The Freidlin-Gärtner formula for general reaction terms

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Abstract

We devise a new geometric approach to study the propagation of disturbance - compactly supported data - in reaction-diffusion equations. The method builds a bridge between the propagation of disturbance and of almost planar solutions. It applies to very general reaction-diffusion equations. The main consequences we derive in this paper are: a new proof of the classical Freidlin-Gärtner formula for the asymptotic speed of spreading for periodic Fisher-KPP equations; extension of the formula to the monostable, combustion and bistable cases; existence of the asymptotic speed of spreading for equations with almost periodic temporal dependence; derivation of multi-tiered propagation for multistable equations.

1 Introduction

We deal with the reaction-diffusion equation

$$\partial_t u = \mathrm{div}(A(x)\nabla u) + q(x)\cdot\nabla u + f(x,u), \quad t>0,\ x\in\mathbb{R}^N. \tag{1}$$

This type of equation is used to model a huge variety of phenomena, from biology, chemistry and physics to social sciences, such as population dynamics, gene diffusion, chemical kinetics, combustion, spread of epidemics and social behaviours. In applications, one typically considers Cauchy problems with compactly supported initial data. Assuming that 0 and 1 are two steady states, 1 being attractive, a natural question is: at which speed does the set where solutions are close to 1 spread? To formulate this question in a precise way one introduces the notion of the *asymptotic speed of spreading*: for a given direction $\xi\in S^{N-1}$, it is a quantity $w(\xi)>0$ such that the solution u to (1) emerging from a compactly supported initial datum $u_0\geq 0, \not\equiv 0$ satisfies

$$\forall c > w(\xi), \quad u(t,x+ct\xi)\to 0 \quad \text{as } t\to+\infty, \tag{2}$$

$$\forall 0\leq c < w(\xi), \quad u(t,x+ct\xi)\to 1 \quad \text{as } t\to+\infty, \tag{3}$$

locally uniformly in $x\in\mathbb{R}^N$.

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Even assuming that $u \to 1$ locally uniformly as $t \to +\infty$, it is not obvious that such a quantity $w(\xi)$ exists and that it does not depend on the initial datum u_0. These properties have been derived by Aronson-Weinberger [2] in the case of the equation $\partial_t u - \Delta u = f(u)$. There, of course, w is independent of ξ. Using large deviation probabilistic techniques, Freidlin and Gärtner have extend the result in [20, 16] to equation (1) under the assumption that A, q, f are periodic in x and that $0 < f(x, u) \leq \partial_u f(x, 0)u$ for $0 < u < 1$. The latter is known as Fisher-KPP condition. The authors obtain the following formula for the speed of spreading:

$$w(\xi) := \min_{\substack{z \in \mathbb{R}^N \\ z \cdot \xi > 0}} \frac{k(z)}{z \cdot \xi},$$

where $k(z)$ is the periodic principal eigenvalue of the linear operator

$$L_z := \text{div}(A\nabla) - 2z \cdot A\nabla + q \cdot z + \big(- \text{div}(Az) - q \cdot z + z \cdot Az + \partial_u f(x, 0)\big).$$

Several years later, in [7] (see also [31]), it has been shown that, for given $e \in S^{N-1}$, the quantity $c^*(e) := \min_{\lambda > 0} k(\lambda e)/\lambda$ coincides with the critical (or minimal) speed of pulsating travelling fronts in the direction e (see Section 1.1 for the definition). Therefore, Freidlin-Gärtner's formula can be rewritten as

$$w(\xi) = \min_{e \cdot \xi > 0} \frac{c^*(e)}{e \cdot \xi}. \tag{4}$$

Namely, $w(\xi)$ is the minimiser of the speed of displacement in the direction ξ among all the fronts, even those in directions $e \neq \xi$.

Pulsating travelling fronts exist not only in the Fisher-KPP case, but also for other classes of reaction terms, though their critical speed no longer fulfils the previous eigenvalue representation. Then one might wonder if the formula (4) holds true beyond the Fisher-KPP case. In the present paper, using a new PDE approach, we show that this is always the case, whenever pulsating travelling fronts are known to exist: monostable, combustion and bistable equations. We point out that in the latter two cases, where $f(x, u)$ is nonpositive in a neighbourhood of $u = 0$, $c^*(e)$ is the unique speed for which a pulsating travelling front in the direction e exists.

The spreading properties for heterogeneous - in particular periodic - reaction-diffusion equations have been widely studied in the literature, with other approaches than the probabilistic one of [20]. One is the viscosity solutions/singular perturbations method of Evans-Souganidis [15] for the Fisher-KPP equation and Barles, Soner and Souganidis [3, 4] for the bistable equation. There the authors characterise the asymptotic propagation of solutions in terms of the evolution of a set governed by a Hamilton-Jacobi equation. An abstract monotone system approach relying on a discrete time-steps formalism is used in Weinberger [31]. It provides a general spreading result for monostable, combustion and bistable periodic equations without assuming the existence of pulsating travelling fronts. Then, in the monostable case, i.e., when $f(x, \cdot)$ is assumed to be positive in $(0, 1)$, the method itself allows the author to show the existence of pulsating fronts and to derive the Freidlin-Gärtner formula (4) in such case. Instead, in the combustion or bistable cases, Theorems 2.1 and 2.2 of [31] assert the existence of the spreading speed but do not relate it with the speeds of pulsating fronts. Finally, a PDE approach is adopted by Berestycki-Hamel-Nadin [6]. This yields the Freidlin-Gärtner formula in the Fisher-KPP case and partially extend it to equations with general space-time dependent coefficients.

Let us describe our method. Property (2) with $w(\xi)$ given by (4) is essentially a direct consequence of the comparison principle between u and the critical pulsating travelling fronts in all directions e satisfying $e \cdot \xi > 0$. A bit of work is however required in order to handle initial data which are not strictly less than 1 and also because we aim to a uniform version of (2) with respect to ξ. The real novelty of this paper consists in the derivation of (3). The reason why this property is harder to obtain than (2) can be explained in the following way: a solution u emerging from a compactly supported initial datum has bounded upper level sets at any time, whereas the upper level sets of a front contain a half-space. This is why one can manage to bound u from above by a suitable translation of any travelling front and eventually get (2), but cannot bound u from below by a front in order to get (3). Nevertheless, assuming that u converges locally uniformly to 1 as $t \to +\infty$, its upper level sets eventually contain arbitrarily large portions of half-spaces, and thus it will be possible to put some front below the limit of translations of u by $\{(t_n, x_n)\}$ with $t_n \to +\infty$. So, supposing by way of contradiction that (3) does not hold, the key is to find a sequence of translations of u, by a suitable $\{(t_n, x_n)\}$, whose limit propagates with an *average speed* slower than a front. To achieve this, we need to deal with all directions of spreading simultaneously, by considering the Wulff shape of the speeds. As a by-product, we derive (3) uniformly with respect to ξ.

Let us point out that, unlike in the singular perturbation approach, we just consider translations of the original equations, without any rescaling. One of the advantages is that the equation we obtain in the limit keeps the same form as the original one, in particular the uniform ellipticity. Another difference with the singular perturbation approach is that, roughly speaking, the latter makes use of the travelling fronts for the original equation in order to obtain the evolution equation for level sets in the limit, whereas our method works the other way around: we start with the analysis of the motion of the level sets and exploit the existence of travelling fronts only at the end.

We now introduce the object we want to study.

Definition 1.1. We say that a closed set $\mathcal{W} \subset \mathbb{R}^N$, coinciding with the closure of its interior, is the *asymptotic set of spreading* for a reaction-diffusion equation if, for any bounded solution u with a compactly supported initial datum $0 \leq u_0 \leq 1$ such that $u(t,x) \to 1$ as $t \to +\infty$ locally uniformly in $x \in \mathbb{R}^N$, there holds

$$\forall \text{ compact set } K \subset \text{int}(\mathcal{W}), \quad \inf_{x \in K} u(t, xt) \to 1 \quad \text{as } t \to +\infty, \tag{5}$$

$$\forall \text{ closed set } C \text{ such that } C \cap \mathcal{W} = \emptyset, \quad \sup_{x \in C} u(t, xt) \to 0 \quad \text{as } t \to +\infty. \tag{6}$$

If only (5) (resp. (6)) holds we say that $\mathcal{W}$ is an *asymptotic subset* (resp. *superset*) *of spreading.*

The above definition essentially says that the upper level sets of u look approximately like $t\mathcal{W}$ for large t. The requirement that the asymptotic set of spreading coincides with the closure of its interior immediately implies its uniqueness. The objective of this paper is to derive the existence of the asymptotic set of spreading and to find an expression for it.

If the asymptotic set of spreading $\mathcal{W}$ is bounded and star-shaped with respect to the origin - all properties that it is natural to expect - we can write

$$\mathcal{W} = \{r\xi \,:\, \xi \in S^{N-1}, \;\; 0 \leq r \leq w(\xi)\},$$

with w upper semicontinuous. If in addition w is strictly positive and continuous then $w(\xi)$ is the asymptotic speed of spreading in the direction ξ, in the sense of (2)-(3). In addition, those limits hold uniformly with respect to $(\xi, c) \in S^{N-1} \times \mathbb{R}_+$ such that $|c - w(\xi)| > \varepsilon$, for any $\varepsilon > 0$.

Remark 1. The requirement in Definition 1.1 that $u \to 1$ as $t \to +\infty$ locally uniformly in space (or equivalently pointwise, due to parabolic estimates and strong maximum principle) is automatically fulfilled by any $u_0 \not\equiv 0$ in the periodic case when f is of KPP type and q is divergence-free with average 0, see [16]. A sharp condition for possibly negative f is used in [8], later extended to the non-periodic setting in [9]. In the non-KPP cases, it may happen that solutions converge uniformly to 0 (when the "hair-trigger" effect fails in the monostable case or when the solution is "quenched" in the combustion or bistable cases, see [2]).

Let us also mention that the hypotheses that u_0 has compact support can be relaxed by a suitably fast exponential decay, and that $u_0 \leq 1$ can be dropped if one assumes that $f(x, s) < 0$ for $s > 1$.

Our approach turns out to apply to general space-time dependent equations provided that front-like solutions are available, yielding an upper and a lower bounds on the asymptotic speed of spreading. Results of this type are derived in the work in progress [11] in the case of Fisher-KPP reaction terms, combining homogenisation techniques with the tool of the generalised principle eigenvalue. It is not always possible to deduce the existence of the asymptotic speed of spreading from such bounds, and there are indeed cases where the speed of spreading does not exist (see [18]). In the present paper, beside the periodic framework, we derive the existence of the asymptotic speed of spreading for combustion and bistable equations with almost periodic dependence in time, which was not previously known. One could wonder if some weaker compactness properties - such as random stationary ergodicity - may guarantee the existence of the speed of spreading, as shown for the Fisher-KPP equation in dimension 1 in [20, 10] and for advection equations in [23]. Problems set in domains with periodic holes, under Neumann boundary condition, may also be envisioned. We have chosen not to treat these cases in the present paper in order to make the presentation as clear as possible.

To sum up, the bridge we build between the propagation of compactly supported data and almost planar solutions provides:

- A new proof of the Freidlin-Gärtner formula.
- Extension of the formula to monostable, combustion, bistable reaction terms.
- Control of the propagation of disturbance in general non-autonomous media in terms of almost planar transition fronts, and in particular the existence of the asymptotic speed of spreading for almost periodic, time dependent equations.
- For very general autonomous or periodic equations, a scheme reducing compactly supported data to *front-like* data, that is, satisfying

  $$\lim_{x\cdot e\to-\infty} u_0(x)=1, \qquad u_0(x)=0 \quad \text{for } x\cdot e \text{ large enough},$$

  which implies that the asymptotic set of spreading is given by the following generalised Freidlin-Gärtner's formula:

  $$\mathcal{W}=\{r\xi\,:\,\xi\in S^{N-1},\ \ 0\leq r\leq w(\xi)\}, \qquad \textit{with} \quad w(\xi)=\inf_{\substack{e\in S^{N-1}\\ e\cdot\xi>0}}\frac{c^*(e)}{e\cdot\xi},$$

  where $c^*(e)$ *is the speed of spreading for front-like data in the direction* e.

We have decided to include here the above generalised Freidlin-Gärtner's formula, even if it is not a theorem, in order to give the flavour of the kind of results one can obtain with the method of this paper. A rigorous statement would require some hypotheses on the operator. However, what it is needed for our scheme to work is just the validity of rather standard structural properties - essentially the comparison principle and a priori estimates - and this is why we believe it will be susceptible of application to a wide class of equations. One application we present here is the multi-tiered propagation of disturbance for multistable equations, which partially extend to higher dimension and to compactly supported data the results of [14, 24] concerning *propagating terraces.*

If the operator is autonomous and rotationally invariant, then in most cases the problem for a truly *planar* datum reduces to an equation in one single space variable. Hence, the generalised Freidlin-Gärtner's formula would imply that $\mathcal{W}$ is a ball with radius independent of N, showing in particular that the propagation of disturbance does not depend on the dimension, at least at the level of the average speed. It is well known that, going beyond the average speed, the location of the interface of the disturbance does depend in general on the dimension (by a $\log t$ order in the autonomous case, see [19, 29]).

The paper is organised as follows: in Section 1.1 we state the result about the asymptotic set of spreading in periodic media, which yields Freidlin-Gärtner's formula for general reaction terms. In Section 1.2 we present the extension to equations depending on both space and time, without any periodicity assumption. In Section 1.3

we present the application to two spatial homogeneous settings: almost periodic temporal-dependent equations and multistable equations. The remaining sections are dedicated to the proofs of these results. Namely, the asymptotic subset and superset of spreading are dealt with in Sections 2 and 3 respectively; in both cases, we start with proving the most general statements, from which we deduce the ones in periodic media. Section 4.1 is dedicated to the derivation of the asymptotic speed of spreading for almost periodic time-dependent equations, while Section 4.2 deals with the multi-tiered propagation for multistable equations.

1.1 Periodic case

We say that a function defined on $\mathbb{R}^N$ is ℓ-periodic, with $\ell = (\ell_1, \dots, \ell_N) \in (0, +\infty)^N$, if it is periodic in each of the directions $e_1, \dots, e_N$ of the canonical basis with period $\ell_1, \dots, \ell_N$ respectively, i.e., if it is invariant under the translations by $\ell_1\mathbb{Z} \times \cdots \times \ell_N\mathbb{Z}$. We let $\mathcal{C}$ denote the periodicity cell $(0, \ell_1) \times \cdots \times (0, \ell_N)$.

Our hypotheses in the periodic case are the ones required to apply the results of [5, 32, 33] concerning the existence of pulsating travelling fronts. The hypotheses intrinsic to our method are weaker (cf. the next subsection).

The matrix field A and the vector field q are smooth [1] and satisfy

$$A \text{ is } \ell\text{-periodic, symmetric and uniformly elliptic,} \tag{7}$$

$$q \text{ is } \ell\text{-periodic,} \qquad \operatorname{div} q = 0, \qquad \int_{\mathcal{C}} q = 0. \tag{8}$$

The function $f : \mathbb{R}^N \times [0, 1] \to \mathbb{R}$ is of class $C^{1+\delta}$, for some $\delta \in (0, 1)$, and satisfies

$$\begin{cases} \forall s \in (0,1), \quad f(\cdot, s) \text{ is } \ell\text{-periodic,} \\ \forall x \in \mathbb{R}^N, \quad f(x, 0) = f(x, 1) = 0, \\ \exists S \in (0,1), \quad \forall x \in \mathbb{R}^N, \quad f(x, \cdot) \text{ is nonincreasing in } [S, 1]. \end{cases} \tag{9}$$

We further assume that f is in one of the following three classes:

$$\textit{Monostable} \qquad \forall s \in (0,1), \quad \min_{x \in \mathbb{R}^N} f(x,s) \geq 0, \quad \max_{x \in \mathbb{R}^N} f(x,s) > 0, \tag{10}$$

$$\textit{Combustion} \qquad \begin{cases} \exists \theta \in (0,1), \quad \forall (x,s) \in \mathbb{R}^N \times [0, \theta], \quad f(x,s) = 0, \\ \forall s \in (\theta, 1), \quad \min_{x \in \mathbb{R}^N} f(x,s) \geq 0, \quad \max_{x \in \mathbb{R}^N} f(x,s) > 0, \end{cases} \tag{11}$$

$$\textit{Bistable} \qquad f(x,s) = s(1-s)(s-\theta), \quad \theta \in (0, 1/2). \tag{12}$$

In the bistable case, in order to apply the results of Xin [32, 33], in addition to f independent of x, we need A, q to be close to constants, in the following sense:

$$\exists h > N+1, \quad \left\| A - \int_{\mathcal{C}} A \right\|_{C^h(\mathcal{C})} < k, \quad \left\| q - \int_{\mathcal{C}} q \right\|_{C^h(\mathcal{C})} < k, \tag{13}$$

[1] More precisely, A is C^3 and q is $C^{1+\delta}$ in the monostable or combustion cases [5], and A, q are C^∞ in the bistable case [32, 33].

where k is a suitable quantity also depending on h.

Under the above hypotheses, it follows from [5] in the cases (10) or (11), and from [32, 33] in the case (12), that (1) admits *pulsating travelling fronts* in any direction $e \in S^{N-1}$. These are entire (i.e., for all times) solutions v satisfying

$$\begin{cases} \forall z \in \ell_1\mathbb{Z} \times \cdots \times \ell_N\mathbb{Z},\ (t,x) \in \mathbb{R}\times\mathbb{R}^N, \quad v(t+\frac{z\cdot e}{c}, x) = v(t, x-z) \\ v(t,x) \to 1 \ \text{ as } x\cdot e \to -\infty, \quad v(t,x) \to 0 \ \text{ as } x \cdot e \to +\infty, \end{cases} \tag{14}$$

for some quantity c, called *speed* of the front. The above limits are understood to hold locally uniformly in $t \in \mathbb{R}$. In the monostable case (10), such fronts exist if and only if c is larger than or equal to a critical value, depending on e, that we call $c^*(e)$. In the other two cases they exist only for a single value of c, still denoted by $c^*(e)$. We further know from [5, 32, 33] that, under the above hypotheses, $c^*(e) > 0$ for all $e \in S^{N-1}$, and any front $v(t,x)$ is increasing in t.

Here is the generalization of Freidlin-Gärtner's result.

Theorem 1.2. *Under the assumptions* (7)-(9) *and either* (10) *or* (11) *or* (12)-(13), *the set*

$$\mathcal{W} := \{r\xi \ : \ \xi \in S^{N-1}, \quad 0 \le r \le w(\xi)\}, \quad \textit{with} \ \ w(\xi) := \inf_{e\cdot\xi>0} \frac{c^*(e)}{e\cdot\xi}, \tag{15}$$

is the asymptotic set of spreading for (1), *in the sense of Definition 1.1.*

Moreover, w is positive and continuous and thus $w(\xi)$ is the asymptotic speed of spreading in the direction ξ.

The infimum in the definition of w is actually a minimum because the function c^* is lower semicontinuous, as we show in Proposition 2.5. The weaker property $\inf c^* > 0$ ensures in general that a function w defined as in (15) is continuous, see Proposition 2.4 below. The continuity of w is crucial for our method to work, and we emphasise that it does not require c^* to be continuous. The continuity of c^* has been derived by Alfaro and Giletti in the monostable and ignition cases in the paper [1], which appeared during the revision process of the present work.

Remark 2. One can readily check, using the positivity of c^*, that the asymptotic set of spreading $\mathcal{W}$ given by (15) can be rewritten in the following way:

$$\mathcal{W} = \{x \ : \ x\cdot e \le c^*(e) \text{ for every } e \in S^{N-1}\},$$

which is the expression of the Wulff shape arising in crystallography. From this expression we deduce that $\mathcal{W}$ is convex. Moreover, calling e_ξ a minimiser in the definition of $w(\xi)$ in (15), there holds

$$\forall \xi' \in S^{N-1}, \quad w(\xi')\xi'\cdot e_\xi \le c^*(e_\xi) = w(\xi)\xi\cdot e_\xi.$$

Namely, e_ξ is an exterior normal to $\mathcal{W}$ at the point $w(\xi)\xi$. Thus, since $w(\xi)\xi\cdot e_\xi = c^*(e_\xi)$, it follows that if $\mathcal{W}$ is smooth then the family $(t\mathcal{W})_{t>0}$ expands in the normal direction ν with speed $c^*(\nu)$, exactly as in the homogeneous case. The results of the next section show that, in a sense, this property holds true in very general contexts.

1.2 Extension to general heterogeneous media

We will derive Theorem 1.2 as a consequence of two results concerning equations with non-periodic space/time dependent coefficients, in the general form

$$\partial_t u = \operatorname{div}(A(t,x)\nabla u) + q(t,x)\cdot\nabla u + f(t,x,u), \quad t>0,\ x\in\mathbb{R}^N, \tag{16}$$

under milder regularity hypotheses. We assume here that there is $\delta > 0$ such that [2]

$$\begin{cases} A \in C^{\delta,1+\delta}(\mathbb{R}^{N+1}) \text{ is symmetric and uniformly elliptic,} \\ q \in C^{\delta}(\mathbb{R}^{N+1}), \\ f \in W^{1,\infty}(\mathbb{R}\times\mathbb{R}^N\times[0,1]). \end{cases} \tag{17}$$

Notice that the regularity of A allows one to write the equation in non-divergence form and to apply Schauder's regularity theory. Further hypotheses on f are:

$$\begin{cases} \forall (t,x)\in\mathbb{R}\times\mathbb{R}^N, \quad f(t,x,0)=f(t,x,1)=0, \\ \exists S\in(0,1), \quad \forall (t,x)\in\mathbb{R}\times\mathbb{R}^N, \quad f(t,x,\cdot) \text{ is nonincreasing in } [S,1], \\ \forall s\in(S,1), \quad \exists\ E \text{ relatively dense in } \mathbb{R}^{N+1}, \quad \inf_{(t,x)\in E} f(t,x,s)>0. \end{cases} \tag{18}$$

We recall that a set E is relatively dense in $\mathbb{R}^{N+1}$ if the function $\operatorname{dist}(\cdot,E)$ is bounded on $\mathbb{R}^{N+1}$. Properties (18) are fulfilled by all classes of reaction terms considered in the previous section; the second condition is needed for the sliding method to work, the last one prevents from having constant solutions between S and 1 (even for limiting equations, see below). In the combustion (11) or bistable (12) cases, the following condition is further satisfied:

$$\exists\theta\in(0,S], \quad \forall (t,x)\in\mathbb{R}\times\mathbb{R}^N, \quad f(t,x,\cdot) \text{ is nonincreasing in } [0,\theta]. \tag{19}$$

This is essentially the condition that yields the uniqueness of the speed of the fronts (cf. Lemma 2.2 below). We extend $f(t,x,s)$ to 0 for $s\notin[0,1]$.

In the generality of the above hypotheses, it may happen that all solutions emerging from compactly supported initial data converge uniformly to 0, as for instance for the equation $u_t - u_{xx} = u(1-u)(u-\theta)$ with $\theta > 1/2$. Then, in such case, one cannot talk about asymptotic set of spreading. The analysis of conditions ensuring the contrary, i.e., eventual invasion for all or some initial data, is adressed in many papers (see the brief discussion in Remark 1) and it is out of the scope of the present one.

Definition 1.3. An (almost planar) *transition front* in the direction $e\in S^{N-1}$ connecting S_2 to S_1 is a bounded solution v for which there exists $X:\mathbb{R}\to\mathbb{R}$ such that

$$\begin{cases} v(t,x+X(t)e)\to S_2 & \text{as } x\cdot e\to-\infty \\ v(t,x+X(t)e)\to S_1 & \text{as } x\cdot e\to+\infty \end{cases} \qquad \text{uniformly in } t. \tag{20}$$

The quantities

$$\liminf_{t\to-\infty}\frac{X(t)}{t}, \qquad \limsup_{t\to+\infty}\frac{X(t)}{t}$$

are called respectively the *past speed* and the *future speed* of the transition front.

[2] For us, $g\in C^{k+\delta}$, $k\in\mathbb{N}$, $\delta\in(0,1)$, means that the derivatives of g of order k are *uniformly* Hölder-continuous with exponent δ; $g=g(t,x)$ is in $C^{k+\delta,h+\gamma}$ if $g(\cdot,x)\in C^{k+\delta}$ and $g(t,\cdot)\in C^{h+\gamma}$.

Although the function X associated with a front is not unique, the past and future speeds are. It is readily seen that a pulsating travelling front with speed c, i.e. satisfying (14), fulfils the Definition 1.3 of transition front with $X(t) = ct$, and thus it has past and future speeds equal to c. The existence of almost planar transition fronts in non-periodic media is an open question, which is very interesting in itself. Owing to Theorems 1.4 and 1.5 below, answering to this question in some particular cases will directly imply the spreading result for compactly supported initial data.

Let us introduce the family of *limiting equations* associated with (16):

$$\partial_t u = \mathrm{div}(A^*(t,x)\nabla u) + q^*(t,x)\cdot\nabla u + f^*(t,x,u), \tag{21}$$

where A^*, q^*, f^* satisfy, for some sequence $(t_n, x_n)_{n\in\mathbb{N}}$ with $t_n \to +\infty$ as $n \to \infty$,

$$A(t+t_n, x+x_n) \to A^*(t,x), \quad q(t+t_n, x+x_n) \to q^*(t,x), \quad f^*(t+t_n, x+x_n, s) \to f^*(t,x,s)$$

locally uniformly in $(t,x,s) \in \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}$. Roughly speaking, the family of limiting equations is the Ω-limit set of the original equation (16).

Theorem 1.4. *Assume that* (17)-(18) *hold. Let* $\underline{c} : S^{N-1} \to \mathbb{R}$ *be such that*

$$\inf_{S^{N-1}} \underline{c} > 0, \tag{22}$$

and, for all $e \in S^{N-1}$, $c < \underline{c}(e)$, $\eta < 1$ *and any limiting equation* (21) *on* $\mathbb{R}_- \times \mathbb{R}^N$, *there is a transition front* v *in the direction* e, *connecting* 1 *to* 0 *if* f *satisfies* (19), *or connecting* 1 *to some* $-\varepsilon < 0$ *otherwise, which has past speed larger than* c *and satisfies* $v(0,0) > \eta$. *Then, the set* $\mathcal{W}$ *given by*

$$\mathcal{W} := \{r\xi \ : \ \xi \in S^{N-1}, \ \ 0 \le r \le w(\xi)\}, \quad \textit{with} \ \ w(\xi) := \inf_{e\cdot\xi>0} \frac{\underline{c}(e)}{e\cdot\xi}, \tag{23}$$

is an asymptotic subset of spreading for (16).

Actually, Theorem 1.4 is in turn a consequence of another result - Theorem 2.3 below - which provides a general criterion for a given set to be an asymptotic subset of spreading. Theorem 2.3 is our most general statement concerning the asymptotic subset of spreading, and it is the building block of the whole paper. However, in all applications presented here - in the previous and in the next sections - the generality of the hypotheses of Theorem 1.4 is sufficient.

Theorem 1.5. *Assume that* (17)-(18) *hold. Let* $\overline{c} : S^{N-1} \to \mathbb{R}$ *be such that*

$$\inf_{S^{N-1}} \overline{c} > 0, \tag{24}$$

and, for all $e \in S^{N-1}$, $\eta < 1$ *and* $R \in \mathbb{R}$, *the equation* (16) *admits a transition front* v *in the direction* e *connecting* 1 *to* 0 *with future speed less than or equal to* $\overline{c}(e)$ *satisfying*

$$\forall t \le 1, \ x\cdot e \le R + \overline{c}(e)t, \qquad v(t,x) \ge \eta. \tag{25}$$

Then, the set $\mathcal{W}$ *given by*

$$\mathcal{W} := \{r\xi \ : \ \xi \in S^{N-1}, \ \ 0 \le r \le w(\xi)\}, \quad \textit{with} \ \ w(\xi) := \inf_{e\cdot\xi>0} \frac{\overline{c}(e)}{e\cdot\xi}, \tag{26}$$

is an asymptotic superset of spreading for (16).

If the functions $\underline{c}$ and $\overline{c}$ in Theorems 1.4 and 1.5 coincide then one obtains the existence of the asymptotic set of spreading. A typical application of Theorem 1.4 is with $\underline{c}(e)$ equal to the minimal speed among all transition fronts in the direction e connecting 1 to 0 for any limiting equation. Instead, $\overline{c}(e)$ in Theorem 1.5 should be the minimal speed among the fronts for the original equation (16). In the periodic case considered in Section 1.1, the two quantities coincide because any limiting equation is simply a translation of the original one. Moreover, in that case, we can restrict to pulsating travelling fronts. This is how we derive Theorem 1.2. However, for equations with arbitrary space-time dependence, it can happen that $\underline{c} < \overline{c}$ and that the asymptotic set of spreading does not exist, cf. [18].

1.3 Further applications

In order to emphasise the versatility of our method, we show the application to two different spatial-invariant equations, deriving the spreading result in cases not covered by the literature.

The first example concerns the equation

$$\partial_t u = \Delta u + f(t, u), \quad t > 0, \ x \in \mathbb{R}^N, \tag{27}$$

under the assumption that $f(t, s)$ is *almost periodic* (a.p. in the sequel) in t uniformly in s, that is, f is uniformly continuous and from any sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ one can extract a subsequence $(t_{n_k})_{k\in\mathbb{N}}$ such that $f(\cdot + t_{n_k}, \cdot)$ converges uniformly in $\mathbb{R}^2$. In this framework, the existence of planar transition fronts is derived by Shen when f is either of combustion type [28] or of bistable type [26] (the precise assumptions are given in Section 4.1 below). Namely, (27) admits a transition front - in the sense of Definition 1.3 - of the form $v(t, x \cdot e)$, $e \in S^{N-1}$, and in addition $v(t, x \cdot e + X(t))$ and $X'(t)$ are a.p. in t uniformly in x. Hence, being a.p., X' satisfies the *uniform average* property:

$$c^* := \lim_{t\to\pm\infty} \frac{1}{t} \int_T^{T+t} X'(s)ds \quad \text{exists uniformly in } T \in \mathbb{R}. \tag{28}$$

In particular, v has past and future speeds equal to c^*. Here is our result.

Proposition 1.6. *Assume that $f(t, s)$ is a.p. in t uniformly in s, that it is uniformly Lipschitz-continuous and fulfils either the combustion or bistable condition (see Section 4.1). Then, if $c^* > 0$, the set $\mathcal{W} := \overline{B}_{c^*}$ is the asymptotic set of spreading for* (27) *in any dimension N.*

If $c^* \leq 0$ (which may happen only in the bistable case) then invasion never occurs and thus the notion of asymptotic set of spreading makes no sense. The question about the spreading speed for (27) when f is monostable remains open: we cannot apply Theorems 1.4, 1.5 because transition fronts are not known to exist in such case. This is only known under the stronger Fisher-KPP condition, c.f. [27, 22]. However, the result of Proposition 1.6 has already been obtained in such case in [22, Proposition 2.6] using a different argument from the one presented here.

The second example concerns the autonomous equation

$$\partial_t u = \Delta u + f(u), \quad t > 0,\ x \in \mathbb{R}^N, \tag{29}$$

without any assumption on the number of steady states between 0 and 1, nor on their stability. The only requirements are:

$$f \in C^1(\mathbb{R}), \qquad f(0) = f(1) = 0, \tag{30}$$

and the existence of an invading solution with initial datum strictly smaller than 1:

$$\begin{cases} \text{there is a solution } \underline{u} \text{ of (29) in dimension 1 with a compactly supported,} \\ \text{continuous initial datum } 0 \le \underline{u}_0 < 1 \text{ such that } \underline{u}(t,x) \to 1 \text{ as } t \to +\infty. \end{cases} \tag{31}$$

These are the hypotheses under which Ducrot-Giletti-Matano [14] derive the existence of a *minimal propagating terrace* for the equation (29) in dimension $N = 1$ (see Section 4.2 below) as well as the convergence of the solution with Heavyside initial datum to such terrace. Our result in arbitrary dimension is the following.

Proposition 1.7. *Under the assumptions* (30)-(31)*, there exist some numbers* $M \in \mathbb{N}$*,* $0 = \theta_0 < \cdots < \theta_M = 1$ *and* $c_1 > \cdots > c_M > 0$ *such that any bounded solution* u *with a compactly supported initial datum* $0 \le u_0 \le 1$ *such that* $u(t,x) \to 1$ *as* $t \to +\infty$ *locally uniformly in* $x \in \mathbb{R}^N$ *satisfies, for* $m = 1, \dots, M$*,*

$$\forall c > c_m, \quad \limsup_{t\to+\infty} \left(\sup_{|x|\ge ct} u(t,x) \right) \le \theta_{m-1}, \tag{32}$$

$$\forall c < c_m, \quad \liminf_{t\to+\infty} \left(\inf_{|x|\le ct} u(t,x) \right) \ge \theta_m. \tag{33}$$

Proposition 1.7 implies that, as $t \to +\infty$,

$$\forall m \in \{1, \dots, M-1\},\ c_{m+1} < c < c' < c_m, \qquad \sup_{ct\le|x|\le c't} |u(t,x) - \theta_m| \to 0,$$

$$\forall c < c_M, \quad \inf_{|x|\le ct} u(t,x) \to 1, \qquad \forall c > c_1, \quad \sup_{|x|\ge ct} u(t,x) \to 0.$$

Namely, u has the following multi-tiered cake shape far from the regions $|x| \sim c_m t$ for large t:

$$u(t,x) \sim \sum_{m=1}^{M-1} \theta_m \mathbb{1}_{B_{c_m t} \setminus B_{c_{m+1} t}}(x) + \theta_M \mathbb{1}_{B_{c_M t}}(x).$$

The states $0 = \theta_0 < \cdots < \theta_M = 1$ and the speeds $c_1 > \cdots > c_M > 0$ are provided by the minimal propagating terrace of [14]. There, the authors deal with spatial-periodic equations in dimension 1 and derive a convergence result which entails Proposition 1.7 above for the initial datum $u_0 = \mathbb{1}_{(-\infty,0]}$. More general front-like data (for the homogeneous equation) are allowed in the recent paper [24]. The restriction to dimension 1 is intrinsic to the methods of [14, 24], which rely on the zero-number principle and a dynamical system approach. We point out that Proposition 1.7 is new even in dimension 1, because no results have been previously obtained for compactly supported initial data. Let us mention that it is also proved in [14, 24], always in dimension 1, that in each region $x \sim c_m t$ the solution develops an interface approaching a planar wave. The extension of this precise convergence result to higher dimension is the object of the work in progress [13].

2 Subset of spreading

2.1 The general sufficient condition

In this subsection we derive a sufficient condition for a compact set $\mathcal{W} \subset \mathbb{R}^N$, which is star-shaped with respect to the origin, to be an asymptotic subset of spreading for (16). A set of this type can be expressed by

$$\mathcal{W} = \{r\xi \,:\, \xi \in S^{N-1},\ \ 0 \le r \le w(\xi)\}, \quad \text{with } \ w \ge 0 \ \text{ upper semicontinuous.} \tag{34}$$

We will assume that $\mathcal{W}$ fulfils the *uniform interior ball condition*, that is, that there exists $\rho > 0$ such that for all $\hat{x} \in \partial\mathcal{W}$, there is $y \in \mathcal{W}$ satisfying

$$|y - \hat{x}| = \rho, \qquad \overline{B}_\rho(y) \subset \mathcal{W}.$$

We say that $\nu(\hat{x}) := (\hat{x} - y)/\rho$ is an exterior unit normal at $\hat{x}$ (possibly not unique).

We will need two auxiliary results. The first one concerns the stability from below of the steady state 1, which is a consequence of the hypothesis (18).

Lemma 2.1. *Under the assumptions* (17)-(18)*, let* $u \in C^{1+\delta/2,2+\delta}(\mathbb{R}_- \times \mathbb{R}^N)$ *be a supersolution of the equation*

$$\partial_t u - \mathrm{div}(A(t,x)\nabla u) + q(t,x)\cdot\nabla u = f(t,x,u), \quad t < 0,\ x \in \mathbb{R}^N, \tag{35}$$

for which there is $H \subset \mathbb{R}^N$ *such that*

$$\sup_{x\in H} \mathrm{dist}(x, \mathbb{R}^N \setminus H) = +\infty, \qquad \inf_{t<0,\, x\in H} u(t,x) > S,$$

where S *is from* (18)*. Then,*

$$\liminf_{\mathrm{dist}(x,\mathbb{R}^N\setminus H)\to+\infty} \Big(\inf_{t<0} u(t,x)\Big) \ge 1.$$

Proof. Assume by contradiction that

$$h := \liminf_{\mathrm{dist}(x,\mathbb{R}^N\setminus H)\to+\infty} \Big(\inf_{t<0} u(t,x)\Big) \in (S,1).$$

Let $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_-$ and $(x_n)_{n\in\mathbb{R}}$ in H be such that

$$\mathrm{dist}(x_n, \mathbb{R}^N \setminus H) \to +\infty \quad \text{and} \quad u(t_n, x_n) \to h \quad \text{as } \ n \to \infty.$$

The functions $u(\cdot + t_n, \cdot + x_n)$ converge as $n \to \infty$ (up to subsequences) locally uniformly on $\mathbb{R}_- \times \mathbb{R}^N$ to a supersolution u_∞ of a limiting equation (21). Furthermore,

$$u_\infty(0,0) = \min_{\mathbb{R}_- \times \mathbb{R}^N} u_\infty = h \in (S, 1).$$

Notice that $f(t,x,s) \geq 0$ if $s \in [S,1]$ by the first two conditions in (18), and then the same is true for f^*. It then follows from the parabolic strong maximum principle that $u_\infty = h$ in $\mathbb{R}_- \times \mathbb{R}^N$, whence $f^*(t,x,h) = 0$ for $t \leq 0$, $x \in \mathbb{R}^N$. Let us check that also the last property of (18) is inherited by f^*. Fix $s \in (S,1)$ and let E be the relatively dense set in $\mathbb{R}^{N+1}$ on which $f(\cdot,\cdot,s)$ has positive infimum. The fact that E is relatively dense means that there is a compact set $K \subset \mathbb{R}^{N+1}$ such that $E \cap (K + \{(\tau,\xi)\}) \neq \emptyset$, for any $(\tau,\xi) \in \mathbb{R} \times \mathbb{R}^N$. Hence, for any $(\tau,\xi) \in \mathbb{R} \times \mathbb{R}^N$,

$$\max_{(t,x)\in(K+\{(\tau,\xi)\})} f^*(t,x,s) = \lim_{n\to\infty} \max_{(t,x)\in(K+\{(\tau+t_n,\xi+x_n)\})} f(t,x,s) \geq \inf_{(t,x)\in E} f(t,x,s) > 0,$$

that is, f^* fulfils the last condition in (18). This is impossible because $f^*(t,x,h) = 0$ for $t \leq 0$, $x \in \mathbb{R}^N$. □

The second auxiliary lemma is a comparison principle. The proof relies on a rather standard application of the sliding method, in the spirit of [12, 5], and it is presented here in the appendix.

Lemma 2.2. *Assume that* (17)-(18) *hold. Let* $\underline{u}, \overline{u} \in C^{1+\delta/2,2+\delta}(\mathbb{R}_- \times \mathbb{R}^N)$ *be respectively a sub and a supersolution of* (35) *satisfying, for some* $e \in S^{N-1}$,

$$\overline{u} > 0, \qquad \liminf_{x\cdot e \to -\infty} \overline{u}(t,x) \geq 1 \quad \text{uniformly in } t \leq 0, \tag{36}$$

$\underline{u} \leq 1$ *and there exists* $\gamma > 0$ *such that either*

$$\forall s > 0,\ \exists L \in \mathbb{R}, \quad \underline{u}(t,x) \leq s \quad \text{for } t \leq 0,\ x\cdot e \geq \gamma t + L \tag{37}$$

if f *satisfies* (19), *or*

$$\exists L \in \mathbb{R}, \quad \underline{u}(t,x) \leq 0 \quad \text{for } t \leq 0,\ x\cdot e \geq \gamma t + L \tag{38}$$

otherwise. Then, $\underline{u}(t,x) \leq \overline{u}(t,x)$ *for* $(t,x) \in \mathbb{R}_- \times \mathbb{R}^N$.

We are now in the position to derive our key result.

Theorem 2.3. *Under the assumptions* (17)-(18), *let* $\mathcal{W} \subset \mathbb{R}^N$ *be a compact set, star-shaped with respect to the origin and satisfying the uniform interior ball condition. Suppose that for all* $\eta, k < 1$, $\hat{x} \in \partial\mathcal{W}$ *and exterior unit normal* $\nu(\hat{x})$ *at* $\hat{x}$, *every limiting equation* (21) *on* $\mathbb{R}_- \times \mathbb{R}^N$ *admits a subsolution* $v \in C^{1+\delta/2,2+\delta}(\mathbb{R}_- \times \mathbb{R}^N)$ *satisfying* $v \leq 1$, $v(0,0) > \eta$, *and, for some* $c > k\hat{x}\cdot\nu(\hat{x})$, *either*

$$\forall s > 0,\ \exists L \in \mathbb{R}, \quad v(t,x) \leq s \quad \text{if } t \leq 0,\ x\cdot\nu(\hat{x}) \geq ct + L \tag{39}$$

if f *satisfies* (19), *or*

$$\exists L \in \mathbb{R}, \quad v(t,x) \leq 0 \quad \text{if } t \leq 0,\ x\cdot\nu(\hat{x}) \geq ct + L \tag{40}$$

otherwise. Then $\mathcal{W}$ *is an asymptotic subset of spreading for* (16).

Proof. First, the interior ball condition implies that $\mathcal{W}$ coincides with the closure of its interior. Let u be as in Definition 1.1. Notice that $u \leq 1$ by the comparison principle. Fix $\eta \in (0,1)$ and $t > 0$. We start dilating $\mathcal{W}$ until touching the level set $\{u(\cdot,t) = \eta\}$. Namely, we define

$$\mathcal{R}^\eta(t) := \sup\{r \geq 0 \,:\, \forall x \in r\mathcal{W},\ u(t,x) > \eta\}.$$

On one hand, the above quantity is finite because it is well known that $u(t,x) \to 0$ as $|x| \to \infty$ (with Gaussian decay, see e.g. [17]), on the other, $\mathcal{R}^\eta(t) \to +\infty$ as $t \to +\infty$ because $u(t,x) \to 1$ as $t \to +\infty$ locally uniformly in $x \in \mathbb{R}^N$. In order to prove that $\mathcal{W}$ satisfies the condition (5) of the asymptotic subsets of spreading it is sufficient to show that

$$\forall \eta \in (0,1), \quad \liminf_{t\to+\infty} \frac{\mathcal{R}^\eta(t)}{t} \geq 1.$$

Indeed, the above condition implies that, for all $\eta, \varepsilon \in (0,1)$, $u(t,xt) > \eta$ for $x \in (1-\varepsilon)\mathcal{W}$ and t larger than some $t_{\eta,\varepsilon}$. Then, for any compact $K \subset \mathrm{int}(\mathcal{W})$, ε can be chosen in such a way that $(1-\varepsilon)^{-1}K \subset \mathcal{W}$, that is, $K \subset (1-\varepsilon)\mathcal{W}$. It follows that, for any $\eta < 1$, $\inf_{x\in K} u(t,xt) > \eta$ if $t > t_{\eta,\varepsilon}$, which is precisely condition (5).

Assume by way of contradiction that there exist $\eta, k \in (0,1)$ such that

$$\liminf_{t\to+\infty} \frac{\mathcal{R}^\eta(t)}{t} < k. \tag{41}$$

Clearly, (41) still holds if one increases η. Then, we can assume without loss of generality that $\eta \in (S,1)$, where S is from (18). Let us drop for simplicity the η in the notation $\mathcal{R}^\eta$. We have that $\liminf_{t\to+\infty}(\mathcal{R}(t) - kt) = -\infty$. We set

$$\forall n \in \mathbb{N}, \qquad t_n := \inf\{t \geq 0 \,:\, \mathcal{R}(t) - kt \leq -n\}.$$

It follows from the continuity of u that the function $\mathcal{R}$ is lower semicontinuous. We then deduce that the above infimum is a minimum, i.e. $\mathcal{R}(t_n) - kt_n \leq -n < \mathcal{R}t - kt$ for all $0 \leq t < t_n$, and that $t_n \to +\infty$ as $n \to \infty$. To sum up, there holds

$$\lim_{n\to\infty} t_n = +\infty, \qquad \forall n \in \mathbb{N},\ t \in [0,t_n), \quad \mathcal{R}(t_n) - k(t_n - t) < \mathcal{R}(t). \tag{42}$$

Take $n \in \mathbb{N}$. By the definition of $\mathcal{R}$, there exists $x_n \in \partial(\mathcal{R}(t_n)\mathcal{W})$ such that $u(t_n,x_n) = \eta$. We know that $|x_n| \to \infty$ as $n \to \infty$. Define the sequence of functions $(u_n)_{n\in\mathbb{N}}$ by $u_n(t,x) := u(t + t_n, x + x_n)$. These functions are equibounded in $C^{1,2}_{loc}$ by standard parabolic interior estimates (see, e.g., [17]), and therefore they converge (up to subsequences) locally uniformly to a solution u^* of some limiting equation (21) which satisfies $u^*(0,0) = \eta$. The strong maximum principle then yields $u^* > 0$.

Take $T \in [0,t_n]$ and $x \in (\mathcal{R}(t_n) - kT)\mathcal{W}$. It follows from (42) that $x \in \mathcal{R}(t_n - T)\mathcal{W}$, whence $u(t_n - T, x) \geq \eta$. We then derive

$$\forall T \in [0,t_n],\ x \in (\mathcal{R}(t_n) - kT)\mathcal{W} - \{x_n\}, \quad u_n(-T,x) \geq \eta. \tag{43}$$

For $n \in \mathbb{N}$, call

$$\hat{x}_n := \frac{x_n}{\mathcal{R}(t_n)} \in \partial \mathcal{W}, \qquad y_n := \hat{x}_n - \rho\nu(\hat{x}_n),$$

where ρ is the radius of the uniform interior ball condition and $\nu(\hat{x}_n)$ is an associated exterior unit normal at $\hat{x}_n$ (and thus y_n is the centre of the ball). The situation is depicted in Figure 1.

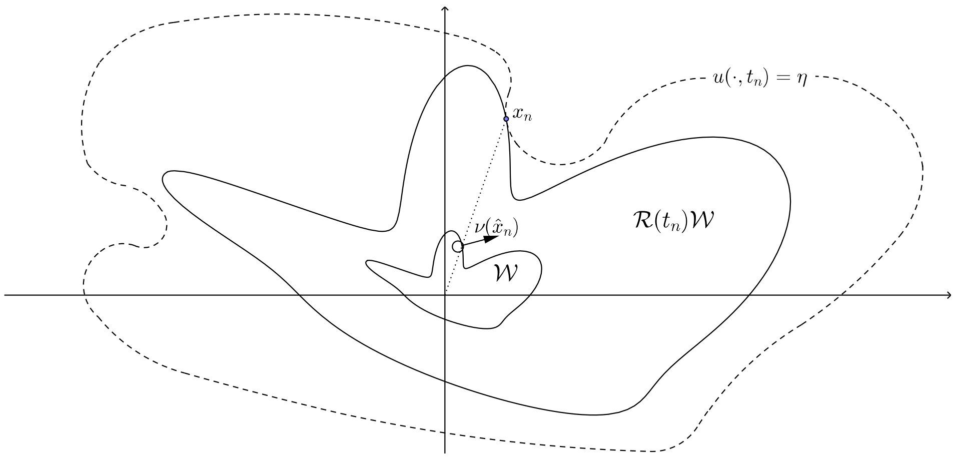

Figure 1: Dialation of $\mathcal{W}$ until touching the level set $\{u(\cdot, t_n) = \eta\}$, at the point x_n.

Let $\hat{x}$, $\nu(\hat{x})$ be the limits of (subsequences of) $(\hat{x}_n)_{n\in\mathbb{N}}$, $(\nu(\hat{x}_n))_{n\in\mathbb{N}}$. Because $\mathcal{W}$ is closed, $\nu(\hat{x})$ is an exterior unit normal at $\hat{x} \in \partial\mathcal{W}$. We claim that, for any $T \geq 0$, as $n \to \infty$, the sets $(\mathcal{R}(t_n) - kT)\mathcal{W} - \{x_n\}$ invade the half-space

$$\mathcal{H}_T := \{x \in \mathbb{R}^N \,:\, x \cdot \nu(\hat{x}) < -k(\hat{x} \cdot \nu(\hat{x}))T\},$$

in the sense that

$$\mathcal{H}_T \subset \bigcup_{M\in\mathbb{N}} \bigcap_{n\geq M} \big((\mathcal{R}(t_n) - kT)\mathcal{W} - \{x_n\}\big) \tag{44}$$

(see Figure 2). This property is a consequence of the fact that these sets satisfy the interior ball condition with radii $(\mathcal{R}(t_n) - kT)\rho$, which tends to ∞ as $n \to \infty$.

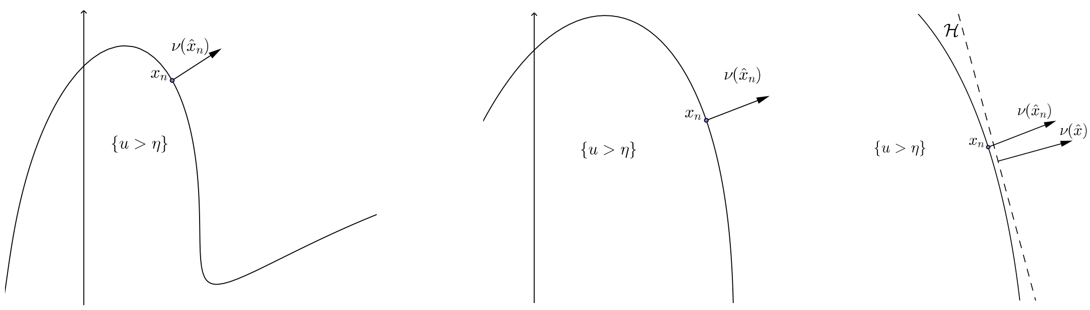

Figure 2: Invasion of the half-space $\mathcal{H}$ by $\mathcal{R}(t_n)\mathcal{W} - \{x_n\}$ as $n \to \infty$.

Let us postpone for a moment the proof of (44) and conclude the proof of the theorem. By (43) and (44) we get

$$\forall T \geq 0,\ x \in \mathcal{H}_T, \quad u^*(-T, x) \geq \eta,$$

and, we recall, $u^*(0,0) = \eta$. Roughly speaking, this means that the set $\{u^* \geq \eta\}$ expands in the direction $\nu(\hat{x})$ with at most speed $k\,\hat{x} \cdot \nu(\hat{x})$, which is smaller than the speed c of the subsolution v given by the hypothesis of the theorem. In order to get a contradiction from this fact, consider the function u^* in the frame moving with speed $k(\hat{x} \cdot \nu(\hat{x}))$ in the direction $\nu(\hat{x})$, i.e.,

$$\overline{u}(t, x) := u^*(t, x + \zeta t), \qquad \text{with} \quad \zeta := k(\hat{x} \cdot \nu(\hat{x}))\nu(\hat{x}).$$

The function $\overline{u}$ satisfies $\overline{u}(t, x) \geq \eta$ if $t \leq 0$ and $x \cdot \nu(\hat{x}) < 0$, together with $\overline{u}(0,0) = \eta$, and it is a solution of the equation

$$\partial_t u - \operatorname{div}(A^*(t, x + \zeta t)\nabla u) + [q^*(t, x + \zeta t) - \zeta] \cdot \nabla u = f^*(t, x + \zeta t, \overline{u}), \quad t < 0,\ x \in \mathbb{R}^N, \tag{45}$$

The nonlinear term $f^*(t, x + \zeta t, s)$ clearly fulfils the first two conditions in (18). Moreover, as we have seen in the last part of the proof of Lemma 2.1, f^* inherits from f the last condition in (18), and then the same is true for $f^*(t, x + \zeta t, s)$. Consequently, since $\overline{u}(t, x) \geq \eta > S$ for $t \leq 0$ and $x \in H := \{x : x \cdot \nu(\hat{x}) < 0\}$, we can apply Lemma 2.1 and infer that $\overline{u}(t, x) \to 1$ as $x \cdot \nu(\hat{x}) \to -\infty$ uniformly in $t \leq 0$. This means that $\overline{u}$ satisfies (36) with $e = \nu(\hat{x})$. Let v and $c > k\hat{x} \cdot \nu(\hat{x})$ be as in the statement of the theorem. The function $\underline{u}$ defined by

$$\underline{u}(t, x) := v(t, x + \zeta t),$$

is a subsolution to (45). We want to apply Lemma 2.2 to $\underline{u}$, $\overline{u}$. To do this, we need to check that $\underline{u}$ satisfies (37) if the nonlinear term in (45) fulfils (19), or the stronger condition (38) otherwise. Properties (37), (38) hold with $e = \nu(\hat{x})$ and $\gamma = c - k\hat{x} \cdot \nu(\hat{x}) > 0$ if v satisfies (39), (40) respectively. On the one hand, (39), which is weaker than (40), always holds by hypothesis. On the other hand, if $f^*(t, x + \zeta t, s)$ does not fulfil (19) then neither does f, because (19) is preserved when passing to the limit of translations. Thus, in such case, v satisfies (40) by hypothesis. We can thereby apply Lemma 2.2 and infer that $\underline{u}(0,0) \leq \overline{u}(0,0)$. This is a contradiction because $\underline{u}(0,0) = v(0,0) > \eta = \overline{u}(0,0)$.

To conclude the proof of the theorem, it remains to derive (44). Take $x \in \mathcal{H}_T$. We compute

$$\begin{aligned}
\left| \frac{x + x_n}{\mathcal{R}(t_n) - kT} - y_n \right| &= \frac{|x + x_n - (\mathcal{R}(t_n) - kT)(\hat{x}_n - \rho\nu(\hat{x}_n))|}{\mathcal{R}(t_n) - kT} \\
&= \frac{|x + kT\hat{x}_n + (\mathcal{R}(t_n) - kT)\rho\nu(\hat{x}_n)|}{\mathcal{R}(t_n) - kT} \\
&= \left| \rho\nu(\hat{x}_n) + \frac{x + kT\hat{x}_n}{\mathcal{R}(t_n) - kT} \right|.
\end{aligned}$$

Calling $z_n := (x + kT\hat{x}_n)/(\mathcal{R}(t_n) - kT)$, which tends to 0 as $n \to \infty$, we rewrite the last term as

$$|\rho\nu(\hat{x}_n) + z_n| = \sqrt{\rho^2 + 2\rho\nu(\hat{x}_n)\cdot z_n + |z_n|^2} = \sqrt{\rho^2 + |z_n|(2\rho\nu(\hat{x}_n)\cdot z_n/|z_n| + |z_n|)}.$$

Since

$$\lim_{n\to\infty}(2\rho\nu(\hat{x}_n)\cdot z_n/|z_n| + |z_n|) = 2\rho\,\frac{x\cdot\nu(\hat{x}) + kT\hat{x}\cdot\nu(\hat{x})}{|x + kT\hat{x}|} < 0,$$

because $x \in \mathcal{H}_T$, we infer that, for sufficiently large n, $|\rho\nu(\hat{x}_n) + z_n| < \rho$ and thus

$$\frac{x + x_n}{\mathcal{R}(t_n) - kT} \subset B_\rho(y_n) \subset \mathcal{W}.$$

Namely, $x + x_n \in (\mathcal{R}(t_n) - kT)\mathcal{W}$, and thus (44) is proved. □

Remark 3. It follows from the proof of Theorem 2.3 that, for given $\hat{x} \in \partial\mathcal{W}$, the only limiting equations for which the existence of the subsolution v is needed are the ones obtained by translations $(t_n, x_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} t_n = +\infty, \qquad \lim_{n\to\infty}\frac{x_n}{|x_n|} = \frac{\hat{x}}{|\hat{x}|}.$$

2.2 Application of the general result

We now prove Theorem 1.4. We cannot apply Theorem 2.3 directly to the set $\mathcal{W}$ defined by (23) because it may not fulfil the uniform interior ball condition. The idea is to consider an interior smooth approximation $\widetilde{\mathcal{W}}$ of $\mathcal{W}$, but to this end we need at least the function w defining $\partial\mathcal{W}$ to be continuous. This is a general consequence of the definition of w.

Proposition 2.4. *Let $c : S^{N-1} \to \mathbb{R}$ satisfy $\inf c > 0$. Then the function $w : S^{N-1} \to \mathbb{R}$ defined by*

$$w(\xi) := \inf_{e\cdot\xi>0}\frac{c(e)}{e\cdot\xi},$$

is positive and continuous.

Proof. There holds the lower bound $w \geq \inf c > 0$. Let us show that w is bounded from above. Consider the family $\mathcal{B} := \{\pm e_1, , \dots, \pm e_N\}$, where $\{e_1, \dots, e_N\}$ is the canonical basis of $\mathbb{R}^N$. Then, calling $\overline{c} := \max_{\mathcal{B}} c$, we find

$$\forall \xi \in S^{N-1}, \quad w(\xi) \leq \min_{\substack{e\in\mathcal{B}\\ e\cdot\xi>0}}\frac{c(e)}{e\cdot\xi} \leq \overline{c}\left(\max_{\substack{e\in\mathcal{B}\\ e\cdot\xi>0}} e\cdot\xi\right)^{-1} \leq \overline{c}\sqrt{N}.$$

Now, fix $\xi \in S^{N-1}$. For $\varepsilon \in (0,1)$, let $e_\varepsilon \in S^{N-1}$ be such that

$$e_\varepsilon\cdot\xi > 0, \qquad w(\xi) > \frac{c(e_\varepsilon)}{e_\varepsilon\cdot\xi} - \varepsilon.$$

Hence, $c(e_\varepsilon)/e_\varepsilon\cdot\xi<\overline{c}\sqrt{N}+1$, from which we deduce

$$c(e_\varepsilon)<\overline{c}\sqrt{N}+1,\qquad e_\varepsilon\cdot\xi>h:=\frac{\inf c}{\overline{c}\sqrt{N}+1}.$$

For $\xi'\in S^{N-1}$ such that $|\xi'-\xi|<h/2$, it holds that $e_\varepsilon\cdot\xi'>h/2$, whence

$$w(\xi')-w(\xi)\leq\frac{c(e_\varepsilon)}{e_\varepsilon\cdot\xi'}-w(\xi)<\frac{c(e_\varepsilon)}{e_\varepsilon\cdot\xi'}-\frac{c(e_\varepsilon)}{e_\varepsilon\cdot\xi}+\varepsilon\leq 2\,\frac{\overline{c}\sqrt{N}+1}{h^2}|\xi-\xi'|+\varepsilon.$$

The latter term is smaller than 2ε for $|\xi'-\xi|$ small enough, independently of ξ,ξ'. This shows that w is (uniformly) continuous. □

Proof of Theorem 1.4. Let w and $\mathcal{W}$ be as in (23). Owing to (22), we can apply Proposition 2.4 and deduce that w is positive and continuous. It follows in particular that $\min w>0$ and that the set $\mathcal{W}$ coincides with the closure of its interior. Moreover, for any $h\in(0,\min w)$, we can consider a smooth approximation $\tilde{w}$ of the function $w-h/2$ satisfying $w-h<\tilde{w}<w$. If we show that, for any $h\in(0,\min w)$, the set

$$\widetilde{\mathcal{W}}:=\{r\xi\ :\ \xi\in S^{N-1},\ \ 0\leq r\leq\tilde{w}(\xi)\},$$

is an asymptotic subset of spreading, the same is true for $\mathcal{W}$, because if $K\Subset\mathrm{int}(\mathcal{W})$ then $K\Subset\mathrm{int}(\widetilde{\mathcal{W}})$ for h small enough. This is achieved by showing that $\widetilde{\mathcal{W}}$ satisfies the hypotheses of Theorem 2.3.

Consider $\eta,k<1$, $\hat{x}\in\partial\widetilde{\mathcal{W}}$, the (unique) exterior unit normal $\nu(\hat{x})$ to $\widetilde{\mathcal{W}}$ and a limiting equation (21). We know that $\hat{x}\neq 0$ and thus we can write $\hat{x}=\tilde{w}(\xi)\xi$, with $\xi:=\hat{x}/|\hat{x}|\in S^{N-1}$. By hypothesis, there is a transition front v in the direction $\nu(\hat{x})$ for (21) on $\mathbb{R}_-\times\mathbb{R}^N$, which connects 0 and 1 if f satisfies (19), or some $-\varepsilon<0$ and 1 otherwise, has speed larger than $c:=k\underline{c}(\nu(\hat{x}))$ and satisfies $v(0,0)>\eta$. Let X be the function for which v satisfies the limits in (20) with $S_1=-\varepsilon$ or 0 and $S_2=1$. It follows from the uniformity of these limits and the strong maximum principle that $v<1$. Moreover, since v has speed larger than c, there holds that $X(t)<ct$ for t less than some $T<0$. On the other hand, we know from [21] that X is locally bounded and thus there exists $K>0$ such that $X(t)<ct+K$ for all $t\leq 0$. As a consequence, by (20), v satisfies (39) if f fulfils (19), or (40) otherwise. Finally, we deduce from the smoothness of $\tilde{w}$ that $\hat{x}\cdot\nu(\hat{x})>0$, i.e. $\xi\cdot\nu(\hat{x})>0$. We can then compute

$$c=k\underline{c}(\nu(\hat{x}))\geq k\nu(\hat{x})\cdot\xi\inf_{\substack{e\in S^{N-1}\\ e\cdot\xi>0}}\frac{\underline{c}(e)}{e\cdot\xi}=k\nu(\hat{x})\cdot\xi\,w(\xi)>k\tilde{w}(\xi)\,\xi\cdot\nu(\hat{x})=k\hat{x}\cdot\nu(\hat{x}).$$

We have shown that v satisfies all the requirements in Theorem 2.3, whence $\widetilde{\mathcal{W}}$ is an asymptotic subset of spreading. □

2.3 The periodic case

We now prove that the set $\mathcal{W}$ in (15) is an asymptotic subset of spreading for equation (1) with periodic coefficients. To do this, it suffices to show that the (minimal) speed c^* for pulsating travelling fronts satisfies the hypotheses for $\underline{c}$ in Theorem 1.4.

We recall the known results about pulsating travelling fronts: they are increasing in time and their critical speed $c^*(e)$ is positive. It is also readily seen that (14) yields the transition front condition (20) with $S_1 = 0$, $S_2 = 1$ and $X(t) = cte$.

The first hypothesis to check in Theorem 1.4 is $\inf c^* > 0$. We derive it from the following result, which is of independent interest.

Proposition 2.5. *Under the assumptions of Theorem 1.2, the function $c^* : S^{N-1} \to \mathbb{R}$ is lower semicontinuous.*

Proof. We need to show that, given a sequence $(e_n)_{n\in\mathbb{N}}$ in S^{N-1} such that

$$e_n \to e \in S^{N-1} \quad \text{and} \quad c^*(e_n) \to c \in [0, +\infty) \quad \text{as } n \to \infty,$$

there holds that $c^*(e) \leq c$. Let v_n be the pulsating travelling front in the direction e_n connecting 1 to 0 with speed $c^*(e_n)$. Take $M \in (\theta, 1)$ if f satisfies either (11) or (12), or set $M := 1/2$ in the case (10). Since $v_n(t, x) \to 0$ or 1 as $t \to -\infty$ or $+\infty$ locally uniformly in $x \in \mathbb{R}^N$, by a temporal translation we reduce to the case where

$$\min_{x\in[0,1]^N} v_n(0, x) = M, \tag{46}$$

where, we recall, $\mathcal{C} = (0, \ell_1) \times \cdots \times (0, \ell_N)$. The v_n converge (up to subsequences) locally uniformly to a solution $0 \leq v \leq 1$ nondecreasing in t and satisfying the normalization (46). Actually, $0 < v < 1$ by the parabolic strong maximum principle.

Case $c > 0$.
Because the v_n satisfy the first condition in (14) with $e = e_n$ and $c = c_n$, passing to the limit as $n \to \infty$ we deduce that v satisfies the first condition in (14). Then, letting $t \to \pm\infty$ in such condition we infer that the functions $v^\pm$ defined by $v^\pm(x) := v(\pm\infty, x)$ are 1-periodic. It follows in particular that

$$\exists x^\pm \in \mathbb{R}^N, \quad v^\pm(x^\pm) = \min_{\mathbb{R}^N} v^\pm =: m^\pm, \qquad 0 \leq m^- \leq M \leq m^+ \leq 1.$$

We further know from parabolic estimates that the convergences of v to $v^\pm$ as $t \to \pm\infty$ hold locally uniformly in $\mathbb{R}^N$, and that the $v^\pm$ are stationary solutions of (1). Since $f \geq 0$ on $\mathbb{R}^N \times [M, 1]$, we have that $f(x, v^+) \geq 0$. The strong maximum principle then yields $v^+ \equiv m^+$ and thus $f(x, m^+) = 0$ for all $x \in \mathbb{R}^N$. We then deduce from the choice of M that $m^+ = 1$, that is, $v^+ \equiv 1$. For $x \in \mathbb{R}^N$, let $z(x) \in \ell_1\mathbb{Z} \times \cdots \times \ell_N\mathbb{Z}$ be such that $x - z(x) \in \overline{\mathcal{C}}$. By the first property in (14), we can write

$$\forall (t, x) \in \mathbb{R} \times \mathbb{R}^N, \quad v(t, x) = v\Big(t - \frac{z(x) \cdot e}{c}, x - z(x)\Big), \quad \text{with} \quad x - z(x) \in \overline{\mathcal{C}}. \tag{47}$$

Whence, since $z(x) \cdot e \to -\infty$ as $x \cdot e \to -\infty$ and $v(+\infty, x) = 1$ locally uniformly in x, we find that

$$v(t, x) \to 1 \quad \text{as } x \cdot e \to -\infty \quad \text{locally uniformly in } t \in \mathbb{R}. \tag{48}$$

Similarly, if f satisfies the monostability hypothesis (10), we derive $m^- = 0$, whence $v^- \equiv 0$ by the strong maximum principle, and (47) eventually yields that both limits in the last condition in (14) hold in such case. That is, v is a pulsating travelling front in the direction e connecting 1 to 0 with speed c, and therefore $c^*(e) \leq c$ by definition. If instead f is of either combustion or bistable type, i.e. (11) or (12) hold, we cannot deduce $m^- = 0$ and infer that v connects 0 and 1. In these cases we resort to Lemma 2.2. Set $\overline{u}(t,x) := v(t, x+cte)$. For $t \in \mathbb{R}$, letting $z(t) \in \ell_1\mathbb{Z} \times \cdots \times \ell_N\mathbb{Z}$ be such that $cte - z(t) \in \overline{\mathcal{C}}$ and using the fact that v verifies the first condition in (14), we get, for $x \in \mathbb{R}^N$,

$$\overline{u}(t,x) = v\Big(t - \frac{z(t)\cdot e}{c}, x + cte - z(t)\Big) = v\Big(\frac{cte - z(t)}{c}\cdot e, x + cte - z(t)\Big).$$

Hence, by (48), $\overline{u}$ satisfies $\overline{u}(t,x) \to 1$ as $x \cdot e \to -\infty$ uniformly in $t \in \mathbb{R}$, and then in particular (36). Next, consider the pulsating travelling front $\tilde{v}$ in the direction e connecting 1 to 0 (with speed $c^*(e)$), translated in time in such a way that $\tilde{v}(0,0) > v(0,0)$, and set $\underline{u}(t,x) := \tilde{v}(t, x + cte)$. If we had $c < c^*(e)$, since

$$\underline{u}(t,x) = \tilde{v}(t, x + c^*(e)te - (c^*(e) - c)te)$$

and $\tilde{v}$ satisfies (20) with $X(t) = c^*(e)t$ and $S_1 = 0$ (and $S_2 = 1$), condition (37) would be fulfilled with $\gamma = c^*(e) - c > 0$. We could then apply Lemma 2.2 to $\underline{u}$ and $\overline{u}$, which satisfy (1) for $t < 0$ with q replaced by $q + ce$, and deduce $\tilde{v} \leq v$ in $\mathbb{R}_- \times \mathbb{R}^N$, in contradiction with $\tilde{v}(0,0) > v(0,0)$. Hence, $c^*(e) \leq c$ in cases (11), (12) too.

Case $c = 0$.
We will show that such case is impossible. The v_n satisfy (47) with $c = c_n$, $e = e_n$. Take $x \cdot e < -|\ell|$. It holds that $z(x) \cdot e < 0$, whence, for any $t \in \mathbb{R}$, we have $t - z(x)\cdot e_n/c_n > 0$ for n large enough because $c_n \searrow 0$. It then follows from the fact that the v_n are increasing in time and from (46) that, for $t \in \mathbb{R}$ and $x \cdot e < -|\ell|$,

$$v(t,x) = \lim_{n\to\infty} v_n\left(t - \frac{z(x)\cdot e}{c_n}, x - z(x)\right) \geq \lim_{n\to\infty} v_n(0, x - z(x)) \geq M.$$

Thus, by Lemma 2.1, $v(t,x) \to 1$ as $x \cdot e \to -\infty$ uniformly in $t \leq 0$, and then $\overline{u} = v$ fulfils (36). If f satisfies either (11) or (12), we get a contradiction as before by applying Lemma 2.2 with $\overline{u} = v$ and $\underline{u}$ equal to the pulsating travelling front $\tilde{v}$ in the direction e connecting 1 to 0. Suppose that f satisfies (10). Setting $f(x,s) = 0$ for $s < 0$, we have that f is of combustion type if considered on, say, $\mathbb{R}^N \times [-1,1]$. Namely, it satisfies hypothesis (11) up to an affine transformation of the second variable. There exists then a pulsating travelling front in the direction e connecting 1 to -1 with a speed $c' > 0$. Let $\underline{u}$ be this front, normalised in such a way that $\underline{u}(0,0) > v(0,0)$. It is an entire solution to (1) satisfying (38). We therefore get a contradiction applying once again Lemma 2.2 with such $\underline{u}$ and $\overline{u} = v$. □

Proposition 2.6. *Under the assumptions of Theorem 1.2, the function w defined in* (15) *is positive and continuous and $\mathcal{W}$ is an asymptotic subset of spreading for* (1).

Proof. The positivity and continuity of w follow from Propositions 2.4, 2.5 and the fact that c^* is positive. In order to apply Theorem 1.4 with $\underline{c} = c^*$, it remains to check the hypothesis concerning the existence of the pulsating travelling front v. To this end, fix $e \in S^{N-1}$, $c < c^*(e)$, $\eta < 1$ and consider a limiting equation (21) associated with (1). By periodicity, the coefficients of such equation are simply translations of A, q, f by the same $\zeta \in \overline{\mathcal{C}}$. We can assume without loss of generality that $\zeta = 0$.

In the case where f is of combustion type (11) or bistable type (12), we take v equal to the pulsating travelling front connecting 1 to 0 in the direction e, normalised in such a way that $v(0,0) > \eta$.

The monostable case (10) is more involved. Let v^* be a pulsating travelling front connecting 1 to 0 in the direction e with (the minimal) speed $c^*(e)$. For $\varepsilon > 0$, the nonlinearity $f : \mathbb{R}^N \times [-\varepsilon, 1] \to \mathbb{R}$ is of combustion type and therefore there exists a unique $c_\varepsilon > 0$ for which (1) admits a pulsating travelling front v_ε in the direction e connecting 1 to $-\varepsilon$. We will show that

$$c_\varepsilon \to c^*(e) \quad \text{as} \;\; \varepsilon \searrow 0. \tag{49}$$

A similar property is derived in [5] using some estimates on the first derivatives of the fronts. Let us present a direct approach based on the comparison result of Lemma 2.2. Recalling that v^* and v_ε satisfy (20) with $X(t) = c^*(e)t\,e$, $S_1 = 0$, $S_2 = 1$ and with $X(t) = c_\varepsilon t\,e$, $S_1 = -\varepsilon$, $S_2 = 1$ respectively, we see that, if we had $c_\varepsilon > c^*(e)$ for some $\varepsilon > 0$, Lemma 2.2 would apply with q replaced by $q + ce$ in equation (35) and

$$\overline{u}(t,x) = v^*(t, x + c^*(e)te), \qquad \underline{u}(t,x) = v_\varepsilon(t, x + c^*(e)te),$$

yielding $v^* \geq v_\varepsilon$ in $\mathbb{R}_- \times \mathbb{R}^N$. This is impossible because, up to a suitable temporal translation, we can always reduce to the case where $v^* < v_\varepsilon$ at, say, $(0,0)$. Hence $c_\varepsilon \leq c^*(e)$. The same argument shows that c_ε is nonincreasing with respect to ε, because, in Lemma 2.2, the conditions $\overline{u} > 0$ in (36) and $\forall s > 0$ in (37) can of course be replaced by $\overline{u} > -\varepsilon$ and $\forall s > -\varepsilon$ respectively. As a consequence, c_ε converges to some value $c_0 \in (0, c^*(e)]$ as $\varepsilon \searrow 0$. Let us normalise the v_ε by $v_\varepsilon(0,0) = 1/2$. As $\varepsilon \to 0$, the v_ε converge (up to subsequences) locally uniformly to an entire solution v_0 of (1) satisfying

$$v_0(0,0) = 1/2, \qquad 0 \leq v_0 \leq 1, \qquad \partial_t v \geq 0.$$

Moreover, v_0 satisfies the first condition in (14) with $c = c_0$. Then, the second condition follows exactly as in the case $c > 0$ of the proof of the Proposition 2.5. This means that v_0 is a pulsating travelling front connecting 1 to 0 in the direction e with speed c_0, which yields $c_*(e) \leq c_0$ by definition and concludes the proof of (49). Finally, by (49), we can choose $\varepsilon > 0$ small enough in such a way that $c_\varepsilon \in (c, c_*(e)]$, and then the associated front v_ε, translated in t in order to have $v_\varepsilon(0,0) > \eta$, satisfies the desired properties for v. The proof of the proposition is thereby achieved. □

3 Asymptotic superset of spreading

Proof of Theorem 1.5. Let w and $\mathcal{W}$ be as in (26). Because of (24), Proposition 2.4 implies that w is positive and continuous. It follows in particular that $\mathcal{W}$ coincides with the closure of its interior. It remains to verify that (6) holds for any u as in Definition 1.1. It is well known (see, e.g., [17]) that $u(1,x)$ decays as a Gaussian as $|x| \to \infty$, because the initial datum u_0 has compact support. Call

$$\eta := \max_{x\in\mathbb{R}^N} u(1,x),$$

which is strictly less than 1 by the parabolic strong maximum principle. Take $e \in S^{N-1}$. Let v be the front given by the hypothesis of the theorem, associated with e, η and some $R \geq 0$ to be chosen. Condition (25) implies that the function v decays at most exponentially in the direction e. This is a consequence of [25, Lemma 3.1], applied to the function $\phi(t,x) := v(t, x+[R-1+\overline{c}(e)t]e)$. Namely, it is shown there that there exists a constant $\lambda > 0$, only depending on $A, q, e, \overline{c}(e)$, such that

$$\forall x\cdot e > R+\overline{c}(e), \qquad v(1,x) \geq \eta e^{-\lambda(x\cdot e-R+1-\overline{c}(e))} \geq \eta e^{-\lambda(x\cdot e+1)}.$$

Since, on the other hand, $v(1,x) \geq \eta$ for $x\cdot e \leq R+\overline{c}(e)$, it follows from the Gaussian decay of $u(1,\cdot)$ that, choosing R large enough, the front v satisfies $v(1,\cdot) \geq u(1,\cdot)$ in the whole $\mathbb{R}^N$. As a consequence of the comparison principle we get $u \leq v$ for all $t \geq 1$, whence, since v satisfies (20) with $S_1 = 0$, and $\limsup_{t\to+\infty} X(t)/t \leq \overline{c}(e)$,

$$\forall c > \overline{c}(e), \quad \sup_{x\cdot e > ct} u(t,x) \to 0 \quad \text{as } t \to +\infty. \tag{50}$$

Using this property in different directions e one easily derives (2) with w as in (26). But the uniform version of (2), property (6), requires some additional work and in particular the continuity of w. We proceed as follows.

Fix $\varepsilon > 0$ and $\xi \in S^{N-1}$. By the definition of w in (26), there is $e \in S^{N-1}$ such that $e\cdot\xi > 0$ and $\overline{c}(e)/e\cdot\xi < w(\xi)+\varepsilon$. It then follows from the continuity of w that there exists $\rho_\xi > 0$ such that

$$\min_{|\xi'-\xi|\leq\rho_\xi} (w(\xi')+\varepsilon)e\cdot\xi' > \overline{c}(e).$$

We can therefore make use of (50) and derive

$$\sup_{\substack{|\xi'-\xi|\leq\rho_\xi \\ r\geq w(\xi')+\varepsilon}} u(t, rt\xi') \to 0 \quad \text{as } t \to +\infty.$$

By compactness, there is a finite covering of S^{N-1} by sets of the type $B_{\rho_\xi}(\xi)$, $\xi \in S^{N-1}$, whence the above limit actually holds taking the sup among all $\xi' \in S^{N-1}$. This concludes the proof of (6), because if C is a closed set such that $C \cap \mathcal{W} = \emptyset$, then $C \subset \{r\xi' : \xi' \in S^{N-1},\ r \geq w(\xi')+\varepsilon\}$ with $\varepsilon = \text{dist}(C, \mathcal{W})$. □

Proof of Theorem 1.2. Let $\mathcal{W}$ be defined by (15). By Proposition 2.6 we know that $\mathcal{W}$ is an asymptotic subset of spreading for (1). In order to prove that it is also an asymptotic superset of spreading, we make use of Theorem 1.5. We need to show that the minimal speed for pulsating travelling fronts c^* fulfils the hypotheses for $\overline{c}$ there. We already know that $\min c^* > 0$ because c^* is positive and it is lower semicontinuous by Proposition 2.5. Fix $e \in S^{N-1}$ and let v be the pulsating travelling front in the direction e connecting 1 to 0 with speed $c^*(e)$. It follows from (14) that v satisfies the transition front condition (20) with $S_1 = 0$, $S_2 = 1$ and $X(t) = c^*(e)t\,e$. Hence, for any $\eta < 1$, there exists $L \in \mathbb{R}$ such that

$$\forall t \in \mathbb{R},\ x \cdot e < L, \quad v(t, x + c^*(e)te) > \eta.$$

For given $R > 1$, let $z \in \ell_1\mathbb{Z} \times \cdots \times \ell_N\mathbb{Z}$ be such that $z \cdot e < L - R$. Hence, the translation v^z of v defined by $v^z(t,x) := v(t, x+z)$, which is still a pulsating travelling front for (1) with speed $c^*(e)$, satisfies

$$\forall t \in \mathbb{R},\ x \cdot e - c^*(e)t \leq R, \quad v^z(t,x) = v(t, x+z) > \eta,$$

because $(x + z - c^*(e)te) \cdot e \leq R + z \cdot e < L$. It follows that v^z fulfils (25). We can therefore apply Theorem 1.5 and conclude the proof. □

4 Application to spatial-homogeneous equations

4.1 Almost periodic, temporal-dependent equations

In this section we deduce Proposition 1.6 from Theorems 1.4, 1.5. The assumptions under which Shen derives the existence of fronts in [28] and [26] are, respectively:

Combustion: $\exists \theta \in (0,1)$, $\forall t \in \mathbb{R}$, $f(t,s) = 0$ for $s \leq \theta$ and $s = 1$, $f(t,s) > 0$ for $s \in (\theta, 1)$; f is of class C^1 with respect to $s \in [\theta, 1]$ and satisfies $\inf_t \partial_s f(t, \theta) > 0$, $\sup_t \partial_s f(t,1) < 0$.

Bistable: the equation $\vartheta'(t) = f(t, \vartheta(t))$ in $\mathbb{R}$ admits an a.p. solution $0 < \theta(t) < 1$, and any other solution satisfies $\vartheta(+\infty) = 0$ if $\vartheta(0) < \theta(0)$ and $\vartheta(+\infty) = 1$ if $\vartheta(0) > \theta(0)$; $f \in C^2$ and its derivatives up to order 2 are a.p. in t uniformly in s; $\sup_t \partial_s f(t,0) < 0$, $\sup_t \partial_s f(t,1) < 0$, $\inf_t \partial_s f(t, \theta(t)) > 0$.

Proof of Proposition 1.6. For $e \in S^{N-1}$, let $v = v(t, x \cdot e)$ be the planar front provided by [28, 26] and let X be the associated function in Definition 1.3. The functions $X'(t)$ and $v(t, x \cdot e + X(t))$ are a.p. in t uniformly in x, and X' has uniform average c^* in the sense of (28). We want to show that the hypotheses of Theorems 1.4, 1.5 are fulfilled with $\underline{c}$ and $\overline{c}$ constantly equal to c^*. The front $v(t, x \cdot e)$ is a transition front in the direction e with future (and past) speed equal to c^*. Moreover, because of the space-invariance of the equation (27), we can translate v in such a way that it fulfils (25) for any given $\eta < 1$ and $R \in \mathbb{R}$.

It remains to show that $\underline{c} \equiv c^*$ satisfies the hypotheses of Theorem 1.4. This is a consequence of the fact that, by the almost periodicity, the limit of translations of a front preserves its average speed. Consider indeed a limiting equation associated with (27). By the almost periodicity of f, this equation is of the form $\partial_t u - \Delta u = f^*(t,u)$, with f^* obtained as the uniform limit of time-translations of f by some diverging sequence $(t_n)_{n \in \mathbb{N}}$. By a priori estimates, the translations of the front $v(t + t_n, x \cdot e + X(t_n))$ converge (up to subsequences) locally uniformly to a solution $v^*(t, x \cdot e)$

of the limiting equation. On the other hand, the almost periodicity implies the existence of w and c such that, as $n \to \infty$ (up to subsequences), there holds

$$v(t+t_n, x\cdot e + X(t+t_n)) \to w(t, x\cdot e), \qquad X'(t+t_n) \to c(t),$$

uniformly in $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$, with $w(t,-\infty) = 1$ and $w(t,+\infty) = 0$ uniformly in $t \in \mathbb{R}$. Next, calling $Y(t) := \int_0^t c(s)ds$ we find that

$$\begin{aligned} v^*(t, x\cdot e + Y(t)) &= \lim_{n\to\infty} v(t+t_n, x\cdot e + Y(t) + X(t_n)) \\ &= \lim_{n\to\infty} v(t+t_n, x\cdot e + \int_0^t \big(c(s) - X'(s+t_n)\big)ds + X(t+t_n)) \\ &= w(t, x\cdot e), \end{aligned}$$

from which we deduce that v^* is a transition front in the sense of Definition 1.3 with $X = Y$. Finally, (28) yields

$$\lim_{t\to\pm\infty} \frac{Y(t)}{t} = \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t c(s)ds = \lim_{t\to\pm\infty}\lim_{n\to\infty} \frac{1}{t}\int_{t_n}^{t_n+t} X'(s)ds = c^*,$$

whence v^* has past speed c^*. It follows that, up to a suitable spatial translation, v^* satisfies the hypotheses of Theorem 1.4 with $\underline{c}(e) = c^*$. □

4.2 Multistable equations

This section is devoted to the proof of Proposition 1.7. In the case of dimension $N = 1$ and Heaviside initial datum $u_0 = \mathbb{1}_{(-\infty,0]}$, Proposition 1.7 is a consequence of the main result of [14]. The states $0 = \theta_0 < \cdots < \theta_M = 1$ and the speeds $c_1 > \cdots > c_M > 0$ are provided by the *minimal propagating terrace* connecting 1 to 0. Namely, for any $m = 1, \dots, M$, there exists a planar wave connecting θ_m to θ_{m-1} with speed c_m, i.e., a decreasing solution of the type $\phi(x - c_m t)$ with $\phi(-\infty) = \theta_m$, $\phi(+\infty) = \theta_{m-1}$. The minimality condition means that these waves are steeper than any other wave ranged in $[0,1]$ (see also [30, § 1.3.2]).

The upper estimate (32) in Proposition 1.7 immediately follows by comparison with the one-dimensional result of [14]. In order to derive the lower estimate (33) from our general characterization of the asymptotic subset of spreading, we need the following ODE result, which is a consequence of the minimality of the terrace.

Lemma 4.1. *For any $m \in \{1, \dots, M\}$, $c < c_m$ and $\eta < \theta_m$, the problem*

$$q'' + cq' + f(q) = 0 \quad \text{in } \mathbb{R}_-$$

admits a solution $q \in C^2((-\infty, 0])$ which is decreasing and satisfies $\eta < q(-\infty) \le \theta_m$ and $q(0) = \theta_{m-1}$.

Proof. Take $c < c_m$ and $\eta \in (\theta_{m-1}, \theta_m)$. Consider the front ϕ connecting θ_m to θ_{m-1} given by the minimal terrace, translated in such a way that $\phi(0) = \eta$. Observe that ϕ' is a nonpositive solution of $(\phi')'' + c_m(\phi')' + f(\phi)'\phi' = 0$, thus it cannot vanish anywhere because otherwise it would be identically equal to 0.

For $\mu < 0$, let q_μ be the solution of $q'' + cq' + f(q) = 0$ in $\mathbb{R}$ with

$$q_\mu(0) = \eta, \qquad q'_\mu(0) = \mu.$$

Let (x_μ, y_μ) be the largest interval (possibly unbounded) containing 0 in which $q'_\mu < 0$. Then call

$$\alpha := \max\{q_\mu(y_\mu), \theta_{m-1}\} < \eta < \beta := \min\{q_\mu(x_\mu), \theta_m\}.$$

Consider the functions $\Phi, Q : (\alpha, \beta) \to \mathbb{R}_-$ defined by $\Phi(u) := \phi'(\phi^{-1}(u))$ and $Q(u) := q'_\mu(q_\mu^{-1}(u))$, where q_μ^{-1} is the inverse of the restriction of q_μ to (x_μ, y_μ). They satisfy

$$\Phi' = -c_m - \frac{f(u)}{\Phi}, \qquad Q' = -c - \frac{f(u)}{Q}, \qquad u \in (\alpha, \beta),$$

$$\Phi(\eta) = \phi'(0), \qquad Q(\eta) = \mu.$$

Hence, if $\mu \le \phi'(0)$ then $Q < \Phi$ in (α, η), and thus from $\Phi(\alpha^+)Q(\alpha^+) = 0$ we get $\Phi(\alpha^+) = 0$, i.e., $q_\mu(y_\mu) \le \alpha = \theta_{m-1}$. Likewise, if $\mu \ge \phi'(0)$ then $Q > \Phi$ in (η, β), whence $q_\mu(x_\mu) = \beta \le \theta_m$.

Let us now define

$$\bar{\mu} := \inf\{\mu \ : \ q_\mu(x_\mu) \le \theta_m\}.$$

Recall that x_μ is allowed to be $-\infty$ and then $q_\mu(x_\mu)$ could be $+\infty$. On one hand, it is clear that $\bar{\mu} > -\infty$, and, on the other, we have seen before that $\bar{\mu} \le \phi'(0)$. Moreover, by the continuity (in C^2_{loc}) of solutions to the Cauchy problem with respect to initial data, it follows that $q_{\bar{\mu}}(x_{\bar{\mu}}) \le \theta_m$. Suppose that $x_{\bar{\mu}} > -\infty$. Then necessarily $f(q_{\bar{\mu}}(x_{\bar{\mu}})) = -q''_{\bar{\mu}}(x_{\bar{\mu}}) > 0$, whence $q_{\bar{\mu}}(x_{\bar{\mu}}) < \theta_m$. It follows that $q'_{\bar{\mu}} < 0$ in a left neighbourhood of $x_{\bar{\mu}}$. From this one readily gets that $x_\mu > -\infty$ and $q_\mu(x_\mu) < \theta_m$ for $\mu < \bar{\mu}$ close enough to $\bar{\mu}$, contradicting the definition of $\bar{\mu}$. We have thereby shown that $x_{\bar{\mu}} = -\infty$.

To sum up, we have found $\bar{\mu} \le \phi'(0)$ such that $q_{\bar{\mu}}$ is decreasing on $(-\infty, y_{\bar{\mu}})$ and satisfies $q_{\bar{\mu}}(-\infty) \le \theta_m$ and $q_{\bar{\mu}}(y_{\bar{\mu}}) \le \theta_{m-1}$. Moreover, the associated function Q satisfies $Q'(\eta) = \Phi'(\eta) - c + c_m > \Phi'(\eta)$, that is, $q'_{\bar{\mu}} \circ q_{\bar{\mu}}^{-1} < \phi' \circ \phi^{-1}$ in a left neighbourhood of η, which means that $q_{\bar{\mu}}$ is steeper than ϕ there. As a consequence, if $q_{\bar{\mu}}(y_{\bar{\mu}}) = \theta_{m-1}$ then $y_{\bar{\mu}} = +\infty$, that is, $q_{\bar{\mu}}$ is a planar wave ranged in $[\theta_{m-1}, \theta_m]$, contraddicting the fact that ϕ is steeper than any of such waves, c.f. [30, § 1.3.2]. Thus, we necessarily have that $q_{\bar{\mu}}(y_{\bar{\mu}}) < \theta_{m-1}$. The function q satisfying the properties stated in the lemma is therefore obtained as a translation of $q_{\bar{\mu}}$. □

Proof of Proposition 1.7. Let u be a bounded solution with a compactly supported initial datum $0 \le u_0 \le 1$ such that $u(t,x) \to 1$ as $t \to +\infty$.

The upper bound (32) is a direct consequence of the 1-dimensional result of [14]. Indeed, taking $R > 0$ such that $\operatorname{supp} u_0 \subset B_R$, we know from [14] that the solution v of (29) in dimension $N = 1$ with initial datum $\mathbb{1}_{(-\infty,R]}$ satisfies

$$\forall m \in \{1, \dots, M\}, \ c > c_m, \quad \limsup_{t \to +\infty} \left(\sup_{r \ge ct} v(t,r) \right) \le \theta_{m-1}.$$

Thus, since by comparison $u(t,x) \le v(t, x\cdot e)$ for all $e \in S^{N-1}$, we see that $u(t,x) \le v(t,|x|)$, and therefore u satisfies (32).

In order to derive the lower bound, we apply recursively Theorem 1.4 to each level of the terrace. For the first level, we take $\underline{c}$ identically equal to c_1 and we just observe that the steady state 1 in Theorem 1.4 can be replaced by θ_1: we only lose the hypothesis that the initial datum is smaller than this new state, but this is needed to get the upper bound on the speed rather than the lower bound. We have that all limiting equations coincide with the original one. As for the front v in a given direction e, we take a translation of $q(x\cdot e - ct)$ with q given by Lemma 4.1 [3]. Then, Theorem 1.4 implies that B_{c_1} is an asymptotic subset of spreading for the state θ_1, i.e.,

$$\forall c < c_1, \quad \liminf_{t\to+\infty} \left(\inf_{|x|\le ct} u(t,x) \right) \ge \theta_1. \tag{51}$$

Next, we want to show that B_{c_2} is an asymptotic subset of spreading for the state θ_2 by applying Theorem 1.4 with $\underline{c} \equiv c_2$ and $v(t,x) = q(x\cdot e - ct)$ given by Lemma 4.1 with $m = 2$. The choice of the steady states 0, 1 in Theorem 1.4 is arbitrary and can be replaced by θ_1, θ_2 respectively, but the problem is that now we lose the condition $u \ge \theta_1$, which is essential for the result to hold. The only passage where such condition is needed is in the proof of Theorem 2.3 - on which Theorem 1.4 relies: in order to compare the limit function $u^*(t,x) = \lim_{n\to\infty} u(t+t_n, x+x_n)$ with the subsolution v one needs $u^* \ge \theta_1$. But then, recalling that $x_n \in \partial(\mathcal{R}^\eta(t_n)\mathcal{W})$, with $\mathcal{W} = B_{c_2}$ and $\mathcal{R}^\eta(t_n) < t_n$ for n large, it follows from (51) and the inequality $c_2 < c_1$ that $u^* \ge \theta_1$. This allows one to conclude the proof of Theorem 2.3, getting (33) for $m = 2$. By iteration, one eventually shows that (33) holds for all $m = 1, \dots, M$. □

Appendix

Proof of Lemma 2.2. We consider the perturbations $(\overline{u}^\varepsilon)_{\varepsilon>0}$ of $\overline{u}$ defined by $\overline{u}^\varepsilon(t,x) := \overline{u}(t,x) + \varepsilon$. By hypothesis, for $\varepsilon > 0$, there exists $T_\varepsilon \le 0$ such that $\overline{u}^\varepsilon(t,x) > \underline{u}(t,x)$ for all $t \le T_\varepsilon$, $x \in \mathbb{R}^N$. Assume by contradiction that there is $\varepsilon_0 > 0$ such that

$$\forall \varepsilon \in (0,\varepsilon_0), \quad \exists t \in (T_\varepsilon, 0],\ x \in \mathbb{R}^N, \quad \overline{u}^\varepsilon(t,x) < \underline{u}(t,x),$$

otherwise the lemma is proved by letting $\varepsilon \to 0$. For $\varepsilon \in (0,\varepsilon_0)$, let $t_\varepsilon \in [T_\varepsilon, 0)$ be the infimum of t for which $u^\varepsilon(t,x) < \underline{u}(t,x)$ for some $x \in \mathbb{R}^N$. Thus, $\overline{u}^\varepsilon \ge \underline{u}$ if $t \le t_\varepsilon$ and, by the uniform continuity of $\underline{u}$ and $\overline{u}$, $\inf_{x\in\mathbb{R}^N}(\overline{u}^\varepsilon - \underline{u})(t_\varepsilon, x) = 0$. The hypotheses on $\underline{u}$ and $\overline{u}$ imply the existence of some $\rho_\varepsilon \in \mathbb{R}$ such that

$$\inf_{x\cdot e = \rho_\varepsilon} (\overline{u}^\varepsilon - \underline{u})(t_\varepsilon, x) = 0. \tag{52}$$

We distinguishing three possible situations.

[3] The fact that this front is only defined in the region where it is nonnegative does not affect the comparison argument with the solution u, because the latter is nonnegative.

Case 1) $(\rho_\varepsilon)_{\varepsilon\in(0,\varepsilon_0)}$ *is bounded.*
Let $(x_\varepsilon)_{\varepsilon\in(0,\varepsilon_0)}$ be such that

$$x_\varepsilon \cdot e = \rho_\varepsilon, \qquad \overline{u}^\varepsilon(t_\varepsilon, x_\varepsilon) - \underline{u}(t_\varepsilon, x_\varepsilon) < \varepsilon.$$

The functions $\underline{u}^\varepsilon(\cdot + t_\varepsilon, \cdot + x_\varepsilon)$ and $\overline{u}(\cdot + t_\varepsilon, \cdot + x_\varepsilon)$ converge (up to subsequences) locally uniformly, as $\varepsilon \to 0$, respectively to a subsolution u_* and a supersolution u^* of a limiting equation (21) on $\mathbb{R}_- \times \mathbb{R}^N$ satisfying

$$\overline{u}^*(0,0) = \underline{u}^*(0,0), \qquad \forall t \leq 0,\ x \in \mathbb{R}^N, \quad \overline{u}^*(t,x) \geq \underline{u}^*(t,x),$$

where the last inequality holds because $\overline{u}^\varepsilon \geq \underline{u}$ if $t \leq t_\varepsilon$. The strong comparison principle then yields $\overline{u}^* = \underline{u}^*$ in $\mathbb{R}_- \times \mathbb{R}^N$. But the boundedness of $x_\varepsilon \cdot e = \rho_\varepsilon$ for $\varepsilon \in (0, \varepsilon_0)$ implies on one hand that $\liminf_{x\cdot e \to -\infty} \overline{u}^*(t,x) \geq 1$ uniformly in $t \leq 0$, by (36), and on the other that

$$\forall x \in \mathbb{R}^N, \quad \limsup_{t\to-\infty} \underline{u}^*(t,x) \leq 0,$$

by (38) or (37). This case is thereby ruled out.
Case 2) $\inf_{\varepsilon\in(0,\varepsilon_0)} \rho_\varepsilon = -\infty$.
Let S be from (18), and take $\varepsilon \in (0, \varepsilon_0)$ such that $-\rho_\varepsilon$ is large enough to have

$$\inf_{\substack{t<0 \\ x\cdot e \leq \rho_\varepsilon + 1}} \overline{u}(t,x) > S.$$

It follows from the second condition in (18) that $\overline{u}^\varepsilon$ is a supersolution of (35) for $x \in \Omega := \{x \,:\, x\cdot e < \rho_\varepsilon + 1\}$. By (52), there is a sequence $(y_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ such that

$$y_n \cdot e = 0, \qquad \lim_{n\to\infty} (\overline{u}^\varepsilon - \underline{u})(t_\varepsilon, y_n + \rho_\varepsilon e) = 0.$$

Passing to the limit on a subsequence of spatial translations by $(y_n)_{n\in\mathbb{N}}$ of $\underline{u}$ and $\overline{u}^\varepsilon$, we end up with a subsolution $\underline{u}_\infty$ and a supersolution $\overline{u}^\varepsilon_\infty$ to some limiting equation (21) in $\mathbb{R}_- \times \Omega$ satisfying

$$\overline{u}^\varepsilon_\infty(t_\varepsilon, \rho_\varepsilon e) = \underline{u}_\infty(t_\varepsilon, \rho_\varepsilon e), \qquad \forall t \leq t_\varepsilon,\ x \in \mathbb{R}^N, \quad \overline{u}^\varepsilon_\infty(t,x) \geq \underline{u}_\infty(t,x).$$

It then follows from the strong comparison principle that $\overline{u}^\varepsilon_\infty = \underline{u}_\infty$ for $t \leq t_\varepsilon$, $x \in \Omega$, which is impossible because, by (36), $\overline{u}^\varepsilon_\infty(t,x) > 1$ if $-x\cdot e$ is large enough, while $\underline{u}_\infty \leq 1$.
Case 3) $\sup_{\varepsilon\in(0,\varepsilon_0)} \rho_\varepsilon = +\infty$.
This case is ruled out when $\underline{u}$ satisfies (38) because, in such case, (52) yields $\rho_\varepsilon < \gamma t_\varepsilon + L < L$. Then, suppose that f satisfies (19) and that $\underline{u}$ satisfies (37). By the latter, there is $\varepsilon \in (0, \varepsilon_0)$ for which ρ_ε is sufficiently large to have

$$\underline{u}(t,x) \leq \theta \quad \text{for} \ \ t \leq 0,\ x \in \Omega := \{x \,:\, x \cdot e > \rho_\varepsilon - 1\},$$

where θ is from assumption (19). It follows from that assumption that the function $\underline{u}^\varepsilon := \underline{u} - \varepsilon$ is a subsolution of (35) for $x \in \Omega$. Moreover, $\overline{u} \geq \underline{u}^\varepsilon$ if $t \leq t_\varepsilon$ and, by (52), $\inf_{x\cdot e=\rho_\varepsilon}(\overline{u} - \underline{u}_\varepsilon)(t_\varepsilon, x) = 0$. Arguing as in the case 2, one finds that the limits $\overline{u}_\infty$, $\underline{u}^\varepsilon_\infty$ of some sequences of translations of $\overline{u}$, $\underline{u}^\varepsilon$ by vectors orthogonal to e coincide for $t \leq t_\varepsilon$, which is impossible because $\underline{u}^\varepsilon < 0$ if $x \cdot e$ is large enough. □

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