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The role of the geometry on the dynamics of reaction-diffusion equations

Habilitation à diriger des recherche

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Résumé

Ce Mémoire porte principalement sur les équations de réaction-diffusion. Il passe en revue l'état de l'art de la recherche dans ce domaine, puis la contribution que j'ai apportée en collaboration avec différentes personnes depuis 2007, c'est-à-dire, après mon doctorat.

La première partie présente les outils théoriques que nous avons utilisés dans l'étude des propriétés qualitatives des solutions d'équations elliptiques et paraboliques. Le plus important est la valeur propre principale généralisée associée à un opérateur elliptique défini sur un domaine non borné. Nous considérons d'abord les cas de l'espace entier et de la condition au bord de Dirichlet. Ensuite, nous discutons des extensions récentes à des conditions plus générales, y compris Neumann et Robin.

La valeur propre principale généralisée nous permet d'étendre la théorie de la stabilité pour les équations paraboliques non linéaires aux problèmes posés dans des domaines non bornés. Un accent particulier est mis ici sur le cadre des équations de réaction-diffusion motivées par des modèles en dynamique des populations. Nous discutons les questions de l' "invasion" et de la "vitesse de propagation" pour différents types de termes de réaction : Fisher-KPP, Monostable, Combustion et Bistable. Notre objectif est de comprendre comment ces propriétés sont affectées par la géométrie du milieu et par ses hétérogénéités. L'analyse des solutions de type front est cruciale à cette fin. Cette notion va des fronts progressifs planaires aux fronts de transition généralisés, en passant par les fronts pulsatoires. Enfin, la notion de terrasse de propagation est considérée dans le cadre des équations Multistables.

Une partie plus appliquée du manuscrit traite d'un modèle en dynamique des populations que nous avons récemment conçu pour tenir compte de l'effet des réseaux de transport sur les invasions biologiques. En partant du cas modèle d'une unique route rectiligne, nous discutons : l'accélération de la vitesse de propagation, la forme de l'ensemble d'invasion, l'effet d'une diffusion non-locale, l'impact d'un changement climatique sur les niches écologiques.

La dernière partie du Mémoire est consacrée à une famille de modèles en sciences sociales. Elle concerne la question de l'éclatement et de la propagation géographique de troubles sociaux, telles que les émeutes ou les révolutions. En termes mathématiques, notre travail se ramène à l'étude d'un système dont les composantes représentent respectivement l'activité de révolte et la tension sociale. Un tel système pourrait être envisagé pour modéliser d'autres phénomènes dans lesquels une variable montre de l'auto-excitation dès que l'autre a atteint un seuil critique. Nous considérons à la fois le cas d'augmentation de la tension ainsi que celui d'inhibition de la tension. Dans une première étude, nous traitons un modèle de "site unique", qui se réduit à un système d'EDO. L'approche que nous utilisons pour le traiter est celle des systèmes dynamiques, qui diffère donc de toutes les autres présentées dans ce manuscrit. Dans un deuxième cadre, nous ajoutons la variable spatiale, ce qui nous ramène à un système plus habituel (pour nous) d'équations paraboliques. Nous retrouvons dans ce cas le célèbre modèle compartimental épidémiologique *SI* (Susceptibles, Infectés).

Mots-clés: *équations de réaction-diffusion, conjecture de Landis, vitesse de propagation, solutions de type front, terrasse de propagation, dynamique des populations, modèle champ-route, dynamique des troubles sociaux.*

Abstract

This Memoir is mainly concerned with reaction-diffusion equations. It reviews the state of the art of the research in this field and then the contribution I have made in collaboration with different people since 2007, that is, after my PhD.

The first part presents the theoretical tools we have employed in the study of qualitative properties of solutions of elliptic and parabolic equations. The most important one is the generalised principal eigenvalue associated with an elliptic operator defined on an unbounded domain. We first consider the cases of the whole space and of the Dirichlet boundary condition. Next, we discuss some recent extensions to more general boundary conditions, including the Neumann and the Robin ones.

The generalised principal eigenvalue allows us to extend the stability theory for nonlinear parabolic equations to problems set in unbounded domains. A particular emphasis is placed here on the framework of reaction-diffusion equations motivated by models in population dynamics. We discuss the questions of "invasion" and of "speed of propagation" for different types of reaction terms : Fisher-KPP, Monostable, Combustion and Bistable. Our goal is to understand how such properties are affected by the geometry of the medium and by its heterogeneities. The analysis of front-like solutions is crucial to this end. This notion ranges from planar travelling fronts to generalised transition fronts, passing through pulsating travelling fronts. Finally, the notion of propagating terrace is considered in the context of Multistable equations.

A more applied part of the manuscript deals with a model in population dynamics that we have recently conceived to account for the effect of transportation networks on biological invasions. Starting from the toy case of a single straight road, we discuss : enhancement of the speed of spreading, the shape of the invasion set, the effect of nonlocal diffusion, the impact of a climate change on ecological niches.

The last part of the Memoir is devoted to a family of models in social sciences. It concerns the question of the outburst and geographical spreading of social unrest, such as riots or revolutions. In mathematical terms, our work amounts to the study of a system whose components represent the rioting activity and the social tension respectively. Such system could be envisioned for modelling other phenomena in which a variable shows self-excitement as soon as the other one has reached a critical threshold. We consider both the tension enhancing and the tension inhibiting cases. In a first study, we deal with a "single site" model, which reduces to an ODE system. The approach we use to treat it is that of dynamical systems and therefore it differs from all the others presented in this manuscript. In a second framework, we add the spatial variable and we are led to a more usual (for us) system of parabolic equations. We recover in such a case the celebrated compartmental model *SI* (Susceptible, Infected) in epidemiology.

Keywords: *reaction-diffusion equations, principal eigenvalue, Landis conjecture, spreading speed, front-like solutions, propagating terrace, population dynamics, road-field model, dynamics of social unrest.*

Table des matières

Introduction

This memoir encompasses my principal research themes since the end of my PhD. Actually, some of my very first objects of study are still part of my current research activity and thus are present in this manuscript. They are : the generalised principal eigenvalue, the long-time behaviour of reaction-diffusion equations. Besides these topics, I have been enlarging my perspectives focusing on some qualitative properties of elliptic and parabolic equations in unbounded domains, as well as on the study of front-like solutions. What drives my research is the fascination towards simple questions that have complex and possibly unexpected answers. This is why I like to work on equations which are as simple as possible and to investigate them in their deepest aspects. It is a great pleasure for me to discover open questions concerning problems that I had previously considered settled, even if often these are hard questions that I am not able to answer. Another direction of research I have been following in the recent years is that of mathematical modelling. This leads me to consider more complicated equations and systems in order to try to capture the complexity of reality. In this context, the questions are motivated by the understanding of the phenomenon described by the model. This type of research allows me to "put into practice" my mathematical knowledge. It also represents a source of inspiration for new theoretical studies.

The memoir is organised in a logical rather than chronological order : from the most abstract topic to the most applied. This structure reflects my point of view of the theoretical results seen as tools to treat problems arising in mathematical models. Indeed, in several occasions, the results presented in this manuscript are used in the following sections.

The first chapter deals with the definition and properties of different notions of generalised principal eigenvalue. The interest for this notion relies on the fact that its sign characterises several important properties concerning elliptic and parabolic equations : maximum principle, stability, bifurcation, existence of front-like solutions, speed of propagation, etc. The problem is that the existence of the principal eigenvalue classically requires some compact properties that are not fulfilled in many interesting cases, such as reaction-diffusion equations in heterogeneous media. This is why one is interested in finding one, or several, quantities whose sign is responsible for any of these properties. We further discuss some extensions of this theory to a class of fully nonlinear operators, with application to control problems. Finally, we describe a link with the unique continuation property at infinity.

Chapter 2 is concerned with the stability analysis for nonlinear parabolic equations, with a particular emphasis on reaction-diffusion equations. These equations have been introduced by R. A. Fisher [F37] and A. N. Kolmogorov, I. G. Petrovskiĭ and N. S. Piskunov [KPP37] in the modelling of population (genetic) dynamics. The questions of extinction or proliferation of a gene or a population are mathematically rephrased there in terms of the stability of solutions. The next question which naturally arises in these models concerns the speed of invasion. According to [KPP37], this speed is asymptotically linear. This result has been corroborated by J. G. Skellam [S51] observing the invasion of muskrat in Eastern Europe. The extension of the classical model to heterogeneous environments leads to some difficult mathematical problems, in some cases still open. This is where the notion of the generalised principal eigenvalue comes into play. The question at stake is to understand how the geometry of the environment affects the propagation and what is the shape of the region invaded by the population at large time.

The way invasion takes place in reaction-diffusion equations is strictly connected with the properties of front-like solutions. The study of such solutions is the object of Chapter 3. What is the appropriate notion of front-like solution depends on the type of heterogeneity of the medium. We start with considering the case where the medium varies in time, in a general fashion. Next, we add the spatial heterogeneity, first of periodic and then of almost periodic type. The last section of the chapter is concerned with reaction-diffusion equations which are no longer of the Fisher-KPP type. Namely, we consider Bistable and Multistable equations. In the latter case, the notion of front-like solution does not suffice to describe the long-time dynamics of the equation. It needs to be replaced by that of propagating terrace.

Chapter 4 is dedicated to a model of population dynamics introduced in a series of papers in collaboration with H. Berestycki and J.-M. Roquejoffre. The model aims at describing biological invasions which are manifestly accelerated by transportation networks. Indeed, it has long been known that fast diffusion on roads can have a driving effect on the spread of epidemics. A classical example is that of the "Black death" plague in the middle of the 14th century. This pandemics spread in Europe at a fast pace along the main commercial roads and then diffused more slowly in the inland, bringing about a dramatic invasion. Our work fits into the general framework of the study of the effect of complex and fragmented habitat on population dynamics. Our aim is to understand whether, and at which extent, the transportation network enhances the classical speed of spreading provided by [KPP37], starting from the toy case of a network composed by a single straight line. The diffusion on the line is modelled by either the Laplace operator with a large coefficient, or by the fractional Laplacian. The cases of a curved road and of two parallel roads has been investigated in some subsequent works. A further study we present concerns the impact of a global environmental change on the dynamics of ecosystems and populations, which is a major challenge in contemporary science.

In Chapter 5 we present some recent works concerning the question, in social sciences, of the outburst and geographical propagation of social unrest. Our model builds on the assumption that the rioting activity occurs when the system is in a sufficiently high level of social tension, which in turns is fuelled by self-reinforcement mechanisms and possibly by exogenous events. This results mathematically in a coupled system of activity and social tension. We consider two scenarios : tension enhancing (cooperative system) and tension inhibiting (activator-inhibitor system). In a first work, we neglect the spatial component and we are reduced to an ODE system, that we treat with dynamical systems tools. Next, we focus on the spatial propagation of the unrest, investigating the speed of spreading as well as front-like solutions. In this last framework, we recover the celebrated *SI* epidemiology models.

Chapitre 1

Generalised principal eigenvalue

The principal eigenvalue is a basic notion associated with an elliptic operator. It essentially encodes the stability properties of solutions of nonlinear equations. This is a very well known fact for problems with an underlying compact structure (bounded domains or periodic domains and solutions). However, stability can be a delicate issue for general problems in unbounded domains.

Another example of use of the principal eigenvalue is the characterization of the existence of the Green function for linear periodic operators (see [A84] and [P95]). Moreover, the principal eigenvalue of an elliptic operator has been shown to play a crucial role in some questions concerning branching processes and it has also been employed in the context of economic models, see e.g. [HS09]. Finally, it is strictly connected with the notion of principal Floquet bundle, as defined by J. Húska, P. Poláčik and M. V. Safonov in [HPS07].

In many applications, typically in reaction-diffusion equations, one is led to consider problems set in an unbounded domain and without a periodic structure. In such a case, the very definition of the principal eigenvalue is arguable. Indeed, the lack of compactness does not allow one to use neither the Krein-Rutman theory nor the Rayleigh quotient, which are the two standard tools employed to define the principal eigenvalue.

1.1 Dirichlet boundary condition ([21])

I have been working on qualitative properties of solutions of Dirichlet problems in unbounded domains since my PhD, mainly in collaboration with H. Berestycki. The goal is to establish a link between a notion of *generalised principal eigenvalue* and the following properties for problems set in an unbounded domain :

- stability of solutions of nonlinear equations ;
- validity of the maximum principle ;
- existence of positive eigenfunctions.

In the case of a bounded smooth domain, it is well known that all these properties are characterised by the sign of the principal eigenvalue. Owing to the work of H. Berestycki, L. Nirenberg and S. R. S. Varadhan [BNV94], we know that this holds true for **bounded non-smooth** domains if one considers a suitable notion of generalised principal eigenvalue. As we will see below, the question is not so quickly settled in the unbounded case.

The papers [33, 32] in collaboration with H. Berestycki, the second one also with F. Hamel, are devoted to the study of the case of an **unbounded** domain. There, we considered the Dirichlet problem

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\mathcal{L}$ is a geneal linear elliptic operator in non-divergence form, that is,

$$\mathcal{L}u = a_{ij}(x)\partial_{ij}u + b_i(x)\partial_i u + c(x)u,$$

and Ω is a general, possibly unbounded, domain in $\mathbb{R}^N$; if $\Omega = \mathbb{R}^N$ then the boundary condition in (1.1) is neglected. The matrix field $(a_{ij})_{ij}$ is assumed to be uniformly elliptic and uniformly continuous, the coefficients b_i, c are just in $L^\infty(\Omega)$. In order to deal with such general operators, we do not adopt a functional analytical point of view. Instead, we used several different notions of generalised principal eigenvalue of $-\mathcal{L}$ relying on pointwise differential inequalities (satisfied a.e. by functions belonging to a suitable Sobolev space). The first one is that of [BNV94] :

$$\lambda_{\mathcal{D}} := \sup\{\lambda \ : \ \exists \phi > 0,\ (\mathcal{L} + \lambda)\phi \leq 0 \text{ in } \Omega\}.$$

An analogous definition had been previously given by S. Agmon in [A82] in the case of operators in divergence form defined on Riemannian manifolds and, for general operators, by R. D. Nussbaum and Y. Pinchover [NP92], building on some ideas of M. H. Protter and H. F. Weinberger [PW66]. We point out that no boundary condition is imposed on the functions ϕ in the definition of $\lambda_{\mathcal{D}}$. The reason is that their positivity automatically implies that they are supersolutions with respect to the Dirichlet boundary condition. The key property of $\lambda_{\mathcal{D}}$ is that it coincides with the limit of the Dirichlet principal eigenvalues associated with a sequence of domains invading Ω.

We showed in [32] that, when related to the linearised operator around a solution of a reaction-diffusion equation, the condition $\lambda_{\mathcal{D}} < 0$ indeed implies that the solution is unstable. However, $\lambda_{\mathcal{D}} > 0$ does not imply that it is stable. We refer to Section 2.1 below for a detailed discussion of this topic. Always in [32], it is shown that the right notion for characterising the stability, at least for concave reaction terms, is

$$\lambda'_{\mathcal{D}} := \inf\{\lambda \ : \ \exists \phi > 0,\ \sup\phi < +\infty,\ (\mathcal{L} + \lambda)\phi \leq 0 \text{ in } \Omega,$$
$$\forall \xi \in \partial\Omega,\ \lim_{x\to\xi} \phi(x) = 0\}.$$

Both $\lambda_{\mathcal{D}}$ and $\lambda'_{\mathcal{D}}$ reduce to the classical Dirichlet principal eigenvalue when Ω is bounded and smooth. In addition, the latter coincides with the periodic principal eigenvalue in the periodic setting.

Let us turn now to the *maximum principle*. This refers to the property that subsolutions of (1.1) which are bounded from above are necessarily nonpositive. It turns out that $\lambda_{\mathcal{D}}$ and $\lambda'_{\mathcal{D}}$ do not suffice to characterise the validity of the maximum principle in unbounded domains. To this purpose we introduce in [21] still another notion of generalised principal eigenvalue :

$$\lambda''_{\mathcal{D}} := \sup\{\lambda \ : \ \exists\phi, \ \inf_{\Omega} \phi > 0, \ (\mathcal{L}+\lambda)\phi \leq 0 \ \text{ in } \Omega\}.$$

Theorem 1.1 ([21])**.** *The maximum principle is satisfied if* $\lambda''_{\mathcal{D}} > 0$ *and only if* $\lambda'_{\mathcal{D}} \geq 0$.

There are examples showing that, in the limiting case where $\lambda'_{\mathcal{D}}$ and $\lambda''_{\mathcal{D}}$ are both equal to 0, the maximum principle might or might not hold.

For the existence of positive eigenfunctions, the right quantity to look at is $\lambda_{\mathcal{D}}$. However, the picture is drastically different from the classical case, in which the principal eigenvalue is the unique eigenvalue admitting a positive eigenfunction.

Theorem 1.2 ([21])**.** *The set of eigenvalues associated with a positive eigenfunction satisfying the Dirichlet boundary condition (if* $\Omega \neq \mathbb{R}^N$*) is* $(-\infty, \lambda_{\mathcal{D}}]$.

The above result is derived in the case of an unbounded **smooth** domain, though the techniques of [BNV94] should allow an extension to the non-smooth case. One could imagine that the discrepancy between the cases of bounded and unbounded domains comes from the fact that no Dirichlet condition is imposed at infinity. We show in [21] that this is not the case, even if one imposes an exponential decay. A positive eigenfunction associated with the eigenvalue $\lambda_{\mathcal{D}}$ is called a *generalised principal eigenfunction*. It is not unique in general. Some sufficient conditions for the simplicity of $\lambda_{\mathcal{D}}$ are derived in [21] using the notion of solution of *minimal growth at infinity*. This is in the spirit – yet a slightly different version – of the notion introduced by S. Agmon in his pioneering and important paper [A82]. The conclusion of Theorem 1.2 was already known in the case that the Dirichlet boundary condition is omitted (see, e.g., [A82]) as well as in the case of **exterior domains**, thanks to the work of Y. Furusho and Y. Ogura [FO81]. Let us further point out that, if $\mathcal{L}$ has Hölder continuous coefficients, the problem of the existence of Dirichlet eigenfunctions could also be approached by using the Green function and the Martin boundary theory (see [P94]).

In view of the relevance of the three different notions of generalised principal eigenvalue, it is useful to determine conditions which yield equality between them, or at least that yield an ordering. If Ω is bounded and smooth, the three notions coincide with the classical principal eigenvalue. In the unbounded case, we have the following.

Theorem 1.3 ([21])**.** *There holds that :*

(i) $\lambda''_{\mathcal{D}} \leq \lambda'_{\mathcal{D}} \leq \lambda_{\mathcal{D}}$ *;*

(ii) if $\mathcal{L}$ *is self-adjoint then* $\lambda_{\mathcal{D}} = \lambda'_{\mathcal{D}}$.

This result improves some relations between $\lambda_{\mathcal{D}}$ and $\lambda'_{\mathcal{D}}$ already established in [32, 33] in low dimension. Actually, the question of whether or not $\lambda'_{\mathcal{D}}$ and $\lambda''_{\mathcal{D}}$ do always coincide is still open. We are able to answer it in some particular cases.

Proposition 1.4 ([21]). *The equalitites $\lambda_{\mathcal{D}} = \lambda'_{\mathcal{D}} = \lambda''_{\mathcal{D}}$ hold in the following cases :*

1) $\mathcal{L}$ is self-adjoint and either $N = 1$ or $\Omega = \mathbb{R}^N$ and $\mathcal{L}$ is radially symmetric ;

2) $\mathcal{L} = \tilde{\mathcal{L}} + \gamma(x)$, the equalities hold for $\tilde{\mathcal{L}}$ and γ is nonnegative and satisfies $\lim\limits_{\substack{x\in\Omega \\ |x|\to\infty}} \gamma(x) = 0$;

3)

$$\lambda_{\mathcal{D}} \leq -\limsup_{\substack{x\in\Omega \\ |x|\to\infty}} c(x);$$

4) $\mathcal{L}$ is either self-adjoint or in non-divergence form with $\lim\limits_{\substack{x\in\Omega \\ |x|\to\infty}} b(x) = 0$, and

$$\forall r > 0,\ \forall \beta < \limsup_{\substack{x\in\Omega \\ |x|\to\infty}} c(x), \quad \exists B_r(x_0) \subset \Omega \ \ s.t. \ \ \inf_{B_r(x_0)} c > \beta.$$

One of the tools used in the proof of these properties is an extension of the boundary Harnack inequality to inhomogeneous Dirichlet problems, that we derive using in a crucial way the results of [BNV94] for bounded non-smooth domains.

Let us mention that it has been recently proved by H. Berestycki and G. Nadin [BN18] that $\lambda'_{\mathcal{D}} = \lambda''_{\mathcal{D}}$ if $\Omega = \mathbb{R}^N$ and the coefficients of $\mathcal{L}$ are *uniquely ergodic*.

As an application of the above results, we are able to extend the basic properties of the classical Dirichlet principal eigenvalue to the case of unbounded domains, provided that c is negative at infinity. In doing so, we recover some of the results by J. Húska and P. Poláčik [HP08] concerning the principal Floquet bundle. In a work in progress in collaboration with H. Berestycki and G. Nadin, we are extending the above definitions of the generalised principal eigenvalue to parabolic operators. One of our objectives is to shed light on the connection between these definitions and the notion of Lyapunov exponent coming from the Floquet theory.

A natural extension of our works is to consider linear elliptic equations on non-compact manifolds. There are only few steps in our arguments that make actually use of the euclidean structure ; for these, some geometric conditions, concerning for instance the volume growth of balls, would be required.

1.2 General boundary condition ([3])

In this section we illustrate how the notions introduced in the previous section need to be adapted if one replaces the Dirichlet boundary condition with more general ones, including the Neumann and Robin conditions. The known results in such case are fewer. In application, the Neumann condition reflects a no-flux condition which is sometimes more relevant than the Dirichlet condition. For instance, in population dynamics, it means that there are regions of the space that are inaccessible to the species (such as a body of water, or a mountain) whereas the Dirichlet condition represents a zone where individuals instantaneously die. So the reason why the theory of the generalised principal eigenvalue with Neumann boundary condition is less developed is not because it is less meaningful, but, rather, that it presents some technical difficulties. The most important one is the lack of monotonicity with respect to the inclusions of domains. Namely, unlike in the Dirichlet case, there is no general relation between the Neumann principal eigenvalue on a domain Ω and that on a subdomain $\Omega' \subset \Omega$.

Let us consider the divergence form operator

$$\mathcal{L}u := \nabla \cdot (A(x)\nabla u) + b(x) \cdot \nabla u + c(x)u,$$

with A uniformly elliptic and b, c bounded, and the associated boundary value problem

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega \\ \mathcal{B}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Here Ω is a smooth – possibly unbounded – domain in $\mathbb{R}^N$. The boundary operator is of the form

$$\mathcal{B}u := \beta(x) \cdot \nabla u + \gamma(x)u,$$

with $\beta : \partial\Omega \to \mathbb{R}^N$ satisfying

$$\beta \cdot \nu > 0 \quad \text{on } \partial\Omega,$$

where ν is the outer unit normal field to Ω. If $\beta \equiv A\nu$ and $\gamma \equiv 0$, we recover the Neumann boundary operator $\nu A \nabla$, while if $\beta \equiv A\nu$ and $\gamma > 0$, we get the Robin operator.

As we have already explained, the Dirichlet boundary condition for the supersolutions ϕ is implicitly imposed in the definition of $\lambda_{\mathcal{D}}$ given in the previous section, because $\phi > 0$ in Ω. Now,instead, the boundary condition needs to be added to the definition of the generalised principal eigenvalue. Namely, we set

$$\lambda_{\mathcal{B}} := \sup\{\lambda \ : \ \exists \phi > 0,\ (\mathcal{L} + \lambda)\phi \le 0 \text{ in } \Omega,\ \mathcal{B}\phi \ge 0 \text{ on } \partial\Omega\}.$$

A similar definition has been given by S. Patrizi [P08] in the case of the Neumann boundary condition for **fully nonlinear** operators in **bounded** domains.

If the domain Ω is **unbounded**, one would like to approximate $\lambda_{\mathcal{B}}$ by a sequence of classical eigenvalues in bounded domains, but this requires a monotonicity property which does not hold under the general boundary condition $\mathcal{B} = 0$. The key observation is that the monotonicity holds for a **mixed** boundary value problem. Namely, one "truncates" Ω and imposes the Dirichlet condition on the new portion of the boundary coming from the truncation. This is what we did with H. Berestycki in [29] in the case where Ω is a cylinder. The general case is more involved because one should avoid the truncated domain to bee "too irregular". This difficulty is bypassed in the Dirichlet case thanks to the results of [BNV94], which apply to arbitrary domains. The Neumann case is dealt with in the paper [9] in collaboration with R. Ducasse, by using the fact that $B_r \cap \Omega$ is a Lipschitz open set for a.e. $r > 0$, as a consequence of the Morse-Sard theorem [M39]. This allows one to invoke the solvability and Hölder-regularity theory of G. M. Lieberman [L86] (or [S60] in the self-adjoint case). Summing up, calling $\Omega_r(y)$ the connected component of $B_r(y) \cap \Omega$ containing a given point

$y \in \Omega$, we consider the mixed eigenvalue problem

$$\begin{cases} -\mathcal{L}\varphi = \lambda\varphi_r & \text{in } \Omega_r(y) \\ \mathcal{B}\varphi = 0 & \text{on } (\partial\Omega_r(y)) \cap B_r(y) \\ \varphi = 0 & \text{on } \partial\Omega_r(y) \cap \partial B_r(y). \end{cases}$$

Although the classical Krein-Rutman theory does not apply to this problem, we show in [3] that, for a.e. $r > 0$, there exists a unique principal eigenvalue $\lambda(y, r)$ associated with a positive eigenfunction φ. Then, using the fact that $r \mapsto \lambda(y, r)$ is decreasing, we prove that $\lambda(y, r) \to \lambda_{\mathcal{B}}$ as $r \to +\infty$ and, as a by-product, the existence of a *generalised principal eigenfunction* associated with $\lambda_{\mathcal{B}}$. Actually, with the same arguments as in [21], one derives the existence of a positive eigenfunction satisfying the general boundary condition for any eigenvalue $\lambda \leq \lambda_{\mathcal{B}}$.

In view of application to the stability analysis, that we shall discuss in detail in Section 2.1, it is of great interest to find conditions that guarantee a sign of the generalised principal eigenvalue. A straightforward condition for the positivity follows by observing that $\lambda_{\mathcal{B}} \geq -\sup c$. Conditions to have $\lambda_{\mathcal{B}} < 0$ are much more involved. We deal with this question in [3] in the case of the self-adjoint operator

$$\mathcal{L}u = \nabla \cdot (A(x)\nabla u) + c(x)u, \tag{1.3}$$

under the Neumann boundary condition

$$\mathcal{B}u = \nu A \nabla u.$$

This is sometimes referred to in the literature as a Schrödinger operator. We write $\lambda_{\mathcal{N}}$ in place of $\lambda_{\mathcal{B}}$, where "$\mathcal{N}$" stands for Neumann. A natural question is : does $c > 0$ in Ω imply $\lambda_{\mathcal{N}} < 0$? The answer is clearly yes if Ω is a bounded domain, but it turns out to be no in general if Ω is unbounded. One can for instance exhibit a counter-example in $\mathbb{R}^N$, $N \geq 4$, with c (which is called "potential" in the terminology of Schrödinger operators) positive and decaying as $|x|^{-1}$ at infinity. More reasonably, one can ask the following.

Question 1. For the operator in (1.3) defined on a smooth domain Ω, does the condition $\inf_\Omega c > 0$ imply $\lambda_{\mathcal{N}} < 0$?

We show in [3] that the answer to Question 1 is affirmative, provided Ω satisfies the *uniform interior ball condition.* The question remains open if such condition is dropped. We actually derive a stronger result, which is expressed in terms of the *least mean* of c, a notion already used in [28] in collaboration with G. Nadin, see Section 3.2 below. It is defined as follows :

$$\lfloor c \rfloor := \liminf_{r \to +\infty} \left(\inf_{y \in \Omega} \frac{\int_{\Omega_r(y)} c}{|\Omega_r(y)|} \right).$$

Theorem 1.5 ([3])**.** *Consider the operator $\mathcal{L}$ defined by* (1.3) *on a smooth domain Ω satisfying the uniform interior ball condition. Then the following inequalities hold :*

$$\lambda_{\mathcal{N}} \leq \lim_{r \to +\infty} \left(\sup_{y \in \Omega} \lambda(y, r) \right) \leq -\lfloor c \rfloor .$$

As we will see in Section 2.1, this result is a powerful tool to derive the stability properties of a solution $\bar{u}$ of a nonlinear equation. Indeed, if $\mathcal{L}$ is the operator issued from the linearisation around $\bar{u}$, and its zero-order coefficient satisfies $\inf_\Omega c > 0$, then Theorem 1.5 implies that $\lambda_\mathcal{N} < 0$, i.e., that $\bar{u}$ is *linearly unstable*. Actually, one infers that

$$\lim_{r\to+\infty}\left(\sup_{y\in\Omega}\lambda(y,r)\right) < 0, \tag{1.4}$$

which in turn implies a much stronger property : that $\bar{u}$ is *uniformly repulsive*, see Theorem 2.2 below. A remarkable consequence of this result is the validity of the *hair-trigger* effect for reaction-diffusion equations of the Fisher-KPP type in general uniformly smooth domains, c.f. Corollary 2.3. Loosely speaking, (1.4) means that any operator in the ω-limit set of $\mathcal{L}$ has a negative generalised principal eigenvalue $\lambda_\mathcal{N}$. Let us finally remark that if Ω is bounded then Theorem 1.5 reduces to the well-known inequality $\lambda_\mathcal{N} \leq -\langle c\rangle$, where $\langle c\rangle$ denotes the average of c. The cornerstone of the proof of Theorem 1.5 is the following geometric result about the growth of balls inside Ω, which is of independent interest.

Lemma 1.6. *Let $\Omega\subset\mathbb{R}^N$ be a measurable set. Then, for any $y\in\Omega$, there holds that*

$$\forall k\in\mathbb{N},\ k\geq 2,\qquad \inf_{1\leq r\leq k}\frac{|\Omega_{r+1}(y)|}{|\Omega_r(y)|} < \Big(Ck(k+1)^N+1\Big)^{1/k},$$

with

$$C=\frac{|B_1(y)|}{|\Omega_1(y)|}.$$

1.3 Fully nonlinear operators and optimisation results ([15, 19])

In the paper [19] in collaboration with H. Berestycki, I. Capuzzo Dolcetta and A. Porretta, we adapt some of the ideas inspired by [BNV94], and developed in [33, 32, 21], to deal with **degenerate elliptic** operators, this time in bounded smooth domains. There, the degeneracy of ellipticity plays the same role as the unboundedness of the domain, in terms of loss of compactness. It turns out that the right framework for treating degenerate elliptic operators – which may include as a limiting case operators of the first order – is that of viscosity solutions. The main difficulty is represented by the lack of ellipticity at the boundary. To overcome it, we approximate the domain from outside by the domains

$$\Omega^\varepsilon := \{x\in\mathbb{R}^N\ :\ \operatorname{dist}(x,\Omega)<\varepsilon\},$$

and we consider the generalised principal eigenvalues $(\lambda^\varepsilon_\mathcal{D})_{\varepsilon>0}$ associated with $(\Omega^\varepsilon)_{\varepsilon>0}$, under Dirichlet boundary conditions, as defined in Section 1.1 (with the differential inequalities understood in the viscosity sense). We then show that the validity of the maximum principle is characterised by the positivity of the limit of $\lambda^\varepsilon_\mathcal{D}$ as $\varepsilon\searrow 0$.

The result is not restricted to linear operators, but it is valid for a class of **fully nonlinear** operators that satisfy an homogeneity condition, such as the Pucci, Bellman, Isaacs operators, as well as the p-Laplacian and the infinity Laplacian. It is remarkable, in my opinion, to have a complete characterisation of the maximum principle in terms of a single quantity, independently of how badly the operator degenerates. The price to pay for such a generality is that the definition of the principal eigenvalue itself is rather implicit and requires the operator to be defined on a larger set. However, if the operator is linear, we show that this coincides with the quantity $\lambda''_{\mathcal{D}}$ defined in Section 1.1, provided the operator satisfies some Fichera-type conditions.

Theorem 1.7 ([19]). *Suppose that in each connected component of $\partial\Omega$ the conditions*

$$D\delta(x)A(x)D\delta(x) = 0, \qquad \mathrm{Tr}(A(x)D^2\delta(x)) + b(x)\cdot D\delta(x) \geq 0$$

are either always satisfied or always violated, where δ denotes the signed distance to $\partial\Omega$. Then there holds that

$$\lim_{\varepsilon\to 0^+} \lambda^\varepsilon_{\mathcal{D}} = \sup\{\lambda \ : \ \exists\phi,\ \inf_\Omega \phi > 0,\ (\mathcal{L}+\lambda)\phi \leq 0 \ \textit{in the viscosity sense in } \Omega\}.$$

Some of the ideas of [19] are applied in the paper [15], in collaboration with M. Bardi and A. Cesaroni, to an **ergodic control problem** under state constraints associated with the stochastic process

$$dX_t^{\alpha.} = b(X_t^{\alpha.}, \alpha_t)dt + \sqrt{2}\sigma(X_t^{\alpha.}, \alpha_t)dW_t, \qquad X_0^{\alpha.} = x \in \overline{\Omega}.$$

The constraint is a condition on the matrix field σ and the vector field b which guarantees the invariance of the domain Ω. We consider the optimisation problem for the infinite-horizon discounted value function, that is, we seek for

$$u_\lambda(x) := \inf_{\alpha.\in\mathcal{A}} \mathbb{E}\left[\int_0^\infty e^{-\lambda t}\ell(X_t^{\alpha.}, \alpha_t)dt\right], \quad x \in \overline{\Omega}.$$

The standard arguments of M. Arisawa and P.-L. Lions [AL98] imply that $\lambda u_\lambda \to c$ and $u_\lambda(x) - u_\lambda(0) \to \chi$ as $\lambda \searrow 0$, where χ is a solution of the Hamilton-Jacobi-Bellman (HJB) equation

$$\sup_\alpha \Big(- b(x,\alpha)\cdot D\chi(x) - \mathrm{Tr}\left((\sigma\sigma^T)(x,\alpha)D^2\chi(x)\right) - \ell(x,\alpha)\Big) = c, \quad x \in \Omega.$$

The above operator fits into the class studied in [19], and the domain invariance Ω leads to a Fichera-type boundary condition. We then derive in [15] the existence and uniqueness of c for which the HJB equation admits a (unique up to an additive constant) solution χ satisfying

$$\lim_{x\to\partial\Omega} \frac{\chi(x)}{-\log\delta(x)} = 0.$$

1.4 The Landis conjecture ([7])

In [KL88], V. A. Kondrat′ev and E. M. Landis asked the following question : if u is a solution of the equation

$$\Delta u + V(x)u = 0 \tag{1.5}$$

in the exterior of a ball in $\mathbb{R}^N$, is it true that the condition

$$\exists \kappa > \sqrt{\sup |V|}, \qquad u(x) \prec e^{-\kappa|x|}, \tag{1.6}$$

necessarily implies $u \equiv 0$? Here, the notation $u \prec v$ means $u(x)/v(x) \to 0$ as $|x| \to \infty$. They also addressed the same question under the stronger requirement that $u(x) \prec e^{-\kappa|x|}$ for all $\kappa > 0$. A positive answer to these questions would imply that two solutions whose difference decays fast enough at infinity must necessarily coincide, that is, a property of *unique continuation at infinity* (**UCI** in the sequel).

The question, known today as Landis' conjecture, is motivated by the trivial observation that in dimension $N = 1$ with V constant, decaying solutions can only exist if $V < 0$, and they decay as $\exp(-\sqrt{|V|}|x|)$. Hence, in such case, one can even take $\kappa = \sqrt{\sup |V|}$ in condition (1.6). This is no longer true as soon as the dimension is greater than 1 : the bounded, radial solution of $\Delta u - u = 0$ outside a ball, which can be expressed in terms of the modified Bessel function of second kind, decays like $|x|^{-\frac{N-1}{2}} e^{-|x|}$. As we will see in the sequel, this discrepancy between one and multidimensional cases holds true for general elliptic equations with variable coefficients.

The Landis conjecture was disproved by V. Z. Meshkov [M91], who exhibited two complex-valued, bounded functions $u, V \not\equiv 0$ satisfying the equation (1.5), with $|u(x)| \leq \exp(-h|x|^{\frac{4}{3}})$ for some $h > 0$. Moreover, the power $4/3$ is shown to be optimal : the **UCI** holds under the requirement $u(x) \prec \exp(-|x|^{\frac{4}{3}+\varepsilon})$ for some $\varepsilon > 0$. These results provide a complete picture in the **complex** case.

The conjecture has been brought back to attention in the 2000s by the works of J. Bourgain and C. E. Kenig [BK05] and C. E. Kenig [K05]. In the former, the authors improve Meshkov's **UCI** result in the case of real-valued functions, pushing the decay condition up to $u(x) \leq \exp(-h|x|^{\frac{4}{3}} \log(|x|))$. However, there is not an analogue of Meshkov's counterexample (nontrivial solutions with exponential decay with power larger than 1) in the real case. This fact led Kenig to ask in [K05, Question 1] whether, in the **real** case, the **UCI** holds for solutions satisfying

$$u(x) \prec e^{-|x|^{1+\varepsilon}} \quad \text{for all } \ \varepsilon > 0.$$

Observe that this is stronger than the original hypothesis (1.6) of [KL88]. However, even this weaker conjecture is still open nowadays, except for some particular situations. C. Kenig, L. Silvestre and J.-N. Wang [KSW15] prove it in dimension $N = 2$, under the additional assumption that $V \leq 0$. The condition on the decay is $u(x) \prec e^{-h|x|(\log|x|)^2}$ for some $h > 0$, hence the result does not imply the Landis conjecture. In the case of equations set in the whole space $\mathbb{R}^2$, the authors are able to handle more general uniformly elliptic operators, still assuming $V \leq 0$, see also [DKW17]. We point out that, for equations in the whole space, the condition $V \leq 0$ implies that $u \equiv 0$ just assuming that $u \prec 1$, as an immediate consequence of the maximum principle. The results of [KSW15, DKW17] are deduced from a quantitative estimate which implies that the set where a nontrivial solution is bounded from below by $e^{-h|x|(\log|x|)^2}$ is relatively dense in $\mathbb{R}^2$.

In the paper [25] in collaboration with L. Ryzhik, we have approached the Landis conjecture motivated by the study of front-like solutions for some heterogeneous reaction-diffusion equations, see Section 3.3 below. I pursued this investigation in [7]. There, I deal with uniformly elliptic operators with real coefficients

$$\mathcal{L}u = \mathrm{Tr}(A(x)D^2u) + q(x) \cdot Du + V(x)u,$$

defined on an *exterior domain* $\Omega \subset \mathbb{R}^N$, i.e., a connected open set with compact complement. For general operators of this type, it is known since the work of A. Pliś [P63] that the **UCI** dramatically fails : there exists an operator $\mathcal{L}$ in $\mathbb{R}^3$ with a Hölder-continuous matrix field A and smooth terms q, V which admits a nontrivial solution vanishing identically outside a ball. Because of this astonishing counterexample, the only hope to derive the **UCI** is by requiring some additional hypotheses on the operator. For instance, the results by Kenig and collaborators are restricted to dimension $N = 2$ and require a sign condition on the potential V. Another possible way to avoid the counterexample of [P63] is by imposing some regularity of the diffusion matrix A. It is indeed shown by N. Garofalo and F.-H. Lin [GL87] that the pathological situation of [P63] cannot arise if A is Lipschitz-continuous. However, this latter restriction does not seem to be useful in an approach based on the comparison principle and Hopf's lemma, which is the one we have adopted in [7].

In the very recent paper [ABG19], A. Arapostathis, A. Biswas and D. Ganguly attack the problem using probabilistic tools. They derive the **UCI** under the additional assumption that $u \geq 0$, or, if $\Omega = \mathbb{R}^N$, that the *generalised principal eigenvalue* of the operator $-\mathcal{L}$ is nonnegative. This is the same notion as the one discussed in Section 1.1 before. We point out that the hypothesis on the generalised principal eigenvalue is more general than $u \geq 0$, and also than $V \leq 0$.

The results we obtain in [7] are of two distinct natures. Among the first set of results, the most interesting one is the **UCI** for radial operators.

Theorem 1.8 ([7]). *Assume that $\mathcal{L}$ is a radial operator of the form*

$$\mathcal{L}u = \Delta u + q(|x|)\frac{x}{|x|} \cdot \nabla u + V(|x|)u.$$

Let u be a nontrivial solution of $\mathcal{L}u = 0$ in an exterior domain Ω. Then,

$$\overline{\lim_{|x|\to+\infty}} |u(x)|e^{\kappa|x|} = +\infty,$$

for all κ satisfying

$$\kappa > \overline{\lim_{r\to+\infty}} \frac{|q|}{2} + \sqrt{\overline{\lim_{r\to+\infty}} \frac{|q|^2}{4} + \overline{\lim_{r\to+\infty}} |V|}.$$

We emphasise that the solution u is not assumed to be radial. This theorem implies that the original Landis conjecture holds for radial potentials V. We remark that if the coefficients q, V are constant with $q \geq 0$ and $V \leq 0$ then the threshold for κ because it is precisely the rate of decay of solutions at $+\infty$, hence sharp. The proof of Theorem 1.8 is achieved by first deriving the **UCI** for 1-dimensional operators and then by applying it to the spherical harmonic decomposition of the solution. Actually, in the 1-dimensional case, we are able to get κ equal to the threshold, which would not be possible in higher dimension, as explained before.

The second set of results concerns positive solutions. This makes the problem much simpler, because the **UCI** could be derived – at least in principle – from a one-side comparison argument. One of the consequences of this is that the results we obtain hold for supersolutions. Next, applying these results to the "test functions" ϕ in the definition of the Dirichlet generalised principal eigenvalue

$$\lambda_{\mathcal{D}} := \sup\{\lambda \ : \ \exists \phi > 0, \ (\mathcal{L} + \lambda)\phi \leq 0 \text{ in } \Omega\},$$

we derive the following.

Theorem 1.9 ([7]). *Let u be a nontrivial solution of $\mathcal{L}u = 0$ in an exterior domain Ω, with $A(x) \geq \alpha(x)I$. Assume that $\lambda_D \geq 0$ and that either $\Omega = \mathbb{R}^N$ or that*

$$\varliminf_{x \to \partial\Omega} u(x) \geq 0.$$

Then,

$$\varlimsup_{|x| \to +\infty} |u(x)| e^{\kappa |x|} = +\infty, \qquad \forall\, \kappa > \varlimsup_{|x| \to \infty} \left(\frac{|q|}{2\alpha} + \sqrt{\frac{|q|^2}{4\alpha^2} + \frac{|V|}{\alpha}} \right).$$

This result is actually a consequence of a more general statement in [7] concerning ancient supersolutions of parabolic equations. The hypothesis $\lambda_D \geq 0$ is satisfied, for instance, if $u > 0$ or if $V \leq 0$. The result in the case $\Omega = \mathbb{R}^N$ has been obtained in a parallel way in [ABG19, Corollary 4.1], using the stochastic representation of solutions, but with a larger threshold for κ.

The following table summarises all the cases in which the **UCI** property is derived in [7], with the corresponding values of the rate of decay κ.

TABLE 1.1 – *Validity of the* **UCI** for $u(x) \prec e^{-\kappa|x|}$

$N = 1$	$\kappa = \sup \frac{\lvert q\rvert}{2\alpha} + \sqrt{\sup \frac{\lvert q\rvert^2}{4\alpha^2} + \sup \frac{\lvert V\rvert}{\alpha}}$
u is radial, or $\mathcal{L}$ is radial	$\forall\, \kappa > \varlimsup_{\lvert x\rvert \to \infty} \frac{\lvert q\rvert}{2\alpha} + \sqrt{\varlimsup_{\lvert x\rvert \to \infty} \frac{\lvert q\rvert^2}{4\alpha^2} + \varlimsup_{\lvert x\rvert \to \infty} \frac{\lvert V\rvert}{\alpha}}$
$u \geq 0$, or $V \leq 0$, or $\Omega = \mathbb{R}^N$ and $\lambda_D \geq 0$, or $\varliminf_{x \to \partial\Omega} u(x) \geq 0$ and $\lambda_D \geq 0$	$\forall\, \kappa > \varlimsup_{\lvert x\rvert \to \infty} \left(\frac{\lvert q\rvert}{2\alpha} + \sqrt{\frac{\lvert q\rvert^2}{4\alpha^2} + \frac{\lvert V\rvert}{\alpha}} \right)$

Chapitre 2

Long-time behaviour for reaction-diffusion equations

Reaction-diffusion equations classically arise in the study of biological phenomena (propagation of genes, biological invasions), in physics (phase transition, combustion) and more recently in social sciences (diffusion of innovations, ideas, social behaviours). They have been extensively studied since the seminal works of R. A. Fisher [F37] and A. N. Kolmogorov, I. G. Petrovskiĭ and N. S. Piskunov [KPP37]. These papers deal with the **homogeneous** equation

$$\partial_t u = \Delta u + f(u), \quad t > 0, \ x \in \mathbb{R}^N, \tag{2.1}$$

where f is a concave function vanishing at 0 and 1. The solution u may represent the density of a population on a territory, or of a genetic trait inside a population. This equation also appears in the theory of branching stochastic processes.

The basic questions addressed in [F37, KPP37] are : will the population go extinct or will it persist and possibly invade the territory ? In mathematical terms, this amounts to understanding the stability properties of the steady states 0 and 1. A subsequent question concerns the speed at which invasion would occur. These questions are solved in [KPP37] for Heaviside initial data, and by D. G. Aronson and H. F. Weinberger [AW78] for general data. The answer is that *invasion* occurs, i.e.

$$u(t,x) \to 1 \quad \text{as } t \to +\infty, \ \text{locally uniformly in } x \in \mathbb{R}^N,$$

for any nontrivial nonnegative initial datum. This property is referred to as the *hair-trigger* effect. Moreover, invasion takes place with an *asymptotic speed of spreading* equal to $c^* := 2\sqrt{f'(0)}$, in the sense that

$$\lim_{t\to+\infty} u(t,xt) = \begin{cases} 1 & \text{locally uniformly in } x \in B_{c^*} \\ 0 & \text{locally uniformly in } x \in \mathbb{R}^N \setminus \overline{B}_{c^*}. \end{cases}$$

Let us emphasise that the asymptotic speed of spreading does not depend on the initial datum, nor on the spatial dimension. Moreover, in this case, it is the same in any direction. This theoretical result has been corroborated by several real observations in ecology. For instance, J. G. Skellam showed in [S51] that the area colonised by muskrats, along their spreading through Europe at the beginning of the 20th century, grew quadratically in time.

It turns out that the asymptotic speed of spreading coincides with the minimal speed of *planar travelling fronts*. These are solutions of the form $\phi(x \cdot e - ct)$, for a given direction $e \in \mathbb{S}^{N-1}$, satisfying

$$\phi(-\infty) = 1, \qquad \phi(+\infty) = 0.$$

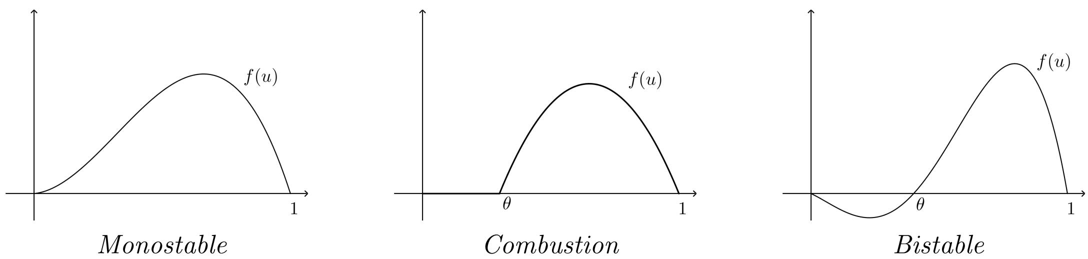
Namely, such solutions exist if and only if $c \geq c^*$.

These results have been extended in [AW78] to more general reaction terms f, with suitable adaptation. Besides the assumption $f(0) = f(1) = 0$, the authors of [AW78] consider three different sets of hypotheses. With the terminology commonly employed in the literature, they are :

Monostable $\quad f > 0 \quad \text{in } (0,1)\,;$

Combustion $\quad \exists \theta \in (0,1), \quad f = 0 \quad \text{in } [0,\theta], \quad f > 0 \quad \text{in } (\theta,1)\,;$

Bistable $\quad \exists \theta \in (0,1), \quad f < 0 \quad \text{in } (0,\theta), \quad f > 0 \quad \text{in } (\theta,1), \quad \int_0^1 f > 0.$



The Monostable case generalises the one considered in [F37, KPP37], the latter being characterised by the so-called *Fisher-KPP* condition :

$$f(u) \leq f'(0)u \quad \text{for all } u \geq 0.$$

There is a deep difference between the Monostable and the Combustion-Bistable cases : in the latter two, there is a unique speed for which a planar travelling front exists. This again coincides with the asymptotic speed of spreading. Another crucial difference between the different types of nonlinearities is that without the condition $f'(0) > 0$ (which holds in the Fisher-KPP case) the hair-trigger effect might fail.

The main limitation of the above results is that in real biological phenomena the propagation can vary from place to place, and also in time, as a result of different environmental conditions. In order to take into account the spatial heterogeneity, equation (2.1) needs to be replaced by a general reaction-diffusion problem of the form

$$\begin{cases} \partial_t u = \nabla \cdot (A(x)\nabla u) + b(x) \cdot \nabla u + f(x,u), & t > 0,\ x \in \Omega \\ \nu A(x)\nabla u = 0, & t > 0,\ x \in \partial\Omega, \end{cases} \tag{2.2}$$

where Ω is an unbounded domain and ν is its outer unit normal. Very few results are available for such a general problem, without supposing some specific structural conditions. Strikingly enough, even for the homogeneous Fisher-KPP equation (2.1), but set on a general domain Ω under Neumann boundary condition, the validity of the hair-trigger effect was an open question. In Section 2.1, we present a result of [3] that answers such question. The invasion property for other types of reaction terms is discussed in Section 2.2, which accounts for the results of [9] obtained in collaboration with R. Ducasse. There we focus on the periodic framework, placing emphasis on the influence of the geometry.

Because of the heterogeneity of (2.2), the asymptotic speed of spreading needs to be defined for any direction $\xi \in \mathbb{S}^{N-1}$. This is a quantity $w(\xi)$ for which **invading** solutions emerging from compactly supported initial data satisfy

$$\lim_{t\to+\infty} u(t, ct\xi) = \begin{cases} 1 & \text{if } 0 \le c < w(\xi) \\ 0 & \text{if } c > w(\xi). \end{cases}$$

The existence of the asymptotic speed of spreading w is derived by J. Gärtner and M. I. Freĭdlin [GF79] in the case $\Omega = \mathbb{R}^N$, $b \equiv 0$ and under the assumption that A, f are **periodic** and f satisfies the Fisher-KPP condition. Using some probabilistic methods in the framework of large deviations, the authors obtain a formula for $w(\xi)$ which enlighten its nontrivial dependence on the direction ξ. It is expressed in terms of the periodic principal eigenvalues of a family of linear operators related to the linearisation of (2.2) around 0. This is a consequence of the fact that the problem is linearly determined due to the Fisher-KPP condition. Subsequent works concerning the speed of spreading from the point of view of probability, PDE, or singular perturbation, essentially rely on this condition.

It turns out that the Freĭdlin-Gärtner formula can be rephrased in terms of the critical (or minimal) speed c^* of *pulsating travelling fronts*, which are the natural extension to the periodic setting of the planar travelling fronts (see Section 3.1 below for the definition). Namely, it is shown in [BHN05, W02] that

$$w(\xi) = \min_{e\cdot\xi>0} \frac{c^*(e)}{e\cdot\xi}. \tag{2.3}$$

Unlike the original formula, this makes sense for general Monostable, Combustion and Bistable reaction terms too. We show in [13] that (2.3) indeed provides the asymptotic speed of spreading in those cases, and actually in all cases where pulsating travelling fronts are available. The result of [13], presented in Section 2.3 below, allows one to describe the asymptotic shape of the level sets of the solution up to an order $o(t)$.

In the case of the homogeneous equation (2.1), J. Gärtner [G82] was able to push the precision of the location of the level sets up to order $O(1)$. This term cannot be got rid of, because it incorporates the everlasting "remembrance" of the initial datum. Thus, the knowledge of the location of the level sets does not allow one to understand whether or not they become rounder and rounder as time goes on (one just infers that this happens after $1/t$ rescaling). At the end of Section 2.3, we present a result of [12] which shows that this is not the case for the Fisher-KPP equation. Finally, we discuss an improvement of Gärtner's result in Section 2.4.

2.1 Stability analysis in unbounded domains ([3])

The stability theory addresses the issue of the behaviour of solutions of a given evolution equation under small perturbations of initial conditions. Consider a solution $\bar{u}(t,x)$ of some nonlinear parabolic equation. Then $\bar{u}$ is said to be *stable* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|u(0,\cdot) - \bar{u}(0,\cdot)\|_\infty < \delta \implies \|u(t,\cdot) - \bar{u}(t,\cdot)\|_\infty < \varepsilon \quad \text{for } t \text{ large.}$$

Otherwise, one says that $\bar{u}$ is *unstable*. A stronger notion is that of *asymptotic stability*, which means that there is $\delta > 0$ such that

$$\|u(0,\cdot) - \bar{u}(0,\cdot)\|_\infty < \delta \implies \|u(t,\cdot) - \bar{u}(t,\cdot)\|_\infty \to 0 \quad \text{as } t \to +\infty.$$

We restrict ourselves to the case where $\bar{u} \equiv \bar{u}(x)$ is a steady state, i.e., a stationary solution. Moreover, up to a modification of the nonlinear operator, we can assume without loss of generality that $\bar{u} \equiv 0$.

In the terminology of reaction-diffusion equations motivated by population dynamics, the asymptotic stability of 0 is referred to as the *extinction* property. In this context, one focuses on nonnegative solutions, that is, one considers the one-side stability. Two other notions classically appear in the literature, as we have seen in the previous section : the *invasion* property and the *hair-trigger* effect. A first step to derive these properties is to show that the null state satisfies an instability property which is not expressed in terms of the previous notions.

Definition 2.1. We say that 0 is *uniformly repulsive* if any solution with an initial datum $u_0 \gneqq 0$ satisfies

$$\inf_x \Big(\liminf_{t\to+\infty} u(t,x) \Big) > 0.$$

Observe that if 0 is uniformly repulsive then it is asymptotically unstable, but it could still be stable.

A useful way to investigate the stability of a steady state is by considering the linearised problem around it. Suppose that for a given problem, the linearisation around the steady state $\bar{u} \equiv 0$ has the form

$$\begin{cases} \partial_t u = \nabla\cdot(A(x)\nabla u) + b(x)\cdot\nabla u + c(x)u, & t>0,\ x\in\Omega \\ \nu A(x)\nabla u = 0, & t>0,\ x\in\partial\Omega. \end{cases} \tag{2.4}$$

As usual, A is a uniformly elliptic (smooth) matrix field, b, c are bounded and Ω is a smooth domain in $\mathbb{R}^N$ with outer unit normal ν.

If Ω is **bounded**, we can consider the classical principal eigenvalue of the operator

$$-\mathcal{L}u := -\nabla\cdot(A(x)\nabla u) - b(x)\cdot\nabla u - c(x)u \quad \text{in } \Omega,$$

under Neumann boundary condition $\nu A\nabla u = 0$ on $\partial\Omega$. Let us call it $\lambda_{\mathcal{N}}$. If $\lambda_{\mathcal{N}} < 0$ (resp. $\lambda_{\mathcal{N}} > 0$) we say that 0 is *linearly unstable* (resp. *stable*). The following implications are readily deduced from the parabolic weak and strong maximum principles (under suitable regularity conditions on the original nonlinear operator) :

$$\lambda_{\mathcal{N}} > 0 \implies 0 \text{ is asymptotically stable,} \tag{2.5}$$
$$\lambda_{\mathcal{N}} < 0 \implies 0 \text{ is uniformly repulsive.} \tag{2.6}$$

Moreover, we have that

$$c \lneqq 0 \text{ in } \Omega \implies \lambda_{\mathcal{N}} > 0, \tag{2.7}$$
$$c \gneqq 0 \text{ in } \Omega \implies \lambda_{\mathcal{N}} < 0, \tag{2.8}$$
$$b \equiv 0 \text{ and } \langle c \rangle > 0 \implies \lambda_{\mathcal{N}} < 0, \tag{2.9}$$

where $\langle c \rangle$ stands for the average of c.

The above properties extend to **periodic** media, that is, if Ω and the coefficients of the operator are periodic, with the same period. Without the periodicity condition, the situation for a **general unbounded** domain is much more delicate. The theory of the generalised principal eigenvalue presented in Chapter 1 is specifically devised to tackle such kind of questions. In what follows, we keep the notation $\lambda_{\mathcal{N}}$ to indicate the generalised principal eigenvalue of $-\mathcal{L}$ in Ω under Neumann boundary condition, as defined in Section 1.2.

We have seen in Section 1.2 that the analogue of the implication (2.7) when Ω is unbounded requires the stronger hypothesis $\sup_\Omega c < 0$. Instead, property (2.8) does not extend to the unbounded case even if $\inf_\Omega c > 0$, because the presence of a large drift b can arbitrarily increase $\lambda_{\mathcal{N}}$. Finally, Theorem 1.5 shows that the implication (2.9) holds true, up to using a suitable notion of average : the least mean.

Let us turn to properties (2.5), (2.6). The former fails, even if one restricts himself to compactly supported initial data. In order to get a sufficient condition for the asymptotic stability one needs to impose the additional hypotheses $\inf_\Omega \phi > 0$, $\sup_\Omega \phi < +\infty$ on the "test functions" in the definition of $\lambda_{\mathcal{N}}$. On the other hand, $\lambda_{\mathcal{N}} < 0$ just guarantees that 0 is unstable, but it is not hard to see that it does not imply that 0 is repulsive, because solutions can still be attracted by 0 at infinity. In order to avoid this, one needs in some sense to impose the negativity of $\lambda_{\mathcal{N}}$ also "at infinity". This is achieved in the self-adjoint case, through a uniform negativity condition on the mixed principal eigenvalues $\lambda(y, r)$ defined in Section 1.2.

Theorem 2.2 ([3])**.** *Assume that the linearised problem around* 0 *has the form* (2.4) *with* $b \equiv 0$ *and that* $\partial\Omega$ *is uniformly smooth. Suppose that the associated mixed principal eigenvalues satisfy*

$$\lim_{r \to +\infty} \left(\sup_{y \in \Omega} \lambda(y, r) \right) < 0.$$

Then 0 *is uniformly repulsive.*

In particular, 0 *is uniformly repulsive if* $\inf_\Omega c > 0$ *or, more in general, if the least mean of c is positive.*

Here, $\partial\Omega$ *uniformly smooth* means that it is locally of class C^N and uniformly of class $C^{1,1}$, i.e., Ω satisfies the uniform interior and exterior ball conditions. The last statement of Theorem 2.2 is a consequence of Theorem 1.5.

An important consequence of Theorem 2.2 is the extension of the hair-trigger effect to heterogeneous equations in general unbounded (uniformly smooth) domains.

Corollary 2.3. *Consider the problem*

$$\begin{cases} \partial_t u = \nabla \cdot (A(x)\nabla u) + f(x,u), & t>0,\ x\in\Omega \\ \nu A(x)\nabla u = 0, & t>0,\ x\in\partial\Omega, \end{cases}$$

with $\partial\Omega$ uniformly smooth and with $u \mapsto f(\cdot,u)$ vanishing at $0,1$ and positive between and satisfying

$$\inf_{x\in\Omega} f_u(x,0) > 0.$$

Then, any nontrivial, nonnegative solution satisfies

$$u(t,x) \to 1 \quad \text{as } t \to +\infty, \quad \text{locally uniformly in } x \in \overline{\Omega}.$$

This result applies in particular to the Fisher-KPP equation, answering a question by N. Nadirashvili [N18]. As a matter of fact, H. Berestycki, F. Hamel and N. Nadirashvili [BHN10] derive the hair-trigger effect under an **hypothesis** on the domain Ω which is closely related to the conclusion of Lemma 1.6 above. What we show in that lemma is that their hypothesis is fulfilled by uniformly smooth domains. We remark that the authors of [BHN10] make directly use of a Rayleigh quotient in truncated domains without invoking the principal eigenvalue, in order to avoid the difficulties arising from the lack of regularity of the boundary. Let us conclude by stressing out that the validity of the hair-trigger effect for non-uniformly smooth domains remains an open question.

2.2 The role of the geometry on propagation ([9])

In the previous section, we have analysed the stability properties of the steady states with respect to small perturbations. We have seen the connection with the linear stability, enlightening the role of the generalised principal eigenvalue. We are interested now in **compactly supported perturbations** which are not necessarily small. We then leave the framework of stability in the sense considered before. For instance, in the homogeneous Bistable case, 0 is stable, but *invasion* can still occur for compactly supported data, provided they are sufficiently large. As usual, invasion refers to locally uniform convergence to 1. The basic question we address here is how the geometry of the medium affects the long-time dynamics for such kind of perturbations.

In order to emphasise the role of the geometry of the domain, we consider the simplest homogeneous equation :

$$\begin{cases} \partial_t u = \Delta u + f(u), & t>0,\ x\in\Omega \\ \nu \cdot \nabla u = 0, & t>0,\ x\in\partial\Omega. \end{cases} \tag{2.10}$$

The domain Ω is assumed to be periodic, i.e.,

$$\Omega + \mathbb{Z}^N = \Omega.$$

We always assume that $f(0) = f(1) = 0$. Between 0 and 1, the function f can be of any of the three types described at the beginning of this chapter : Monostable, Combustion, Bistable. In particular,

$$\exists \theta \in [0,1), \quad f > 0 \quad \text{in} \ \ (\theta, 1). \tag{2.11}$$

We consider initial data satisfying $0 \leq u_0 \leq 1$.

The literature typically focuses on two classes of initial data : compactly supported and *front-like*, i.e., satisfying

$$\lim_{x\cdot e\to-\infty} u_0(x) = 1, \qquad \lim_{x\cdot e\to+\infty} u(x) = 0, \tag{2.12}$$

for some $e \in \mathbb{S}^{N-1}$. One could expect that these two classes do not behave too differently for large times. However, while it is clear that the validity of the invasion property for compactly supported data implies that for front-like data, the reverse is not obvious. This is the first topic we deal with in the paper [9] in collaboration with R. Ducasse.

Theorem 2.4 ([9]). *Consider the problem* (2.10) *in a periodic domain* Ω*. Then the following properties are equivalent :*

(i) *Invasion occurs for all front-like initial data ;*

(ii) *For any* $\eta \in (\theta, 1)$*, where* θ *is as in* (2.11)*, there is* $r > 0$ *such that invasion occurs for any initial datum satisfying*

$$u_0 > \eta \quad \textit{in} \ \ \Omega \cap B_r.$$

The interesting implication in the above theorem is the following : if invasion occurs for all front-like data then it occurs for "large enough", possibly compactly supported, initial data too. The proof of this result relies on the method developed in [13], which is illustrated in Section 2.3 below.

Next, owing to the existence of *pulsating travelling fronts* provided by [BH02], we deduce that properties (i)-(ii) of Theorem 2.4 hold if f is of the Combustion type. We point out that the same conclusion can be reached by combining the results of [BH02] with those obtained by H. F. Weinberger in [W02] using a discrete dynamical system approach. Besides the periodic setting, the invasion property holds for large initial data for the problem (2.10) in an exterior domain, with f of the Combustion type. We further show in [9] that it holds true in the whole space even if the Laplace operator is replaced by a diffusion operator in non-divergence form depending in a general fashion on space.

A natural question is whether condition (i) is necessary to have (ii). It would be tempting to conjecture that the existence of a single front-like datum for which invasion occurs would imply property (ii). We show in [9] that this is not the case. We achieve this in the Bistable case, exploiting a phenomenon of **blocking** first exhibited by H. Matano [M79] in a bounded domain, then extended by H. Berestycki, F. Hamel and H. Matano [BHM09] to exterior domains and by H. Berestycki, J. Bouhours, and G. Chapuisat [BBC16] to cylindrical domains. The idea there is that invasion can be "blocked" by a narrow passage followed by an abrupt opening. Exploiting this mechanism, we are able to construct a periodic domain where *oriented invasion* occurs, in the sense of the following.

Theorem 2.5 ([9]). *Let f be of the Bistable type. There exists a periodic domain Ω such that, for any $\eta \in (\theta, 1)$, there is $r > 0$ such that for any initial datum satisfying*

$$u_0 > \eta \quad \text{in} \ \ \Omega \cap B_r,$$

the limit

$$\hat{u}(x) := \lim_{t \to +\infty} u(t, x)$$

exists locally uniformly in $x \in \overline{\Omega}$ and satisfies the following :

- Invasion in the direction e_1 :

$$\hat{u}(x) \to 1 \quad \text{as} \ \ x \cdot e_1 \to +\infty;$$

- Blocking in the direction $-e_1$:

$$\hat{u}(x) \to 0 \quad \text{as} \ \ x \cdot e_1 \to -\infty;$$

Moreover, invasion occurs for any initial datum which satisfies the front-like condition (2.12) *with $e = e_1$.*

The function $\hat{u}$ in Theorem 2.5 is therefore a nontrivial steady state which is asymptotically stable from below. This means that the "bistability" character of the equation is broken by the geometry of Ω. The domain Ω of Theorem 2.5 is depicted in Figure 2.1.

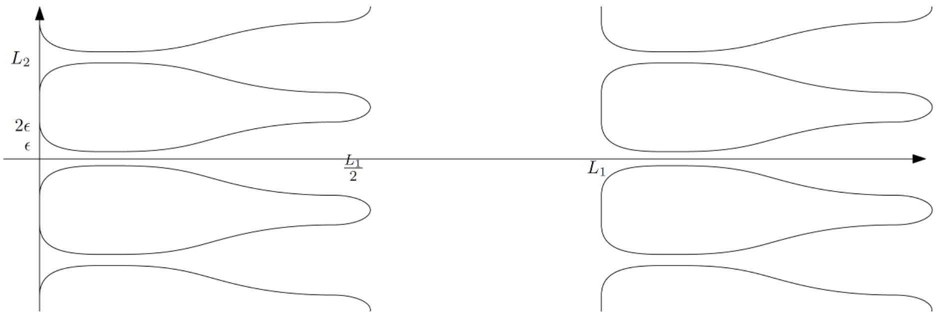


FIG. 2.1. The domain exhibiting the *oriented invasion*.

Concerning problem (2.10) when f is of the Monostable type, we know that the invasion property holds for large data in periodic and exterior domains, by comparison with the Combustion case. Actually, if $f(u) \geq u^p$ for $0 < u \ll 1$, where $p = 1 + \frac{2}{N}$ is the Fujita exponent, and Ω is an exterior domain, then the *hair-trigger* effect holds, namely, invasion occurs for any nontrivial initial datum $u_0 \gneqq 0$, see [AW78]. On the other hand, if p is sufficiently large, it is possible to construct some non-periodic domain for which blocking occurs no matter how large the support of u_0 is.

The known results in the literature about the invasion property for the problem (2.10) are summarised in the following table.

TABLE 2.1 – *Validity of the invasion property*

	$\mathbb{R}^N$	Periodic	Exterior	General
Fisher-KPP	✔	✔	✔	✔
Monostable	✓	✓	✓	×
Combustion	✓	✓	✓	×
Bistable	✓	×	×	×

✔ : the hair-trigger effect holds
✓ : invasion occurs if $u_0 > \eta > \theta$ in a large ball
× : invasion may fail even if $u_0 > \eta > \theta$ in a large ball

We recall that the result in the Fisher-KPP case is a consequence of Corollary 2.3 ; there, the domain can be "general", but still uniformly smooth.

2.3 The shape of expansion ([12, 13])

In the previous section, we have discussed several conditions guaranteeing the invasion property – locally uniform convergence to 1 – for compactly supported initial data. Here we want to describe the speed at which invasion takes place in any direction. This would provide us with a picture of the asymptotic shape of the level sets of solutions. This problem has been dealt in [13], through a new geometric approach which can be applied to general heterogeneous reaction-diffusion equations of the form

$$\partial_t u = \nabla \cdot (A(x)\nabla u) + b(x) \cdot \nabla u + f(x,u), \quad t > 0, \ x \in \mathbb{R}^N. \tag{2.13}$$

The method builds a bridge between the propagation of compactly supported data and of front-like solutions, i.e., satisfying (2.12). The first consequences derived in [13] are a new proof of the classical Freĭdlin-Gärtner formula for periodic Fisher-KPP equations, as well as its extension to Monostable, Combustion and Bistable reaction terms. We recall that this formula provides the asymptotic speed of spreading in any direction $\xi \in \mathbb{S}^{N-1}$. It can be expressed as follows :

$$w(\xi) = \min_{e \cdot \xi > 0} \frac{c^*(e)}{e \cdot \xi},$$

where $c^*(e)$ is the minimal speed of *pulsating travelling fronts* in the direction e. We actually derive a uniform version of the formula with respect to ξ, by introducing the following.

Definition 2.6. We say that a closed set $\mathcal{W} \subset \mathbb{R}^N$, coinciding with the closure of its interior, is the *asymptotic set of spreading* for (2.13) if for any invading solution with a compactly supported initial datum, there holds that

$$\forall \text{ compact set } K \subset \text{int}(\mathcal{W}), \quad \inf_{x \in K} u(t, xt) \to 1 \quad \text{as } t \to +\infty,$$

$$\forall \text{ closed set } C \text{ such that } C \cap \mathcal{W} = \emptyset, \quad \sup_{x\in C} u(t,xt) \to 0 \quad \text{as } t \to +\infty.$$

The method requires the existence of some front-like solutions propagating with a positive speed. In the periodic case (i.e., if the terms in (2.13) are periodic with the same period in any given direction), these are naturally provided by pulsating travelling fronts, see the discussion in Section 3.1 below.

Theorem 2.7 ([13])**.** *Assume that the terms in* (2.13) *are periodic with the same periods and that the (minimal) speed* $c^*(e)$ *of pulsating travelling fronts in any direction* e *is positive. Then the set*

$$\mathcal{W} := \{r\xi \ : \ \xi \in \mathbb{S}^{N-1}, \ \ 0 \leq r \leq w(\xi)\}, \quad \textit{with} \ \ w(\xi) := \min_{e\cdot\xi>0} \frac{c^*(e)}{e\cdot\xi},$$

is the asymptotic set of spreading for (2.13)*.*

Moreover, w *is positive and continuous and thus* $w(\xi)$ *is the asymptotic speed of spreading in the direction* ξ*.*

This result recovers the existence of the asymptotic set of spreading derived by H. F. Weinberger in [W02], through the analysis of the Poincaré map from the point of view of discrete dynamical systems. Let us point out that the formula involving the speeds of fronts is obtained in [W02] only in the Monostable case.

Other consequences of the method developed in [13] are : the existence of the asymptotic speed of spreading for equations with almost periodic temporal dependence ; derivation of multi-tiered propagation for Multistable equations. R. Ducasse extended it in [D18b] to equations set in periodic domains, under Neumann boundary conditions.

Theorem 2.7 provides a description of the level sets of solutions emerging from compactly supported initial data up to order $o(t)$. Namely, it implies that every level set between 0 and 1 is located at the position $\big(w(e)t + o(t)\big)e$, $e \in \mathbb{S}^{N-1}$. As we shall discuss in detail in the next section, more precise descriptions are available in some particular cases, typically under the Fisher-KPP hypothesis, or for the homogeneous equation (2.1). In the latter case, w is constantly equal to the (minimal) speed of planar fronts c^* and the location of level sets of solutions in any given direction is $c^*t - C_N \log t + O(1)$, where $C_N \geq 0$ depends on the dimension and on the type of reaction term, see [G82, U85, D15]. This allows one to conclude that, for a solution u to (2.1) with a compactly supported initial datum, the Hausdorff distance between the upper level set

$$\mathcal{U}_\eta(t) := \{x \in \mathbb{R}^N \ : \ u(t,x) > \eta\}, \quad \eta \in (0,1),$$

and the ball $B_{c^*t-C_N\log t}$ remains bounded in time. The same conclusion, but without specifying the radius, that is, replacing the ball $B_{c^*t-C_N\log t}$ with a ball of some radius $r(t)$, is derived by C. K. R. T. Jones in [J83] through a simple and beautiful reflection argument.

However, the above results leave open the question of whether or not $\mathcal{U}_\eta(t)$ actually converges to a ball as $t \to +\infty$. Namely, introducing the quantities

$$\mathcal{R}^i_\eta(t) := \sup\{r > 0 \,:\, \exists x_0 \in \mathbb{R}^N,\ B_r(x_0) \subset \mathcal{U}_\eta(t)\},$$

$$\mathcal{R}^e_\eta(t) := \inf\{r > 0 \,:\, \exists x_0 \in \mathbb{R}^N,\ B_r(x_0) \supset \mathcal{U}_\eta(t)\},$$

is it true that $\mathcal{R}^e_\eta(t) - \mathcal{R}^i_\eta(t) \to 0$ as $t \to +\infty$? A negative answer has been given by H. Yagisita [Y01] and V. Roussier [R04], both in the Bistable case. The common idea there is to construct a solution which looks like a planar front when followed along a given direction, shifted by different values depending on the direction. This method relies on the strong stability of the unique (up to shift) front for the Bistable equation. The question remained open in many relevant cases, such as, strikingly, the linear one, as well as for the Fisher-KPP equation. We give a negative answer in those cases in [12], as a consequence of a general non-symmetrization property.

Theorem 2.8 ([12])**.** *Assume that f is positive in $(0,1)$ and that $f(u)/u$ is nonincreasing in $(0,+\infty)$. Let u_1, u_2 be two nonnegative, not identically equal to 0, continuous functions with compact support. Then, for $|\xi|$ large enough, the solution to (2.1) with initial datum*

$$u_0(x) = u_1(x) + u_2(x+\xi)$$

satisfies

$$\forall \eta \in (0,1), \quad \mathcal{R}^e_\eta(t) - \mathcal{R}^i_\eta(t) \not\to 0 \quad \text{as } t \to +\infty.$$

The cornerstone of the proof of Theorem 2.8 is an estimate on the width of the interface between two distinct level sets of solutions. This can be viewed as a steepness property, which is of independent interest.

2.4 Lag behind the front in the Fisher-KPP case ([8])

The results presented up to now concern the asymptotic speed of spreading of solutions emerging from compactly supported data. The knowledge of this speed allows one to localise the position of the level sets up to an $o(t)$ term. We discuss now the improvements of this description in the case of the Fisher-KPP equation. We recall that for the homogeneous Fisher-KPP equation (2.1), the asymptotic speed of spreading is equal to $c^* := 2\sqrt{f'(0)}$, which coincides with the speed of the slowest planar travelling front $\phi^*(x \cdot e - c^* t)$.

The main result of the pioneering paper [KPP37] is that, in dimension $N = 1$, the solution starting from the Heaviside initial datum $u_0 = \mathbb{1}_{(-\infty,0]}$ converges to the slowest travelling front in the following sense :

$$u(t,x) = \phi_*(x - \sigma_\infty(t)) + o(1) \quad \text{as } t \to +\infty,$$

uniformly with respect to $x \in \mathbb{R}$, for some function

$$\sigma_\infty(t) = c^* t + o(t) \quad \text{as } t \to +\infty.$$

Hence, the $o(t)$ term above represents the **shift** between the solution and the front. An important issue is then to understand how this $o(t)$ looks like. R. A. Fisher had already made an informal argument in [F37] showing that $o(t) = O(\ln t)$. This has been confirmed by M. Bramson in the important papers [B78, B83]. Namely, there exists a constant x_∞, depending on u_0, such that

$$\sigma_\infty(t) = c^* t - \frac{3}{c^*} \ln t - x_\infty + o(1) \quad \text{as} \;\; t \to +\infty.$$

As a consequence, the actual position of the level set of solutions lags behind the front by a $\ln t$ order. The above formula is obtained in [B83] by applying elaborate probabilistic arguments to the branching Brownian motion from which the Fisher-KPP emerges. A weaker version of Bramson's result, precise up to the $O(1)$ term, has been derived with purely PDE arguments by F. Hamel, J. Nolen, J.-M. Roquejoffre and L. Ryzhik [HNRR13]. The general strategy of [HNRR13] turned out to be extremely flexible, and adaptable to more intricate situations, such as the one-dimensional spatially periodic case, see [HNRR16]. The basic idea of the method of [HNRR13] consists in finding a suitable moving frame $Y(t)$ where the solution of the Dirichlet problem

$$\begin{cases} \partial_t u - \partial_{xx} u = f'(0) u, & t > 0, \; x > Y(t) \\ u(t, Y(t)) = 0, & t > 0 \end{cases}$$

neither tends to 0 nor to $+\infty$ as $t \to +\infty$. This would allow one to construct some sub and supersolutions providing a control of the original solution. It turns out that the good choice is $Y(t) = c^* t - \frac{3}{c^*} \ln t$.

In spatial dimension N larger than 1, the asymptotics has been pushed less far. Aronson-Weinberger's result is made precise up to $O(1)$ by J. Gärtner [G82]. Namely, for every $\eta \in (0,1)$, the η-level set of a solution to (2.1) emerging from a compactly supported initial datum is trapped between two spheres of radius

$$R(t) = c^* t - \frac{N+2}{c^*} \ln t + O(1) \quad \text{as} \;\; t \to +\infty.$$

Gärtner's contribution is probabilistic, and a PDE proof of his result is provided by A. Ducrot [D15], whose method is inspired by [HNRR13]. We point out that it is not possible to get rid of the terms $O(1)$ in the above expansion, because, as we have seen in the previous section, it is shown in [12] that generally the difference between the radii of the outer and inner spheres does not tend to zero as $t \to +\infty$. So, it is a natural question to investigate whether it is possible to make precise the $O(1)$ in Gärtner's expansion in terms of a function s^∞ depending on the spherical variable. We have managed to do this in the recent work [8], in collaboration with J.-M. Roquejoffre and V. Roussier-Michon.

Theorem 2.9 ([8])**.** *Let u be a solution of* (2.1) *with a compactly supported initial datum. There is a function $s^\infty \in W^{1,\infty}(\mathbb{S}^{N-1})$ such that*

$$u(t,x) = \phi^* \left(|x| - c^* t + \frac{N+2}{c^*} \ln t + s^\infty \left(\frac{x}{|x|} \right) \right) + o(1) \quad \textit{as} \;\; t \to +\infty,$$

uniformly with respect to $x \in \mathbb{R}$.

This completes the result of [G82], providing at the same time the description of the profile of the solution at large time. The logarithmic shift observed here can be decomposed into two parts having different origins : the one-dimensional shift $\frac{3}{c^*}\ln t$ described before and an additional $\frac{N-1}{c^*}\ln t$ shift. The latter is due to the curvature term and it systematically arises in reaction-diffusion equations, the nonlinearity f does not need to be of the Fisher-KPP type (see [Y01, R04]).

Chapitre 3

Travelling fronts

3.1 The notion of front ([18])

Consider the homogeneous equation

$$\partial_t u - \Delta u = f(u), \quad t \in \mathbb{R}, \; x \in \mathbb{R}^N, \tag{3.1}$$

A *planar travelling front* in a direction $e \in \mathbb{S}^{N-1}$ is a solution of the form $u(x,t) = \phi(x \cdot e - ct)$, for some $c \in \mathbb{R}$, called *speed*, and some function $0 \leq \phi \leq 1$, called *profile*, satisfying $\phi(-\infty) = 1$ and $\phi(+\infty) = 0$. It is known since the seminal paper [KPP37] that under the hypothesis

$$f(0) = f(1) = 0, \qquad 0 < f(u) \leq f'(0)u \quad \text{for all } u \in (0,1), \tag{3.2}$$

planar travelling fronts exist if and only if $c \geq c^* := 2\sqrt{f'(0)}$. Moreover, the profile ϕ, which satisfies the ordinary differential equation

$$-\phi'' - c\phi' = f(\phi),$$

is unique up to shift.

When the terms in the equation depend on space or time, planar travelling fronts no longer exist. If the dependence is periodic, the relevant notion is that of *pulsating travelling front*, introduced in parallel ways by N. Shigesada, K. Kawasaki and E. Teramoto [SKT86] and J. X. Xin [X92]. This is an entire in time solution of the type

$$u(t,x) = \phi(x, x \cdot e - ct),$$

with $0 < \phi(x,z) < 1$ periodic with respect to x and satisfying

$$\phi(\cdot, -\infty) \equiv 1, \qquad \phi(\cdot, +\infty) \equiv 0.$$

Then the level sets of u are trapped between two hyperplanes orthogonal to e and moving with speed c. In the framework of periodic spatial dependence, pulsating travelling fronts have been shown to exist in the Monostable case, for a half-line of speeds, and in the Combustion and Bistable cases, for one single speed, by H. Berestycki and F. Hamel [BH02], H. F. Weinberger [W02] and J. X. Xin [X91, X93]. The papers by J. Nolen, M. Rudd and J. X. Xin [NRX05] and G. Nadin [N09] deal with space-time periodic equations of Fisher-KPP type and N. Alikakos, P. W. Bates and X. Chen [ABC99] treat the Bistable case. As a matter of fact, the hypotheses in the Bistable case are quite restrictive. This is a consequence of the fact that the invasion property, as the notion of stability itself, is a very delicate issue in heterogeneous media, as we have seen in Section 2.2. In Section 3.4, we discuss a dynamical system point of view we have adopted to tackle it, in collaboration with T. Giletti [5].

In order to deal with general space-time dependent equations, the notion of *generalised transition front* has been introduced by H. Berestycki and F. Hamel in [BH07, BH12].

Definition 3.1. A *generalised transition front* (in the direction $e \in \mathbb{S}^{N-1}$) is a positive time-global solution u for which there exists a function $X \in W^{1,\infty}_{loc}(\mathbb{R})$ such that

$$\lim_{x\cdot e\to-\infty} u(t, X(t)e + x) = 1, \qquad \lim_{x\cdot e\to+\infty} u(t, X(t)e + x) = 0,$$

uniformly with respect to $t \in \mathbb{R}$.

The function X therefore reflects the positions of a transition front as time runs, whence $X'(t)$ could be interpreted as the instantaneous speed of the front. This is however misleading because X is only defined up to an additive bounded function. Despite this fact, we show in [18] that the local oscillations of X are uniformly bounded, which implies in particular that $X(t)/t$ is bounded for, say, $|t| \geq 1$.

Definition 3.1 encloses the notion of pulsating travelling front. It is actually a particular case of a more general one given in [BH07, BH12] (see also the works of W. Shen [S01, S04] for the one-dimensional case) referred to as "almost planar" transition fronts, which is related to another notion involving the continuity with respect to the environment, given by H. Matano.

As a matter of fact, the notion of generalised transition front turns out to be meaningful even for the homogeneous equation (3.1) under the Fisher-KPP hypothesis (3.2). Indeed, what was unexpected until quite recently, was the existence of front-like solutions connecting 0 and 1 which are not planar travelling fronts and whose speed changes in time. More precisely, it was proved by F. Hamel and N. Nadirashvili [HN01] that, under the stronger assumption that f is concave, for any real numbers $c_- < c_+$ larger than or equal to c^*, there exists a solution converging to the standard planar fronts $\phi_{c_\pm}(x \cdot e - c_\pm t)$ as $t \to \pm\infty$. Roughly speaking, this is a front-like solution whose speed increases in time. The goal of the joint paper [18] with F. Hamel is precisely to describe the set of transition fronts for (3.1), focusing in particular on their asymptotic speed. The previous example shows that in general one cannot expect a transition front to have a *global mean speed*, which is defined by

$$\lim_{t-s\to+\infty} \frac{X(t) - X(s)}{t - s}.$$

This is why we introduce the notions of *asymptotic past* and *future speeds*, defined as the limits of

$$\frac{X(t)}{t} \quad \text{as } t \to -\infty \text{ (past)}, \quad \text{and as } t \to +\infty \text{ (future)}.$$

If these limits exist, they are uniquely determined, independently of the choice of X.

Theorem 3.2. *Assume that f is positive and concave (in the large sense) in $(0,1)$. If a generalised transition front u for (3.1) admits some asymptotic past speed c_- and future speed c_+, then*

$$c^* \le c_- \le c_+ < +\infty.$$

Moreover, if $c_- = c_+ > c^*$, then u is necessarily a standard planar front $\phi(x - c_\pm t)$.

This result shows that transition fronts can never decelerate, that is, the behaviour observed in [HN01] is the only possible one, except for standard fronts. The question that naturally arises is wether transition fronts always admit the asymptotic past and future speeds. We prove in [18] that this is the case for "supercritical" fronts, that is, satisfying

$$\liminf_{t \to -\infty} \frac{X(t)}{t} > c^*.$$

Among other things, our arguments use the decomposition of entire solutions as superposition of planar travelling fronts provided by [HN01]. This requires the concavity of f. A conjecture in [HN01] related to this decomposition would imply the existence of the past and future speeds for any front, not just for the supercritical ones.

3.2 General time-dependent media ([22, 28])

In the papers [28, 22] in collaboration with G. Nadin and F. Hamel respectively, we consider a class of temporal dependent reaction-diffusion equations of Fisher-KPP type. Both works deal with the case of spatial dimension 1. A model equation considered there is

$$\partial_t u - \partial_{xx} u = \mu(t) f(u), \quad t \in \mathbb{R},\ x \in \mathbb{R}, \tag{3.3}$$

where μ is a positive function and f satisfies the Fisher-KPP condition (3.2) (if there is a diffusion coefficient depending on time, one can get rid of it by a change of the temporal variable). In [22] we aim at characterising the generalised transition fronts in the case where μ admits limits at $\pm\infty$.

Theorem 3.3 ([22])**.** *Assume that f is concave and that the limits*

$$\mu_\pm := \mu(\pm\infty)$$

exist and are positive. Then, a generalised transition front for (3.3) with asymptotic past and future speeds $c_\pm$ exists if and only if

$$c_- \ge 2\sqrt{\mu_-} \quad \textit{and} \quad c_+ \ge \kappa + \frac{\mu_+}{\kappa}, \quad \textit{with} \quad \kappa = \min\left(\sqrt{\mu_+}, \frac{c_- - \sqrt{c_-^2 - 4\mu_-}}{2}\right).$$

We further show in [22] that, in all cases, except possibly when $\mu_+ > \mu_-$ and $c_\pm$ satisfy $c_- = 2\sqrt{\mu_-}$ and $c_+ = \sqrt{\mu_-} + \frac{\mu_+}{\sqrt{\mu_-}}$, the generalised transition front converges as $t \to \pm\infty$ to two planar fronts $\phi_{c_\pm}(x - c_\pm t)$ for the limiting equations with nonlinearities $\mu_\pm f(u)$.

The set of asymptotic speeds $c_\pm$ provided by Theorem 3.3 can be equivalently expressed by

$$c_\pm = \kappa_\pm + \frac{\mu_\pm}{\kappa_\pm}, \quad \kappa_- \in \big(0, \sqrt{\mu_-}\big], \quad \kappa_+ \in \big(0, \min(\kappa_-, \sqrt{\mu_+})\big].$$

This expression yields a clear interpretation : $\mu_\pm$ reflect the characteristics of the medium as $t \to \pm\infty$, while $\kappa_\pm$ are the exponential rates of decay of the asymptotic profiles $\phi_{c_\pm}$ of the front as $t \to \pm\infty$. Thus, the asymptotic rate of decay of the front as $t \to -\infty$ is larger than or equal to the one as $t \to +\infty$. Since the slower the decay, the faster the front, this can be viewed as the reason behind the fact that transition fronts always globally accelerate when $\mu_+ \geq \mu_-$, whence in particular in the case of Theorem 3.2. Let us mention that the existence of generalised transition fronts in the case $\kappa_+ = \kappa_-$ was already derived in [BH12, 28].

In [28] we deal with a class of equations which also includes (3.3) as a model case. But, in contradistinction with [22], we do not make any assumption on μ except that it is a bounded function with positive infimum. The goal of [28] is to construct a family of fronts capturing a range of speeds as large as possible, rather than characterising the whole class of transition fronts, which seems to be a task out of reach still at the present time. In the paper [S11a], appeared shortly before [28], W. Shen proves the existence of some generalised transition fronts in the case where μ is *uniquely ergodic*, a notion which implies in particular the existence of the uniform mean

$$\langle \mu \rangle = \lim_{T \to +\infty} \frac{1}{T} \int_t^{t+T} \mu(s) ds \quad \text{uniformly with respect to } t \in \mathbb{R}.$$

This hypothesis excludes, for example, **random stationary ergodic** coefficients, which is a type of dependence we specifically focus on in [28].

In the study of equations whose terms depend in a general fashion on time, our first task was to identify a notion replacing the uniform mean used in the uniquely ergodic and almost periodic cases. To this end, we introduce in [28] the following.

Definition 3.4. The *least mean* of a function $g \in L^\infty(\mathbb{R})$ is given by

$$\lfloor g \rfloor := \lim_{T \to +\infty} \left(\inf_{t \in \mathbb{R}} \frac{1}{T} \int_t^{t+T} g(s) ds \right).$$

One can actually replace the $\lim_{T \to +\infty}$ with $\sup_{T>0}$ in the above definition, whence the least mean is always well defined. We then refer to the *least mean speed* of a generalised transition front as the least mean of the derivative of the function X in Definition 3.1, that is,

$$\lim_{T \to +\infty} \left(\inf_{t \in \mathbb{R}} \frac{X(t+T) - X(t)}{T} \right).$$

This is again independent of the choice of X. We are then able to characterise the whole class of admissible speeds in terms of this notion.

Theorem 3.5 ([28])**.** *A generalised transition front for* (3.3) *with least mean speed equal to c exists if and only if* $c \geq 2\sqrt{\lfloor\mu\rfloor}$.

The arguments we employ are constructive and imply in particular that in the case where the coefficients satisfy some properties such as almost periodicity or stationary ergodicity, these are inherited by the profile of the front. Actually, the existence of the front with the critical speed $2\sqrt{\lfloor\mu\rfloor}$ is not proved in [28], but follows from the results contained there, as shown in [N15]. The notion of least mean speed turns out to be very useful in the study of generalised transition fronts in the absence of some self-averaging properties (almost periodicity, unique ergodicity, etc.), see, e.g., [SS18a, SS18b]. The reason is that it enjoys the following key characterisation derived in [28] :

$$\lfloor g\rfloor = \sup_{\sigma\in W^{1,\infty}(\mathbb{R})}\Big(\inf(g+\sigma')\Big).$$

This means that, even though the least mean of a function does not always coincide with its infimum, this is true up to a perturbation with bounded primitive.

3.3 Extensions to spatially inhomogeneous media ([11, 23, 25])

The study of generalised transition fronts in spatially varying media is much more delicate than in the temporal-dependent case. The reason is not just technical. Some obstructions to the existence of fronts can indeed arise due to the spatial heterogeneities. J. Nolen, J.-M. Roquejoffre, L. Ryzhik and A. Zlatoš exhibit in [NRRZ12] some spatial heterogeneous KPP nonlinearities for which no generalised transition fronts exist. This is in contrast with the combustion case, where generalised transition fronts have been shown to exist in dimension 1 in parallel ways by J. Nolen and L. Ryzhik [NR09] and A. Mellet, J.-M. Roquejoffre and J. Sire [MRS10]. This is why, when dealing with heterogeneous Fisher-KPP equations of the type

$$\partial_t u - a_{ij}(t,x)\partial_{ij}u - b_i(t,x)\partial_i u = f(t,x,u),\quad t\in\mathbb{R},\ x\in\mathbb{R}^N, \tag{3.4}$$

some hypotheses concerning the spatial dependence of the terms should be imposed. The KPP condition in the heterogeneous case reads

$$\inf_{\mathbb{R}^{N+1}} f(\cdot,\cdot,u)>0,\quad f(t,x,u)\leq\partial_u f(t,x,0)u\quad\text{for all }(t,x)\in\mathbb{R}^{N+1},\ u\in(0,1).$$

W. Shen [S11b] deals with the case where (a_{ij}) is the identity matrix and b, f are periodic in x and uniquely ergodic in t. We recall that the latter hypothesis implies that the function admits a uniform mean. This allows the author to use the principal Lyapunov exponent and to construct transition fronts with speeds admitting a uniform mean larger than some threshold. We show in the paper [25] in collaboration with L. Ryzhik, that the methods of [28] allows one to drop the unique ergodicity assumption, provided that $a_{i,j}, b_i$ are independent of t and f is independent of x. However, the most interesting contribution of [28] is a non-existence result for generalised transition fronts providing a sharp lower bound for the least mean speed. This is achieved through an estimate of the rate of decay of entire solutions of general parabolic equations, which is related to the Landis conjecture presented in Section 1.4 above.

We were able to handle the general equation (3.4) in collaboration with G. Nadin. We derive the following.

Theorem 3.6 ([23])**.** *Assume that the terms in* (3.4) *are periodic with respect to x. Then, there exist $0 < c_* \leq c^*$ such that a generalised transition front with least mean speed equal to c exists if $c > c^*$ and only if $c \geq c_*$.*

The thresholds c_* has an explicit expression in terms of the generalised principal eigenvalues of a family of parabolic operators associated with (3.4). Instead, c^* is a principal Lyapunov exponent for the equation. We are not able to prove that $c_* = c^*$ in general, but we show that equality holds in all the cases previously known in the literature, recovering in particular the results of [28, 25] and also completing [S11b] with a sharp non-existing result.

The cornerstone of the construction in [23] is represented by a uniform Harnack-type inequality. The derivation of this inequality is the only point where we exploit the spatial periodicity of the equation. We then investigated with G. Nadin the validity of the Harnack inequality beyond the periodic framework. In [11] we manage to derive it in the case of the 1-dimensional equation

$$\partial_t u - \partial_x\big(a(x)\partial_x u\big) = c(x)f(u), \quad t \in \mathbb{R},\ x \in \mathbb{R}, \tag{3.5}$$

with f of Fisher-KPP type and a, a', c *almost periodic*, in the sense of Bochner.

Definition 3.7. A function $g : \mathbb{R} \to \mathbb{R}$ is *almost periodic* if from any sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ one can extract a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $g(x_{n_k} + x)$ converges uniformly in $x \in \mathbb{R}$.

Our main result relies on the hypothesis that the linearised operator

$$-\mathcal{L}u = -\partial_x\big(a(x)\partial_x u\big) - c(x)f'(0)$$

admits an almost periodic positive eigenfunction. This should necessary be a generalised principal eigenfunction, whose existence is provided by [21] as discussed in Section 1.1 above. However, its almost periodicity is not granted, because it is shown in [30] that a linear elliptic equation with almost periodic coefficients may admit positive bounded solutions which are not almost periodic.

Theorem 3.8 ([11])**.** *Assume that $-\mathcal{L}$ admits an almost periodic generalised principal eigenfunction. Then there exists $c^* > 0$ such that a generalised transition front with global mean speed c exists if and only if $c \geq c^*$.*

In the supercritical case $c > c^*$, the fronts we construct can be written as $u(t,x) = U(\int_0^x \sigma - t, x)$, where σ is an almost periodic function with uniform mean $1/c$ and $U(z,x)$ is almost periodic in x.

Let us comment on the hypothesis of the existence of an almost periodic positive eigenfunction, which is a very delicate issue. On one hand, it is guaranteed in two relevant cases, where generalised transition fronts were not known to exist :

1. c is constant.

2. a and c are *quasi periodic* and their periods satisfy some non-degeneracy Diophantine condition.

The first condition trivially implies that $-\mathcal{L}$ admits the constant eigenfunction, whereas the second case follows from a result by S. M. Kozlov [K83]. On the other hand, we show that the hypothesis can be slightly relaxed using the **criticality theory** for linear elliptic operators, see, e.g., [P88, P95], which in dimension 1 is equivalent to the validity of the Liouville property, as shown by S. Agmon [A82] and M. Murata [M86].

3.4 Bistable and Multistable periodic equations ([5])

In the previous sections we have discussed the existence of front-like solutions for Fisher-KPP equations in various settings. The question in the Bistable case is a long-time standing open problem. In the periodic framework, the only available results until quite recently were those of J. X. Xin, see [X91, X92, X93], essentially of the "perturbation" type. They are valid when the heterogeneity is restricted to the diffusion coefficient, which must be close to a constant. In these works the author also shows examples of periodic Bistable equations which do not admit pulsating fronts, as a consequence of a phenomenon called *quenching*. We can now say that these examples are not really "Bistable", because some non-trivial stable stationary states appear due to the spatial heterogeneity, exactly as for the blocking phenomenon described in Section 2.2.

More recently, the existence of front-like solutions has been derived for some classes of spatial-dependent Bistable equations in dimension 1. J. Nolen and L. Ryzhik [NR09] and A. Zlatoš [Z17] deal with the case where the etherogeneity lies in the reaction terms, whereas W. Ding, F. Hamel and X.-Q. Zhao [DHZ17] show the existence of pulsating travelling fronts when the period of the coefficients approaches either 0 or ∞. The latter results are also of a perturbation type, in a sense. With a completely different approach, J. Fang and X.-Q. Zhao [FZ15] obtain pulsating fronts under the abstract hypothesis that the equation does not admit non-trivial stable steady states, always in dimension 1. They use the same method as H. F. Weinberger [W82, W02], which is based on the study of the dynamical system generated by the discrete-time evolution map. In the recent work [D16], A. Ducrot was able to construct the pulsating travelling front in arbitrary dimension under the same assumption as in [FZ15], but using the PDE method of [BH02].

In collaboration with T. Giletti, we have been working on a development of Weinberger's dynamical system method with the objective of tackling, at the same time, general periodic equations in higher dimension, as well as equations of the *Multistable* type, i.e., having an arbitrary (but finite) number of stable steady states. We consider the equation

$$\partial_t u = \nabla \cdot (A(x)\nabla u) + f(x,u), \quad t \in \mathbb{R},\ x \in \mathbb{R}^N, \tag{3.6}$$

with A uniformly elliptic and A, f periodic in x, with the same period. We assume that the equation admits an asymptotically stable, positive, periodic steady state $\bar{p}$, which generalises the extremal state 1 of the previous sections. The different notions of stability are recalled in Section 2.1. The dynamical system approach requires the stability properties of the intermediate states to be expressed in an abstract way, rather than by some explicit conditions on the coefficients.

Hypotheses 1.

Bistable : any periodic steady state between 0 and $\bar{p}$ is linearly unstable.

Multistable :

(i) There is a finite number of linearly stable periodic steady states

$$\bar{p} \equiv p_0 > p_1 > \cdots > p_K \equiv 0;$$

(ii) any other periodic steady state between 0 and $\bar{p}$ is linearly unstable.

It follows from the "order interval trichotomy" of E. N. Dancer and P. Hess [DH91] that between any pair of ordered stable periodic steady states there exists another steady state. Hypotheses 1 further imply that there are no ordered periodic states between two consecutive linearly stable states. We actually consider in [5] a weaker assumption than linear instability, which is referred to as *counter-propagation*. Because of this, our existence result under the Bistability hypothesis extends that of [D16].

The notion of a single front is not sufficient to describe the dynamical properties of a Multistable equation, for which a so-called *propagating terrace* may appear. This was already observed by P. C. Fife and J. B. McLeod [FM77], who referred to it as "minimal decomposition".

Definition 3.9. A *propagating terrace* connecting $\bar{p}$ to 0 in the direction $e \in \mathbb{S}^{N-1}$ is a couple of two finite sequences $(q_j)_{0\le j\le J}$ and $(U_j)_{1\le j\le J}$ such that :

- the functions q_j are periodic steady states of (3.6) and satisfy

$$\bar{p} \equiv q_0 > q_1 > \cdots > q_J \equiv 0;$$

- for any $1 \le j \le J$, the function U_j is a pulsating travelling front connecting q_{j-1} to q_j with speed $c_j \in \mathbb{R}$ and direction e ;
- the sequence $(c_j)_{1\le j\le J}$ satisfies

$$c_1 \le c_2 \le \cdots \le c_J.$$

A propagating terrace is then a family of stacked fronts connecting intermediate steady states whose speeds are ordered. Its existence and uniqueness (up to temporal shift) has been derived by A. Ducrot, T. Giletti and H. Matano [DGM14] in the 1-dimensional case. It is also shown there that the terrace completely describes the long-time behaviour of solutions of the Cauchy problem, in the sense depicted in Figure 3.1. For this latter property, the ordering of the speeds of the fronts is essential. The method of [DGM14] relies on an intrinsically one-dimensional argument : the "zero number principle".

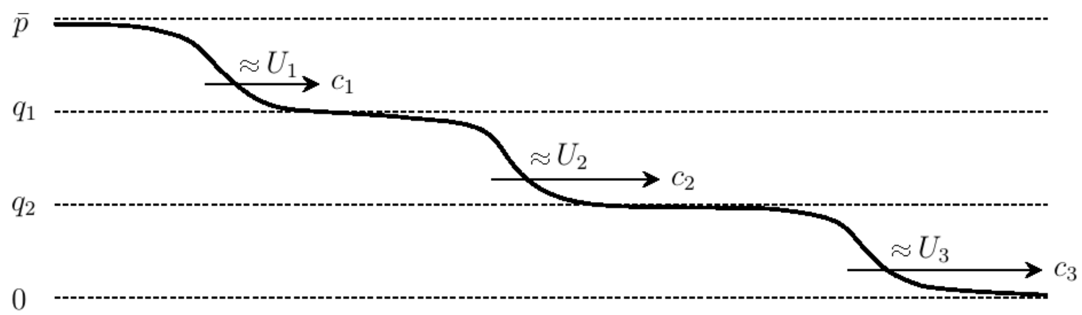


FIG. 3.1. The convergence of the solution towards the terrace.

Let us sketch our dynamical system approach. We consider the mapping

$$\mathcal{F}_c[U](x, z + x \cdot e - c) = \mathcal{E}[y \mapsto U(y, z + y \cdot e)](x),$$

where $\mathcal{E}[u_0]$ denotes the evolution by (3.6) of the datum u_0 after time 1, the direction e is fixed and $c \in \mathbb{R}$ acts as a free parameter. A (discrete) pulsating travelling front is a fixed point for $\mathcal{F}_{e,c}$ which connects two ordered steady states as $z \to \pm\infty$. In order to find such a fixed point, we consider a function $\chi(x, z)$ such that $\chi(x, z)$ is close to $\bar{p}$ for $-z \gg 1$ and equal to 0 for $z \gg 1$, then we define by recurrence

$$a_0^c := \chi,$$
$$a_{n+1}^c := \max\{\chi, \mathcal{F}_c[a_n^c]\}.$$

Then, because the (monotone) sequence $(a_n^c)_{n\in\mathbb{N}}$ converges to $\bar{p}$ for $-c$ large enough, the idea is to define c^* as the largest c for which this occurs. This is what is done in [W02] and indeed allows the author to obtain a front with speed c^* in the Monostable case. In the Bistable and Multistable cases, we introduce a variation on the method which consists in capturing the sequence $(a_n^c)_{n\in\mathbb{N}}$ at a suitable iteration and then passing to the limit as $c \nearrow c^*$. This provides us with the uppermost discrete front of the terrace. The remaining tasks are : letting the time step of $\mathcal{E}$ go to 0 in order to obtain a continuous front, iterating the argument to construct the lower fronts, showing that the steady states "selected" by the fronts are stable and finally that the speeds are ordered. In the end, we derive the following.

Theorem 3.10 ([5]). *For any $e \in \mathbb{S}^{N-1}$, there exists a propagating terrace $((q_j)_j, (U_j)_j)$ connecting $\bar{p}$ to 0 in the direction e.*

Furthermore, all the q_j are stable steady states and all the fronts U_j are monotonic in time.

It is clear that the shape of the terrace depends on the direction e, because so do the profiles of the fronts U_j. What was a priori not expected is that even the intermediate states involved in the terrace may vary, as well as the number of “floors”.

Proposition 3.11. *There exists an equation of the type* (3.6) *in dimension* $N = 2$ *for which :*

(i) *the propagating terrace connecting* $\bar{p}$ *to 0 in the direction* $(1, 0)$ *consists of two travelling fronts ;*

(ii) *the propagating terrace connecting* $\bar{p}$ *to 0 in the direction* $(0, 1)$ *consists of a single travelling front.*

Figure 3.2 represents a numerical simulation showing the shape of the solution of the associated Cauchy problem.

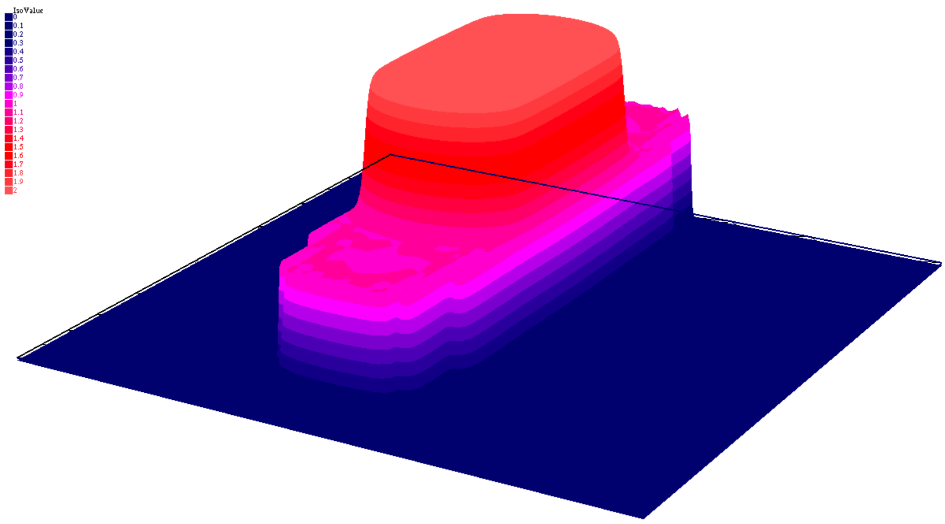


FIG. 3.2. The steady state 1 (purple) belongs to the terrace only for some of the directions.

The uniqueness and stability of the propagating terrace is the subject of a work in progress with T. Giletti. There, we aim at extending the results of [13] concerning the homogeneous case. This is not a mere adaptation of the method of [13] once we dispose of the propagating terrace provided by [5]. The reason is that the kind of asymmetric scenario depicted by Figure 3.2, related to Proposition 3.11, represents a real obstruction for the method to work.

Chapitre 4

Population dynamics in the presence of transportation networks

Several observations in ecology and epidemiology show that the invasion rate for some biological phenomena is increased by the presence of roads. In collaboration with H. Berestycki and J.-M. Roquejoffre we propose a new system to account for these observations and possibly quantify them. One of the examples we have in mind when designing the model is the spread of the "Black Death" in the middle of the 14$^{\text{th}}$ century. This pandemic was one of the most devastating in human history. It propagated along the Silk Road and, once reached the port of Marseille by merchant ships from Crimea, it spread across Europe at a very fast pace, first along the main trade routes and then to the interior of the territory (see for example A. Siegfried [S60]). Other examples are the concentration and spreading in correspondence of *seismic lines* of populations of wolves in western Canadian forests, or the proliferation of certain insects, such as the tiger mosquito or the pine processionary moth in South-western France (see [24]), whose speed is supposed to be accelerated by vehicle transportations on the roads.

In the series of articles [27, 26, 17], we study a simplified model where there is only one straight road. The dynamics in the surrounding environment, called "the field", is governed by a standard Fisher-KPP equation. The diffusion coefficient on the road, denoted by D, is larger than the one in the field, d. A first idea is to use two unknown functions, u and v, for representing individuals on the road $\mathbb{R} \times \{0\}$ and in the field $\mathbb{R}^2$ respectively , assuming there are instantaneous exchanges by some fractions μ and ν between the two. By symmetry, one can consider the problem in the half-space $\mathbb{R} \times \mathbb{R}_+$. This leads us to a system of two evolution equations and an exchange condition on the line :

$$\begin{cases} \partial_t u - D\partial_{xx} u = \nu v(x,0,t) - \mu u & t>0,\ x\in\mathbb{R}, \\ \partial_t v - d\Delta v = f(v) & t>0,\ (x,y)\in\mathbb{R}\times\mathbb{R}_+, \\ -d\partial_y v(x,0,t) = \mu u(x,t) - \nu v(x,0,t) & t>0,\ x\in\mathbb{R}. \end{cases}$$

The function f is of the Fisher-KPP type. This is an unconventional system because of the coupling between evolution equations in different dimensions.

Our works have given rise to many recent developments (it is the core topic of three PhD theses supervised by H. Berestycki and J.-M. Roquejoffre, as well as some post-doctoral fellowships within the ERC project "ReaDi"). The ultimate goal of this program is to deal with the case of general *networks* with fast diffusion - including of fractional type.

4.1 Enhancement of the speed ([16, 17, 26, 27])

In [27], once the well-posedness of the problem being settled, we investigate the question of whether there is an asymptotic speed of spreading in the direction of the road and, in such case, its comparison with the standard one in the absence of the road, that is, $c^* = 2\sqrt{df'(0)}$. Our main result can be summarised as follows.

Theorem 4.1 ([27]). *There exists an asymptotic speed of spreading w in the direction of the road. Moreover, we have :*

- $w = c^*$ *if* $D \leq 2d$ *;*
- $w > c^*$ *if* $D > 2d$ *;*
- $w = O(\sqrt{D})$ *as* $D \to +\infty$.

Therefore, the road enhances the speed of spreading of the population as soon as the ratio D/d is larger than 2. The threshold 2 does not reflect the spatial dimension, as one might have imagined, but it is rather a consequence of the absence of reproduction on the road. In fact, we show in our subsequent work [26] that if one adds a term $f(u)$ on the road then the threshold becomes 1. The last property of the theorem shows that the spreading speed becomes arbitrarily large as the diffusion on the road diverges. The key argument for characterising the speed of spreading w is the analysis of the transition from real to complex of exponential solutions for the linearised system, the complex ones being obtained using the Rouché theorem. This argument is very general and it sheds a new light also on the Freidlin-Gärtner formula discussed in Section 2.3. Our result has been recently extended by R. Ducasse [D18a] to the case of a curved road.

Once the result on the speed of spreading in the direction of the road had been established, we wandered what happens in the other directions. Namely, we have investigated the speed of spreading $w(\vartheta)$ in any given direction $(\cos\vartheta, \sin\vartheta)$. It is natural to expect $w(\vartheta)$ to be a decreasing function of the angle ϑ between the direction and the road, which reaches the value c^* at $\vartheta = \pi/2$ – that is, no enhancement of the speed in the direction orthogonal to the road. Our result confirms this intuition.

Theorem 4.2 ([17]). *The asymptotic speed of spreading $w(\vartheta)$ exists in any direction $(\cos\vartheta, \sin\vartheta)$ and it is of class C^1. Moreover, if $D/d > 2$ then there exists $\vartheta_0 \in (0, \pi/2)$ such that*

- $w'(\vartheta) < 0$ *for* $\vartheta \in (0, \vartheta_0)$ *;*
- $w(\vartheta) \equiv c^*$ *for* $\vartheta \in [\vartheta_0, \pi/2]$.

This means that the speed of spreading is strictly decreasing up to a *critical angle* ϑ_0, but starting from this angle it becomes constantly equal to c^*. We therefore derive a highly non-trivial asymptotic form of the invasion set $\{v \sim 1\}$. Indeed such set is given, up to an error of order $o(t)$, by $t\mathcal{W}$, where $\mathcal{W}$ is the hull of the speeds of spreading :

$$\mathcal{W} = \{w(\vartheta)\,(\cos\vartheta, \sin\vartheta) \;:\; \vartheta \in [0,\pi]\}.$$

The phenomenon of the critical angle may evoke the analogy with the *Huygens principle*, that is, every point of a wavefront acts as a new source for the wave. Then, considering the invasion set $\{v \sim 1\}$ as a wave, this would imply that it evolves with a normal speed equal to c^* and therefore its curvature is either $(c^*)^{-1}$ (a circle) or 0 (a straight line). Namely, $\mathcal{W}$ should coincide with the boundary of the convex hull of the union of the disk B_{c^*} and the segment $[-w(0), w(0)] \times \{0\}$, exhibiting indeed a critical angle. The unexpected result of [17] is that $\mathcal{W}$ is actually larger than this set, as shown in Figure 4.1.

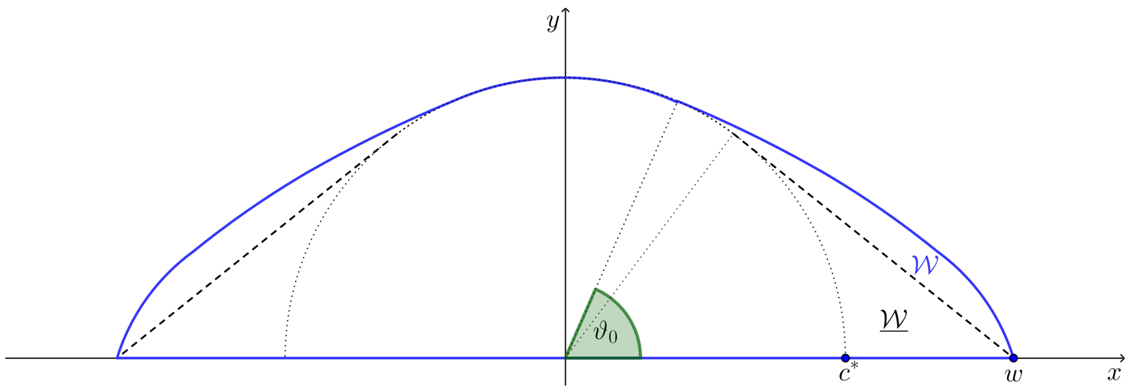


FIG. 4.1. The hull of the speeds $\mathcal{W}$ (blue line) and the set given by the Huygens principle $\underline{\mathcal{W}}$ (dashed line).

This analytical result has been numerically confirmed by A. C. Coulon, see Figure 4.2. The way we interpret it is that the presence of the road affects the solution even at a great distance, by means of a modification of its exponential tail.

A further work on the model that we have carried on, always in collaboration with H. Berestycki and J.-M. Roquejoffre, concerns the existence of travelling fronts for the system. In [16] we derive the same type of result as in the case of the standard Fisher-KPP equation. Namely, travelling fronts with a speed c exist if and only if $c \geq w$, where w is the asymptotic speed of spreading given by Theorem 4.1.

4.2 A nonlocal diffusion on the line ([20])

The starting point of the model considered in [27] is that the displacement of individuals on the road, although faster than usual, is still of the Brownian type, which leads, by applying Fick's law on the propagation of particles, to a macroscopic diffusion term given by the Laplace operator multiplied by a large constant D.

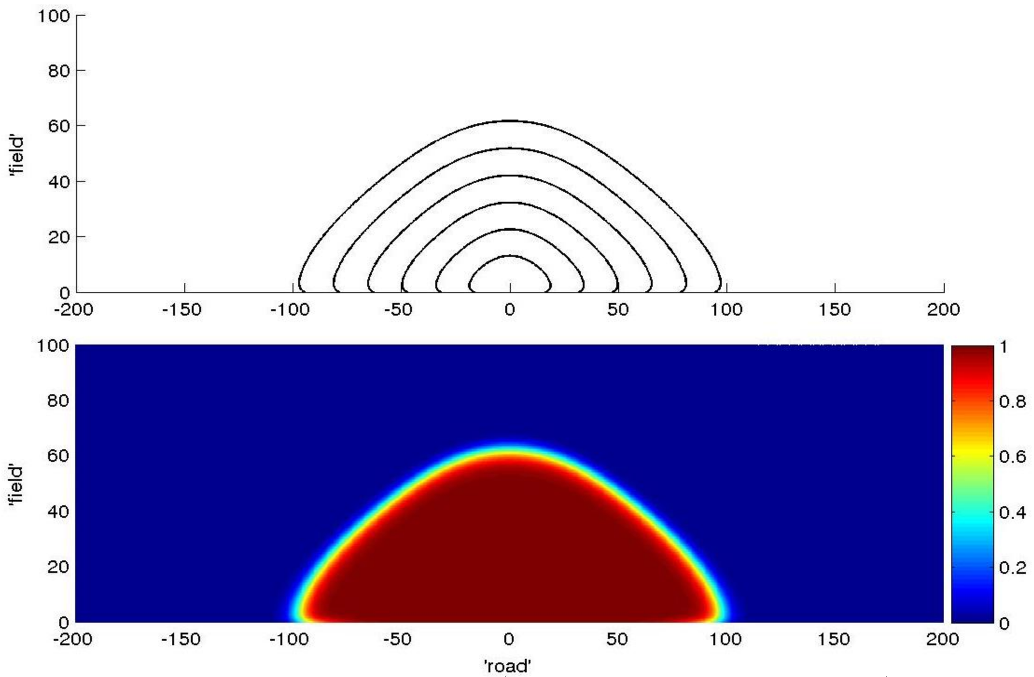


FIG. 4.2. Level sets of v (simulation by A. C. Coulon)

More general random walks, the Lévy flights, would allow one to describe the situation where individuals are able to displace very quickly, very far away. This is called "non-local diffusion". An example of this is the movement of individuals in an urban environment in the presence of metropolitan lines, or, on a larger scale, air connections. Another example of non-local diffusion that is of particular interest for us is the mutual influence on the social behaviour between communities which are not necessarily geographically close, due to the transmission of information through old and new media. The macroscopic realisation of this non-local diffusion is the *fractional Laplacian*, defined by

$$(-\partial_{xx})^s u(x) = P.V. \int_{\mathbb{R}} \frac{u(x)-u(y)}{|x-y|^{1+2s}} dy,$$

where $P.V.$ stands for the Cauchy Principal Value. In the collaboration [20] with H. Berestycki, A. C. Coulon and J.-M. Roquejoffre, we consider the variation of the model of [27] in which the diffusion on the road is given by the fractional Laplacian of order $s \in (0.1)$, that is,

$$\begin{cases} \partial_t u + (-\partial_{xx})^s u = \nu v(x,0,t) - \mu u & t>0,\ x \in \mathbb{R}, \\ \partial_t v - d\Delta v = f(v) & t>0,\ (x,y) \in \mathbb{R}\times\mathbb{R}_+, \\ -d\partial_y v(x,0,t) = \mu u(x,t) - \nu v(x,0,t) & t>0,\ x\in\mathbb{R}. \end{cases}$$

The study of this new system required a important preliminary effort to obtain the results about existence and uniqueness of solutions, achieved in A. C. Coulon's PhD thesis [C14] using the theory of sectoral operators. Next, owing to our results in [27], it is natural to expect that the road enhances the propagation, but the question is to determine to what extent, and to compare the result with the "catastrophic" scenario where the diffusion is of fractional type in the whole space. In such case, it is shown by X. Cabré and J.-M. Roquejoffre [CR13] that the speed of spreading grows exponentially in time, in the sense that any solution of the Cauchy problem (in dimension 1) with a compactly supported initial datum satisfies

$$u(t, \pm e^{ct}) \xrightarrow[t\to+\infty]{} \begin{cases} 1 & \text{if } c < \gamma \\ 0 & \text{if } c > \gamma \end{cases}, \qquad \text{where } \gamma := \frac{f'(0)}{1+2s}.$$

We show that, for the above system, this property holds true for the u component, as well as for the v component, locally in y, with the same value γ as above.

Theorem 4.3 ([20])**.** *Any solution starting from a nonnegative, nontrivial, compactly supported initial datum satisfies*

- $\forall c < \gamma, \quad \big(u(t,x), v(t,x,y)\big) \xrightarrow[t\to+\infty]{} (\nu/\mu, 1)$ *uniformly in* $|x| \le e^{ct}$ *and locally uniformly in* $y \ge 0$ *;*
- $\forall c > \gamma, \quad \big(u(t,x), v(t,x,y)\big) \xrightarrow[t\to+\infty]{} (0,0)$ *uniformly in* $|x| \ge e^{ct}$*,* $y \ge 0$*.*

So the horizontal spreading is completely governed by the diffusion on the road. On the other hand, the speed of spreading for v in the other directions only depends on the vertical component of the direction – with respect to which it takes place at the standard speed c^* – because the horizontal displacement, being exponentially fast, is negligible. Namely, the following holds.

Theorem 4.4 ([20])**.** *For all* $\vartheta \in (0, \pi)$*, we have :*

- $\forall c < c^*/\sin\vartheta, \quad v(t, r\cos\vartheta, r\sin\vartheta) \xrightarrow[t\to+\infty]{} 1$ *uniformly in* $r \in [0, ct]$ *;*
- $\forall c > c^*/\sin\vartheta, \quad v(t, r\cos\vartheta, r\sin\vartheta) \xrightarrow[t\to+\infty]{} 0$ *uniformly in* $r \ge ct$*.*

This means that the level lines of v become asymptotically parallel to the road, see Figure 4.3.

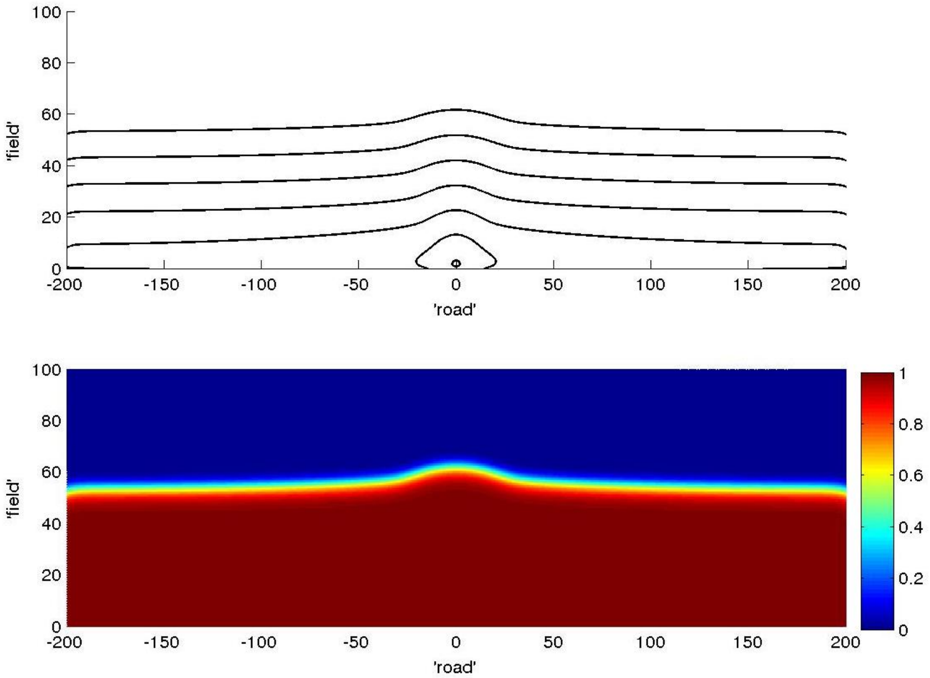


FIG. 4.3. Level sets of v with fractional diffusion on the road.

4.3 Higher dimension and different geometries ([2, 14])

The results of [27, 26, 17] show how a structure of dimension 1 (the "road") can affect in a crucial fashion the overall dynamics of the propagation process in the plane (the "field"). From a mathematical point of view, it is natural to wonder if this property extends to higher dimensions with different geometries, that is, if the fast diffusion is localised on a set of codimension 1. In the joint work [14] with A. Tellini and E. Valdinoci, we consider the system

$$\begin{cases} \partial_t u - D\Delta_{\partial\Omega} u = \nu v - \mu u & t > 0,\ x \in \partial\Omega, \\ \partial_t v - d\Delta v = f(v) & t > 0,\ x \in \Omega, \\ d\partial_n v = \mu u - \nu v & t > 0,\ x \in \partial\Omega. \end{cases}$$

where Ω is the cylinder $\mathbb{R} \times B_R \subset \mathbb{R}^{N+1}$, $\Delta_{\partial\Omega}$ stands for the Laplace-Beltrami operator on $\partial\Omega$ and ∂_n is the exterior normal derivative at Ω. The case of dimension $N+1 = 2$ describes an environment bounded by two parallel roads, which is therefore particularly interesting. Exploiting some fine properties of Bessel's functions, we derive the existence of the asymptotic speed of spreading w in the direction of the axis of the cylinder. We further investigate its dependence with respect to the diffusion coefficient D on $\partial\Omega$ and to the radius R of the cylinder.

Theorem 4.5 ([14])**.** *The asymptotic speed of spreading satisfies, as a function of D,*

- *$D \mapsto w(D)$ is strictly increasing and $w(0^+) = w_0 > 0$, $w(D) = O(\sqrt{D})$ as $D \to +\infty$;*

and, as a function of R,

- *if $D \le 2d$, $R \mapsto w(R)$ is increasing ;*
- *if $D > 2d$, there exists $R^* > 0$ such as $R \mapsto w(R)$ is increasing for $R < R^*$ and decreasing for $R > R^*$, and there holds*

$$w(R^*) = \sqrt{\frac{D^2}{D-d} f'(0)}.$$

The dependency of the speed of spreading in terms of the geometry of the domain is intriguing : when D/d exceeds the threshold 2 (which is the same as in [27]), the speed ceases to be a monotonous function of the radius R and a critical radius R^* appears, which maximises it. This phenomenon is a consequence of the balance between two opposite effects : the smaller the radius of the cylinder, the lesser the distance of interior points from the fast diffusion boundary, on one hand, but the smaller the region where the population can proliferate, on the other. One might expect that such a lack of monotonicity should not occur if one incorporates the reaction term on the boundary too. Finally, as $R \to +\infty$, we recover the speed of spreading of [27] in the half-space.

In the recent work [2] in collaboration with H. Berestycki and A. Tellini, we consider a reaction-diffusion system for two densities lying in adjacent domains of $\mathbb{R}^N$. As in [14], we consider the case where one domain is the cylinder $\Omega = \mathbb{R} \times B_R$ and the other domain is its complement. Diffusion and reaction terms for the two densities are considered, and an exchange occurs through the separating boundary. The system reads

$$\begin{cases} u_t - D\Delta u = g(u) & t > 0,\ x \in \Omega \\ v_t - d\Delta v = f(v) & t > 0,\ x \in \mathbb{R}^N \setminus \overline{\Omega} \\ D\,\partial_n u = \nu v - \mu u & t > 0,\ x \in \partial\Omega \\ -d\,\partial_n v = \mu u - \nu v & t > 0,\ x \in \partial\Omega. \end{cases}$$

With respect to the model in [27], the above system with $N = 2$ represents the case in which the road of fast diffusion is replaced by a *strip*. Hence, the former can be viewed as a *singular limit* of the latter. The purpose of [2] is to study whether the properties of the road-field model hold true with a thick region, and what is the behaviour for higher spatial dimensions. We are able to prove the existence of an asymptotic speed of propagation w up to dimension $N = 5$. We then analyse when an enhancement with respect to the homogeneous case takes place, as well as the behavior of w as the diffusion D and the radius of the cylinder vary. Some of our results are summarised in the following.

Theorem 4.6 ([2]). *Let $2 \le N \le 5$. The asymptotic speed of spreading w exists and it is greater than or equal to the Fisher-KPP speed in $\mathbb{R}^N \setminus \Omega$, which is $c_f^* := 2\sqrt{df'(0)}$. Moreover, we have :*

- $$w > c_f^* \quad \iff \quad \frac{D}{d} > 2 - \frac{g'(0)}{f'(0)} + C_N,$$
  where $C_N = 0$ *if* $N = 2, 3$ *and* $C_N > 0$ *if* $N = 4, 5$ *;*
- *as a function of* R, w *is strictly increasing whenever* $w > c_f^*$ *and satisfies*

$$w(R) \xrightarrow[R\to 0]{} c_f^*, \qquad w(R) \xrightarrow[R\to+\infty]{} \begin{cases} c_f^* & \text{if } \frac{D}{d} \leq 2 - \frac{g'(0)}{f'(0)} \\ c_g^* & \text{if } \frac{d}{D} \leq 2 - \frac{f'(0)}{g'(0)} \\ c_a^* & \text{otherwise,} \end{cases}$$

where $c_g^* := 2\sqrt{Dg'(0)}$ *denotes the Fisher-KPP speed in* Ω, *while*

$$c_a^* := \frac{|Df'(0) - dg'(0)|}{\sqrt{(D-d)(f'(0) - g'(0))}}.$$

The above notation c_a^* stands for **anomalous speed**. Such a name is used in the context of cooperative systems of equations set on the same domain ; it refers to a speed which is greater than both of the Fisher-KPP speeds for the densities considered separately. Indeed, it turns out that our limit of w as $R \to +\infty$ coincides with the asymptotic speed of spreading for another system, recently studied by A. Morris, L. Börger and E. Crooks [MBC19], describing the evolution of two densities that represent two parts of a population with different phenotypes. The mutation between the two densities plays the same role as the exchange condition in our system. However, a crucial difference is that the two densities share the same environment and that mutation occurs at every point, whereas in our case the exchange is concentrated on the interface between the two domains. For this reason, the fact the asymptotic speeds for the two models coincide is far from being obvious.

4.4 An ecological niche facing a climate change ([4, 6])

In all the models presented in the previous sections of this chapter, the environment outside the road is homogeneous, that is, it does not change from a place to another. This “homogeneity” hypothesis does not hold in several situations. For instance, the spreading of invasive species can occur only in regions that are favourable enough. This is the case of the tiger mosquito : it is believed that cold temperatures are responsible for stopping its northward progression in North America. This means that its **ecological niche** is limited by the climate conditions. From a biological perspective, the niche can be characterised by a temperature range, or by a localisation of resources. An important feature of an ecological niche is that it can evolve over time. For instance, **global warming** raising the temperature in the territory occupied by the tiger mosquito, leads to a northward displacement of its ecological niche and eventually entails the further spreading of the mosquito into places that were previously inaccessible, see the account by I. Rochlin, D. V. Ninivaggi, M. L. Hutchinson and A. Farajollahi [RNHF13]. A model which describes the dynamics of a population facing a climate change has been introduced by H. Berestycki, O. Diekmann, C. J. Nagelkerke and P. A. Zegeling in [BDNZ09]. The authors show that if the change does not occur too quickly, then the population manages to persist by tracking the favourable zone. On the contrary, if the speed of the climate change is larger than a certain value, then the population will eventually go extinct. In the papers [31, 29] in collaboration with H. Berestycki, we have extended the results of [BDNZ09] to higher dimension as well as to different geometries and periodic temporal varying media.

In [4], we introduce and study a model of population dynamics which takes into account the two phenomena presented above :

1. the presence of a line with fast diffusion (the road) ;
2. an ecological niche, possibly moving in time, as a consequence for instance of a climate change.

This phenomena are in some sense in competition : the road enhances the diffusion of the species, while the ecological niche confines its spreading. Two questions naturally arise.

Question 1. Does the presence of the road help or, on the contrary, inhibit the persistence of a species living in an ecological niche ?

Question 2. Does the picture change if the niche is moving ?

We consider the same system as in [27], but with a nonlinear term f which now depends on space and time :

$$\begin{cases} \partial_t u - D\partial_{xx}u = \nu v(x,0,t) - \mu u & t>0,\ x\in\mathbb{R} \\ \partial_t v - d\Delta v = f(x-ct,y,v) & t>0,\ (x,y)\in\mathbb{R}\times\mathbb{R}_+ \\ -d\partial_y v(x,0,t) = \mu u(x,t) - \nu v(x,0,t) & t>0,\ x\in\mathbb{R}. \end{cases}$$

The ecological niche is characterised by $f > 0$; we assume that it is bounded. The temporal dependence through the shift $-ct$ means that the niche is moving with constant speed c. We then study the *extinction* (uniform convergence to 0) or *persistence* (its negation) for this system, that we refer to as the model "with the road", in comparison with the one "without the road". We first consider the case $c = 0$ that is, the niche is not moving (there is no climate change).

Theorem 4.7 ([4])**.** *Assume that* $c = 0$*.*

(*i*) *If extinction occurs for the model "without the road", then extinction occurs for the one "with the road" as well.*

(*ii*) *There exist some terms* d, D, μ, ν, f *for which persistence occurs for the model "without the road", but extinction occurs for the one "with the road".*

This theorem answers Question 1. Indeed, statement (*i*) shows that the presence of the road can never entail the persistence of a population which would be doomed to extinction without the road. In other words, the road never improves the chances of survival of a population living in an ecological niche. Statement (*ii*) asserts that the road can actually make things worse : there are situations where the population would persist in an ecological niche, but the introduction of a road drives it to extinction. This is due to the "leakage" effect that the road causes to the population.

We then turn to Question 2, that is, we consider the case $c > 0$ corresponding to a moving niche due to a climate change. We start with analysing the influence of c on the survival of the species for the system "with the road".

Theorem 4.8 ([4]). *There exist* $0 \leq c_\star \leq c^\star \leq 2\sqrt{\max\{d, D\}[\sup \partial_v f|_{v=0}]^+}$, *such that the following hold :*

- *Persistence occurs if* $0 \leq c < c_\star$ *;*
- *Extinction occurs if* $c \geq c^\star$.

Moreover, if persistence occurs for $c = 0$ *then* $c_\star > 0$.

We do not know whether the thresholds $c_\star$, $c^\star$ actually always coincide. This is left as an open question. The proof of Theorem 4.8 relies on the analysis of the properties of the generalised principal eigenvalue that we have introduced for this class of systems in our previous paper [6]. While it was somehow hidden in the previous works concerning the homogeneous model, this notion turns out to be essential to handle heterogeneous road-field systems, as already pointed out by T. Giletti, L. Monsaingeon and M. Zhou [GMZ15] in a simpler case.

Theorem 4.9 ([4]). *Assume that* $D > 2d$. *There exist* f *and* $0 < c_1 < c_2$ *such that :*

- *If* $c \in [0, c_1)$, *persistence occurs for both models "without the road" and "with the road".*
- *If* $c \in [c_1, c_2)$, *extinction occurs for the model "without the road", but persistence occurs for the one "with the road".*

This result answers Question 2. Indeed, it shows that there are cases where the road can help the population to survive faster climate change than it would if there were no road. We remark that the threshold $D > 2d$ on the diffusion coefficients is the same as in [38] for the road to induce an enhancement of the asymptotic speed of spreading.

Chapitre 5

The dynamics of social unrest

We are currently working on a class of models concerning the question, in social sciences, of understanding the dynamics of social unrest, such as rioting activities or civil disobedience. Our approach is inspired by the works of H. Berestycki, J.-P. Nadal and N. Rodíguez [BN10, BNR15]. We do not aim to discuss the sociological origins of social unrest. Instead, we propose a model built on simple features to account for recurrent patterns observed in real life. The starting point is the observation that the mutual influence of neighbouring communities can lead to "waves" of manifestations of discontent that spread geographically. To model this, we assume that there is an implicit field of so called **social tension** which fuels the **rioting activity**. The social tension accounts for the resentment of a population towards society, would it be for political, economic, or sociological reasons. It may be affected by several factors, of different natures, such as economy, level of education, political awareness, trust of the community in government, etc. The social tension and the rioting activity are assumed to follow coupled dynamics and to influence the surrounding people towards a diffusion mechanism.

We focus on two distinct phenomena :

1. The outburst of a social unrest as a response of one or more triggering events ;
2. The spatial propagation of a social unrest already present in a given location.

In the first scenario, the system is supposed to be in a relaxed state (no unrest and low social tension) until some exogenous event takes place ; this increases the level of social tension and may eventually lead to the burst of the unrest. In the second case, the rioting activity is already present, but it is localised in space, and we investigate whether and how it will propagate.

5.1 Outbursts triggered by exogenous events ([10])

Some striking examples of outburst of social unrest are the one occurred in the French suburbs in 2005 – triggered in Clichy-sous-Bois by the death of two young people prosecuted by the police, which was widespread after a few days throughout the country – the London riots in 2011 and the manifestations of civil disobedience that broke out in Ferguson, Missouri (USA) after a fatal shooting by a police officer.

A question that arises is : why do some riots spread while others remain localised ? A common feature of these riots is that they are triggered by a single event (episodes of brutality or police shootings, some political decisions such as the introduction of a new tax, etc.). However, the reaction to these events is not always the same. Not every shooting incident necessarily triggers a revolt, nor does any revolt turns into a revolution. For this to happen, the system must be "ripe", in the sense that social tension must be sufficiently high. When this is not the case, the event is followed by a prompt resumption of calm. This suggests, from the modelling point of view, that an intrinsic mechanism of relaxation occurs on social unrest in a context of low social tension (due to fatigue, police repression, incarceration, etc.). A relevant analogy is that of fires, which require an event with large enough energy to activate the flames, then a favourable environment for their propagation. One indeed talks about a "spark that triggers a revolt". This analogy supports a modelling through reaction-diffusion equations, that are classically invoked in combustion theory. The model designed in [BNR15] focuses on self-reinforcement and spatial propagation mechanisms. The authors first introduce a discrete model and then the following continuous version :

$$\begin{cases} \partial_t u - D\Delta u = r(v)f(u) - \kappa u \\ \partial_t v + \mathcal{L}v = \sum_n A_n \delta_{(t_n,x_n)} - \left(\frac{\theta}{(1+u)^p} - \eta\right) v. \end{cases} \tag{5.1}$$

The unknown function u represents the level of rioting activity, whereas v measures the social tension. The latter affects the dynamics of u through the term $r(v)f(u)$, where r is an approximation of the indicator function $\mathbb{1}_{[a,+\infty)}$ and f is a nonlinearity of the Fisher-KPP type. The value $a > 0$ is the critical threshold of social tension starting from which the mechanism of self-reinforcement takes place ; κ is the natural rate of decline of rioting activity. The exponent p in the other coupling term $-\theta v/(1+u)^p$ can be positive or negative, depending on the phenomenon one aims at describing :

$p > 0$: **tension enhancing** (the system is *cooperative*) ;

$p < 0$: **tension inhibiting** (the system is of *activator-inhibitor* type).

The operator $\mathcal{L}$ incorporates the diffusion of the social tension and it is typically given by the Laplace operator, but a non-local operator such as the fractional Laplacian could also be envisioned. The source term $\sum_n A_n \delta_{(t_n,x_n)}$, where δ indicates the Dirac function, accounts for the exogenous factors, which act as "shocks" occurring at times t_n and places x_n.

The authors of [BNR15] show that, for suitable values of the parameters, if there is a finite number of shocks then the riot tends to dissipate and disappear. They then observe through numerical simulations that different scenarios can occur in the presence of an infinite number of shocks. Some of these results are also derived analytically in the case where the spatial component is omitted, and thus the model reduces to a system of ordinary differential equations :

$$\begin{cases} \partial_t u = r(v)f(u) - \kappa u \\ \partial_t v = \sum_n A_n \delta_{t_n} - \left(\frac{\theta}{(1+u)^p} - \eta\right) v. \end{cases}$$

The analytical study of this "single site" model is pushed further in our joint paper [10] in collaboration with H. Berestycki and N. Rodríguez. We consider there the case of a sequence of identical shocks : $A\sum_n \delta_{nT}$. We start with the analysis of the existence of cycles, i.e., temporal-periodic solutions (u, v). Next, we study their attractiveness for the Cauchy problem. In the tension enhancing case $p > 0$, we can rely on the existing literature concerning monotone dynamical systems. However, according to the examples by E. N. Dancer and P. Hess [DH91], we know that the general theory does not guarantee the convergence of solutions towards cycles. We are able to prove this convergence for our specific system, and also to characterise whether the limit cycle $(\hat{u}, \hat{v})$ is *relaxed* (i.e., $\hat{u} \equiv 0$) or *excited* ($\hat{u} > 0$).

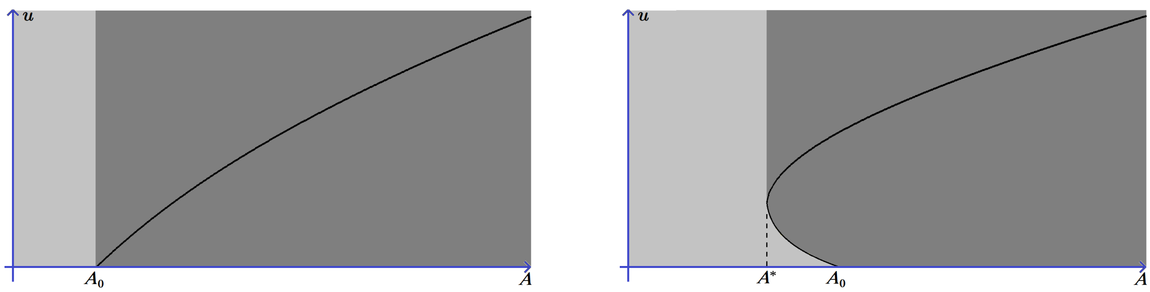
Theorem 5.1 ([10], Tension enhancing case)**.** *For any initial datum* (u_0, v_0)*, the solution approaches a periodic cycle* $(\hat{u}, \hat{v})$ *as* $t \to +\infty$*. Moreover, there exist* $0 < A^* \leq A_0$ *such that the following hold if* $u_0 > 0$ *:*

- *if* $A < A^*$ *then* $(\hat{u}, \hat{v})$ *is relaxed ;*
- *if* $A^* < A < A_0$ *then* $(\hat{u}, \hat{v})$ *is*

$$\begin{cases} \textit{relaxed} & \textit{if } v_0 < k \\ \textit{excited} & \textit{if } v_0 \geq k \end{cases}$$

  with k *depending on* u_0 *;*
- *if* $A > A_0$ *then* $(\hat{u}, \hat{v})$ *is excited.*

Depending on the other parameters, it may happen that $A^* = A_0$, in which case the system is Monostable ; if instead $A^* < A_0$ then it is (at least) Bistable, see the bifurcation diagrams below.



Monostable case $A^* = A_0$ *Bistable case* $A^* < A_0$

FIG. 5.1. Bifurcation diagrams in the tension enhancing case.

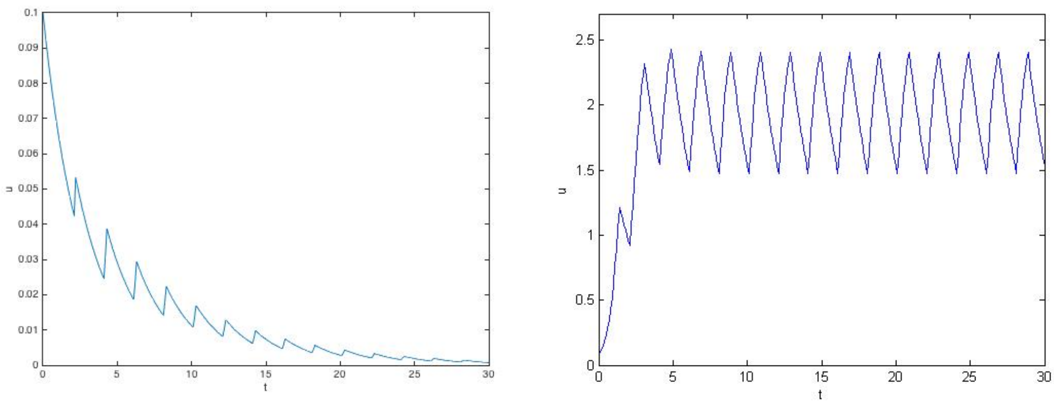
One of the key steps of our proof is represented by a property which is a reminiscence of the "southeast ordering" for 2×2 dynamical systems used by M. W. Hirsch and H. Smith [HS05].

The tension inhibiting case $p < 0$ is more difficult to treat and the result we obtain is not as complete. There are very few general stability results in the literature for systems of activator-inhibitor type, because they can exhibit in principle chaotic behaviour and strange attractors.

Theorem 5.2 ([10], Tension inhibiting case)**.** *There exists $A_0 > 0$ such that the following hold :*

- *if $A < A_0$ then any solution approaches the relaxed cycle as $t \to +\infty$;*
- *if $A > A_0$ then any solution with $u_0 > 0$ satisfies $\inf_{t>0} u > 0$; moreover there exists an excited cycle.*

The question of the convergence towards a cycle for any solution remains open. Numerical simulations seem to suggest it, see Figure 5.2.



$A < A_0$: convergence to the quiet cycle $\quad$ $A > A_0$: convergence to the excited cycle

FIG. 5.2. The graph of u in the tension inhibiting case, with initial condition $(u(0), v(0)) = (0.1, 0.1)$ and varying values of A.

We keep working on this type of systems. They can be envisioned for modelling other phenomena in which a variable shows self-excitement as soon as the other variable has reached a critical threshold.

5.2 Spreading of the unrest ([1])

Another important feature usually observed in movements of social unrest is the **geographical spreading**. This phenomenon can be interpreted as the result of both rioters' movement or the word to mouth communications, as well as the long-range influence through the media. In a work in progress in collaboration with H. Berestycki and S. Nordmann, we aim at developing a good mathematical framework to study such phenomenon. A key assumption we make is to discard the intrinsic dynamics of social tension in the absence of social unrest. In other words, we consider a period where no triggering exogenous events take place and we assume that, in a normal situation, the evolution of social tension occurs on a larger time scale than episodes of social unrest. We also suppose that some (possibly very small) rioting activity is initially in place, localised in space. These assumptions allow us to focus more clearly on the interplay between unrest and social tension and to analyse under which condition does an outbreak of revolt take place.

Even if social movements can take many different forms, a first naive classification would be to distinguish a “riot”, which lasts a couple of weeks and then fades, from a “revolution”, which lasts much longer and can result in significant political or sociological changes (think of the French Revolution, or the Arab Spring). A riot can be interpreted as a burst of social unrest which **decreases** social tension by letting people venting their anger. Once social tension falls below a threshold value, social unrest fades and eventually stops. This is the case referred to as **tension inhibiting** in the previous section. It is qualitatively comparable to the outburst of a disease, which propagates until the number of susceptible individuals falls below a certain threshold. This behaviour is well captured by the celebrate ***SI*** (Susceptible, Infected) epidemiology model of W.O Kermack and A.G McKendrick [KM27]. Instead, a revolution can be seen as a manifestation of social unrest which **increases** social tension. This dynamics of positive feedback escalates towards a sustainable state of high social unrest. This is the **tension enhancing**, which results from the mathematical point of view in a monotone, cooperative system.

Epidemiology models and monotone systems are studied quite separately in the literature, because they rely on different mathematical structures. Our goal is to propose a single framework that encompasses both.

We consider the following system of reaction-diffusion equations :

$$\begin{cases} \partial_t u = d_1 \Delta u + \Phi(u,v), & t>0,\ x\in\mathbb{R}^N \\ \partial_t v = d_2 \Delta v + \Psi(u,v), & t>0,\ x\in\mathbb{R}^N, \end{cases} \tag{5.2}$$

with initial conditions $u(0,x) = u_0(x) \gneqq 0$ compactly supported and $v(0,x) \equiv v_0$ positive constant, and with $d_1 > 0$, $d_2 \geq 0$. Our goals are :

- establish whether or not the rioting activity propagates ;
- in case this happens, determine its speed ;
- investigate travelling wave solutions.

Our main assumption is

$$\Psi(0,v) = 0 \quad \text{for all} \ \ v\in\mathbb{R},$$

that is, if $u \equiv 0$ then the dynamics of v consists of pure diffusion. Typical nonlinearities we have in mind are

$$\Phi(u,v) = r(v)u(1-u) - \gamma u,$$
$$\Psi(u,v) = f(u)v(1-v),$$

where $\gamma > 0$ and the function r is nondecreasing and nonnegative and f satisfies $f(0) = 0$. The tension inhibiting case corresponds to $f \leq 0$. In such a case, for any initial datum $v(0,x) \equiv v_0 \in (0,1)$ and $u(0,x) = u_0(x) \gneqq 0$, the solution satisfies $v \leq v_0$ for all times and therefore

$$\Phi(u,v) \leq \partial_u \Phi(0,v_0) u. \tag{5.3}$$

So, we refer to condition (5.3) as the *tension inhibiting* case. In this framework, we recover the *SI* system

$$\begin{cases} \partial_t I = \Delta I + \beta SI - \gamma I, \\ \partial_t S = -\beta SI, \end{cases}$$

with $\beta, \gamma > 0$ and the social tension playing the role of the Susceptible S and the rioting activity that of the Infected I. There is a vast literature on this model and its variants. Nevertheless, most of the adopted methods rely on the specific form of the system and can hardly be extended to more general problems. In this regard, our work can be seen as a new approach on the *SI* model which extends some classical results to a broader class of systems and allows one to model a wide variety of phenomena. We also mention that, even for the classical *SI* system, our approach provides some simpler proofs than those available in the literature. Above all, some of our results do not require hypothesis (5.3), that is, they apply to systems which are not necessarily tension inhibiting.

It turns out that the dynamics of system (5.2) is governed by the sign of the following quantity :

$$K_0 := \partial_u \Phi(0, v_0).$$

Theorem 5.3 ([1]). *Assume that* (5.3) *holds.*

- *If* $K_0 < 0$ *then*

$$\lim_{t\to+\infty} \left(\sup_{x\in\mathbb{R}^N} |(u(t,x), v(t,x)) - (0, v_0)| \right) = 0.$$

- *If* $K_0 > 0$ *then, calling* $c^* := 2\sqrt{d_1 K_0}$*, we have :*

$$\forall c < c^*, \quad \limsup_{t\to+\infty} \left(\sup_{|x|\geq ct} |(u(t,x), v(t,x)) - (0, v_0)| \right) > 0,$$

$$\forall c > c^*, \quad \lim_{t\to+\infty} \left(\sup_{|x|\geq ct} |(u(t,x), v(t,x)) - (0, v_0)| \right) = 0.$$

The condition $K_0 > 0$ is analogous to the condition $S_0 > \frac{\gamma}{\beta}$ in the *SI* model. If $K_0 > 0$ then the system enjoys the **hair-trigger** effect, with an asymptotic speed of spreading equal to c^*. If we drop the assumption (5.3), the dichotomy of Theorem 5.3 holds true, but up to now we are not able to identify an asymptotic speed of spreading ; we only get some upper and lower estimates. Loosely speaking, u behaves like a solution of a single equation of Fisher-KPP type if (5.3) holds, or of general Monostable type if (5.3) does not hold.

Next, we study travelling front solutions, i.e., solutions of the form

$$\big(U(x\cdot e - ct), V(x\cdot e - ct)\big),$$

satisfying the condition

$$\big(U(+\infty), V(+\infty)\big) = (0, v_0),$$

but without any prescribed condition at $-\infty$.

Theorem 5.4 ([1]). *Assume that* (5.3) *holds.*

- *If* $K_0 < 0$ *then there exists no transition wave.*
- *If* $K_0 > 0$ *then there exists no transition wave with speed* $c < c^*$ *and there exists a transition wave for any speed* $c > c^*$.

The speed c^* is the same as in Theorem 5.3. Again, in the general case where (5.3) is dropped, we do not derive a sharp critical threshold speed for the existence of fronts, but just some lower and upper bounds. In the tension inhibiting case, the fronts satisfy $U(-\infty) = 0$, $V(-\infty) < v_0$, exactly as for the *SI* system, which means that U is a "bump". Instead, if the system is tension enhancing, then $U(-\infty) > 0$, $V(-\infty) > v_0$ and the front is componentwise decreasing. The analysis of the value of $V(-\infty)$ is an interesting question that we are currently investigating.

List of publications

Appearing in the memoir

[1] H. Berestycki, S. Nordmann, and L. Rossi. Generalized epidemiology models for the dynamics of social unrest. *Preprint*, 2019.

[2] H. Berestycki, L. Rossi, and A. Tellini. Coupled reaction-diffusion equations on adjacent domains. *Preprint*, 2019.

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