SYMMETRIZATION AND ANTI-SYMMETRIZATION IN PARABOLIC EQUATIONS

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Abstract. We derive some symmetrization and anti-symmetrization properties of parabolic equations. First, we deduce from a result by Jones [6] a quantitative estimate of how far the level sets of solutions are from being spherical. Next, using this property, we derive a criterion providing solutions whose level sets do not converge to spheres for a class of equations including linear equations and Fisher-KPP reaction-diffusion equations.

1. Introduction

We are concerned with the spherical symmetrization feature of the equation

$$\partial_t u = \Delta u + f(u), \quad t > 0,\ x \in \mathbb{R}^N. \tag{1.1}$$

This is a semilinear reaction-diffusion equation, but we do not exclude the case where f is linear; our results are new even in such case. A result by Jones [6] asserts that solutions emerging from compactly supported initial data look more and more spherical as t increases, in the following sense: the normal to the level sets at nonsingular points always intersect the convex hull of the support of the datum and therefore, if the level sets go to infinity, their normal approaches the radial direction. This result is derived for reaction-diffusion equations of bistable type, but the very elegant proof, based on a reflection argument, actually applies to much more general equations, also time-dependent. Using some geometrical arguments, we will show that Jones' result implies more than the convergence of the normal to the radial direction: it provides an explicit estimate of the distance between the upper level sets and suitable balls. Namely, if $\operatorname{supp} u_0 \subset B_\delta$ then the upper level set

$$\mathcal{U}_\theta(t) := \{x \in \mathbb{R}^N \,:\, u(t,x) > \theta\} \tag{1.2}$$

satisfies

$$B_{r(t)} \subset \mathcal{U}_\theta(t) \subset B_{r(t)+\delta\pi}, \tag{1.3}$$

for some function r and for t sufficiently large. The precise statement is given in Theorem 1.2 below. This means that the Hausdorff distance between the upper level set $\mathcal{U}_\theta(t)$ and a suitable ball is bounded by the constant $\delta\pi/2$ for large t. In the case where f is of KPP-type and time-independent, this property has already been obtained by Ducrot [3], with in addition the explicit expression for $r(t)$, but with a generic constant instead of the precise value $\delta\pi/2$.

Then, the question that naturally arises is whether property (1.3) is sharp or not, that is, does the difference between the radii of the internal and external balls tends to 0 as $t \to +\infty$? Of course, for this question to make sense we have to consider all possible balls, not just the ones centred at the origin. This leads us to define

$$\mathcal{R}^i_\theta(t) := \sup\{r > 0 \,:\, \exists x_0 \in \mathbb{R}^N,\ u(t,x) > \theta \text{ for all } x \in B_r(x_0)\}, \tag{1.4}$$

$$\mathcal{R}^e_\theta(t) := \inf\{r > 0 \,:\, \exists x_0 \in \mathbb{R}^N,\ u(t,x) \le \theta \text{ for all } x \in (B_r(x_0))^c\}, \tag{1.5}$$

which are the radii respectively of the largest ball contained in $\mathcal{U}_\theta(t)$ and of the smallest ball containing $\mathcal{U}_\theta(t)$. Another notion of symmetrization is the convergence to a radial function. We consider both.

Definition 1.1. We say that a function $u : [0,+\infty) \times \mathbb{R}^N \to \mathbb{R}$ is *asymptotically spherical* if it fulfils one of the following properties:

(i)
$$\forall \theta \in (0,1), \quad \lim_{t\to+\infty} \big(\mathcal{R}^e_\theta(t) - \mathcal{R}^i_\theta(t)\big) = 0;$$

(ii) there exist two functions $\phi : [0,+\infty)^2 \to \mathbb{R}$, $\Gamma : [0,+\infty) \to \mathbb{R}^N$ such that
$$\lim_{t\to+\infty} \big(u(t,x) - \phi(t, |x - \Gamma(t)|)\big) = 0.$$

The symmetrization properties (i) and (ii) are not related in general, unless the function $r \mapsto \phi(t,r)$ has a strictly monotonic character.

Two counter-examples to the spherical symmetrization are known in the literature for reaction-diffusion equations of bistable type: Yagisita [10] and Roussier [7], the latter in dimension 2. The common idea there is to construct a solution which looks like a planar front when followed along a given direction, shifted by different values depending on the direction. This is possible due to the strong stability of the (unique up to shift) front. Hence, those examples have a rather specific form. The question remains open in many relevant cases, such as, strikingly, the linear one.

In the present paper we focus on concave (in a weak sense) terms f, including the case of linear equations and Fisher-KPP equations, possibly time-dependent. One cannot proceed as in the bistable case because of the lack of strong stability of fronts. Using a different method, we build a large class of non-asymptotically spherical solutions. We find in particular that the set of initial data for which the solution is not asymptotically spherical is dense in the space of compactly supported continuous functions. Curiously, to achieve this we use our previous symmetrization result. We also need to derive some estimates of the distance between level sets of solutions - c.f. Theorems 3.1, 3.3 below - that we believe are of independent interest.

1.1. **Hypotheses and main results.** Consider the problem

$$\partial_t u = \Delta u + f(t,u), \quad t > 0,\ x \in \mathbb{R}^N. \tag{1.6}$$

We will always assume in the sequel that $f(t,z)$ is Hölder continuous in t and uniformly Lipschitz continuous in z, uniformly with respect to t, and satisfies

$$\forall t > 0, \quad f(t,0) = 0.$$

The initial datum will always be nonnegative and continuous and solutions will be classical and locally bounded in time. We are concerned with *invading* solutions, that is, solutions satisfying

$$\exists Z \in (0,+\infty], \quad \forall K \Subset \mathbb{R}^N, \quad \liminf_{t\to+\infty} \Big(\min_{x\in K} u(t,x)\Big) \geq Z. \tag{1.7}$$

If Z is finite, it can be viewed as the saturation level for the problem. Typically, $Z = 1$ in reaction-diffusion equations; if f satisfies the KPP hypothesis (1.9) below, it is known that any nontrivial solution is invading, but this may not be the case for other classes of reaction terms, c.f. [1]. In the case where f is linear and increasing in z, we have that all nontrivial solutions fulfil the invasion condition with $Z = +\infty$.

Our symmetrization result holds as soon as Jones' technique applies and thus it does not require any specific assumption on f.

Theorem 1.2. *Let u be a solution to* (1.6) *with initial datum supported in a ball B_δ satisfying the invasion property* (1.7). *Then, for $\theta \in (0, Z)$ and t large enough, the upper level set $\mathcal{U}_\theta(t)$ defined by* (1.2) *is star-shaped with respect to the origin and satisfies*

$$B_{r_\theta(t)} \subset \mathcal{U}_\theta(t) \subset B_{r_\theta(t)+\delta\pi}, \tag{1.8}$$

for some positive function r_θ. In particular,

$$0 \leq \mathcal{R}^e_\theta(t) - \mathcal{R}^i_\theta(t) \leq \delta\pi.$$

Next, we build non-asymptotically spherical solutions under the following positivity and weak concavity assumption on f:

$$\begin{cases} \exists Z > 0, \quad \forall z \in (0, Z), \quad \inf_{t>0} f(t, z) > 0, \\ z \mapsto \dfrac{f(t, z)}{z} \quad \text{is nonincreasing in } (0, +\infty), \text{ for all } t > 0. \end{cases} \tag{1.9}$$

This hypothesis holds in the linear case $f(t, z) = \zeta(t)z$ with $\inf \zeta > 0$, or when f is a reaction term of KPP-type, such as $z(1 - z)$.

Theorem 1.3. *Assume that f satisfies* (1.9). *Let u_1, u_2 be two nonnegative, not identically equal to* 0*, continuous functions with compact support. Then, for $|\xi|$ large enough, the solution to* (1.6) *with initial datum*

$$u_0(x) = u_1(x) + u_2(x + \xi)$$

is not asymptotically spherical.

This theorem roughly says that if the support of the initial datum has two components which are far apart enough then the solution is not asymptotically spherical. Thus, adding to a compactly supported initial datum u_1 a compactly supported perturbation u_2, as small as wanted but sufficiently far, will give rise to a solution which is not asymptotically spherical.

2. The symmetrization result

Proof of Theorem 1.2. We know from Jones [6] that, for t sufficiently large, if $u(t, x_0) = \theta$ and $\nabla u(t, x_0) \neq 0$ then the normal line to the upper level set $\mathcal{U}_\theta(t)$ through the point x_0 intersects the convex hull of the support of the initial datum u_0. This property is derived in [6] using a reflection argument (see also a simplified proof by Berestycki [2, Theorem 2.9]) inspired by Serrin [8] and Gidas, Ni, Nirenberg [5]. We repeat the first step of the argument, in order to see that u is radially decreasing outside $B_\delta \supset \operatorname{supp} u_0$. More precisely, we will show that

$$\forall x_0 \notin B_\delta, \quad x_0 \cdot \nabla u(t, x_0) < 0. \tag{2.1}$$

Take $x_0 \notin B_\delta$ and, calling $\mathcal{T}$ the reflection with respect to the hyperplane $\{x \in \mathbb{R}^N : x \cdot x_0 = |x_0|^2\}$, define

$$v(t, x) := u(t, \mathcal{T}(x)).$$

The function v satisfies the same equation as u, because the Laplace operator is invariant under reflection, together with the initial and boundary conditions

$$v(0, x) = 0 \quad \text{for } x \cdot x_0 < |x_0|^2, \qquad v(t, x) = u(t, x) \quad \text{for } t > 0,\ x \cdot x_0 = |x_0|^2.$$

It follows from the parabolic strong comparison principle that, for $t > 0$ and $x \cdot x_0 < |x_0|^2$, it holds $v(t,x) < u(t,x)$. Then, by Hopf's lemma,

$$\partial_{x_0} u(t,x_0) < \partial_{x_0} v(t,x_0).$$

From this, because $\partial_{x_0} v(t,x_0) = -\partial_{x_0} u(t,x_0)$, we get $\partial_{x_0} u(t,x_0) < 0$, that is (2.1).

Fix $\theta \in (0,Z)$. The invasion property (1.7) implies that $\mathcal{U}_\theta(t) \supset B_\delta$ for t larger than some t_0. Therefore, for $t > t_0$ and $e \in S^{N-1}$, the function $\rho \mapsto u(t,\rho e)$ is larger than θ for $\rho \in (0,\delta)$. Then, by (2.1), it is strictly decreasing for $\rho \geq \delta$ and moreover it tends to 0 at infinity because $u(t,x) \to 0$ as $|x| \to \infty$ by standard parabolic decay. Consequently, the set $\mathcal{U}_\theta(t)$ is star-shaped with respect to the origin for $t > t_0$.

Now, take $t > t_0$ large enough so that Jones' result applies. Pick two points $P, Q \in \partial\mathcal{U}_\theta(t)$. By considering the plane H through P, Q and the origin, we can reduce to the bidimensional case: ignoring the other $N-2$ directions, we write

$$H \cap \partial\mathcal{U}_\theta(t) = \{\varphi(\alpha)(\cos\alpha, \sin\alpha) \,:\, \alpha \in [0,2\pi)\},$$

for some positive function φ. The points P, Q are obtained for two angles α_P, α_Q. It is not restrictive to assume that $|\alpha_P - \alpha_Q| \leq \pi$. Since $B_\delta \subset \mathcal{U}_\theta(t)$, we know from (2.1) that φ is of class C^1. For $\alpha \in [0,2\pi)$, we set for short $\mathbf{x} = \varphi(\alpha)(\cos\alpha, \sin\alpha)$, $\mathbf{v} = \nabla u(\mathbf{x})$ and we compute

$$0 = \frac{d}{d\alpha} u(\varphi(\alpha)(\cos\alpha,\sin\alpha)) = \mathbf{v} \cdot (\varphi'(\alpha)\cos\alpha - \varphi(\alpha)\sin\alpha, \varphi'(\alpha)\sin\alpha + \varphi(\alpha)\cos\alpha). \tag{2.2}$$

Jones' result implies that the distance from the line $s \mapsto \mathbf{x} - s\mathbf{v}$ and the origin is less than $\tilde{\delta} := \max\{|x| \,:\, x \in \operatorname{supp} u_0\} < \delta$. Namely,

$$\mathbf{w} := \mathbf{x} - \frac{\mathbf{x}\cdot\mathbf{v}}{|\mathbf{v}|^2}\,\mathbf{v}$$

satisfies $|\mathbf{w}| \leq \tilde{\delta}$. In order to get an estimate on φ' we observe that, by (2.2),

$$\begin{aligned} 0 &= (\mathbf{x} - \mathbf{w}) \cdot (\varphi'(\alpha)\cos\alpha - \varphi(\alpha)\sin\alpha, \varphi'(\alpha)\sin\alpha + \varphi(\alpha)\cos\alpha) \\ &= \varphi(\alpha)\varphi'(\alpha) - \mathbf{w} \cdot (\varphi'(\alpha)\cos\alpha - \varphi(\alpha)\sin\alpha, \varphi'(\alpha)\sin\alpha + \varphi(\alpha)\cos\alpha), \end{aligned}$$

whence, because $|\mathbf{w}| \leq \tilde{\delta}$,

$$\big(\varphi(\alpha)\varphi'(\alpha)\big)^2 \leq \tilde{\delta}^2[(\varphi'(\alpha))^2 + (\varphi(\alpha))^2].$$

We eventually find that

$$(\varphi'(\alpha))^2 \leq \frac{\tilde{\delta}^2}{1 - \tilde{\delta}^2/(\varphi(\alpha))^2}.$$

Notice that, by the invasion condition, $\min\varphi \to +\infty$ as $t \to +\infty$. Hence, for t large enough, $|\varphi'(\alpha)| \leq \delta$ for all $\alpha \in [0,2\pi)$. Reverting to the points P, Q, this implies that $\big||P| - |Q|\big| \leq \pi\delta$. This concludes the proof of the theorem. □

3. Non-asymptotically spherical solutions

In order to make the construction of the counter-example of Theorem 1.3 as transparent as possible, we start with the particular instance where $u_1 \equiv u_2$.

We will need to control the distance between level sets of solutions. Because of its independent interest, we derive it under weaker assumptions than (1.9). Namely,

$$\begin{cases} \exists g \in C([0,+\infty)), \quad \forall t,z \geq 0,\ f(t,z) \geq g(z), \\ \exists Z > \theta_0 > 0, \quad g \leq 0 \ \text{ in } (0,\theta_0), \quad g > 0 \ \text{ in } (\theta_0, Z), \quad \displaystyle\int_0^Z g > 0. \end{cases} \tag{3.1}$$

Besides the case (1.9), this hypothesis is fulfilled when f is a reaction term of any of the classical types considered in the literature: monostable, combustion, bistable [1], but also in much wider cases where f has several zeroes. It is only required here to ensure invasion for solutions with large enough, compactly supported initial data.

Theorem 3.1. *Under the assumption* (3.1)*, let u be a solution of* (1.6) *with compactly supported initial datum for which the invasion property* (1.7) *holds with the same Z as in* (3.1)*. Then, for $\theta_0 < \theta' < \theta < Z$, the functions r_θ, $r_{\theta'}$ provided by Theorem 1.2 satisfy*

$$\liminf_{t\to+\infty} \big(r_{\theta'}(t) - r_\theta(t)\big) < +\infty.$$

Let us comment on this statement before giving the proof. Combined with (1.8), it implies that

$$\liminf_{t\to+\infty} \operatorname{dist}\big(\mathcal{U}_{\theta'}(t), \mathcal{U}_\theta(t)\big) < +\infty.$$

This means that the width of the interface $\{\theta' < u < \theta\}$ is bounded along a sequence of times, which can be viewed as a steepness property of the profile. We do not know whether the estimate holds true with lim inf replaced by sup. This is left as an open question. Let us point out that the interface $\{\theta' < u < \theta\}$ may not have bounded width if the initial datum does not decay sufficiently fast at infinity, see [9, Theorem 8.4]. Also, the restriction on the levels θ, θ' to belong to the same positivity region of f cannot be dropped, since we know from [4, Theorem 3.3] that, for multistable nonlinearities, different level sets can spread with different speeds [1].

Proof of Theorem 3.1. We first show that the variation of $r_\theta(t)$ with respect to t cannot tend to infinity, next we see that the variation with respect to θ is controlled by that with respect to t.

Step 1. Control of the variation with respect to t.
We claim that

$$\forall \theta \in (\theta_0, 1),\ T > 0, \quad \liminf_{t\to+\infty} \big(r_\theta(t+T) - r_\theta(t)\big) < +\infty. \tag{3.2}$$

The function u is a subsolution of the linear equation

$$\partial_t u = \Delta u + \zeta u, \quad t > 0,\ x \in \mathbb{R}^N,$$

with time-independent zero-order term $\zeta := \sup_{t,z>0} f(t,z)/z$. It is straightforward to check (using for instance the heat kernel) that such equation admits a finite speed

[1] This is proved for front-like initial data, but then the same phenomenon is expected to occur for invading solutions with compactly supported initial data.

of spreading c^*, which can be actually computed: $c^* = 2\sqrt{\zeta}$. Namely, being u a subsolution with compactly supported initial datum, there holds

$$\forall c > c^*, \quad \lim_{t\to+\infty} \sup_{|x|\geq ct} u(t,x) = 0.$$

Since $u(t,x) > \theta$ if $|x| < r_\theta(t)$, this implies that, for any $c > c^*$, $r_\theta(t) < ct$ for t sufficiently large. Suppose now that (3.2) does not hold. Then there exist $\theta \in (\theta_0, 1)$ and $T, \tau > 0$ such that

$$\forall t \geq \tau, \quad r_\theta(t+T) - r_\theta(t) > (c^*+1)T,$$

which, applied recursively yields

$$\liminf_{n\to\infty} \frac{r_\theta(\tau + nT)}{nT} \geq c^* + 1.$$

This contradicts the fact that $r_\theta(t) < (c^* + 1/2)t$ for large t.

Step 2. Control of the variation with respect to θ.
Let θ_0, Z, g be from (3.1) and take $\theta_0 < \theta' < \theta < Z$. It is clear that, up to perturbing the function g in $(0, \theta_0)$ and then in a neighbourhood of 0 and Z, it is not restrictive to assume that $g < 0$ in $(0, \theta_0)$ and $g(0) = g(Z) = 0$, still preserving property (3.1). Then, it is known that the invasion occurs for solutions to the equation

$$\partial_s v = \Delta v + g(v), \quad s > 0, \ x \in \mathbb{R}^N, \tag{3.3}$$

with large initial data. Namely, by [1, Remark 6.5] there exists $R > 0$ such that the solution v with initial datum $v_0(x) = \theta' \mathbb{1}_{B_R}(x)$ converges locally uniformly to 1 as $s \to +\infty$. In particular, $v(T, 0) > \theta$ for some $T > 0$. Since u invades, there exists $\tau > 0$ such that $r_{\theta'}(t) > R$ for $t \geq \tau$. Take $t \geq \tau$, $\rho \in (R, r_{\theta'}(t))$ and $\xi \in B_{\rho-R}$. The inequality $u(t, \xi + x) \geq v_0(x)$ holds for all $x \in \mathbb{R}^N$, because if $|x| < R$ then $|\xi + x| < \rho < r_{\theta'}(t)$ and hence $u(t, \xi + x) > \theta' \geq v_0(x)$, whereas $v_0(x) = 0 \leq u(t, \xi + x)$ if $|x| \geq R$. Therefore, since any space/time translation of u is a supersolution to (3.3), the parabolic comparison principle yields

$$\forall s \geq 0, \ x \in \mathbb{R}^N, \quad u(t+s, \xi + x) \geq v(s, x).$$

Computed at $s = T$, $x = 0$, this inequality gives $u(t+T, \xi) \geq v(T, 0) > \theta$. This means that $\xi \in \mathcal{U}_\theta(t+T)$, which by (1.8) is contained in $B_{r_\theta(t+T)+\delta\pi}$, where B_δ contains the support of the initial datum of u. Thus, by the arbitrariness of $\xi \in B_{\rho-R}$ and $\rho \in (R, r_{\theta'}(t))$, we deduce

$$\forall t \geq \tau, \quad r_{\theta'}(t) - R \leq r_\theta(t+T) + \delta\pi. \tag{3.4}$$

Owing to (3.2), this concludes the proof of the lemma. □

Remark 1. In the case where f is independent of t, or more in general satisfies $f(\cdot + T, \cdot) \geq f$ for some $T > 0$, Theorem 3.1 can be improved to obtain a relation between the level sets of two distinct solutions, relaxing at the same time the restriction on the initial datum. Namely, if u^1, u^2 are invading solutions with initial data u_0^1, u_0^2 smaller than Z and decaying at most as $e^{-|x|^2}$, then for all $\theta, \theta' \in (\theta_0, Z)$, the functions r_θ^1, $r_{\theta'}^2$ associated with u^1, u^2 satisfy

$$\liminf_{t\to+\infty} \left| r_\theta^1(t) - r_{\theta'}^2(t) \right| < +\infty. \tag{3.5}$$

Indeed, by (1.7), at time nT with n large enough, both u^1 and u^2 are larger than the function v_0 used in the step 2 of the proof of Theorem 3.1. Thus, if f fulfils $f(\cdot + T, \cdot) \geq f$ then the solution u emerging from v_0 satisfies

$$\forall t \geq 0,\ x \in \mathbb{R}^N, \quad u(t,x) \leq \min\{u^1(t+nT,x), u^2(t+nT,x)\}.$$

On the other hand, u is invading and, by comparison with the heat equation, decays not faster than $e^{-\frac{|x|^2}{t}}$. As a consequence, at time mT with m large, it is greater than u_0^1, u_0^2 and therefore

$$\forall t \geq 0,\ x \in \mathbb{R}^N, \quad u(t+mT,x) \geq \max\{u^1(t,x), u^2(t,x)\}.$$

These estimates imply that the level sets of u^1 and u^2 are trapped between those of u, up to a time shift. One can then derive (3.5) applying Theorem 3.1 - more precisely properties (3.4) and (3.2) - to u.

Proposition 3.2. *The conclusion of Theorem 1.3 holds if* $u_1 \equiv u_2$.

Proof. First, under the standing assumptions, the existence of a unique classical bounded solution for the Cauchy problem associated with (1.6) follows from the standard parabolic theory. Furthermore, by comparison with the equation with reaction term $\inf_{t>0} f(t,z)$, we infer that under the KPP assumption (1.9) any nontrivial solution to (1.6) satisfies the invasion property (1.7) with Z given by (1.9), see [1].

Let w be the solution to (1.6) with initial datum u_1, and let $\tilde{\delta}$ be such that $\operatorname{supp} u_1 \subset B_{\tilde{\delta}}$. Call $(\tilde{r}_\theta)_{\theta\in(0,Z)}$ the family of functions r_θ provided by Theorem 1.2 for the solution w. We know from the parabolic strong comparison principle that $u(t,x) > w(t,x)$ for all $t>0$, $x \in \mathbb{R}^N$. Thus, for given $\theta \in (0,Z)$ and t large enough so that $\tilde{r}_\theta(t)$ is defined, we have that

$$\min_{|x|\leq \tilde{r}_\theta(t)} u(t,x) > \min_{|x|\leq \tilde{r}_\theta(t)} w(t,x) \geq \theta,$$

whence, in particular, $u(t, \tilde{r}_\theta(t)\xi/|\xi|) > \theta$. Comparing now u with $w(t,x+\xi)$ we get

$$\min_{|x+\xi|\leq \tilde{r}_\theta(t)} u(t,x) > \min_{|x+\xi|\leq \tilde{r}_\theta(t)} w(t,x+\xi) \geq \theta,$$

and then $u(t, -\xi - \tilde{r}_\theta(t)\xi/|\xi|) > \theta$. It follows that the diameter of the upper level set $\mathcal{U}_\theta(t)$ is at least $|\xi| + 2\tilde{r}_\theta(t)$, which means that

$$\mathcal{R}^e_\theta(t) \geq \tilde{r}_\theta(t) + \frac{|\xi|}{2}. \tag{3.6}$$

Next, we derive an upper bound for $\mathcal{R}^i_\theta(t)$ by considering the expansion of the level set in a direction $\eta \in S^{N-1}$ orthogonal to ξ. It is just here that we use the concavity hypothesis (1.9): it implies that the sum of supersolutions is a supersolution, because

$$\forall\, 0 < \alpha \leq \beta, \quad f(t,\alpha+\beta) \leq \frac{f(t,\beta)}{\beta}(\alpha+\beta) = \frac{f(t,\beta)}{\beta}\alpha + f(t,\beta) \leq f(t,\alpha) + f(t,\beta). \tag{3.7}$$

Hence, the function $w(t,x) + w(t,x+\xi)$ is a supersolution of (1.6), coinciding with u at $t=0$, and then, by comparison, $u(t,x) \leq w(t,x) + w(t,x+\xi)$ for $t \geq 0$, $x \in \mathbb{R}^N$. We deduce that if $x\cdot\eta \geq \tilde{r}_{\theta/2}(t) + \tilde{\delta}\pi$ then $|x|, |x+\xi| \geq \tilde{r}_{\theta/2}(t) + \tilde{\delta}\pi$ and thus

$$u(t,x) \leq w(t,x) + w(t,x+\xi) \leq \theta.$$

This means that the width of the set $\mathcal{U}_\theta(t)$ is at most $2\tilde{r}_{\theta/2}(t) + 2\tilde{\delta}\pi$, whence

$$\mathcal{R}^i_\theta(t) \leq \tilde{r}_{\theta/2}(t) + \tilde{\delta}\pi. \tag{3.8}$$

Gathering together (3.6) and (3.8) we eventually obtain

$$\liminf_{t\to+\infty}\big(\mathcal{R}^i_\theta(t)-\mathcal{R}^e_\theta(t)\big)\leq\liminf_{t\to+\infty}\big(\tilde r_{\theta/2}(t)-\tilde r_\theta(t)\big)+\tilde\delta\pi-\frac{|\xi|}{2}.$$

Now, we know from Theorem 3.1 that the first term of the right-hand side is finite (and independent of ξ) and therefore u does not fulfil condition (i) of Definition 1.1 provided $|\xi|$ is sufficiently large.

Finally, if we use (3.8) with θ replaced by any $\theta'\in(0,\theta)$, we get

$$\liminf_{t\to+\infty}\big(\mathcal{R}^i_{\theta'}(t)-\mathcal{R}^e_\theta(t)\big)<0,$$

for a possibly larger $|\xi|$. This contradicts the property (ii) of Definition 1.1, because the latter implies that $\mathcal{R}^i_{\theta'}(t)>\mathcal{R}^e_\theta(t)$ for t large enough. To see this, consider ϕ and Γ given by Definition 1.1 (ii). Take $\tilde\theta\in(\theta',\theta)$ and define

$$\rho(t):=\max\{r>0\,:\,\phi(t,r)=\tilde\theta\}.$$

This quantity is well defined for t large enough because $u(t,x)-\phi(t,|x-\Gamma(t)|)\to0$ as $t\to+\infty$ uniformly in x and u is invading. There holds that

$$\phi(t,r)\begin{cases}=\tilde\theta & \text{for } r=\rho(t)\\ <\tilde\theta & \text{for } r>\rho(t),\end{cases}$$

and therefore, for t sufficiently large,

$$\partial B_{\rho(t)}(\Gamma(t))\subset\mathcal{U}_{\theta'}(t),\qquad \mathcal{U}_\theta(t)\subset\overline{B}_{\rho(t)}(\Gamma(t)).$$

The second inclusion implies that $\mathcal{R}^e_\theta(t)\leq\rho(t)$. On the other hand, for t large enough, we have that $\mathcal{U}_{\theta'}(t)$ is star-shaped owing to Theorem 1.2 and thus the first inclusion yields $\overline{B}_{\rho(t)}(\Gamma(t))\subset\mathcal{U}_{\theta'}(t)$, whence $\mathcal{R}^i_{\theta'}(t)\geq\rho(t)>\mathcal{R}^e_\theta(t)$. □

3.1. **The general construction.** This section is devoted to the proof of Theorem 1.3. First, we need an improvement of Theorem 3.1 which allows one to compare the position of level sets of distinct solutions. We have seen in Remark 1 that this can be achieved with minor modification in the time-independent case. For the time-dependent equation (1.6), an alternative argument is required. We are able to perform it under the KPP hypothesis.

Theorem 3.3. *Assume that f satisfies* (1.9)*. Let u^1, u^2 be two solutions of* (1.6) *with compactly supported initial data $u^1_0,u^2_0\geq0,\not\equiv0$. Then, for all $\theta,\theta'\in(0,1)$, the functions r^1_θ, $r^2_{\theta'}$ given by Theorem 1.2 with $u=u_1$ and $u=u_2$ respectively, satisfy*

$$\liminf_{t\to+\infty}\big|r^1_\theta(t)-r^2_{\theta'}(t)\big|<+\infty.$$

Proof. We let τ_y denote the translation acting on a (possibly time independent) function u as $\tau_yu(t,x):=u(t,x-y)$. Take $\zeta\in\mathbb{R}^N$ such that $\underline{u}_0:=\min\{\tau_\zeta u^1_0,u^2_0\}$ is not identically equal to 0, and let $\underline{u}$ be the solution of (1.6) with initial datum $\underline{u}_0$. Then, let $\overline{u}$ be the solution of (1.6) with initial datum $\overline{u}_0:=\max\{u^1_0,u^2_0\}$. By comparison, we have that

$$\underline{u}\leq\tau_\zeta u^1,u^2,\qquad \overline{u}\geq u^1,u^2. \tag{3.9}$$

Set $\underline{\theta}:=\min\{\theta,\theta'\}$, $\overline{\theta}:=\max\{\theta,\theta'\}$ and let $\underline{r}_{\overline{\theta}},\overline{r}_{\underline{\theta}}$ denote the functions provided by Theorem 1.2 with $u=\underline{u}$ and $u=\overline{u}$ respectively (recall that invasion always occurs by (1.9)). Our goal is to bound $r^1_\theta,r^2_{\theta'}$ from below by $\underline{r}_{\overline{\theta}}$ (up to additive constants) and from above by $\overline{r}_{\underline{\theta}}$ and finally to control the difference between $\overline{r}_{\underline{\theta}}$ and $\underline{r}_{\overline{\theta}}$. Of course, the last step will be the most involved, and this is where we require the KPP hypothesis (1.9), together with Theorem 3.1.

Let t be large enough so that (1.8) applies for the various functions and values of θ in play. From the first inequality in (3.9) and (1.8) we infer from one hand that

$$|x| = r^2_{\theta'}(t) + \delta_2\pi \implies \underline{u}(t,x) \leq u^2(t,x) \leq \theta' \leq \overline{\theta} \implies |x| \geq \underline{r}_{\overline{\theta}}(t),$$

that is,

$$\underline{r}_{\overline{\theta}}(t) \leq r^2_{\theta'}(t) + \delta_2\pi. \tag{3.10}$$

From the other hand,

$$|x| = |\zeta| + r^1_{\theta}(t) + \delta_1\pi \implies \underline{u}(t,x) \leq \tau_\zeta u^1(t,x) \leq \theta \leq \overline{\theta} \implies |x| \geq \underline{r}_{\overline{\theta}}(t),$$

whence

$$\underline{r}_{\overline{\theta}}(t) \leq |\zeta| + r^1_{\theta}(t) + \delta_1\pi. \tag{3.11}$$

Similarly, the second inequality in (3.9) and (1.8) yield

$$r^1_{\theta}(t), r^2_{\theta'}(t) \leq \overline{r}_{\underline{\theta}}(t) + \overline{\delta}\pi, \tag{3.12}$$

where $\overline{\delta} := \max\{\delta_1, \delta_2\}$ (observe that $\operatorname{supp} \overline{u}_0 \subset B_{\overline{\delta}}$).

We now estimate $\overline{r}_{\underline{\theta}}$ in terms of $\underline{r}_{\underline{\theta}}$. Since $\underline{u}_0 \not\equiv 0$ and $\overline{u}_0$ is compactly supported, one can find a family of points $\{x_1, \dots, x_n\}$ such that

$$\overline{u}_0 \leq \sum_{j=1}^{n} \tau_{x_j} \underline{u}_0.$$

Recall that the KPP hypothesis yields (3.7), which, applied recursively, implies that the sum of supersolutions is a supersolution. Therefore, by comparison,

$$\overline{u} \leq \sum_{j=1}^{n} \tau_{x_j} \underline{u}$$

holds true for all $t > 0$. Then, with the same argument as before (notice that $\operatorname{supp} \underline{u}_0 \subset B_{\delta_2}$) we find that

$$|x| = \max_{j=1,\dots,n} |x_j| + \underline{r}_{\underline{\theta}/n}(t) + \delta_2\pi \implies \overline{u}(t,x) \leq \sum_{j=1}^{n} \tau_{x_j} \underline{u}(t,x) \leq \sum_{j=1}^{n} \underline{\theta}/n = \underline{\theta},$$

whence

$$\overline{r}_{\underline{\theta}}(t) \leq \max_{j=1,\dots,n} |x_j| + \underline{r}_{\underline{\theta}/n}(t) + \delta_2\pi.$$

Therefore, owing to Theorem 3.1,

$$\liminf_{t\to+\infty} \left(\overline{r}_{\underline{\theta}}(t) - \underline{r}_{\overline{\theta}}(t)\right) \leq \max_{j=1,\dots,n} |x_j| + \delta_2\pi + \liminf_{t\to+\infty} \left(\underline{r}_{\underline{\theta}/n}(t) - \underline{r}_{\overline{\theta}}(t)\right) < +\infty.$$

The proof is thereby concluded, because, by (3.10)-(3.12),

$$\left|r^1_{\theta} - r^2_{\theta'}\right| = \max\{r^1_{\theta}, r^2_{\theta'}\} - \min\{r^1_{\theta}, r^2_{\theta'}\} \leq \overline{r}_{\underline{\theta}} + \overline{\delta}\pi - \underline{r}_{\overline{\theta}} + \delta_2\pi + |\zeta| + \delta_1\pi. \qquad \square$$

Proof of Theorem 1.3. We want to adapt the arguments of the proof of Proposition 3.2. First, we consider a nonnegative, not identically equal to 0, continuous functions $\underline{w}_0$, with compact support, satisfying, for some $x_1, x_2 \in \mathbb{R}^N$,

$$\forall x \in \mathbb{R}^N, \quad \underline{w}_0(x) \leq u_1(x + x_1)\,,\, u_2(x + x_2).$$

Let $\underline{w}$ be the solution emerging from $\underline{w}_0$ and $\overline{w}$ be the one emerging from $\max\{u_1, u_2\}$, and call $(\underline{r}_\theta)_{\theta\in(0,1)}$, $(\overline{r}_\theta)_{\theta\in(0,1)}$ the families of functions provided by Theorem 1.2 associated with these solutions. The comparison principle yields, for $t > 0$ and $x \in \mathbb{R}^N$,

$$u(t,x) \leq \overline{w}(t,x) + \overline{w}(t,x+\xi), \qquad u(t,x) \geq \max\big\{\underline{w}(t,x-x_1)\,,\,\underline{w}(t,x-x_2+\xi)\big\}.$$

For $\theta \in (0,1)$, using the first inequality above together with the argument employed to derive (3.8) we find that $\mathcal{R}^i_\theta(t) \leq \overline{r}_{\theta/2}(t) + \overline{\delta}\pi$, while from the second inequality, computed at $x = x_1 + \underline{r}_\theta(t)\xi/|\xi|$ and $x = x_2 - \xi - \underline{r}_\theta(t)\xi/|\xi|$, we deduce

$$\mathcal{R}^e_\theta(t) \geq \underline{r}_\theta(t) + \frac{|\xi| - |x_1| - |x_2|}{2}.$$

Consequently,

$$\liminf_{t\to+\infty}\big(\mathcal{R}^i_\theta(t) - \mathcal{R}^e_\theta(t)\big) \leq \liminf_{t\to+\infty}\big(\overline{r}_{\theta/2}(t) - \underline{r}_\theta(t)\big) + \overline{\delta}\pi - \frac{|\xi| - |x_1| - |x_2|}{2}.$$

We now make use of Theorem 3.3. It entails that the first term of the right-hand side is finite and therefore u does not fulfil condition (i) of Definition 1.1 if $|\xi|$ is sufficiently large. Clearly, one can get the above inequality with $\mathcal{R}^i_\theta(t)$ and $\overline{r}_{\theta/2}$ replaced by $\mathcal{R}^i_{\theta'}(t)$ and $\overline{r}_{\theta'/2}$ respectively, for any $\theta' \in (0,\theta)$. Then, choosing $|\xi|$ very large one infers that condition (ii) of Definition 1.1 is violated as well because, as seen at the end of the proof of Proposition 3.2, it entails $\mathcal{R}^i_{\theta'}(t) > \mathcal{R}^e_\theta(t)$ for t large enough. □

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