P}$ in $\mathbb{R}\times\Omega$, which is the unique constant μ such that the eigenvalue problem

$$\begin{cases} \mathcal{P}\psi = \mu\psi & \text{a. e. in } \mathbb{R}\times\Omega \\ \psi = 0 & \text{on } \mathbb{R}\times\partial\Omega \\ \psi \ \text{ is } T\text{-periodic in } t \end{cases}$$

admits a positive solution ψ (called Dirichlet principal eigenfunction) which is unique up to multiplication. For the existence and uniqueness of the Dirichlet principal eigenvalue for time-periodic parabolic operators with smooth coefficients see [10]. From now on, $\mu(R)$ will denote the Dirichlet principal eigenvalue of $\mathcal{P}$ in $\mathbb{R}\times B_R$ and ψ_R the associated principal eigenfunction such that $\psi_R(0) = 1$. The following result is the analogue of Proposition 1.

Proposition 6. ([6]) *The function $\mu(R) : \mathbb{R}^+ \to \mathbb{R}$ decreases and satisfies*

$$\lim_{R\to\infty} \mu(R) = \mu_1.$$

Furthermore, there exists a generalized principal eigenfunction of $\mathcal{P}$ in $\mathbb{R}^{N+1}$, that is, a T-periodic in t positive function $\psi \in W^{1,2}_{p,loc}(\mathbb{R}^{N+1})$, for any $1 \le p < \infty$, such that

$$\mathcal{P}\psi = \mu_1\psi \quad \textit{a. e. in } \mathbb{R}^{N+1}.$$

As in Section 2, we start by a change of function: U is a solution of (4) if and only if $v(t,x) := U(t,x)e^{\frac{c}{2}x\cdot e}$ is a solution of

$$\begin{cases} \partial_t v = \triangle v + h(t,x,v(t,x)e^{-\frac{c}{2}x\cdot e})e^{\frac{c}{2}x\cdot e} - \dfrac{c^2}{4}v, & (t,x)\in\mathbb{R}^{N+1} \\ v > 0 \quad \text{in } \mathbb{R}^{N+1} \\ v(t,x)e^{-\frac{c}{2}x\cdot e} \quad \text{is bounded} \\ v \quad \text{is } T\text{-periodic in } t. \end{cases} \tag{36}$$

Under assumption (13), the linearized operator about $w \equiv 0$ associated with the equation in (36) is

$$\tilde{\mathcal{P}}w := \partial_t w - \triangle w - (h_s(t,x,0) - c^2/4)w.$$

Since $\tilde{\mathcal{P}}\phi = (\mathcal{P}(\phi e^{-\frac{c}{2}x\cdot e}))e^{\frac{c}{2}x\cdot e}$, from definition (17) it follows that

Proposition 7. *The T-periodic principal eigenvalue of $\mathcal{P}$ and $\tilde{\mathcal{P}}$ coincide in any domain $\mathbb{R}\times\Omega$, with $\Omega\subset\mathbb{R}^N$. That is, $\mu_1(\mathcal{P}, \mathbb{R}\times\Omega) = \mu_1(\tilde{\mathcal{P}}, \mathbb{R}\times\Omega)$.*

Since $h_s(t,x,0) = f_s(x,0) + g_s(x+cte,0)$, the dependence on c of the operator $\tilde{\mathcal{P}}$ is more complicated than that of the operator $\tilde{\mathcal{L}}$ introduced in Section 2.1. Thus, the existence of a critical speed is not clear in this framework. However, we can define a subcritical speed $\underline{c}$ and a supercritical speed $\overline{c}$. To this end, set

$$\mathcal{P}_0 w := \partial_t w - \triangle w - f_s(x,0)w,$$

and $\mu_0 := \mu_1(\mathcal{P}_0, \mathbb{R}^{N+1})$, where $\mu_1(\mathcal{P}_0, \mathbb{R}^{N+1})$ is the generalized T-periodic principal eigenvalue of $\mathcal{P}_0$ in the whole space.

Definition 3.1. (Sub and supercritical speeds) We define

$$\underline{c} := \begin{cases} 2\sqrt{\inf\limits_{x\in\mathbb{R}^N} g_s(x,0) - \mu_0} & \text{if } \mu_0 < \inf\limits_{x\in\mathbb{R}^N} g_s(x,0) \\ 0 & \text{otherwise,} \end{cases}$$

$$\overline{c} := \begin{cases} 2\sqrt{\sup\limits_{x\in\mathbb{R}^N} g_s(x,0) - \mu_0} & \text{if } \mu_0 < \sup\limits_{x\in\mathbb{R}^N} g_s(x,0) \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 8. *If $c < \underline{c}$, then $\mu_1 < 0$. If $c \geq \overline{c}$, then $\mu_1 \geq 0$.*

Proof. For any function $\phi \in W^{1,2}_{N,loc}(\mathbb{R}^{N+1})$, T-periodic in t and positive, the following inequalities hold:

$$\left(\mathcal{P}_0 - \sup_{x\in\mathbb{R}^N} g_s(x,0) + \frac{c^2}{4}\right)\phi \leq \tilde{\mathcal{P}}\phi \leq \left(\mathcal{P}_0 - \inf_{x\in\mathbb{R}^N} g_s(x,0) + \frac{c^2}{4}\right).$$

Hence, using definition (17) we derive

$$\mu_0 - \sup_{x\in\mathbb{R}^N} g_s(x,0) + \frac{c^2}{4} \leq \mu_1(\tilde{\mathcal{P}}, \mathbb{R}^{N+1}) \leq \mu_0 - \inf_{x\in\mathbb{R}^N} g_s(x,0) + \frac{c^2}{4}.$$

The statement then follows, because $\mu_1(\tilde{\mathcal{P}}, \mathbb{R}^{N+1}) = \mu_1$ by Proposition 7. $\square$

3.2. **Exponential decay.** The next result is the parabolic version of Lemma 2.2.

Lemma 3.2. *Let $v \in W^{1,2}_{N+1,loc}(\mathbb{R}^{N+1})$ and let $C, \gamma > 0$ be such that*

$$\forall\ (t,x) \in \mathbb{R}^{N+1}, \quad 0 < v(t,x) \leq Ce^{\sqrt{\gamma}|x|},$$

$$\limsup_{|x|\to\infty} \frac{\partial_t v(t,x) - \triangle v(t,x)}{v(t,x)} < -\gamma,$$

uniformly in $t \in \mathbb{R}$. Then, there exists a constant $\kappa > 0$ such that

$$\forall\ (t,x) \in \mathbb{R}^{N+1}, \quad v(t,x) \leq \kappa e^{-\sqrt{\gamma}\,|x|}.$$

Proof. We consider $v \in W^{1,2}_{N+1,loc}(\mathbb{R}^{N+1})$ so that we may apply the maximum principle (see e. g. [14]). The embedding theorem yields $v \in C^0(\mathbb{R}^{N+1})$. Let $\varepsilon, R > 0$ be such that $\partial_t v - \triangle v < (-\gamma - 2\varepsilon)v$ for a. e. $t \in \mathbb{R}$ and $|x| > R$. For $\rho, a > 0$, consider the same functions $\vartheta_{\rho,a}$ as in Lemma 2.2. Fix $\tau \in \mathbb{R}$ and define

$$\theta_{\rho,a}(t,x) := \vartheta_{\rho,a}(|x|) + (\tau - t)\delta_{\rho,a},$$

where $\delta_{\rho,a} = \frac{\varepsilon}{2}\min_{[\rho,\rho+a]}\vartheta_{\rho,a} > 0$. By computation, for $t \in \mathbb{R}$ and $\rho < |x| < \rho + a$ one gets

$$\begin{aligned}\partial_t\theta_{\rho,a} - \triangle\theta_{\rho,a}(x) &= -\delta_{\rho,a} - (\gamma+\varepsilon)\vartheta_{\rho,a}(|x|) - \frac{N-1}{|x|}\vartheta'_{\rho,a}(|x|) \\ &\geq -\left(\gamma + \frac{3}{2}\varepsilon\right)\vartheta_{\rho,a}(|x|) - \frac{N-1}{|x|}\vartheta'_{\rho,a}(|x|).\end{aligned}$$

Since $\vartheta'_{\rho,a}(|x|) \leq \sqrt{\gamma+\varepsilon}\vartheta_{\rho,a}(|x|)$, we can find $\tilde\rho > R$ in such a way that, for any $a > 0$,

$$\partial_t\theta_{\tilde\rho,a} - \triangle\theta_{\tilde\rho,a} > -(\gamma+2\varepsilon)\vartheta_{\tilde\rho,a}(|x|) > -(\gamma+2\varepsilon)\theta_{\tilde\rho,a} \quad \text{in } (-\infty,\tau)\times(B_{\tilde\rho+a}\setminus B_{\tilde\rho}).$$

We apply the comparison principle between v and $\theta_{\tilde\rho,a}$ in a cylinder $(t_0,\tau)\times(B_{\tilde\rho+a}\setminus B_{\tilde\rho})$. Let $t_0 < \tau$ be such that $\theta_{\tilde\rho,a}(t_0,x) \geq v(t_0,x)$ for $x \in B_{\tilde\rho+a}\setminus B_{\tilde\rho}$. For $t < \tau$ and $x \in \partial(B_{\tilde\rho+a}\setminus B_{\tilde\rho})$ we see that

$$\theta_{\tilde\rho,a}(t,x) \geq \vartheta_{\tilde\rho,a}(|x|) = Ce^{\sqrt{\gamma}|x|} \geq v(t,x).$$

Therefore, the parabolic weak maximum principle implies $\theta_{\tilde\rho,a} \geq v$ in $(t_0,\tau)\times(B_{\tilde\rho+a}\setminus B_{\tilde\rho})$, for any $a > 0$. Then, in particular, for $|x| > \tilde\rho$

$$v(\tau,x) \leq \lim_{a\to\infty}\theta_{\tilde\rho,a}(\tau,x) = Ce^{(\sqrt{\gamma}+\sqrt{\gamma+\varepsilon})\tilde\rho}e^{-\sqrt{\gamma+\varepsilon}\,|x|},$$

Since τ can be chosen arbitrarily, this concludes the proof. □

We can now derive the exponential decay of solutions of (4).

Proposition 9. *Let U be a solution of* (4) *and assume that* (13), (15), (16) *hold. Then, there exist two positive constants k,ε such that*

$$\forall\,(t,x)\in\mathbb{R}^{N+1}, \quad U(t,x) \leq ke^{-\varepsilon|x|}.$$

Proof. The function $v(t,x) := U(t,x)e^{\frac{c}{2}x\cdot e}$ solves (36) and by parabolic interior estimates we know that it belongs to $W^{1,2}_{N+1,loc}(\mathbb{R}^{N+1})$. Proceeding as in the proof of Proposition 4, we see that v satisfies the hypotheses of Lemma 3.2, for any $\gamma\in(c^2/4,\zeta+c^2/4)$ and some $C>0$. Therefore, for any $\gamma\in(c^2/4,\zeta+c^2/4)$ there exists $\kappa_\gamma > 0$ such that

$$\forall\,(t,x)\in\mathbb{R}^{N+1}, \quad v(t,x) \leq \kappa_\gamma e^{-\sqrt{\gamma}\,|x|}.$$

This concludes the proof because

$$U(t,x) = v(t,x)e^{-\frac{c}{2}x\cdot e} \leq \kappa_\gamma e^{(-\sqrt{\gamma}+\frac{c}{2})|x|}.$$

□

3.3. **Comparison principle.** In this section, we establish a comparison result that will be used to derive the necessary condition for the existence of pulsating travelling wave solutions as well as their uniqueness.

Theorem 3.3. *Assume that* (13), (15), (16) *hold. Let $\underline{U},\overline{U}\in W^{1,2}_{N+1,loc}(\mathbb{R}^{N+1})$ be respectively a nonnegative subsolution and a supersolution of the parabolic equation in* (4)*, T-periodic in t and satisfying*

$$\overline{U} > 0, \qquad \limsup_{|x|\to\infty}\underline{U}(t,x) \leq 0 \quad \textit{uniformly in } t\in\mathbb{R}.$$

Then $\underline{U}\leq\overline{U}$.

Proof. For $\varepsilon > 0$ define the set

$$K_\varepsilon := \{k > 0 \ :\ k\overline{U} \geq \underline{U} - \varepsilon \text{ in } \mathbb{R}^{N+1}\}.$$

Let us first show that it is nonempty. Note that $\underline{U},\overline{U}\in C^0(\mathbb{R}^{N+1})$ thanks to the embedding theorem. By hypothesis, for any $\varepsilon>0$ there exists $r(\varepsilon)>0$ such that

$$\forall\, t\in\mathbb{R},\ |x|\geq r(\varepsilon), \quad \underline{U}(t,x)-\varepsilon \leq 0. \tag{37}$$

Since $\overline{U}$ is positive and

$$\sup_{\substack{t\in\mathbb{R}\\ |x|<r(\varepsilon)}} \underline{U}(t,x) < \infty, \qquad \inf_{\substack{t\in\mathbb{R}\\ |x|<r(\varepsilon)}} \overline{U}(t,x) > 0,$$

because $\underline{U}$ and $\overline{U}$ are continuous and T-periodic in t, it follows that $k\overline{U} \geq \underline{U} - \varepsilon$ for k large enough, that is K_ε is nonempty.

For $\varepsilon > 0$ set $k(\varepsilon) := \inf K_\varepsilon$. Clearly, the function $k(\varepsilon)$ is nonincreasing in ε. Let us assume, by way of contradiction, that

$$k^* := \lim_{\varepsilon\to 0^+} k(\varepsilon) > 1$$

(with possibly $k^* = \infty$). For any $0 < \varepsilon < \sup_{\mathbb{R}^{N+1}} \underline{U}$, one sees that $k(\varepsilon) > 0$, $k(\varepsilon)\overline{U} - \underline{U} + \varepsilon \geq 0$ in $\mathbb{R}^{N+1}$ and there exists a sequence $(t_n^\varepsilon, x_n^\varepsilon)_{n\in\mathbb{N}}$ in $\mathbb{R}^{N+1}$ such that

$$\left(k(\varepsilon) - \frac{1}{n}\right)\overline{U}(t_n^\varepsilon, x_n^\varepsilon) < \underline{U}(t_n^\varepsilon, x_n^\varepsilon) - \varepsilon.$$

Owing to the periodicity in t, it is not restrictive to assume that the sequence $(t_n^\varepsilon)_{n\in\mathbb{N}}$ is contained in $[0,T)$. Moreover, $x_n^\varepsilon \in B_{r(\varepsilon)}$ for n large enough because, otherwise, it would mean that $\overline{U}(t_n^\varepsilon, x_n^\varepsilon) < 0$ by (37). Consequently, up to extraction of a suitable subsequence, $(t_n^\varepsilon, x_n^\varepsilon)$ converges to some $(t(\varepsilon), x(\varepsilon)) \in [0,T] \times \overline{B}_{r(\varepsilon)}$ as $n \to \infty$. Hence, $k(\varepsilon)\overline{U}(t(\varepsilon), x(\varepsilon)) \leq \underline{U}(t(\varepsilon), x(\varepsilon)) - \varepsilon$. Thus, for any $\varepsilon > 0$ the following properties hold:

$$k(\varepsilon)\overline{U} - \underline{U} + \varepsilon \geq 0 \ \text{ in } \mathbb{R}^{N+1}, \qquad (k(\varepsilon)\overline{U} - \underline{U} + \varepsilon)(t(\varepsilon), x(\varepsilon)) = 0. \tag{38}$$

We consider separately two different situations.

Case 1: $\liminf\limits_{\varepsilon\to 0^+} |x(\varepsilon)| < \infty$.

Then, there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+$ such that

$$\lim_{n\to\infty} \varepsilon_n = 0, \qquad \lim_{n\to\infty} t(\varepsilon_n) = \tau \in [0,T], \qquad \lim_{n\to\infty} x(\varepsilon_n) = \xi \in \mathbb{R}^N.$$

From (38) it follows that $k^* < \infty$ and that the function $W := k^*\overline{U} - \underline{U}$ is nonnegative and vanishes at (τ, ξ). Moreover,

$$\partial_t W - \triangle W - ce\cdot\nabla W \geq k^* h(t,x,\overline{U}) - h(t,x,\underline{U}) \quad \text{a. e. in } \mathbb{R}^{N+1}.$$

Since $k^* > 1$, condition (9) yields

$$\partial_t W - \triangle W - ce\cdot\nabla W \geq h(t,x,k^*\overline{U}) - h(t,x,\underline{U}) = z(t,x)W \quad \text{a. e. in } \mathbb{R}^{N+1},$$

with strict inequality a. e. in D, where the function z is defined by

$$z(t,x) := \begin{cases} \frac{h(t,x,k^*\overline{U}) - h(t,x,\underline{U})}{k^*\overline{U} - \underline{U}} & \text{if } k^*\overline{U} \neq \underline{U} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $z \in L^\infty_{loc}(\mathbb{R}^{N+1})$. Thus, the parabolic strong maximum principle yields $W = 0$ in $(-\infty, \tau) \times \mathbb{R}^N$ and then $W \equiv 0$ in $\mathbb{R}^{N+1}$ by periodicity in t. This is impossible because W is a strict supersolution of a linear equation in D.

Case 2: $\lim\limits_{\varepsilon\to 0^+} |x(\varepsilon)| = \infty$.

For $\varepsilon > 0$ set $W_\varepsilon := k(\varepsilon)\overline{U} - \underline{U} + \varepsilon$. Then, by (38), $W_\varepsilon \geq 0$ and $W_\varepsilon(t(\varepsilon), x(\varepsilon)) = 0$. Furthermore, for $\varepsilon > 0$ small and for a. e. $(t,x) \in \mathbb{R}^{N+1}$,

$$\begin{aligned} \partial_t W_\varepsilon - \triangle W_\varepsilon - ce\cdot\nabla W_\varepsilon &\geq k(\varepsilon)h(t,x,\overline{U}) - h(t,x,\underline{U}) \\ &\geq h(t,x,k(\varepsilon)\overline{U}) - h(t,x,\underline{U}). \end{aligned}$$

We claim that, for ε small enough, there exists a cylindrical domain $\mathcal{C} \subset \mathbb{R}^{N+1}$ containing $(t(\varepsilon), x(\varepsilon))$ such that

$$h(t,x,k(\varepsilon)\overline{U}) - h(t,x,\underline{U}) > 0 \quad \text{a. e. in } \mathcal{C}. \tag{39}$$

This is a contradiction, because in this case $\partial_t W_\varepsilon - \triangle W_\varepsilon - ce\cdot\nabla W_\varepsilon > 0$ a. e. in $\mathcal{C}$, and the parabolic strong maximum principle yields $W_\varepsilon = 0$ for $(t,x) \in \mathcal{C}$, $t < t(\varepsilon)$. Let us prove the claim. By (16) there exists a positive constant R such that $h_s(t,x,0) < 0$ for a. e. $t \in \mathbb{R}$, $|x| > R$. Hence, we can find $\varepsilon, \rho > 0$ small enough in such a way that $h_s(t,x,0) < 0$ a. e. in $\mathbb{R} \times B_\rho(x(\varepsilon))$. Taking a smaller ρ if need be, it is not restrictive to assume that $k(\varepsilon)\overline{U} < \underline{U}$ in the cylinder

$$\mathcal{C}_\rho(t(\varepsilon), x(\varepsilon)) := (t(\varepsilon) - \rho, t(\varepsilon) + \rho) \times B_\rho(x(\varepsilon)).$$

Now, we make use of the following property which is a consequence of assumptions (13) and (15):

$$\forall\, 0 \le s_1 \le s_2, \quad h(t,x,s_2) - h(t,x,s_1) \le h_s(t,x,0)(s_2 - s_1) \quad \text{for a. e. } (t,x) \in \mathbb{R}^{N+1}.$$

Hence,

$$h(t,x,k(\varepsilon)\overline{U}) - h(t,x,\underline{U}) \ge h_s(t,x,0)(k(\varepsilon)\overline{U} - \underline{U}) > 0 \quad \text{a. e. in } \mathcal{C}_\rho(t(\varepsilon), x(\varepsilon)),$$

i. e. (39) holds with $\mathcal{C} = \mathcal{C}_\rho(t(\varepsilon), x(\varepsilon))$.

We have shown that $k^* := \lim_{\varepsilon\to 0^+} k(\varepsilon) \le 1$. Consequently, from (38) it follows that

$$\underline{U} \le \lim_{\varepsilon\to 0^+} (k(\varepsilon)\overline{U} + \varepsilon) \le \overline{U} \quad \text{in } \mathbb{R}^{N+1}.$$

The proof is thus complete. □

3.4. **Convergence results for general time-periodic parabolic operators.** Since the next results are of independent interest, we state them for more general semilinear parabolic equations:

$$\partial_t u = \partial_i(a_{ij}(t,x)\partial_j u) + b_i(t,x)\partial_i u + h(t,x,u), \qquad t > 0,\ x \in \mathbb{R}^N. \tag{40}$$

Here $a_{ij} \in C^{0,1}(\mathbb{R}^{N+1})$, $b_i \in C^{0,\theta}(\mathbb{R}^{N+1})$ for some $0 < \theta < 1$, h is a Carathéodory function such that $h(t,x,0) \in L^\infty(\mathbb{R}^{N+1})$ and $s \mapsto h(t,x,s)$ is locally Lipschitz continuous, uniformly for a. e. $(t,x) \in \mathbb{R}^{N+1}$. We further assume that the matrix field $(a_{ij})_{ij}$ is symmetric and uniformly elliptic:

$$\forall\, (t,x) \in \mathbb{R}^{N+1},\ \xi \in \mathbb{R}^N, \quad \underline{a}|\xi|^2 \le a_{ij}(t,x)\xi_i\xi_j \le \overline{a}|\xi|^2,$$

for some constants $0 < \underline{a} \le \overline{a}$. The functions a_{ij}, b_i and h are also assumed to be T-periodic in t.

When the coefficients in the equation (40) are independent of time, it is well known that if $u(t,x)$ is the solution with initial datum $u(0,x) = u_0(x)$ which is a subsolution of the stationary equation, then $t \mapsto u(t,x)$ is nondecreasing. This result is extremely useful in analyzing the long term dynamics of parabolic equations and it has been used here in Section 2.5. It does not hold for equations with time dependent coefficients. However, we will prove next that it can be extended to equations whose coefficients are periodic in time, with the same period.

Theorem 3.4. *Let $v \in L^\infty(\mathbb{R}^{N+1})$ be a T-periodic in t subsolution (resp. supersolution) of (40). Assume that the solution u of (40) with initial datum $u(0,x) = v(0,x)$ exists for every $t > 0$. Then,*

$$\forall\, t \ge 0,\ x \in \mathbb{R}^N, \quad u(t+T,x) - u(t,x) \ge 0 \quad \textit{(resp. } \le 0\textit{)}.$$

If in addition $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^N)$ *then there exists a solution* U *of* (40) *which is bounded,* T*-periodic in* t *and satisfies* $U \geq v$ *(resp.* $U \leq v$*) in* $\mathbb{R}^{N+1}$ *and*

$$\lim_{t\to\infty} (u(t,x) - U(t,x)) = 0,$$

locally uniformly with respect to $x \in \mathbb{R}^N$*.*

Proof. We prove the result when v is a subsolution, the other case being analogous. Since v and u are respectively a subsolution and a solution of problem (40), with the same initial datum, the comparison principle yields $v \leq u$ a. e. in $\mathbb{R}^+ \times \mathbb{R}^N$. Then, in particular,

$$\text{for a. e. } x \in \mathbb{R}^N, \quad u(T,x) \geq v(T,x) = v(0,x) = u(0,x).$$

Thus, thanks to the periodicity of the terms in (40), the function $u(t+T,x)$ is again a solution of (40), with initial condition $u(T,x) \geq u(0,x)$. Therefore, applying again the comparison principle, we derive $u(t+T,x) \geq u(t,x)$.

To prove the second statement, consider the sequence of functions $u_n(t,x) := u(t+nT,x)$. By hypothesis it is bounded, and we have shown that it is nondecreasing. It follows that the u_n converge pointwise to some bounded function $U(t,x)$ such that $U \geq v$. Moreover, U is T-periodic in t because

$$U(t+T,x) = \lim_{n\to\infty} u(t+T+nT,x) = U(t,x).$$

Since the u_n are solutions of

$$\partial_t u_n = \partial_i(a_{ij}(t,x)\partial_j u_n) + b_i(t,x)\partial_i u_n + h(t,x,u_n), \qquad t > -nT,\ x \in \mathbb{R}^N, \tag{41}$$

with initial datum $u_n(-nT,x) = v(0,x)$, standard parabolic estimates imply that for any $\rho > 0$ there exists a constant $C_\rho > 0$ such that, for $n > \rho/T + 1$,

$$\|u_n\|_{N+1,(-\rho,\rho)\times B_\rho} \leq C_\rho \left(\|h(t,x,u)\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^N)} + \|u\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^N)} \right).$$

Consequently, using the compact injection theorems we find that the u_n converge to U locally uniformly in $\mathbb{R}^{N+1}$ and, passing to the weak limit in (41), that U solves (40). For any $t > 0$ let $n(t) \in \mathbb{N}$ be such that $\tau(t) := t - n(t)T \in [0,T)$. Then, the periodicity of U yields

$$|u(t,x) - U(t,x)| = |u_{n(t)}(\tau(t),x) - U(\tau(t),x)|,$$

which goes to zero as $n(t) \to \infty$, i. e. as $t \to \infty$, locally uniformly in $x \in \mathbb{R}^N$. □

Let us now suppose that $s \mapsto h(t,x,s) \in C^1([0,\delta])$ for some $\delta > 0$, uniformly for a. e. $(t,x) \in \mathbb{R}^{N+1}$. Under assumptions (13) and (14), Theorem 3.4 yields a sufficient condition for the existence of pulsating travelling wave solutions associated with (40), that is, solutions of

$$\begin{cases} \partial_t U = \partial_i(a_{ij}(t,x)\partial_j U) + b_i(t,x)\partial_i U + h(t,x,U), & (t,x) \in \mathbb{R}^{N+1} \\ U > 0 \quad \text{in } \mathbb{R}^{N+1} \\ U \quad \text{is bounded} \\ U \quad \text{is } T\text{-periodic in } t. \end{cases} \tag{42}$$

This condition is $\mu_1(P,\mathbb{R}^{N+1}) < 0$, where

$$Pw := \partial_t w - \partial_i(a_{ij}(t,x)\partial_j w) - b_i(t,x)\partial_i w - h_s(t,x,0)w.$$

is the associated linearized operator about $w \equiv 0$ and $\mu_1(P,\mathbb{R}^{N+1})$ is the generalized T-periodic principal eigenvalue of P in $\mathbb{R}^{N+1}$ defined by (17). We point out that neither (15) nor (16) are required in the next two theorems.

Theorem 3.5. *If* (13)*,* (14) *hold and* $\mu_1(P,\mathbb{R}^{N+1})<0$ *then* (42) *admits at least a solution.*

Proof. The idea of the proof is to find a subsolution v of (42) such that $v\leq S$, where S is the constant in (14), and then apply Theorem 3.4. Such a function v can be constructed by using the fact that $\mu_1(P,\mathbb{R}^{N+1})<0$, in an analogous way to the method in Section 2.4 to prove Theorem 1.1. Indeed, by Proposition 6 (which holds even for general operators such as P), there exists $R>0$ such that $\tilde{\mu}(R)<0$, where $\tilde{\mu}(R)$ is the T-periodic Dirichlet principal eigenvalue of P in $\mathbb{R}\times B_R$. Let us denote by $\tilde{\psi}_R$ the associated principal eigenfunction. Then, with the same arguments as in the proof of Theorem 1.1, we can find $\kappa>0$ small enough in such a way that the function

$$v(t,x):=\begin{cases}\kappa\tilde{\psi}_R(t,x) & x\in\mathbb{R}\times B_R\\ 0 & \text{otherwise}\end{cases}$$

is a subsolution of (40) and satisfies $v\leq S$ in $\mathbb{R}^{N+1}$. Consider the solution u of (40) with initial datum $u(0,x)=v(0,x)$. By standard parabolic theory of weak solutions, such a solution exists for every $t>0$ and satisfies $v\leq u\leq S$ thanks to the maximum principle. Thus, applying Theorem 3.4, we infer the existence of a T-periodic in t bounded solution U of (40) such that $U\geq v$. Since $U>0$ by the strong maximum principle, the proof is concluded. □

Another consequence of Theorem 3.4 is the following result concerning the large time behavior of solutions of (40), with arbitrary positive bounded initial datum, that will be used in the next section to prove Theorem 1.4.

Theorem 3.6. *Let* $u(t,x)$ *be the solution of* (40) *with an initial condition* $u(0,x)=u_0(x)\in L^\infty(\mathbb{R}^N)$ *which is nonnegative and not identically equal to zero. Under assumptions* (13)*,* (14) *the following properties hold:*
(i) if (42) *does not admit any solution then*

$$\lim_{t\to\infty}u(t,x)=0,$$

locally uniformly with respect to $x\in\mathbb{R}^N$*;*
(ii) if $\mu_1(P,\mathbb{R}^{N+1})<0$ *and* (42) *admits a unique solution* U *then*

$$\lim_{t\to\infty}(u(t,x)-U(t,x))=0,$$

locally uniformly with respect to $x\in\mathbb{R}^N$*.*

Proof. Set $S':=\max(S,\|u_0\|_{L^\infty(\mathbb{R}^N)})$, where S is the positive constant in (14). The function $v\equiv S'$ is a T-periodic in t supersolution of (40) and, by (13), $w\equiv 0$ is a subsolution. Let $\overline{u}$ be the solution of (40) with initial datum S'. From the comparison principle it follows that the functions u and $\overline{u}$ satisfy $0\leq u\leq\overline{u}\leq S'$ in $\mathbb{R}^+\times\mathbb{R}^N$ and then, in particular, they exist for every $t>0$. Therefore, we can apply Theorem 3.4 to the function $\overline{u}$ and infer that

$$0=\lim_{t\to\infty}\left(\overline{u}(t,x)-\overline{U}(t,x)\right)\geq\limsup_{t\to\infty}\left(u(t,x)-\overline{U}(t,x)\right), \tag{43}$$

locally uniformly with respect to $x\in\mathbb{R}^N$, where $\overline{U}$ is a T-periodic in t solution of (40) such that $0\leq\overline{U}\leq S'$.

(i) By hypothesis, $\overline{U}$ has to vanish at some $(\tau,\xi)\in\mathbb{R}^{N+1}$. Hence, $\overline{U}=0$ in $(-\infty,\tau)\times\mathbb{R}^N$ by the parabolic strong maximum principle and then $\overline{U}\equiv 0$ in $\mathbb{R}^{N+1}$ by periodicity in t. The statement then follows from (43).

(ii) First, note that the strong maximum principle yields $u(T,x) > 0$ in $\mathbb{R}^N$. Proceeding as in the proof of Theorem 3.5, we can find a T-periodic in t subsolution $\tilde{v} \in L^\infty(\mathbb{R}^{N+1})$ of (40) such that $\tilde{v} \geq 0$, $\tilde{v} \not\equiv 0$ and $\tilde{v}(0,x) < u(T,x)$ in $\mathbb{R}^N$. Let $\underline{u}$ be the solution of (40) with initial datum $\tilde{v}(0,x)$. The comparison principle yields

$$\forall\, t \geq 0,\ x \in \mathbb{R}^N, \quad \tilde{v}(t,x) \leq \underline{u}(t,x) \leq u(t+T,x) \leq S'.$$

Applying Theorem 3.4, with $v = \tilde{v}$ and $u = \underline{u}$, we derive

$$\begin{aligned} 0 = \lim_{t\to\infty} (\underline{u}(t,x) - \underline{U}(t,x)) &\leq \liminf_{t\to\infty} (u(t+T,x) - \underline{U}(t,x)) \\ &= \liminf_{t\to\infty} (u(t,x) - \underline{U}(t,x)), \end{aligned} \tag{44}$$

locally uniformly with respect to $x \in \mathbb{R}^N$, where $\underline{U}$ is a T-periodic in t bounded solution of (40) such that $\underline{U} \geq \tilde{v}$. The strong maximum principle yields $\underline{U} > 0$ in $\mathbb{R}^{N+1}$. By (43), (44) and the periodicity in t, we see that $\overline{U} \geq \underline{U} > 0$ and then both $\underline{U}$ and $\overline{U}$ coincide with the unique solution U of (42). Therefore, the statement follows from (43) and (44). □

Remark 1. Let us point out, without going into details, that the results of Theorems 3.4-3.6 also hold for the Dirichlet problem

$$\begin{cases} \partial_t u = \partial_i(a_{ij}(t,x)\partial_j u) + b_i(t,x)\partial_i u + h(t,x,u), & t > 0,\ x \in \Omega \\ u(t,x) = 0, \quad t > 0,\ x \in \partial\Omega, \end{cases}$$

with Ω smooth domain in $\mathbb{R}^N$.

3.5. **Conclusion of the proofs.**
Proof of Theorem 1.3. If $\mu_1 < 0$ then the existence of a solution of (4) follows from Theorem 3.5. Assume now, by way of contradiction, that $\mu_1 \geq 0$ and that (4) admits a solution U. Let ψ be a generalized principal eigenfunction of $\mathcal{P}$ in $\mathbb{R}^{N+1}$, given by Proposition 6, normalized in such a way that $\psi(0,0) < U(0,0)$. Using (15) we derive

$$\partial_t\psi - \triangle\psi - ce\cdot\nabla\psi = (h_s(t,x,0) + \mu_1)\psi \geq h(t,x,\psi) \quad \text{a. e. in } \mathbb{R}^{N+1}.$$

Since Proposition 9 implies $\lim_{|x|\to\infty} U(t,x) = 0$, uniformly in $t \in \mathbb{R}$, applying Theorem 3.3 with $\underline{U} = U$ and $\overline{U} = \psi$ we get the following contradiction: $U \leq \psi$.
The uniqueness result follows directly from Proposition 9 and the comparison principle Theorem 3.3. □

Proof of Theorem 1.4. Let u be the solution of (3) with nonnegative bounded initial datum $u_0 \not\equiv 0$. The function $\tilde{u}(t,x) := u(t, x + cte)$ is a solution of

$$\partial_t\tilde{u} = \triangle\tilde{u} + ce\cdot\nabla\tilde{u} + h(t,x,\tilde{u}), \quad t > 0,\ x \in \mathbb{R}^N,$$

with initial datum u_0. The parabolic maximum principle shows that $0 \leq \tilde{u} \leq S'$, where $S' := \max(S, \|u_0\|_\infty)$ and S is the constant in (14). Applying the convergence result for general operators, given by Theorem 3.6, together with Theorem 1.3, we infer that

$$\lim_{t\to\infty} (\tilde{u}(t,x) - U(t,x)) = 0,$$

locally uniformly in $x \in \mathbb{R}^N$, where either $U \equiv 0$ if $\mu_1 \geq 0$, or U is the unique solution of (4) if $\mu_1 < 0$. Thus, to conclude the proof of Theorem 1.4 it only remains to show that the above limit is uniform with respect to $x \in \mathbb{R}^N$. Assume by contradiction that this is not the case. Then, there exist a positive constant ε and two sequences $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+$ and $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ such that

$$\lim_{n\to\infty} t_n = \infty, \qquad \lim_{n\to\infty} |x_n| = \infty, \qquad \forall\, n \in \mathbb{N}, \quad |\tilde{u}(t_n, x_n) - U(t_n, x_n)| \geq \varepsilon.$$

Since $\lim_{n\to\infty} U(t_n, x_n) = 0$ by Proposition 9 and $\tilde{u}$ is positive, it follows that

$$\liminf_{n\to\infty} \tilde{u}(t_n, x_n) \geq \varepsilon. \tag{45}$$

For $n \in \mathbb{N}$ define the function $\tilde{u}_n(t,x) := \tilde{u}(t+t_n, x+x_n)$. It holds that $0 \leq \tilde{u}_n \leq S'$ and

$$\partial_t \tilde{u}_n = \triangle \tilde{u}_n + ce \cdot \nabla \tilde{u}_n + h(t+t_n, x+x_n, \tilde{u}_n) \quad \text{for a. e. } t > -t_n,\ x \in \mathbb{R}^N.$$

Thanks to (15) and (35), parabolic estimates and embedding theorems imply that (a subsequence of) the $\tilde{u}_n$ converge locally uniformly in $\mathbb{R}^{N+1}$ to some function $\tilde{u}_\infty$ satisfying

$$\partial_t \tilde{u}_\infty \leq \triangle \tilde{u}_\infty + ce \cdot \nabla \tilde{u}_\infty - \zeta \tilde{u}_\infty \quad \text{in } \mathbb{R}^{N+1}.$$

Furthermore, $\tilde{u}_\infty(0,0) \geq \varepsilon$ by (45). Define the function $\theta(t) := S' e^{-\zeta(t-t_0)}$, where $t_0 \in \mathbb{R}$ will be chosen later. One sees that $\partial_t \theta = -\zeta\theta$ in $\mathbb{R}$ and $\theta(t_0) = S' \geq \tilde{u}_\infty(t_0, x)$ for any $x \in \mathbb{R}^N$. As a consequence of the maximum principle we get $\theta(t) \geq \tilde{u}_\infty(t,x)$ for $t \geq t_0,\ x \in \mathbb{R}^N$. In particular, if $t_0 < 0$, we obtain

$$\varepsilon \leq \tilde{u}_\infty(0,0) \leq S' e^{\zeta t_0}.$$

This is a contradiction for $-t_0$ large enough and the proof is concluded. □

Remark 2. All arguments in the proofs of Theorems 1.3 and 1.4 still work if one replaces h with

$$\tilde{h}(t,x,s) = f(x,s) + g(x + (c-c')te, s)$$

and sets $T := l/(c-c')$. That is, Theorem 1.5 holds true.

References

[1] H. Berestycki, O. Diekmann, C. J. Nagelkerke and P. A. Zegeleing, *Can a species keep pace with a shifting climate ?*, Bull. Math. Biol., to appear.

[2] H. Berestycki, F. Hamel and L. Rossi, *Liouville-type results for semilinear elliptic equations in unbounded domains*, Ann. Mat. Pura Appl., **186** (2007), 469–507.

[3] H. Berestycki and P.-L. Lions, *Some applications of the method of super and subsolutions*, in “Bifurcation and nonlinear eigenvalue problems” (Proc., Session, Univ. Paris XIII, Villetaneuse, 1978), Lecture Notes in Math., 782, Springer, Berlin, 1980, 16–41.

[4] H. Berestycki, L. Nirenberg and S. R. S. Varadhan, *The principal eigenvalue and maximum principle for second-order elliptic operators in general domains*, Comm. Pure Appl. Math., **47** (1994), 47–92.

[5] H. Berestycki and L. Rossi, *On the principal eigenvalue of elliptic operators in $\mathbb{R}^N$ and applications*, J. Eur. Math. Soc. (JEMS), **8** (2006), 195–215.

[6] H. Berestycki and L. Rossi, *Generalized principal eigenvalues for elliptic and parabolic linear operators*, In preparation.

[7] H. Berestycki and L. Rossi, *Reaction-diffusion equations for population dynamics with forced speed. II-Cylindrical-type domains*, In preparation.

[8] E. N. Dancer, *On the existence and uniqueness of positive solutions for competing species models with diffusion*, Trans. Amer. Math. Soc., **326** (1991), 829–859.

[9] D. Gilbarg and N. S. Trudinger, “Elliptic partial differential equations of second order,” 2^{nd} edition, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 244, Springer-Verlag, Berlin, 1983.

[10] P. Hess, “Periodic-parabolic boundary value problems and positivity,” Pitman Research Notes in Mathematics Series, 247, Longman Scientific & Technical, Harlow, 1991.

[11] M. A. Krasnosel′skij, E. A. Lifshits and A. V. Sobolev, *Positive linear systems*, Sigma Series in Applied Mathematics, 5, Heldermann Verlag, Berlin, 1989, The method of positive operators, Translated from the Russian by J. Appell.

[12] M. G. Kreĭn and M. A. Rutman, *Linear operators leaving invariant a cone in a Banach space*, Amer. Math. Soc. Translation, **1950** (1950), no. 26, 128.

[13] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural′ceva, "Linear and quasilinear equations of parabolic type," Translated from the Russian by S. Smith. Translations of Mathematical Monographs, 23, American Mathematical Society, Providence, R.I., 1967.

[14] G. M. Lieberman, "Second order parabolic differential equations," World Scientific Publishing Co. Inc., River Edge, NJ, 1996.

[15] G. Nadin, *The principal eigenvalue of a space-time periodic parabolic operator*, Preprint, 2007.

[16] M. H. Protter and H. F. Weinberger, "Maximum principles in differential equations," Prentice-Hall Inc., Englewood Cliffs, N.J., 1967.

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