Civil Wars: A New Lotka-Volterra Competitive System and Analysis of Winning Strategies*

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Abstract

We introduce a new model in population dynamics that describes two species sharing the same environmental resources in a situation of open hostility. The interactions among these populations is described not in terms of random encounters but via the strategic decisions of one population that can attack the other according to different levels of aggressiveness.

This leads to a non-variational model for the two populations at war, taking into account structural parameters such as the relative fit of the two populations with respect to the available resources and the effectiveness of the attack strikes of the aggressive population.

The analysis that we perform is rigorous and focuses on the dynamical properties of the system, by detecting and describing all the possible equilibria and their basins of attraction.

Moreover, we will analyze the strategies that may lead to the victory of the aggressive population, i.e. the choices of the aggressiveness parameter, in dependence of the structural constants of the system and possibly varying in time in order to optimize the efficacy of the attacks, which take to the extinction in finite time of the defensive population.

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The model that we present is flexible enough to include also technological competition models of aggressive companies releasing computer viruses to set a rival companies out of the market.

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1 Introduction

Among the several models dealing with the dynamics of biological systems, the case of populations engaging into a mutual conflict seems to be unexplored. This work aims at laying the foundations of a new model describing two populations competing for the same resource with one aggressive population which may attack the other: concretely, one may think of a situation in which two populations live together in the same territory and share the same environmental resources, till one population wants to prevail and try to kill the other. We consider this situation as a "civil war", since the two populations share land and resources; the two populations may be equally fit to the environment (and, in this sense, they are "indistinguishable", up to the aggressive attitude of one of the populations), or they can have a different compatibility to the resources (in which case one may think that the conflict could be motivated by the different accessibility to environmental resources).

Given the lack of reliable data related to civil wars, a foundation of a solid mathematical theory for this type of conflicts may only leverage on the deduction of the model from first principles: we follow this approach to obtain the description of the problem in terms of a system of two ordinary differential equations, each describing the evolution in time of the density of one of the two populations.

The method of analysis that we adopt is a combination of techniques from different fields, including ordinary differential equations, dynamical systems and optimal control.

This viewpoint will allow us to rigorously investigate the model, with a special focus on a number of mathematical features of concrete interest, such as the possible extinction of one of the two populations and the analysis of the strategies that lead to the victory of the aggressive population.

In particular, we will analyze the *dynamics of the system*, characterizing the equilibria and their features (including possible basins of attraction) in terms of the different parameters of the model (such as relative fitness to the environment, aggressiveness and effectiveness of strikes). Also, we will study the initial configurations which may lead to the victory of the aggressive population, also taking into account different possible *strategies* to achieve the victory: roughly speaking, we suppose that the aggressive population may adjust the parameter describing the aggressiveness in order to either dim or exacerbate the conflict with the aim of destroying the second population (of course, the war has a cost in terms of life for both the populations, hence the aggressive population must select the appropriate strategy in terms of the structural parameters of the system). We will show that the initial data allowing the victory of the aggressive population does not exhaust the all space, namely *there* *exists initial configurations for which the aggressive population cannot make the other extinct*, regardless the strategy adopted during the conflict.

Furthermore, *for identical populations with the same fit to the environment the constant strategies suffices* for the aggressive population to possibly achieve the victory: namely, if an initial configuration admits a piecewise continuous in time strategy that leads to the victory of the aggressive population, then it also admits a constant in time strategy that reaches the same objective (and of course, for the aggressive population, the possibility of focusing only on constant strategies would entail concrete practical advantages).

Conversely, *for populations with different fit to the environment, the constant strategies do not exhaust all the winning strategies*: that is, in this case, there are initial conditions which allow the victory of the aggressive population only under the exploitation of a strategy that is not constant in time.

In any case, we will also prove that *strategies with at most one jump discontinuity are sufficient* for the aggressive population: namely, independently from the relative fit to the environment, if an initial condition allows the aggressive population to reach the victory through a piecewise continuous in time strategy, then the same goal can be reached using a "bang-bang" strategy with at most one jump.

We will also discuss the *winning strategies that minimize the duration of the war*: in this case, we will show that jump discontinuous strategies may be not sufficient and interpolating arcs have to be taken into account.

We now describe in further detail our model of conflict between the two populations and the attack strategies pursued by the aggressive population. Our idea is to modify the Lotka-Volterra competitive system for two populations with density u and v, adding to the usual competition for resources the fact that both populations suffer some losses as an outcome of the attacks. The key point in our analysis is that the clashes do not depend on the chance of meeting of the two populations, given by the quantity uv, as it happens in many other works in the literature (starting from the publications of Lotka and Volterra, [11] and [25]), but they are sought by the first population and depend only on the size u of the first population and on its level of aggressiveness a. The resulting model is

$$\begin{cases} \dot{u} &= u(1-u-v) - acu, \quad \text{for } t > 0, \\ \dot{v} &= \rho v(1-u-v) - au, \quad \text{for } t > 0, \end{cases} \tag{1.1}$$

where a, c and ρ are nonnegative real numbers. Here, the coefficient ρ models the fitness of the second population with respect of the first one when resources are abundant for both; it is linked with the exponential growth rate of the two species. The parameter c here stands for the quotient of endured per inflicted damages for the first population. Deeper justifications to the model (1.1) will be given in Subsection 1.1.

Notice that the size of the second population v may become negative in finite time while the first population is still alive. The situation where $v = 0$ and $u > 0$ represents the extinction of the second population and the victory of the first one.

To describe our results, for communication convenience (and in spite of our personal fully pacifist believes) we take the perspective of the first population, that is, the aggressive one; the objective of this population is to win the war, and, to achieve that, it can influence the system by tuning the parameter a.

From now on, we may refer to the parameter a as the *strategy*, that may also depend on time, and we will say that it is *winning* if it leads to victory of the first population.

The main problems that we deal with in this paper are:

1. The characterization of the *initial conditions for which there exists a winning strategy.*

2. The *success of the constant strategies*, compared to all possible strategies.

3. The *construction of a winning strategy* for a given initial datum.

4. The *existence of a single winning strategy independently of the initial datum.*

We discuss all these topics in Subsection 1.4, presenting concrete answers to each of these problems.

Also, since to our knowledge this is the first time that system (1.1) is considered, in Subsections 1.2 and 1.3 we will discuss the dynamics and some interesting results about the dependence of the basins of attraction on the other parameters.

It would also be extremely interesting to add the space component to our model, by considering a system of reaction-diffusion equations. This will be the subject of a further work.

1.1 Motivations and derivation of the model

The classic Lotka-Volterra equations were first introduced for modelling population dynamics between animals [25] and then used to model other phenomena involving competition, for example in technology substitution [13]. The competitive Lotka-Volterra system concerns the sizes $u_1(t)$ and $u_2(t)$ of two species competing for the same resources. The system that the couple $(u_1(t), u_2(t))$ solves is

$$\begin{cases} \dot{u}_1 = r_1 u_1 \left(\sigma - \dfrac{u_1 + \alpha_{12} u_2}{k_1} \right), & t > 0, \\ \dot{u}_2 = r_2 u_2 \left(\sigma - \dfrac{u_2 + \alpha_{21} u_1}{k_2} \right), & t > 0, \end{cases} \tag{1.2}$$

where r_1, r_2, σ, α_{12}, α_{21}, k_1 and k_2 are nonnegative real numbers.

Here, the coefficients α_{12} and α_{21} represent the competition between individuals of different species, and indeed they appear multiplied by the term $u_1 u_2$, which represents a probability of meeting.

The coefficient r_i is the exponential growth rate of the $i-$th population, that is, the reproduction rate that is observed when the resources are abundant. The parameters k_i are called carrying capacity and represent the number of individuals of the $i-$th population that can be fed with the resources of the territory, that are quantified by σ. It is however usual to rescale the system in order to reduce the number of parameters. In general, u_1 and u_2 are rescaled so that they vary in the interval $[0, 1]$, thus describing densities of populations.

The behavior of the system depends substantially on the values of α_{12} and α_{21} with respect to the threshold given by the value 1, see e.g. [1]: if $\alpha_{12} < 1 < \alpha_{21}$, then the first species u_1 has an advantage over the second one u_2 and will eventually prevail; if α_{12} and α_{21} are both strictly above or below the threshold, then the first population that penetrates the environment (that is, the one that has a greater size at the initial time) will persist while the other will extinguish.

Some modification of the Lotka-Volterra model were made in stochastic analysis by adding a noise term of the form $-f(t)u_i$ in the $i-$th equation, finding some interesting phenomena of phase transition, see e.g. [9].

The ODE system in (1.2) is of course the cornerstone to study the case of two competitive populations that diffuse in space. Many different types of diffusion have been compared and one can find a huge literature on the topic, see [15, 4, 12] for some examples and [14] for a more general overview. We point out that other dynamic systems presenting finite time extinction of one or more species have been generalised for heterogeneous environments, see for example the model in [7] for the predator-prey behaviour of cats and birds, that has been thereafter widely studied.

In this paper, we will focus not only on *basic competition for resources*, but also on *situations of open hostility*. In social sciences, war models are in general little studied; indeed, the collection of data up to modern times is hard for the lack of reliable sources. Also, there is still much discussion about what factors are involved and how to quantify them: in general, the outcome of a war does not only depend on the availability of resources, but also on more subtle factors as the commitment of the population and the knowledge of the battlefield, see e.g. [21]. Instead, the causes of war were investigated by the statistician L.F. Richardson, who proposed some models for predicting the beginning of a conflict, see [18].

In addition to the human populations, behavior of hostility between groups of the same species has been observed in chimpanzee. Other species with complex social behaviors are able to coordinate attacks against groups of different species: ants versus termites, agouti versus snakes, small birds versus hawk and owls, see e.g. [23].

The model that we present here is clearly a simplification of reality. Nevertheless, we tried to capture some important features of conflicts between rational and strategic populations, introducing in the mathematical modeling the new idea that a conflict may be sought and the parameters that influence its development may be conveniently adjusted.

Specifically, in our model, the interactions between populations are not merely driven by chance and the strategic decisions of the population play a crucial role in the final outcome of the conflict, and we consider this perspective as an interesting novelty in the mathematical description of competitive environments.

At a technical level, our aim is to introduce a model for conflict between two populations u and v, starting from the model when the two populations compete for food and modifying it to add the information about the clashes. We imagine that each individual of the first population u decides to attack an individual of the second population with some probability a in a given period of time. We assume that hostilities take the form of "duels", that is, one-to-one fights. In each duel, the individual of the first population has a probability ζ_u of being killed and a probability ζ_v of killing his or her opponent; notice that in some duel the two fighters might be both killed. Thus, after one time-period, the casualties for the first and second populations are $a\zeta_u u$ and $a\zeta_v u$ respectively. The same conclusions are found if we imagine that the first population forms an army to attack the second, which tries to resist by recruting an army of proportional size. At the end of each battle, a ratio of the total soldiers is dead, and this is again of the form $a\zeta_u u$ for the first population and $a\zeta_v u$ for the second one.

Another effect that we take into account is the drop in the fertility of the population during wars. This seems due to the fact that families suffer some income loss during war time, because of a lowering of the average productivity and lacking salaries only partially compensated by the state; another reason possibly discouraging couples to have children is the increased chance of death of the parents during war. As pointed out in [24], in some cases the number of lost births during wars are comparable to the number of casualties. However, it is not reasonable to think that this information should be included in the exponential growth rates r_u and r_v, because the fertility drop really depends on the intensity of the war. For this reason, we introduce the parameters $c_u \geqslant 0$ and $c_v \geqslant 0$ that are to be multiplied by au for both populations.

Moreover, for simplicity, we also suppose that the clashes take place apart from inhabited zone, without having influence on the harvesting of resources.

Now we derive the system of equations from a microlocal analysis. As in the Lotka-Volterra model, it is assumed that the change of the size of the population in an interval of time Δt is proportional to the size of the population $u(t)$, that is

$$u(t + \Delta t) - u(t) \approx u(t) f(u, v)$$

for some appropriate function $f(u, v)$. In particular, $f(u, v)$ should depend on resources that are available and reachable for the population. The maximum number of individuals that can be fed with all the resources of the environment is k; taking into account all the individuals of the two populations, the available resources are

$$k - u - v.$$

Notice that we suppose here that each individual consumes the same amount of resources, independently of its belonging. In our model, this assumption is reasonable since all the individuals belong to the same species. Also, the competition for the resources depends only on the number of individuals, independently on their identity.

Furthermore, our model is sufficiently general to take into account the fact that the growth rate of the populations can be possibly different. In practice, this possible difference could be the outcome of a cultural distinction, or it may be also due to some slight genetic differentiation, as it happened for Homo Sapiens and Neanderthal, see [6].

Let us call r_u and r_v the fertility of the first and second populations respectively. The contribution to the population growth rate is given by

$$f(u, v) := r_u \left(1 - \frac{u + v}{k}\right),$$

and these effects can be comprised in a typical Lotka-Volterra system.

Instead, in our model, we also take into account the possible death rate due to casualties. In this way, we obtain a term such as $-a\zeta_u$ to be added to $f(u,v)$. The fertility losses give another term $-ac_u$ for the first population. We also perform the same analysis for the second population, with the appropriate coefficients.

With these considerations, the system of the equations that we obtain is

$$\begin{cases} \dot{u} &= r_u u \left(1 - \dfrac{u+v}{k}\right) - a(c_u + \zeta_u)u, \quad t > 0, \\ \dot{v} &= r_v v \left(1 - \dfrac{v+u}{k}\right) - a(c_v + \zeta_v)u, \quad t > 0. \end{cases} \tag{1.3}$$

As usual in these kinds of models, we can rescale the variables and the coefficients in order to find an equivalent model with fewer parameters. Hence, we perform the changes of variables

$$\begin{aligned} &\tilde{u}(\tilde{t}) = \frac{u(t)}{k}, \quad \tilde{v}(\tilde{t}) = \frac{v(t)}{k}, \quad \text{where } \tilde{t} = r_u t, \\ &\tilde{a} = \frac{a(c_v + \zeta_v)}{r_u}, \quad \tilde{c} = \frac{c_u + \zeta_u}{c_v + \zeta_v} \quad \text{and} \quad \rho = \frac{r_v}{r_u}, \end{aligned} \tag{1.4}$$

and, dropping the tildas for the sake of readability, we finally get the system in (1.1). We will also refer to it as the civil war model (CW).

From the change of variables in (1.4), we notice in particular that a may now take values in $[0, +\infty)$.

The competitive Lotka-Volterra system is already used to study some market phenomena as technology substitution, see e.g. [13, 2, 26], and our model aims at adding new features to such models.

Concretely, in the technological competition model, one can think that u and v represent the capitals of two computer companies. In this setting, to start with, one can suppose that the first company produces a very successful product, say computers with a certain operating system, in an infinite market, reinvesting a proportion r_u of the profits into the production of additional items, which are purchased by the market, and so on: in this way, one obtains a linear equation of the type $\dot{u} = r_u u$, with exponentially growing solutions. The case in which the market is not infinite, but reaches a saturation threshold k, would correspond to the equation

$$\dot{u} = r_u u \left(1 - \frac{u}{k}\right).$$

Then, when a second computer company comes into the business, selling computers with a different operating system to the same market, one obtains the competitive system of equations

$$\begin{cases} \dot{u} = r_u u \left(1 - \dfrac{u+v}{k}\right), \\ \dot{v} = r_v v \left(1 - \dfrac{v+u}{k}\right). \end{cases}$$

At this stage, the first company may decide to use an "aggressive" strategy consisting in spreading a virus attacking the other company's operating system, with the aim of setting the other company out of the market (once the competition of the second company is removed, the first company can then exploit the market in a monopolistic regime). To model this strategy, one can suppose that the first company invests a proportion of its capital in the project and diffusion of the virus, according to a quantifying parameter $a_u \geqslant 0$, thus producing the equation

$$\dot{u} = r_u u \left(1 - \frac{u+v}{k}\right) - a_u u. \tag{1.5}$$

This directly impacts the capital of the second company proportionally to the virus spread, since the second company has to spend money to project and release antiviruses, as well as to repay unsatisfied customers, hence resulting in a second equation of the form

$$\dot{v} = r_v v \left(1 - \frac{v+u}{k}\right) - a_v u. \tag{1.6}$$

The case $a_u = a_v$ would correspond to an "even" effect in which the costs of producing the virus is in balance with the damages that it causes. It is also realistic to take into account the case $a_u < a_v$ (e.g., the first company manages to produce and diffuse the virus at low cost, with high impact on the functionality of the operating system of the second company) as well as the case $a_u > a_v$ (e.g., the cost of producing and diffusing the virus is high with respect to the damages caused).

We remark that equations (1.5) and (1.6) can be set into the form (1.3), thus showing the interesting versatility of our model also in financial mathematics.

1.2 Some notation and basic results on the dynamics of system (1.1)

We denote by $(u(t), v(t))$ a solution of (1.1) starting from a point $(u(0), v(0)) \in [0,1] \times [0,1]$. We will also refer to the *orbit* of $(u(0), v(0))$ as the collection of points $(u(t), v(t))$ for $t \in \mathbb{R}$, thus both positive and negative times, while the *trajectory* is the collection of points $(u(t), v(t))$ for $t \geqslant 0$.

As already mentioned in the discussion below formula (1.1), v can reach the value 0 and even negative values in finite time. However, we will suppose that the dynamics stops when the value $v = 0$ is reached for the first time. At this point, the conflict ends with the victory of the first population u, that can continue its evolution with a classical Lotka-Volterra equation of the form

$$\dot{u} = u(1-u)$$

and that would certainly fall into the attractive equilibrium $u = 1$. The only other possibility is that the solutions are constrained in the set $[0,1] \times (0,1]$.

In order to state our first result on the dynamics of the system (1.1), we first observe that, in a real-world situation, the value of a would probably be non-constant and discontinuous, so we allow this coefficient to take values in the class $\mathcal{A}$ defined as follows:

$$\begin{aligned}\mathcal{A} := \big\{ a : [0,+\infty) \to [0,+\infty) \text{ s.t. } a \text{ is continuous} \\ \text{except at most at a finite number of points} \big\}.\end{aligned} \tag{1.7}$$

A *solution related to a strategy* $a(t) \in \mathcal{A}$ is a pair $(u(t), v(t)) \in C_0(0, +\infty) \times C_0(0, +\infty)$, which is C^1 outside the discontinuous points of $a(t)$ and solves system (1.1). Moreover, once the initial datum is imposed, the solution is assumed to be continuous at $t = 0$.

In this setting, we establish the existence of the solutions of problem (1.1) and we classify their behavior with respect to the possible exit from the domain $[0, 1] \times [0, 1]$:

Proposition 1.1. *Let $a(t) \in \mathcal{A}$. Given $(u(0), v(0)) \in [0, 1] \times [0, 1]$, there exists a solution $(u(t), v(t))$ with $a = a(t)$ of system (1.1) starting at $(u(0), v(0))$.*

Furthermore, one of the two following situations occurs:

(1) The solution $(u(t), v(t))$ issued from $(u(0), v(0))$ belongs to $[0, 1] \times (0, 1]$ for all $t \geqslant 0$.

(2) There exists $T \geqslant 0$ such that the solution $(u(t), v(t))$ issued from $(u(0), v(0))$ exists unique for all $t \leqslant T$, and $v(T) = 0$ and $u(T) > 0$.

As a consequence of Proposition 1.1, we can define the the *stopping time* of the solution $(u(t), v(t))$ as

$$T_s(u(0), v(0)) = \begin{cases} +\infty & \text{if situation (1) occurs,} \\ T & \text{if situation (2) occurs.} \end{cases} \tag{1.8}$$

From now on, we will implicitly consider solutions $(u(t), v(t))$ only for $t \leqslant T_s(u(0), v(0))$.

Now we are going to analyze the dynamics of (1.1) with a particular focus on possible strategies. To do this, we now define the *basins of attraction*. The first one is the basin of attraction of the point $(0, 1)$, that is

$$\begin{aligned} \mathcal{B} := \Big\{ &(u(0), v(0)) \in [0, 1] \times [0, 1] \text{ s.t.} \\ &T_s(u(0), v(0)) = +\infty, \ (u(t), v(t)) \stackrel{t \to \infty}{\longrightarrow} (0, 1) \Big\}, \end{aligned} \tag{1.9}$$

namely the set of the initial points for which the first population gets extinct (in infinite time) and the second one survives. The other one is

$$\mathcal{E} := \{(u(0), v(0)) \in ([0, 1] \times [0, 1]) \setminus (0, 0) \text{ s.t. } T_s(u(0), v(0)) < +\infty\}, \tag{1.10}$$

namely the set of initial points for which we have the victory of the first population and the extinction of the second one.

Of course, the sets $\mathcal{B}$ and $\mathcal{E}$ depend on the parameters a, c, and ρ; we will express this dependence by writing $\mathcal{B}(a, c, \rho)$ and $\mathcal{E}(a, c, \rho)$ when it is needed, and omit it otherwise for the sake of readability. The dependence on parameters will be carefully studied in Subsection 3.

1.3 Dynamics of system (1.1) for constant strategies

The first step towards the understanding of the dynamics of the system in (1.1) is is to analyze the behavior of the system for constant coefficients.

To this end, we introduce some notation. Following the terminology on pages 9-10 in [27], we say that an equilibrium point (or fixed point) of the dynamics is a (hyperbolic) *sink* if all the eigenvalues of the linearized map have strictly negative real parts, a (hyperbolic) *source* if all the eigenvalues of the linearized map have strictly positive real parts, and a (hyperbolic) *saddle* if some of the eigenvalues of the linearized map have strictly positive real parts and some have negative real parts (since in this paper we work in dimension 2, saddles correspond to linearized maps with one eigenvalue with strictly positive real part and one eigenvalue with strictly negative real part). We also recall that sinks are asymptotically stable (and sources are asymptotically stable for the reversed-time dynamics), see e.g. Theorem 1.1.1 in [27].

With this terminology, we state the following theorem:

Theorem 1.2 (Dynamics of system (1.1))**.** *For $a > 0$ and $\rho > 0$ the system (1.1) has the following features:*

(i) When $0 < ac < 1$, the system has 3 equilibria: $(0,0)$ is a source, $(0,1)$ is a sink, and

$$(u_s, v_s) := \left(\frac{1-ac}{1+\rho c}\rho c, \frac{1-ac}{1+\rho c}\right) \in (0,1) \times (0,1) \tag{1.11}$$

is a saddle.

(ii) When $ac > 1$, the system has 2 equilibria: $(0,1)$ is a sink and $(0,0)$ is a saddle.

(iii) When $ac = 1$, the system has 2 equilibria: $(0,1)$ is a sink and $(0,0)$ corresponds to a strictly positive eigenvalue and a null one.

(iv) We have

$$[0,1] \times [0,1] = \mathcal{B} \cup \mathcal{E} \cup \mathcal{M} \tag{1.12}$$

where $\mathcal{B}$ and $\mathcal{E}$ are defined in (1.9) and (1.10), respectively, and $\mathcal{M}$ is a smooth curve.

(v) The trajectories starting in $\mathcal{M}$ tend to (u_s, v_s) if $0 < ac < 1$, and to $(0,0)$ if $ac \geqslant 1$ as t goes to $+\infty$.

More precisely, one can say that the curve $\mathcal{M}$ in Theorem 1.2 is the stable manifold of the saddle point (u_s, v_s) when $0 < ac < 1$, and of the saddle point $(0,0)$ when $ac > 1$. The case $ac = 1$ needs a special treatment, due to the degeneracy of one eigenvalue, and in this case the curve $\mathcal{M}$ corresponds to the center manifold of $(0,0)$, and an ad-hoc argument will be exploited to show that also in this degenerate case orbits that start in $\mathcal{M}$ are asymptotic in the future to $(0,0)$.

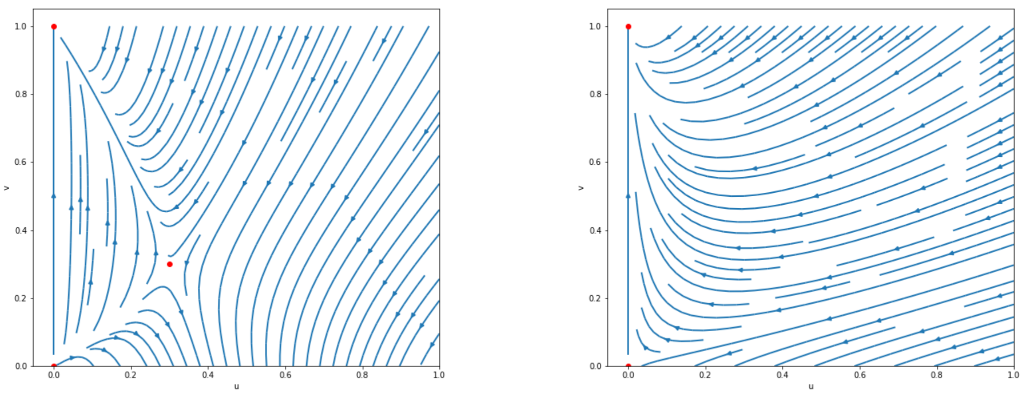
As a matter of fact, $\mathcal{M}$ acts as a dividing wall between the two basins of attraction, as described in (iv) of Theorem 1.2 and in the forthcoming Proposition 2.9.

Moreover, in the forthcoming Propositions 2.1 and 2.7 we will show that $\mathcal{M}$ can be written as the graph of a function. This is particularly useful because, by studying the properties of this function, we gain relevant pieces of information on the sets $\mathcal{B}$ and $\mathcal{E}$ in (1.9) and (1.10).

We point out that in Theorem 1.2 we find that the set of initial data $[0,1]\times[0,1]$ splits into three part: the set $\mathcal{E}$, given in (1.10), made of points going to the extinction of the second population in finite time; the set $\mathcal{B}$, given in (1.9), which is the basin of attraction of the equilibrium $(0,1)$; the set $\mathcal{M}$, which is a manifold of dimension 1 that separates $\mathcal{B}$ from $\mathcal{E}$.

In particular, Theorem 1.2 shows that, also for our model, the Gause principle of exclusion is respected; that is, in general, two competing populations cannot coexist in the same territory, see e.g. [5].

One peculiar feature of our system is that, if the aggressiveness is too strong, the equilibrium $(0,0)$ changes its "stability" properties, passing from a source (as in (i) of Theorem 1.2) to a saddle point (as in (ii) of Theorem 1.2). This shows that the war may have self-destructive outcomes, therefore it is important for the first population to analyze the situation in order to choose a proper level of aggressiveness. Figure 1 shows one example of dynamics for each case.



(a) $a = 0.8$, $c = 0.5$, $\rho = 2$ (b) $a = 0.8$, $c = 3$, $\rho = 2$

Figure 1: *The figures show a phase portrait for the indicated values of the coefficients. In blue, the orbits of the points. The red dots represent the equilibria.*

1.4 Dynamics of system (1.1) for variable strategies and optimal strategies for the first population

We now deal with the problem of choosing the strategy a such that the first population wins, that is a problem of *target reachability* for a control-affine system. As we will see, the problem is not *controllable*, meaning that, starting from a given initial point, it is not always possible to reach a given target.

We now introduce some terminology, that we will use throughout the paper.

Recalling (1.7), for any $\mathcal{T} \subseteq \mathcal{A}$, we set

$$\mathcal{V}_{\mathcal{T}} := \bigcup_{a(\cdot)\in\mathcal{T}} \mathcal{E}(a(\cdot)), \tag{1.13}$$

where $\mathcal{E}(a(\cdot))$ denotes the set of initial data (u_0, v_0) such that $T_s(u_0, v_0) < +\infty$, when the coefficient a in (1.1) is replaced by the function $a(t)$.

Namely, $\mathcal{V}_{\mathcal{T}}$ represents the set of initial conditions for which u is able to win by choosing a suitable strategy in $\mathcal{T}$; we call $\mathcal{V}_{\mathcal{T}}$ the *victory set* with admissible strategies in $\mathcal{T}$. We also say that $a(\cdot)$ is a *winning strategy* for the point (u_0, v_0) if $(u_0, v_0) \in \mathcal{E}(a(\cdot))$.

Moreover, we will call

$$(u_s^0, v_s^0) := \left(\frac{\rho c}{1+\rho c}, \frac{1}{1+\rho c}\right). \tag{1.14}$$

Notice that (u_s^0, v_s^0) is the limit point as a tends to 0 of the sequence of saddle points $\{(u_s^a, v_s^a)\}_{a>0}$ defined in (1.11).

With this notation, the first question that we address is for which initial configurations it is possible for the population u to have a winning strategy, that is, to characterize the victory set. For this, we allow the strategy to take all the values in $[0, +\infty)$. In this setting, we have the following result:

Theorem 1.3. *(i) For $\rho = 1$, we have that*

$$\begin{aligned}\mathcal{V}_{\mathcal{A}} = \Big\{(u,v) \in [0,1]\times[0,1] \ \ s.t. \ \ & v - \frac{u}{c} < 0 \ \ \text{if } u \in [0,c] \\ & \text{and} \ \ v \leqslant 1 \ \ \text{if } u \in (c,1]\Big\},\end{aligned} \tag{1.15}$$

with the convention that the last line in (1.15) is not present if $c \geqslant 1$.

(ii) For $\rho < 1$, we have that

$$\begin{aligned}\mathcal{V}_{\mathcal{A}} = \Big\{&(u,v) \in [0,1]\times[0,1] \ \ s.t. \ \ v < \gamma_0(u) \ \text{if } u \in [0, u_s^0], \\ & v < \frac{u}{c} + \frac{1-\rho}{1+\rho c} \ \text{if } u \in \left[u_s^0, \frac{\rho c(c+1)}{1+\rho c}\right] \\ & \text{and} \ \ v \leqslant 1 \ \text{if } u \in \left(\frac{\rho c(c+1)}{1+\rho c}, 1\right]\Big\},\end{aligned} \tag{1.16}$$

where

$$\gamma_0(u) := \frac{u^\rho}{\rho c (u_s^0)^{\rho-1}},$$

and we use the convention that the last line in (1.16) is not present if $\frac{\rho c(c+1)}{1+\rho c} \geqslant 1$.

(iii) For $\rho > 1$, we have that

$$\begin{aligned}\mathcal{V}_{\mathcal{A}} = \Bigg\{ &(u,v) \in [0,1]\times[0,1] \ \ s.t. \ \ v < \frac{u}{c} \ \text{if } u \in [0, u_\infty], \\ &v < \zeta(u) \ \text{if } u \in \left(u_\infty, \frac{c}{(c+1)^{\frac{\rho-1}{\rho}}} \right] \\ &\text{and } \ v \leqslant 1 \ \text{if } u \in \left(\frac{c}{(c+1)^{\frac{\rho-1}{\rho}}}, 1 \right] \Bigg\},\end{aligned} \tag{1.17}$$

where

$$u_\infty := \frac{c}{c+1} \qquad \text{and} \qquad \zeta(u) := \frac{u^\rho}{c\, u_\infty^{\rho-1}}. \tag{1.18}$$

and we use the convention that the last line in (1.17) is not present if $\frac{c}{(c+1)^{\frac{\rho-1}{\rho}}} \geqslant 1$.

In practice, constant strategies could be certainly easier to implement and it is therefore natural to investigate whether or not it suffices to restrict to constant strategies without altering the possibility of victory. The next result addresses this problem by showing that when $\rho = 1$ constant strategies are as good as all strategies, but instead when $\rho \neq 1$ victory cannot be achieved by only exploiting constant strategies:

Theorem 1.4. *Let $\mathcal{K} \subset \mathcal{A}$ be the set of constant functions. Then the following holds:*

(i) For $\rho = 1$, we have that $\mathcal{V}_{\mathcal{A}} = \mathcal{V}_{\mathcal{K}} = \mathcal{E}(a)$ for all $a > 0$;

(ii) For $\rho \neq 1$, we have that $\mathcal{V}_{\mathcal{K}} \subsetneq \mathcal{V}_{\mathcal{A}}$.

The result of Theorem 1.4, part (i), reveals a special rigidity of the case $\rho = 1$ in which, no matter which strategy u chooses, the victory depends only on the initial conditions, but it is independent of the strategy $a(t)$. Instead, as stated in Theorem 1.4, part (ii), for $\rho \neq 1$ the choice of $a(t)$ plays a crucial role in determining which population is going to win and constant strategies do not exhaust all the possible winning strategies. We stress that $\rho = 1$ plays also a special role in the biological interpretation of the model, since in this case the two populations have the same fit to the environmental resource, and hence, in a sense, they are indistinguishable, up to the possible aggressive behavior of the first population.

Next, we show that the set $\mathcal{V}_{\mathcal{A}}$ can be recovered if we use piecewise constant functions with at most one discontinuity, that we call Heaviside functions.

Theorem 1.5. *There holds that $\mathcal{V}_{\mathcal{A}} = \mathcal{V}_{\mathcal{H}}$, where $\mathcal{H}$ is the set of Heaviside functions.*

The proof of Theorem 1.5 solves also the third question mentioned in the Introduction. As a matter of fact, it proves that for each point we either have a constant winning strategy or a winning strategy of type

$$a(t) = \begin{cases} a_1 & \text{if } t < T, \\ a_2 & \text{if } t \geqslant T, \end{cases}$$

for some $T \in (0, T_s)$, and for suitable values a_1, $a_2 \in (0, +\infty)$ such that one is very small and the other one very large, the order depending on ρ. The construction that we give also puts in light the fact that the choice of the strategy depends on the initial datum, answering also our fourth question.

It is interesting to observe that the winning strategy that switches abruptly from a small to a large value could be considered, in the optimal control terminology, as a "bang-bang" strategy. Even in a target reachability problem, the structure predicted by Pontryagin's Maximum Principle is brought in light: the bounds of the set $\mathcal{V}_{\mathcal{A}}$, as given in Theorem 1.3, depend on the bounds that we impose on the strategy, that are, $a \in [0, +\infty)$.

It is natural to consider also the case in which the level of aggressiveness is constrained between a minimal and maximal threshold, which corresponds to the setting $a \in [m, M]$ for suitable $M \geqslant m \geqslant 0$, with the hypothesis that $M > 0$. In this setting, we denote by $\mathcal{A}_{m,M}$ the class of piecewise continuous strategies $a(\cdot)$ in $\mathcal{A}$ such that $m \leqslant a(t) \leqslant M$ for all $t > 0$ and we let

$$\mathcal{V}_{m,M} := \mathcal{V}_{\mathcal{A}_{m,M}} = \bigcup_{\substack{a(\cdot)\in\mathcal{A} \\ m\leqslant a(t)\leqslant M}} \mathcal{E}(a(\cdot)) = \bigcup_{a(\cdot)\in\mathcal{A}_{m,M}} \mathcal{E}(a(\cdot)). \tag{1.19}$$

Then we have the following:

Theorem 1.6. *Let M and m be two real numbers such that $M \geqslant m \geqslant 0$. Then, for $\rho \neq 1$ we have the strict inclusion $\mathcal{V}_{m,M} \subsetneq \mathcal{V}_{\mathcal{A}}$.*

Notice that for $\rho = 1$, Theorem 1.4 gives instead that $\mathcal{V}_{m,M} = \mathcal{V}_{\mathcal{A}}$ and we think that this is a nice feature, outlining a special role played by the parameter ρ (roughly speaking, when $\rho = 1$ constant strategies suffice to detect all possible winning configurations, thanks to Theorem 1.4, while when $\rho \neq 1$ non-constant strategies are necessary to detect all winning configurations).

1.4.1 Time minimizing strategy

Once established that it is possible to win starting in a certain initial condition, we are interested in knowing which of the possible strategies is best to choose. One condition that may be taken into account is the duration of the war. Now, this question can be written as a minimization problem with a proper functional to minimize and therefore the classical Pontryagin theory applies.

To state our next result, we recall the setting in (1.19) and define

$$\mathcal{S}(u_0, v_0) := \Big\{ a(\cdot) \in \mathcal{A}_{m,M} \;\text{ s.t. }\; (u_0, v_0) \in \mathcal{E}(a(\cdot)) \Big\},$$

that is the set of all bounded strategies for which the trajectory starting at (u_0, v_0) leads to the victory of the first population. To each $a(\cdot) \in \mathcal{S}(u_0, v_0)$ we associate the stopping time defined in (1.8), and we express its dependence on $a(\cdot)$ by writing $T_s(a(\cdot))$. In this setting, we provide the following statement concerning the strategy leading to the quickest possible victory for the first population:

Theorem 1.7. *Given a point* $(u_0, v_0) \in \mathcal{V}_{m,M}$*, there exists a winning strategy* $\tilde{a}(t) \in \mathcal{S}(u_0, v_0)$*, and a trajectory* $(\tilde{u}(t), \tilde{v}(t))$ *associated with* $\tilde{a}(t)$*, for* $t \in [0, T]$*, with* $(\tilde{u}(0), \tilde{v}(0)) = (u_0, v_0)$*, where* T *is given by*

$$T = \min_{a(\cdot) \in \mathcal{S}} T_s(a(\cdot)).$$

Moreover,

$$\tilde{a}(t) \in \{m,\ M,\ a_s(t)\},$$

where

$$a_s(t) := \frac{(1 - \tilde{u}(t) - \tilde{v}(t))[\tilde{u}(t)\,(2c + 1 - \rho c) + \rho c]}{\tilde{u}(t)\,2c(c+1)}. \tag{1.20}$$

The surprising fact given by Theorem 1.7 is that the minimizing strategy is not only of bang-bang type, but it may assume some values along a *singular arc*, given by $a_s(t)$. This possibility is realized in some concrete cases, as we verified by running some numerical simulations, whose results can be visualized in Figure 2.

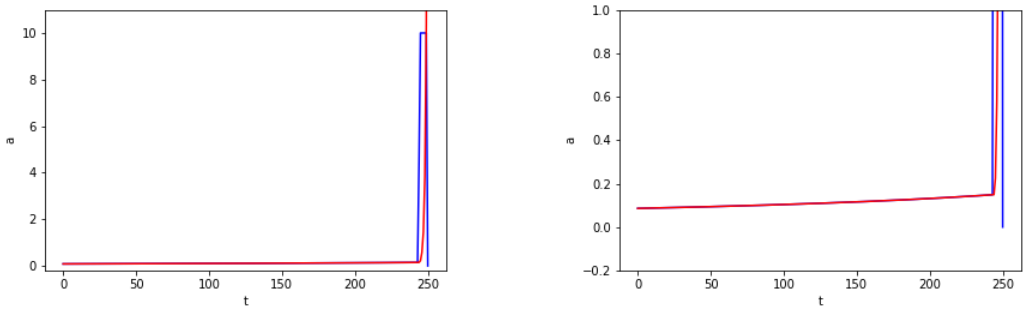


Figure 2: *The figure shows the result of a numerical simulation searching a minimizing time strategy* $\tilde{a}(t)$ *for the problem starting in* $(0.5, 0.1875)$ *for the parameters* $\rho = 0.5$*,* $c = 4.0$*,* $m = 0$ *and* $M = 10$*. In blue, the value found for* $\tilde{a}(t)$*; in red, the value of* $a_s(t)$ *for the corresponding trajectory* $(u(t), v(t))$*. As one can observe,* $\tilde{a}(t) \equiv a_s(t)$ *in a long trait. The simulation was done using AMPL-Ipopt on the server NEOS and pictures have been made with Python.*

1.5 Organization of the paper

In the forthcoming Section 2 we will exploit methods from ordinary differential equations and dynamical systems to describe the equilibria of the system and their possible basins of attraction. The dependence of the dynamics on the structural parameters, such as fit to the environment, aggressiveness and efficacy of attacks, is discussed in detail in Section 3.

Section 4 is devoted to the analysis of the strategies that allow the first population to eradicate the second one (this part needs an original combination of methods from dynamical systems and optimal control theory).

2 First results on the dynamics and proofs of Proposition 1.1 and Theorem 1.2

In this section we provide some useful results on the behavior of the solutions of (1.1) and on the basin of attraction. In particular, we provide the proofs of Proposition 1.1 and Theorem 1.2 and we state a characterization of the sets $\mathcal{B}$ and $\mathcal{E}$ given in (1.9) and (1.10), respectively, see Propositions 2.9.

This material will be extremely useful for the analysis of the strategy that we operate later.

We start with some preliminary notation. Given a close set $\mathcal{S} \subseteq [0,1] \times [0,1]$, we say that a trajectory $(u(t), v(t))$ originated in $\mathcal{S}$ *exits* the set $\mathcal{S}$ at some time $T \geqslant 0$ if

- $(u(t), v(t)) \in \mathcal{S}$ for $t \leqslant T$,
- $(u(T), v(T)) \in \partial\mathcal{S}$,
- for any vector ν normal to $\partial\mathcal{S}$ at the point $(u(T), v(T))$, it holds that

$$(\dot{u}(T), \dot{v}(T)) \cdot \nu > 0.$$

Now, we prove Proposition 1.1, which is fundamental to the well-definition of our model:

Proof of Proposition 1.1. We consider the function $a(t) \in \mathcal{A}$, which is continuous except in a finite number of points $0 < t_1 < \cdots < t_n$. In all the intervals $(0, t_1)$, $(t_i, t_{i+1}]$, for $i \in \{1, \cdots, n-1\}$, and $(t_n, +\infty)$, the equations in (1.1) have smooth coefficients, and therefore a solution does exist. Now, it is sufficient to consider $(u(t_i), v(t_i))$ as the initial datum for the dynamics in $(t_i, t_{i+1}]$ to construct a solution $(u(t), v(t))$ for all $t > 0$ satisfying system (1.1). This is a rather classical result and we refer to [16] for more details.

Now, we prove that either the possibility in (1) or the possibility in (2) can occur. For this, by using the equation for v in (1.1), we notice that for $v = 1$ the inward pointing normal derivative is

$$-\dot{v}|_{v=1} = \left(-\rho v(1-u-v) + au\right)|_{v=1} = u(\rho + a) \geqslant 0.$$

This means that no trajectory can exit $[0,1] \times [0,1]$ on the edge $v = 1$. Similarly, using the equation for u in (1.1), we see that for $u = 1$ the normal derivative inward pointing is

$$-\dot{u}|_{u=1} = \left(-u(1-u-v) + acu\right)|_{u=1} = v + ac \geqslant 0,$$

and therefore no trajectory can exit $[0,1] \times [0,1]$ on the edge $u = 1$.

Moreover, it is easy to see that all points on the line $u = 0$ go to the equilibrium $(0,1)$, thus trajectories do not cross the line $u = 0$. The only remaining possibilities are that the trajectories stay in $[0,1] \times (0,1]$, that is possibility (1), or they exit the square on the side $v = 0$, that is possibility (2). □

Now, we give the proof of (i), (ii) and (iii) of Theorem 1.2.

Proof of (i), (ii) and (iii) of Theorem 1.2. We first consider equilibria with first coordinate $u = 0$. In this case, from the second equation in (1.1), we have that the equilibria must satisfy $\rho v(1-v) = 0$, thus $v = 0$ or $v = 1$. As a consequence, $(0,0)$ and $(0,1)$ are two equilibria of the system.

Now, we consider equilibria with first coordinate $u > 0$. Equilibria of this form must satisfy $\dot{u} = 0$ with $u \neq 0$, and therefore, from the first equation in (1.1),

$$1 - u - v - ac = 0. \tag{2.1}$$

Moreover from the condition $\dot{v} = 0$ and the second equation in (1.1), we see that

$$\rho v(1 - u - v) - au = 0. \tag{2.2}$$

Putting together (2.1) and (2.2), we obtain that the intersection point must lie on the line $\rho cv - u = 0$. Since the equilibrium is at the intersection between two lines, it must be unique. One can easily verify that the values given in (1.11) satisfy (2.1) and (2.2).

From now on, we distinguish the three situations in (i), (ii) and (iii) of Theorem 1.2.

(i) If $0 < ac < 1$, we have that the point (u_s, v_s) given in (1.11) lies in $(0,1)\times(0,1)$. As a result, in this case the system has 3 equilibria, given by $(0,0)$, $(0,1)$ and (u_s, v_s).

Now, we observe that the Jacobian of the system (1.1) is

$$J(u,v) = \begin{pmatrix} 1 - 2u - v - ac & -u \\ -\rho v - a & \rho(1 - u - 2v) \end{pmatrix}. \tag{2.3}$$

At the point $(0,0)$, the matrix has eigenvalues $\rho > 0$ and $1 - ac > 0$, thus $(0,0)$ is a source. At the point $(0,1)$, the Jacobian (2.3) has eigenvalues $-ac < 0$ and $-\rho < 0$, thus $(0,1)$ is a sink. At the point (u_s, v_s), by exploiting the relations (2.1) and (2.2) we have that

$$J(u_s, v_s) = \begin{pmatrix} -u_s & -u_s \\ -\rho v_s - a & \rho(ac - v_s) \end{pmatrix},$$

which, by the change of basis given by the matrix

$$\begin{pmatrix} -\frac{1}{u_s} & 0 \\ -\frac{1}{u_s}\left[\left(\frac{u_s}{c} + a\right)\left(\frac{\rho c - c}{1+\rho c}\right) + ac\right] & \frac{\rho c - c}{1+\rho c} \end{pmatrix},$$

becomes

$$\begin{pmatrix} 1 & 1 \\ ac & \rho ac \end{pmatrix}. \tag{2.4}$$

The characteristic polynomial of the matrix in (2.4) is $\lambda^2 - \lambda(1 + \rho ac) + \rho ac - ac$, that has two real roots, as one can see by inspection. Hence, $J(u_s, v_s)$ has two real eigenvalues. Moreover, the determinant of $J(u_s, v_s)$ is $-\rho acu_s - au_s < 0$, which implies that $J(u_s, v_s)$ has one positive and one negative eigenvalues. These considerations give that (u_s, v_s) is a saddle point, as desired. This completes the proof of (i) in Theorem 1.2.

(ii) and (iii) We assume that $ac \geqslant 1$. We observe that the equilibrium described by the coordinates (u_s, v_s) in (1.11) coincides with $(0,0)$ for $ac = 1$, and lies outside $[0,1] \times [0,1]$ for $ac > 1$. As a result, when $ac \geqslant 1$ the system has 2 equilibria, given by $(0,0)$ and $(0,1)$.

Looking at the Jacobian in (2.3), one sees that at the point $(0,1)$, it has eigenvalues $-ac < 0$ and $-\rho < 0$, and therefore $(0,1)$ is a sink when $ac \geqslant 1$.

Furthermore, from (2.3) one finds that if $ac > 1$ then $J(0,0)$ has the positive eigenvalue ρ and the negative eigenvalue $1 - ac$, thus $(0,0)$ is a saddle point.

If instead $ac = 1$, then $J(0,0)$ has one positive eigenvalue and one null eigenvalue, as desired. □

To complete the proof of Theorem 1.2, we will deal with the cases $ac \neq 1$ and $ac = 1$ separately. This analysis will be performed in the forthcoming Sections 2.1 and 2.2. The completion of the proof of Theorem 1.2 will then be given in Section 2.3.

2.1 Characterization of $\mathcal{M}$ when $ac \neq 1$

We consider here the case $ac \neq 1$. The case $ac = 1$ is degenerate and it will be treated separately in Section 2.2.

We point out that in the proof of (i) and (ii) in Theorem 1.2 we found a saddle point in both cases. By the Stable Manifold Theorem (see for example [16]), the point (u_s, v_s) in (1.11) in the case $0 < ac < 1$ and the point $(0,0)$ in the case $ac > 1$ have a stable manifold and an unstable manifold. These manifolds are unique, they have dimension 1, and they are tangent to the eigenvectors of the linearized system. We will denote by $\mathcal{M}$ the stable manifold associated with these saddle points. Since we are interested in the dynamics in the square $[0,1] \times [0,1]$, with a slight abuse of notation we will only consider the restriction of $\mathcal{M}$ in $[0,1] \times [0,1]$.

In order to complete the proof of Theorem 1.2, we now analyze some properties of $\mathcal{M}$:

Proposition 2.1. *For $ac \neq 1$ the set $\mathcal{M}$ can be written as the graph of a unique increasing C^2 function $\gamma : [0, u_{\mathcal{M}}] \to [0, v_{\mathcal{M}}]$ for some $(u_{\mathcal{M}}, v_{\mathcal{M}}) \in \big(\{1\} \times [0,1]\big) \cup \big((0,1] \times \{1\}\big)$, such that $\gamma(0) = 0$, $\gamma(u_{\mathcal{M}}) = v_{\mathcal{M}}$ and*

- *if $0 < ac < 1$, $\gamma(u_s) = v_s$;*
- *if $ac > 1$, in $u = 0$ the function γ is tangent to the line $(\rho - 1 + ac)v - au = 0$.*

As a byproduct of the proof of Proposition 2.1, we also obtain some useful information on the structure of the stable manifold and the basins of attraction, that we summarize here below:

Corollary 2.2. *Suppose that $0 < ac < 1$. Then, the curves (2.1) and (2.2), loci of the points such that $\dot{u} = 0$ and $\dot{v} = 0$ respectively, divide the square $[0,1] \times [0,1]$ into*

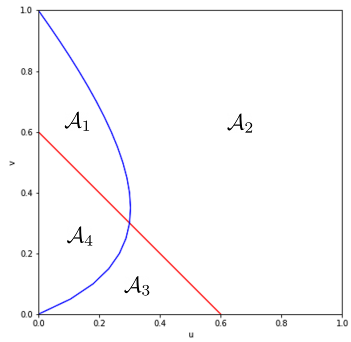


Figure 3: *Partition of* $[0,1]\times[0,1]$ *in the case* $a = 0.8$, $c = 0.5$, $\rho = 2$, *as given by* (2.5). *In red, the curve* $\dot{u} = 0$. *In blue, the curve* $\dot{v} = 0$, *parametrized by the function* σ *in* (2.6).

four regions:

$$\begin{aligned} \mathcal{A}_1 &:= \big\{(u,v)\in[0,1]\times[0,1] \;\; s.t \;\; \dot{u}\leqslant 0,\ \dot{v}\geqslant 0\big\}, \\ \mathcal{A}_2 &:= \big\{(u,v)\in[0,1]\times[0,1] \;\; s.t \;\; \dot{u}\leqslant 0,\ \dot{v}\leqslant 0\big\}, \\ \mathcal{A}_3 &:= \big\{(u,v)\in[0,1]\times[0,1] \;\; s.t \;\; \dot{u}\geqslant 0,\ \dot{v}\leqslant 0\big\}, \\ \mathcal{A}_4 &:= \big\{(u,v)\in[0,1]\times[0,1] \;\; s.t \;\; \dot{u}\geqslant 0,\ \dot{v}\geqslant 0\big\}. \end{aligned} \tag{2.5}$$

Furthermore, the sets $\mathcal{A}_1\cup\mathcal{A}_4$ *and* $\mathcal{A}_2\cup\mathcal{A}_3$ *are separated by the curve* $\dot{v} = 0$, *given by the graph of the continuous function*

$$\sigma(v) := 1 - \frac{\rho v^2 + a}{\rho v + a}, \tag{2.6}$$

that satisfies $\sigma(0) = 0$, $\sigma(1) = 0$, *and* $0 < \sigma(v) < 1$ *for all* $v \in (0,1)$.

In addition,

$$\mathcal{M}\setminus\{(u_s,v_s)\} \text{ is contained in } \mathcal{A}_2\cup\mathcal{A}_4, \tag{2.7}$$

$$(\mathcal{A}_3\setminus\{(0,0),(u_s,v_s)\}) \subseteq \mathcal{E}, \tag{2.8}$$

and

$$\mathcal{A}_1\setminus\{(u_s,v_s)\}\subset\mathcal{B}, \tag{2.9}$$

where the notation in (1.9) *and* (1.10) *has been utilized.*

To visualize the statements in Corollary 2.2, one can see Figure 3.

Corollary 2.3. *Suppose that* $ac > 1$. *Then , we have that* $\dot{u}\leqslant 0$ *in* $[0,1]\times[0,1]$, *and the curve* (2.2) *divides the square* $[0,1]\times[0,1]$ *into two regions:*

$$\begin{aligned} \mathcal{A}_1 &:= \big\{(u,v)\in[0,1]\times[0,1] \;\; s.t. \;\; \dot{u}\leqslant 0,\ \dot{v}\geqslant 0\big\}, \\ \mathcal{A}_2 &:= \big\{(u,v)\in[0,1]\times[0,1] \;\; s.t. \;\; \dot{u}\leqslant 0,\ \dot{v}\leqslant 0\big\}. \end{aligned} \tag{2.10}$$

Furthermore, the sets $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *are separated by the curve* $\dot{v} = 0$, *given by the graph of the continuous function* σ *given in* (2.6).

In addition,

$$\mathcal{M}\subset\mathcal{A}_2. \tag{2.11}$$

Proposition 2.1 and Corollaries 2.2 and 2.3 are a bit technical, but provide fundamental information to obtain a characterization of the sets $\mathcal{E}$ and $\mathcal{B}$, given in the forthcoming Proposition 2.9.

We now provide the proof of Proposition 2.1 (and, as a byproduct, of Corollaries 2.2 and 2.3).

Proof of Proposition 2.1 and Corollaries 2.2 and 2.3. We treat separately the cases $0 < ac < 1$ and $ac > 1$. We start with the case $0 < ac < 1$, and divide the proof in three steps.

Step 1: localizing $\mathcal{M}$. With the notation introduced in (2.5), we prove that

$$\begin{aligned}&\text{all trajectories starting in } \mathcal{A}_3 \setminus \{(0,0),(u_s,v_s)\}\\ &\text{exit the set } \mathcal{A}_3 \text{ on the side } v=0.\end{aligned} \tag{2.12}$$

To this aim, we first observe that

$$\text{there are no cycles entirely contained in } \mathcal{A}_3, \tag{2.13}$$

because $\dot{u}$ and $\dot{v}$ have a sign. Furthermore,

$$\text{there are no equilibria where a trajectory in the interior of } \mathcal{A}_3 \text{ can converge.} \tag{2.14}$$

Indeed, no point in $\mathcal{A}_3$ with positive first coordinate can be mapped in $(0,0)$ without exiting the set, because $\dot{u} \geqslant 0$ in $\mathcal{A}_3$. Also, for all $(u_0,v_0) \in \mathcal{A}_3 \setminus (u_s,v_s)$, we have that $v_0 < v_s$. On the other hand, $\dot{v} \leqslant 0$ in $\mathcal{A}_3$, so no trajectory that is entirely contained in $\mathcal{A}_3$ can converge to (u_s,v_s). These observations prove (2.14).

As a consequence of (2.13), (2.14) and the Poincaré-Bendixson Theorem (see e.g. [20]), we have that all the trajectories in the interior of $\mathcal{A}_3$ must exit the set at some time.

We remark that the side connecting $(0,0)$ and (u_s,v_s) can be written as the of points belonging to

$$\big\{(u,v) \in [0,1] \times (0,v_s) \ \text{ s.t. } \ u = \sigma(v)\big\},$$

where the function σ is defined in (2.6). In this set, it holds that $\dot{v} = 0$ and $\dot{u} > 0$, thus the normal derivative pointing outward $\mathcal{A}_3$ is negative, so the trajectories cannot go outside $\mathcal{A}_3$ passing through this side.

Furthermore, on the side connecting (u_s,v_s) with $(1-ac,0)$, that lies on the straight line $v = 1-ac-u$, we have that $\dot{u} = 0$ and $\dot{v} < 0$ for $(u,v) \neq (u_s,v_s)$, so also here the outer normal derivative is negative. Therefore, the trajectories cannot go outside $\mathcal{A}_3$ passing through this side either.

These considerations complete the proof of (2.12). Accordingly, recalling the definition of $\mathcal{E}$ in (1.10), we see that

$$\big(\mathcal{A}_3 \setminus \{(0,0),(u_s,v_s)\}\big) \subseteq \mathcal{E}. \tag{2.15}$$

In a similar way one can prove that all trajectories starting in $\mathcal{A}_1 \setminus \{(u_s,v_s)\}$ must converge to $(0,1)$, which, recalling the definition of $\mathcal{B}$ in (1.9), implies that

$$\mathcal{A}_1 \setminus \{(u_s,v_s)\} \subset \mathcal{B}. \tag{2.16}$$

Thanks to (2.15) and (2.16), we have that the stable manifold $\mathcal{M}$ has no intersection with $\mathcal{A}_1 \setminus \{(u_s, v_s)\}$ and $\mathcal{A}_3 \setminus \{(0,0),(u_s,v_s)\}$, and therefore $\mathcal{M}$ must lie in $\mathcal{A}_2 \cup \mathcal{A}_4$.

Also, we know that $\mathcal{M}$ is tangent to an eigenvector in (u_s, v_s), and we observe that

$$(1,-1) \text{ is not an eigenvector of the linearized system.} \tag{2.17}$$

Indeed, if $(1,-1)$ were an eigenvector, then

$$\begin{pmatrix} 1-ac-2u_s-v_s & -u_s \\ -\rho v_s - a & \rho - \rho u_s - 2\rho v_s \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

which implies that $1-ac-a-\rho = (u_s+v_s)(1-\rho)$. Hence, recalling (1.11), we obtain that $-a = \rho a c$, which is impossible. This establishes (2.17).

In light of (2.17), we conclude that $\mathcal{M} \setminus \{(u_s, v_s)\}$ must have intersection with both $\mathcal{A}_2$ and $\mathcal{A}_4$.

Step 2: defining $\gamma(u)$. Since $\dot{u} > 0$ and $\dot{v} > 0$ in the interior of $\mathcal{A}_4$, the portion of $\mathcal{M}$ in $\mathcal{A}_4$ can be described globally as the graph of a monotone increasing smooth function $\gamma_1 : U \to [0, v_s]$, for a suitable interval $U \subseteq [0, u_s]$ with $u_s \in U$, and such that $\gamma_1(u_s) = v_s$.

We stress that, for $u > u_s$, the points $(u,v) \in \mathcal{M}$ belong to $\mathcal{A}_2$.

Similarly, in the interior of $\mathcal{A}_2$ we have that $\dot{u} < 0$ and $\dot{v} < 0$. Therefore, we find that $\mathcal{M}$ can be represented in $\mathcal{A}_2$ as the graph of a monotone increasing smooth function $\gamma_2 : V \to [v_s, 1]$, for a suitable interval $V \subseteq [u_s, 1]$ with $u_s \in V$, and such that $\gamma_2(u_s) = v_s$. Notice that in the second case the trajectories and the parametrization run in opposite directions.

Now, we define

$$\gamma(u) := \begin{cases} \gamma_1(u) & \text{if } u \in U, \\ \gamma_2(u) & \text{if } u \in V, \end{cases}$$

and we observe that it is an increasing smooth function locally parametrizing $\mathcal{M}$ around (u_s, v_s) (thanks to the Stable Manifold Theorem).

We point out that, in light of the Stable Manifold Theorem, the stable manifold $\mathcal{M}$ is globally parametrized by an increasing smooth function on a set $W \subset [0,1]$.

Step 3: $\gamma(0) = 0$ and $\gamma(u_{\mathcal{M}}) = v_{\mathcal{M}}$ for some $(u_{\mathcal{M}}, v_{\mathcal{M}}) \in \partial\big([0,1]\times[0,1]\big)$. We first prove that

$$\gamma(0) = 0. \tag{2.18}$$

For this, we claim that

$$\text{orbits in the interior of } \mathcal{A}_4 \text{ do not come from outside } \mathcal{A}_4. \tag{2.19}$$

Indeed, it is easy to see that points on the half axis $\{u = 0\}$ converge to $(0,1)$, and therefore a trajectory cannot enter $\mathcal{A}_4$ from this side.

As for the side connecting $(0,0)$ to (u_s, v_s), here one has that $\dot{u} \geqslant 0$ and $\dot{v} = 0$, and so the inward pointing normal derivative is negative. Therefore, no trajectory can enter $\mathcal{A}_4$ on this side.

Moreover, on the side connecting (u_s, v_s) to $(0, 1-ac)$ the inward pointing normal derivative is negative, because $\dot{u} = 0$ and $\dot{v} \geqslant 0$, thus we have that no trajectory can enter $\mathcal{A}_4$ on this side either. These considerations prove (2.19).

Furthermore, we have that

$$\text{no cycles are allowed in } \mathcal{A}_4, \tag{2.20}$$

because $\dot{u} \geqslant 0$ and $\dot{v} \geqslant 0$ in $\mathcal{A}_4$.

From (2.19), (2.20) and the Poincaré-Bendixson Theorem (see e.g. [20]), we conclude that, given a point $(\tilde{u}, \tilde{v}) \in \mathcal{M}$ in the interior of $\mathcal{A}_4$, the α-limit set of $(\tilde{u}, \tilde{v})$, that we denote by $\alpha_{(\tilde{u},\tilde{v})}$, can be

$$\begin{aligned}&\text{either an equilibrium or a union of (finitely many)}\\&\text{equilibria and non-closed orbits connecting these equilibria.}\end{aligned} \tag{2.21}$$

We stress that, being $(\tilde{u}, \tilde{v})$ in the interior of $\mathcal{A}_4$, we have that

$$\tilde{u} < u_s. \tag{2.22}$$

Now, we observe that

$$\alpha_{(\tilde{u},\tilde{v})} \text{ cannot contain the saddle point } (u_s, v_s). \tag{2.23}$$

Indeed, suppose by contradiction that $\alpha_{(\tilde{u},\tilde{v})}$ does contain (u_s, v_s). Then, we denote by $\phi_{(\tilde{u},\tilde{v})}(t) = \big(u_{(\tilde{u},\tilde{v})}(t), v_{(\tilde{u},\tilde{v})}(t)\big)$ the solution of (1.1) with $\phi_{(\tilde{u},\tilde{v})}(0) = (\tilde{u}, \tilde{v})$, and we have that there exists a sequence $t_j \to -\infty$ such that $\phi_{(\tilde{u},\tilde{v})}(t_j)$ converges to (u_s, v_s) as $j \to +\infty$. In particular, in light of (2.22), there exists j_0 sufficiently large such that

$$u_{(\tilde{u},\tilde{v})}(0) = \tilde{u} < u_{(\tilde{u},\tilde{v})}(t_{j_0}).$$

Consequently, there exists $t_\star \in (t_{j_0}, 0)$ such that $\dot{u}_{(\tilde{u},\tilde{v})}(t_\star) < 0$.

As a result, it follows that $\phi_{(\tilde{u},\tilde{v})}(t_\star) \notin \mathcal{A}_4$. This, together with the fact that $\phi_{(\tilde{u},\tilde{v})}(0) \in \mathcal{A}_4$, is in contradiction with (2.19), and the proof of (2.23) is thereby complete.

Thus, from (2.21) and (2.23), we deduce that $\alpha_{(\tilde{u},\tilde{v})} = \{(0,0)\}$. This gives that $(0,0)$ lies on the stable manifold $\mathcal{M}$, and therefore the proof of (2.18) is complete.

Now, we show that

$$\text{there exists } (u_{\mathcal{M}}, v_{\mathcal{M}}) \in \partial\big([0,1] \times [0,1]\big) \text{ such that } \gamma(u_{\mathcal{M}}) = v_{\mathcal{M}}. \tag{2.24}$$

To prove it, we first observe that

$$\text{orbits in } \mathcal{A}_2 \text{ converging to } (u_s, v_s) \text{ come from outside } \mathcal{A}_2. \tag{2.25}$$

Indeed, we suppose by contradiction that

$$\text{an orbit in } \mathcal{A}_2 \text{ converging to } (u_s, v_s) \text{ stays confined in } \mathcal{A}_2. \tag{2.26}$$

We remark that, in this case,

$$\text{an orbit in } \mathcal{A}_2 \text{ cannot be a cycle,} \tag{2.27}$$

because $\dot{u}$ and $\dot{v}$ have a sign in $\mathcal{A}_2$. Then, by the Poincaré-Bendixson Theorem (see e.g. [20]), we conclude that, given a point $(\tilde{u},\tilde{v}) \in \mathcal{M}$ in the interior of $\mathcal{A}_2$, the α-limit set of $(\tilde{u},\tilde{v})$, that we denote by $\alpha_{(\tilde{u},\tilde{v})}$, can be either an equilibrium or a union of (finitely many) equilibria and non-closed orbits connecting these equilibria. We notice that the set $\alpha_{(\tilde{u},\tilde{v})}$ cannot contain $(0,1)$, since it is a stable equilibrium. We also claim that

$$\alpha_{(\tilde{u},\tilde{v})} \text{ cannot contain } (u_s,v_s). \tag{2.28}$$

Indeed, we suppose by contradiction that $\alpha_{(\tilde{u},\tilde{v})}$ does contain (u_s,v_s). We observe that, since $\dot{u} \leqslant 0$ in $\mathcal{A}_2$,

$$\tilde{u} > u_s. \tag{2.29}$$

We denote by $\phi_{(\tilde{u},\tilde{v})}(t) = \big(u_{(\tilde{u},\tilde{v})}(t), v_{(\tilde{u},\tilde{v})}(t)\big)$ the solution of (1.1) with $\phi_{(\tilde{u},\tilde{v})}(0) = (\tilde{u},\tilde{v})$, and we have that there exists a sequence $t_j \to -\infty$ such that $\phi_{(\tilde{u},\tilde{v})}(t_j)$ converges to (u_s,v_s) as $j \to +\infty$. In particular, in light of (2.29), there exists j_0 sufficiently large such that

$$u_{(\tilde{u},\tilde{v})}(0) = \tilde{u} > u_{(\tilde{u},\tilde{v})}(t_{j_0}).$$

Consequently, there exists $t_\star \in (t_{j_0}, 0)$ such that $\dot{u}_{(\tilde{u},\tilde{v})}(t_\star) > 0$. Accordingly, we have that $\phi_{(\tilde{u},\tilde{v})}(t_\star) \notin \mathcal{A}_2$. This and the fact that $\phi_{(\tilde{u},\tilde{v})}(0) \in \mathcal{A}_2$ give a contradiction with (2.26), and therefore this establishes (2.28).

These considerations complete the proof of (2.25).

Now, we observe that the inward pointing normal derivative at every point in $\mathcal{A}_2 \cap \mathcal{A}_3 \setminus \{(u_s,v_s)\}$ is negative, since $\dot{u} = 0$ and $\dot{v} \leqslant 0$. Hence, no trajectory can enter from this side. Also, the inward pointing normal derivative at every point in $\mathcal{A}_1 \cap \mathcal{A}_2 \setminus \{(u_s,v_s)\}$ is negative, since $\dot{u} \leqslant 0$ and $\dot{v} = 0$. Hence, no trajectory can enter from this side either.

These observations and (2.25) give the desired result in (2.24), and thus Proposition 2.1 is established in the case $ac < 1$.

Now we treat the case $ac > 1$, using the same ideas. In this setting, $\mathcal{M}$ is the stable manifold associated with the saddle point $(0,0)$. We point out that, in this case, for all points in $[0,1] \times [0,1]$ we have that $\dot{u} \leqslant 0$. Hence, the curve of points satisfying $\dot{v} = 0$, that was also given in (2.2), divides the square $[0,1] \times [0,1]$ into two regions $\mathcal{A}_1$ and $\mathcal{A}_2$, defined in (2.10).

Now, one can repeat verbatim the arguments in *Step 1* with obvious modifications, to find that $\mathcal{M} \subset \mathcal{A}_2$.

Since the derivatives of u and v have a sign in $\mathcal{A}_2$, and the set $\mathcal{M}$ in this case is the trajectory of a point converging to $(0,0)$, the set $\mathcal{M}$ can be represented globally as the graph of a smooth increasing function $\gamma : U \to [0,1]$ for a suitable interval $U \subseteq [0,1]$ containing the origin. As a consequence, the condition $\gamma(0) = 0$ is trivially satisfied in this setting. The existence of a suitable $(u_{\mathcal{M}}, v_{\mathcal{M}})$ can be derived reasoning as in *Step 3* with obvious modifications.

Now, we prove that

$$\text{at } u = 0 \text{ the function } \gamma \text{ is tangent to the line } (\rho - 1 + ac)v - au = 0. \tag{2.30}$$

For this, we recall (2.3) and we see, by inspection, that the Jacobian matrix $J(0,0)$ has two eigenvectors, namely $(0,1)$ and $(\rho - 1 + ac, a)$. The first one is tangent to the line $u = 0$, that is the unstable manifold of $(0, 0)$, as one can easily verify. Thus, the second eigenvector is the one tangent to $\mathcal{M}$, as prescribed by the Stable Manifold Theorem (see e.g. [16]). Hence, in $(0, 0)$ the manifold $\mathcal{M}$ is tangent to the line $(\rho - 1 + ac)v - au = 0$ and so is the function γ in $u = 0$. This proves (2.30), and thus Proposition 2.1 is established in the case $ac > 1$ as well. □

2.2 Characterization of $\mathcal{M}$ when $ac = 1$

Here we will prove the counterpart of Proposition 2.1 in the degenerate case $ac = 1$.

To this end, looking at the velocity fields, we first observe that

$$\begin{aligned} &\text{trajectories starting in } (0,1) \times (-\infty, 1) \text{ at time } t = 0 \\ &\text{remain in } (0,1) \times (-\infty, 1) \text{ for all time } t > 0. \end{aligned} \tag{2.31}$$

We also point out that

$$\begin{aligned} &\text{trajectories entering the region } \mathcal{R} := \{u \in (0,1),\ u + v < 0\} \\ &\text{at some time } t_0 \in \mathbb{R} \\ &\text{remain in that region for all time } t > t_0, \end{aligned} \tag{2.32}$$

since $\dot{v} = \rho v(1 - u - v) - au = -\rho u - au < 0$ along $\{u \in (0, 1),\ u + v = 0\}$.

Also, by the Center Manifold Theorem (see e.g. Theorem 1 on page 16 of [3] or pages 89-90 in [17]), there exists a collection $\mathcal{M}_0$ of invariant curves, which are all tangent at the origin to the eigenvector corresponding to the null eigenvalue, that is the straight line $\rho v - au = 0$. Then, we define $\mathcal{M} := \mathcal{M}_0 \cap ([0, 1] \times [0, 1])$ and we observe that this intersection is nonvoid, given the tangency property of $\mathcal{M}_0$ at the origin.

In what follows, for every $t \in \mathbb{R}$, we denote by $(u(t), v(t)) = \phi_p(t)$ the orbit of $p \in \mathcal{M} \setminus \{(0, 0)\}$. We start by providing an observation related to negative times:

Lemma 2.4. *If $p \in \mathcal{M} \setminus \{(0, 0)\}$ then $\phi_p(t)$ cannot approach the origin for negative values of t.*

Proof. We argue by contradiction and denote by $t_1, \ldots, t_n, \ldots$ a sequence of such negative values of t, for which $t_n \to -\infty$ and

$$\lim_{n \to +\infty} \phi_p(t_n) = (0, 0).$$

Up to a subsequence, we can also suppose that

$$u(t_{n+1}) < u(t_n). \tag{2.33}$$

In light of (2.32), we have that, for all $T \leqslant 0$,

$$\phi_p(T) \notin \mathcal{R}. \tag{2.34}$$

Indeed, if $\phi_p(T) \in \mathcal{R}$, we deduce from (2.32) that $\phi_p(t) \in \mathcal{R}$ for all $t \geqslant T$. In particular, we can take $t = 0 \geqslant T$ and conclude that $p = \phi_p(0) \in \mathcal{R}$, and this is in contradiction with the assumption that $p \in \mathcal{M} \setminus \{(0, 0)\}$.

As a byproduct of (2.34), we obtain that, for all $T \leqslant 0$,

$$\phi_p(T) \in \{u \in (0,1),\, u+v \geqslant 0\} \subseteq \{\dot{u} = -u(u+v) \leqslant 0\}.$$

In particular

$$u(t_n) - u(t_{n+1}) = \int_{t_{n+1}}^{t_n} \dot{u}(\tau)\, d\tau \leqslant 0,$$

which is in contradiction with (2.33), and consequently we have established the desired result. ☐

Now we show that the ω-limit of any point lying on the global center manifold coincides with the origin, according to the next result:

Lemma 2.5. *If $p \in \mathcal{M}$, then its ω-limit is $(0,0)$.*

Proof. We observe that, for every $t > 0$,

$$\phi_p(t) \in [0,1] \times [0,1]. \tag{2.35}$$

Indeed, by (2.31), one sees that, for $t > 0$, $\phi_t(p)$ cannot cross $\{0\} \times [0,1]$, $\{1\} \times [0,1]$ and $[0,1] \times \{1\}$, hence the only possible escape side is given by $[0,1] \times \{0\}$. Therefore, to prove (2.35), we suppose, by contradiction, that there exists $t_0 \geqslant 0$ such that $\phi_p(t_0) \in [0,1] \times \{0\}$, that is $v(t_0) = 0$. Since $(0,0)$ is an equilibrium, it follows that $u(t_0) \neq 0$. In particular, $u(t_0) > 0$ and accordingly $\dot{v}(t_0) = -au(t_0) < 0$. This means that $v(t_0 + \varepsilon) < 0$ for all $\varepsilon \in (0, \varepsilon_0)$ for a suitable $\varepsilon_0 > 0$. Looking again at the velocity fields, this entails that $\phi_p(t) \in (0,1) \times (-\infty, 0)$ for all $t > \varepsilon_0$. Consequently, $\phi_p(t)$ cannot approach the straight line $\rho v - au = 0$ for $t > \varepsilon_0$.

This, combined with Lemma 2.4, says that the trajectory emanating from p can never approach the straight line $\rho v - au = 0$ at the origin, in contradiction with the definition of $\mathcal{M}$, and thus the proof of (2.35) is complete.

From (2.35) and the Poincaré-Bendixson Theorem (see e.g. [20]), we deduce that the ω-limit of p can be either a cycle, or an equilibrium, or a union of (finitely many) equilibria and non-closed orbits connecting these equilibria. We observe that the ω-limit of p cannot be a cycle, since $\dot{u}$ has a sign in $[0,1] \times [0,1]$. Moreover, it cannot contain the sink $(0,1)$, due to Lemma 2.4. Hence, the only possibility is that the ω-limit of p coincides with $(0,0)$, which is the desired result. ☐

As a consequence of Lemma 2.5 and the fact that $\dot{u} < 0$ in $(0,1] \times [0,1]$, we obtain the following statement:

Corollary 2.6. *Every trajectory in $\mathcal{M}$ has the form $\{\phi_p(t),\, t \in \mathbb{R}\}$, with*

$$\lim_{t \to +\infty} \phi_p(t) = (0,0)$$

and there exists $t_p \in \mathbb{R}$ such that $\phi_p(t_p) \in \big(\{1\} \times [0,1]\big) \cup \big([0,1] \times \{1\}\big)$.

The result in Corollary 2.6 can be sharpened in view of the following statement (which can be seen as the counterpart of Proposition 2.1 in the degenerate case $ac = 1$): namely, since the center manifold can in principle contain many different trajectories (see e.g. Figure 5.3 in [3]), we provide a tailor-made argument that excludes this possibility in the specific case that we deal with.

Proposition 2.7. *For $ac = 1$ $\mathcal{M}$ contains one, and only one, trajectory, which is asymptotic to the origin as $t \to +\infty$, and that can be written as a graph $\gamma : [0, u_{\mathcal{M}}] \to [0, v_{\mathcal{M}}]$, for some $(u_{\mathcal{M}}, v_{\mathcal{M}}) \in \big(\{1\}\times[0,1]\big)\cup\big((0,1]\times\{1\}\big)$, where γ is an increasing C^2 function such that $\gamma(0) = 0$, $\gamma(u_{\mathcal{M}}) = v_{\mathcal{M}}$ and the graph of γ at the origin is tangent to the line $\rho v - au = 0$.*

Proof. First of all, we show that

$$\mathcal{M} \text{ contains one, and only one, trajectory.} \tag{2.36}$$

Suppose, by contradiction, that $\mathcal{M}$ contains two different orbits, that we denote by $\mathcal{M}_-$ and $\mathcal{M}_+$. Using Corollary 2.6, we can suppose that $\mathcal{M}_+$ lies above $\mathcal{M}_-$ and

$$\begin{gathered}\text{the region } \mathcal{P} \subset [0,1]\times[0,1] \text{ contained between } \mathcal{M}_+ \text{ and } \mathcal{M}_- \\ \text{lies in } \{\dot{u} < 0\}.\end{gathered} \tag{2.37}$$

Consequently, for every $p \in \mathcal{P}$, it follows that

$$\lim_{t\to+\infty} \phi_p(t) = (0,0). \tag{2.38}$$

In particular, we can take an open ball $B \subset \mathcal{P}$ in the vicinity of the origin, denote by $\mu(t)$ the Lebesgue measure of $\mathcal{S}(t) := \{\phi_p(t),\ p \in B\}$, and write that $\mu(0) > 0$ and

$$\lim_{t\to+\infty} \mu(t) = 0. \tag{2.39}$$

We point out that $\mathcal{S}(t)$ lies in the vicinity of the origin for all $t \geqslant 0$, thanks to (2.37). As a consequence, for all t, $\tau > 0$, changing variable

$$y := \phi_x(\tau) = x + \int_0^\tau \frac{d\phi_x(\theta)}{d\theta}\,d\theta = x + \tau\frac{d\phi_x(0)}{dt} + O(\tau^2),$$

we find that

$$\begin{aligned}
\mu(t+\tau) &= \int_{\mathcal{S}(t+\tau)} dy \\
&= \int_{\mathcal{S}(t)} \big|\det\big(D_x\phi_x(\tau)\big)\big|\,dx \\
&= \int_{\mathcal{S}(t)} \left|\det D_x\left(x + \tau\frac{d\phi_x(0)}{dt} + O(\tau^2)\right)\right|\,dx \\
&= \int_{\mathcal{S}(t)} \left(1 + \tau\,\mathrm{Tr}\left(D_x\frac{d\phi_x(0)}{dt}\right) + O(\tau^2)\right)\,dx \\
&= \mu(t) + \tau\int_{\mathcal{S}(t)} \mathrm{Tr}\left(D_x\frac{d\phi_x(0)}{dt}\right)\,dx + O(\tau^2),
\end{aligned}$$

where Tr denotes the trace of a (2×2)-matrix.

As a consequence,

$$\frac{d\mu}{dt}(t) = \int_{\mathcal{S}(t)} \mathrm{Tr}\left(D_x\frac{d\phi_x(0)}{dt}\right)\,dx. \tag{2.40}$$

Also, using the notation $x = (u, v)$, we can write (1.1) when $ac = 1$ in the form

$$\frac{d\phi_x}{dt}(t) = \dot{x}(t) = \begin{pmatrix} \dot{u}(t) \\ \dot{v}(t) \end{pmatrix} = \begin{pmatrix} -u(t)(u(t)+v(t)) \\ \rho v(t)(1-u(t)-v(t)) - au(t) \end{pmatrix}.$$

Accordingly,

$$D_x \frac{d\phi_x(0)}{dt} = \begin{pmatrix} -\partial_u\big(u(u+v)\big) & -\partial_v\big(u(u+v)\big) \\ \partial_u\big(\rho v(1-u-v) - au\big) & \partial_v\big(\rho v(1-u-v) - au\big) \end{pmatrix},$$

whence

$$\begin{aligned} \mathrm{Tr}\left(D_x \frac{d\phi_x(0)}{dt}\right) &= -\partial_u\big(u(u+v)\big) + \partial_v\big(\rho v(1-u-v) - au\big) \\ &= -2u - v + \rho(1-u-v) - \rho v \\ &= \rho + O(|x|) \end{aligned} \tag{2.41}$$

for x near the origin.

As a result, recalling (2.38), we can take t sufficiently large, such that $\mathcal{S}(t)$ lies in a neighborhood of the origin, exploit (2.41) to write that $\mathrm{Tr}\left(D_x \frac{d\phi_x(0)}{dt}\right) \geqslant \frac{\rho}{2}$ and then (2.40) to conclude that

$$\frac{d\mu}{dt}(t) \geqslant \frac{\rho}{2} \int_{\mathcal{S}(t)} dx = \frac{\rho}{2}\mu(t).$$

This implies that $\mu(t)$ diverges (exponentially fast) as $t \to +\infty$, which is in contradiction with (2.39). The proof of (2.36) is thereby complete.

Now, we check the other claims in the statement of Proposition 2.7. The asymptotic property as $t \to +\infty$ is a consequence of Corollary 2.6. Also, the graphical property as well as the monotonicity property of the graph follow from the fact that $\mathcal{M} \subset \{\dot{u} < 0\}$. The smoothness of the graph follows from the smoothness of the center manifold. The fact that $\gamma(0) = 0$ and $\gamma(u_{\mathcal{M}}) = v_{\mathcal{M}}$ follow also from Corollary 2.6. The tangency property at the origin is a consequence of the tangency property of the center manifold to the center eigenspace. □

As a byproduct of the proof of Proposition 2.7 we also obtain the following information:

Corollary 2.8. *Suppose that $ac = 1$. Then , we have that $\dot{u} \leqslant 0$ in $[0, 1] \times [0, 1]$, and the curve (2.2) divides the square $[0, 1] \times [0, 1]$ into two regions $\mathcal{A}_1$ and $\mathcal{A}_2$, defined in (2.10).*

Furthermore, the sets $\mathcal{A}_1$ and $\mathcal{A}_2$ are separated by the curve $\dot{v} = 0$, given by the graph of the continuous function σ given in (2.6).

In addition,

$$\mathcal{M} \subset \mathcal{A}_2. \tag{2.42}$$

2.3 Completion of the proof of Theorem 1.2

We observe that, by the Stable Manifold Theorem and the Center Manifold Theorem, the statement in (v) of Theorem 1.2 is obviously fulfilled.

Hence, to complete the proof of Theorem 1.2, it remains to show that the statement in (iv) holds true. To this aim, exploiting the useful pieces of information in Propositions 2.1 and 2.7, we first give a characterization of the sets $\mathcal{E}$ and $\mathcal{B}$:

Proposition 2.9. *The following characterizations of the sets in* (1.9) *and* (1.10) *are true:*

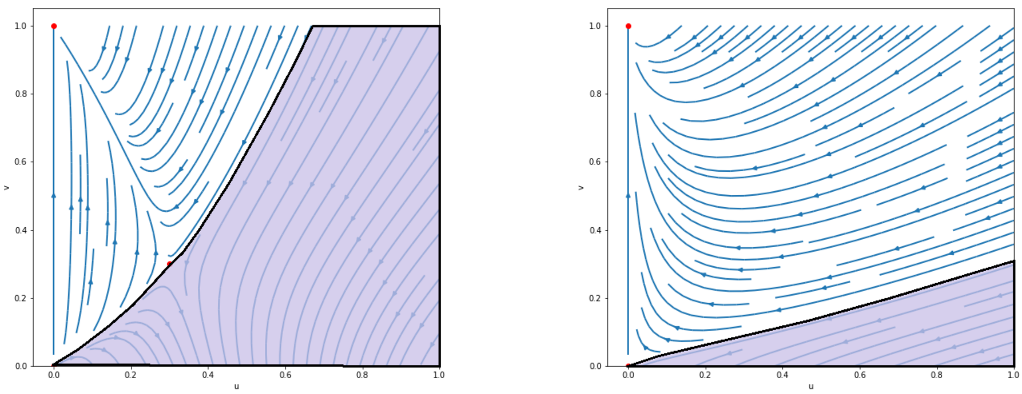
$$\begin{aligned}\mathcal{E} = \Big\{(u,v) \in [0,1]\times[0,1] \;\; s.t. \;\; v < \gamma(u) \;\, if\; u \in [0, u_{\mathcal{M}}] \\ and\; v \leqslant 1 \;\; if\; u \in (u_{\mathcal{M}}, 1]\Big\},\end{aligned} \tag{2.43}$$

and

$$\mathcal{B} = \Big\{(u,v) \in [0,u_{\mathcal{M}}]\times[0,1] \;\; s.t. \;\; v > \gamma(u) \;\, if\; u \in [0, u_{\mathcal{M}}]\Big\}, \tag{2.44}$$

for some $(u_{\mathcal{M}}, v_{\mathcal{M}}) \in \partial\left([0,1]\times[0,1]\right)$.

One can visualize the appearance of the set $\mathcal{E}$ in (2.43) in two particular cases in Figure 4.



(a) $a = 0.8$, $c = 0.5$, $\rho = 2$ (b) $a = 0.8$, $c = 3$, $\rho = 2$

Figure 4: *The figures show the phase portrait for the indicated values of the coefficients. In blue, the orbits of the points. The red dots show the equilibria. In violet, the set* $\mathcal{E}$.

Proof of Proposition 2.9. We let γ be the parametrization of $\mathcal{M}$, as given by Propositions 2.1 (when $ac \neq 1$) and 2.7 (when $ac = 1$), and we consider the sets

$$\begin{aligned} &\mathcal{X} := \Big\{(u,v) \in [0,1]\times[0,1] \;\; \text{s.t.} \;\; v < \gamma(u)\Big\} \\ \text{and} \qquad &\mathcal{Y} := \Big\{(u,v) \in [0,1]\times[0,1] \;\; \text{s.t.} \;\; v > \gamma(u)\Big\}.\end{aligned}$$

The goal is to prove that $\mathcal{X} \equiv \mathcal{E}$ and $\mathcal{Y} \equiv \mathcal{B}$. We observe that, when $u_{\mathcal{M}} = 1$, then $\mathcal{X} \cup \mathcal{Y} \cup \mathcal{M} = [0,1]\times[0,1]$. When instead $u_{\mathcal{M}} \in (0,1)$, then $\mathcal{X} \cup \mathcal{Y} \cup \mathcal{M} \cup$

$\big((u_{\mathcal{M}},1]\times[0,1]\big)=[0,1]\times[0,1]$. Accordingly, if we show that

$$\mathcal{X}\cup\big((u_{\mathcal{M}},1]\times[0,1]\big)\subseteq\mathcal{E} \tag{2.45}$$
$$\text{and}\qquad \mathcal{Y}\subseteq\mathcal{B}, \tag{2.46}$$

we are done.

Hence, we now focus on the proof of (2.45). Namely, recalling (1.10), we show that

$$\text{all trajectories starting in } \mathcal{X} \text{ exit the set on the side } (0,1]\times\{0\}. \tag{2.47}$$

For this, we first notice that, gathering together (2.8), (2.9), (2.19), (2.20) and (2.27), we find that

$$\text{no limit cycle exists in } [0,1]\times[0,1] \tag{2.48}$$

(in the case $0<ac<1$, and the same holds true in the case $ac\geqslant 1$ since $\dot u$ has a sign).

In addition,

$$\text{the } \omega\text{-limit of any point in } \mathcal{X} \text{ cannot contain an equilibrium.} \tag{2.49}$$

Indeed, by Propositions 2.1 (when $(ac\neq 1)$ and 2.7 (when $ac=1$), we have that $\gamma(0)=0<1$, and therefore $(0,1)\notin\overline{\mathcal{X}}$. Moreover, if $ac<1$, a trajectory in $\mathcal{X}$ cannot converge to (u_s,v_s), since $\mathcal{X}$ does not contain points of the stable manifold $\mathcal{M}$, nor to $(0,0)$, since this is a repulsive equilibrium and no trajectory converges here. If instead $ac\geqslant 1$, then it cannot converge to $(0,0)$, since $\mathcal{X}$ does not contain points of $\mathcal{M}$. These observations completes the proof of (2.49).

From (2.48), (2.49) and the Poincaré-Bendixson Theorem (see e.g. [20]), we have that every trajectory starting in $\mathcal{X}$ leaves the set (possibly in infinite time).

If the trajectory leaves at $t=+\infty$, then it converges to some equilibrium on $\partial\mathcal{X}$, which is in contradiction with (2.49).

As a consequence a trajectory in $\mathcal{X}$ leaves the set in finite time. Suppose that a trajectory leaves $\mathcal{X}$ at a point $(u,v)\in\partial\mathcal{X}$; then either $(u,v)\in\mathcal{M}$ or $(u,v)\in\partial([0,1]\times[0,1])$. The first possibility is impossible, otherwise the starting point of the trajectory would converge to (u_s,v_s). Hence, the only possibility is that the trajectory leaves $\mathcal{X}$ at $(u,v)\in\partial([0,1]\times[0,1])$. By Proposition 1.1 this is possible only if $u>0$ and $v=0$, which proves (2.47). As a consequence of (2.47) we obtain that

$$\mathcal{X}\subseteq\mathcal{E}. \tag{2.50}$$

We now claim that

$$\big((u_{\mathcal{M}},1]\times[0,1]\big)\subseteq\mathcal{E}. \tag{2.51}$$

To this end, we observe that there are neither cycles nor equlibria in $(u_{\mathcal{M}},1]\times[0,1]$, and therefore we can use the Poincaré-Bendixson Theorem (see e.g. [20]) to conclude that any trajectory starting in $(u_{\mathcal{M}},1]\times[0,1]$ must exit the set. Also, the inward normal velocity along the sides $\{1\}\times(0,1]$ and $(u_{\mathcal{M}},1)\times\{1\}$ is positive, and thus no trajectory can exit from these sides. Now, if a trajectory exits $(u_{\mathcal{M}},1]\times[0,1]$ from the side $\{u_{\mathcal{M}}\}\times(0,1)$, then it enters the set $\mathcal{X}$, and therefore (2.51) is a consequence of (2.50) in this case. If instead a trajectory exits $(u_{\mathcal{M}}, 1] \times [0, 1]$ from the side $(0, 1) \times \{0\}$, then we directly obtain (2.51).

From (2.50) and (2.51) we obtain (2.45), as desired.

We now prove (2.46), namely we show that

$$\text{for all } (u_0, v_0) \in \mathcal{Y} \text{ we have that } (u(t), v(t)) \to (0, 1) \text{ as } t \to +\infty. \tag{2.52}$$

To this end, we observe that (u_s, v_s) (if $0 < ac < 1$) and $(0, 0)$ are not in $\mathcal{Y}$. Moreover, no trajectory starting in $\mathcal{Y}$ converges to (u_s, v_s) (if $0 < ac < 1$), nor to $(0, 0)$, since $\mathcal{Y}$ does not contain points on $\mathcal{M}$.

In addition, recalling (2.48), we have that there are no limit cycles in $\mathcal{Y}$. As a consequence, by the Poincaré-Bendixson Theorem (see e.g. [20]), we have that every trajectory starting in $\mathcal{Y}$ either go to $(0, 1)$ or it exits the set at some point of $\partial\mathcal{Y}$.

In the latter case, since no trajectory can cross $\mathcal{M}$, the only possibility is that the trajectory exits $\mathcal{Y}$ at some point $(u, v) \in \partial\big([0, 1] \times [0, 1]\big)$. We notice that, since γ is increasing, we have that $\gamma(u) > 0$ for all $u > 0$. As a consequence,

$$\text{if } (u, v) \in \overline{\mathcal{Y}}, \text{ then } v > \gamma(u) > 0 \text{ for all } u > 0. \tag{2.53}$$

Now, thanks to Proposition 1.1, the only possibility that a trajectory exits $\mathcal{Y}$ at some point $(u, v) \in \partial\big([0, 1] \times [0, 1]\big)$ is for $u > 0$ and $v = 0$, which would contradict (2.53).

As a result, the only remaining possibility is that a trajectory in $\mathcal{Y}$ converges to $(0, 1)$, which proves (2.52). Hence, the proof of (2.46) is complete as well. □

With this, we are now able to complete the proof of Theorem 1.2:

Proof of (iv) of Theorem 1.2. The statement in (iv) of Theorem 1.2 is a direct consequence of the parametrization of the manifold $\mathcal{M}$, as given by Proposition 2.1 for $ac \neq 1$ and by Proposition 2.7 for $ac = 1$, and the characterization of the sets $\mathcal{B}$ and $\mathcal{E}$, as given by Proposition 2.9. □

3 Dependence of the dynamics on the parameters

In this section we discuss the dependence on the parameters involved in the system (1.1).

The dynamics of the system in (1.1) depends qualitatively only on ac, but of course the position of the saddle equilibrium and the size and shape of the basins of attraction depend quantitatively upon all the parameters. Here we perform a deep analysis on each parameter separately.

We notice that the system in (1.1) does not present a variational structure, due to the presence of the terms $-acu$ in the first equation and $-au$ in the second one, that are of first order in u. Thus, the classical methods of the calculus of variations cannot be used and we have to make use of ad-hoc arguments, of geometrical flavour.

3.1 Dependence of the dynamics on the parameter c

We start by studying the dependence on c, that represents the losses (soldier death and missing births) caused by the war for the first population with respect to the second one. In the following proposition, we will express the dependence on c of the basin of attraction $\mathcal{E}$ in (1.10) by writing explicitly $\mathcal{E}(c)$.

Proposition 3.1 (Dependence of the dynamics on c). *With the notation in* (1.10)*, we have that*

(i) If $0 < c_1 < c_2$*, then* $\mathcal{E}(c_2) \subset \mathcal{E}(c_1)$ *.*

(ii) It holds that

$$\bigcap_{c>0} \mathcal{E}(c) = (0, 1] \times \{0\}. \tag{3.1}$$

We remark that the behavior for c sufficiently small is given by (i) of Theorem 1.2: in this case, there is a saddle point inside the domain $[0, 1] \times [0, 1]$, thus $\mathcal{E}(c) \neq (0, 1] \times [0, 1]$. On the other hand, as $c \to +\infty$, the set $\mathcal{E}(c)$ gets smaller and smaller until the first population has no chances of victory if the second population has a positive size.

The parameter c appears only in the first equation and it is multiplied by $-au$, that is always negative in the domain we are interested in. Thus, the dependence on c is independent of the other parameters. As one would expect, Proposition 3.1 tells us that the greater the cost of the war for the first population, the fewer possibilities of victory there are for it.

Proof of Proposition 3.1. (i) We take $c_2 > c_1 > 0$. According to Theorem 1.2, we denote by (u_s^2, v_s^2) the coexistence equilibrium for the parameter c_2 if $ac_2 < 1$, otherwise we set $(u_s^2, v_s^2) = (0, 0)$; similarly, we call (u_s^1, v_s^1) the coexistence equilibrium for the parameter c_1 if $ac_1 < 1$, and in the other cases we set $(u_s^1, v_s^1) = (0, 0)$.

We observe that

$$v_s^2 \leqslant v_s^1. \tag{3.2}$$

Indeed, if $ac_2 < 1$ then also $ac_1 < 1$, and therefore, using the characterization in (1.11),

$$\frac{\partial v_s}{\partial c} = \frac{-a(1 + \rho c) - \rho(1 - ac)}{(1 + \rho c)^2} = \frac{-a - \rho}{(1 + \rho c)^2} < 0,$$

which implies (3.2) in this case. If instead $ac_2 \geqslant 1$ then the inequality in (3.2) is trivially satisfied, thanks to (i), (ii) and (iii) of Theorem 1.2.

Now, in the notation of Propositions 2.1 (if $ac \neq 1$) and 2.7 (if $ac = 1$), thanks to the characterization in (2.43), if we prove that

$$\gamma_{c_1}(u) > \gamma_{c_2}(u) \quad \text{for any } u \in (0, \min\{u_{\mathcal{M}}^{c_1}, u_{\mathcal{M}}^{c_2}\}], \tag{3.3}$$

then the inclusion in (i) is shown. Hence, we now focus on the proof of (3.3).

To this end, we observe that, since γ_{c_1} is an increasing function, its inverse function $f_{c_1} : [0, v_{\mathcal{M}}^{c_1}] \to [0, u_{\mathcal{M}}^{c_1}]$ is well defined and is increasing as well. In an analogue fashion, we define $f_{c_2}(v)$ as the inverse of $\gamma_{c_2}(u)$. We point out that the inequality in (3.3) holds true if

$$f_{c_1}(v) < f_{c_2}(v) \quad \text{for any } v \in (0, \min\{v_{\mathcal{M}}^{c_1}, v_{\mathcal{M}}^{c_2}\}]. \tag{3.4}$$

Accordingly, we will show (3.4) in three steps.

First, in light of (3.2), we show that

$$\text{the claim in (3.4) is true in the interval } [v_s^2, v_s^1] \cap (0, +\infty). \tag{3.5}$$

For this, if $ac_1 \geqslant 1$, then also $ac_2 \geqslant 1$, and therefore $v_s^1 = v_s^2 = 0$, thanks to (ii) and (iii) in Theorem 1.2. Accordingly, in this case the interval $[v_s^2, v_s^1]$ coincides with the singleton $\{0\}$, and so there is nothing to prove.

Otherwise, we recall that the curve $u = \sigma(v)$, given in (2.6) and representing the points where $\dot{v} = 0$, is independent of c. Moreover, thanks to formula (2.7) in Corollary 2.2 if $ac < 1$, formula (2.11) in Corollary 2.3 if $ac > 1$, and formula (2.42) in Corollary 2.8 if $ac = 1$ (see also Figure 3), we have that $f_{c_1}(v) < \sigma(v)$ for $v < v_s^1$ and $f_{c_2}(v) > \sigma(v)$ for $v > v_s^2$, which proves (3.5) in the open interval (v_s^2, v_s^1).

Moreover, it holds that

$$f_{c_1}(v_s^1) = \sigma(v_s^1) < f_{c_2}(v_s^1), \tag{3.6}$$

and (if $ac_2 < 1$, otherwise $v_s^2 = 0$ and there is no need to perform this computation)

$$f_{c_1}(v_s^2) < \sigma(v_s^2) = f_{c_2}(v_s^2). \tag{3.7}$$

This completes the proof of (3.5).

Next we show that

$$\text{the claim in (3.4) is true in the interval } (0, v_s^2). \tag{3.8}$$

If $ac_2 \geqslant 1$, then $v_s^2 = 0$, and so the claim in (3.8) is trivial. Hence, we suppose that $ac_2 < 1$ and we argue by contradiction, assuming that for some $v \in (0, v_s^2)$ it holds that $f_{c_1}(v) \geqslant f_{c_2}(v)$. As a consequence, we can define

$$\bar{v} := \sup\left\{v \in (0, v_s^2) \ \text{ s.t. } \ f_{c_1}(v) \geqslant f_{c_2}(v)\right\}.$$

We observe that, by continuity, we have that $f_{c_1}(\bar{v}) = f_{c_2}(\bar{v})$, and therefore, by (3.5), we see that $\bar{v} < v_s^2$. As a result, since $f_{c_1}(v) < f_{c_2}(v)$ for every $v \in (\bar{v}, v_s^2]$, then it holds that

$$\frac{df_{c_1}}{dv}(\bar{v}) < \frac{df_{c_2}}{dv}(\bar{v}). \tag{3.9}$$

On the other hand, we can compute the derivatives by exploiting the fact that γ_{c_1} and γ_{c_2} follow the flux for the system (1.1). Namely, setting $\bar{u} := f_{c_1}(\bar{v})$, we have that

$$\begin{aligned} &\frac{df_{c_1}}{dv}(\bar{v}) = \frac{\dot{u}}{\dot{v}}(\bar{v}) = \frac{\bar{u}(1-\bar{u}-\bar{v}) - ac_1\bar{u}}{\rho\bar{v}(1-\bar{u}-\bar{v}) - a\bar{u}} \\ \text{and} \qquad &\frac{df_{c_2}}{dv}(\bar{v}) = \frac{\dot{u}}{\dot{v}}(\bar{v}) = \frac{\bar{u}(1-\bar{u}-\bar{v}) - ac_2\bar{u}}{\rho\bar{v}(1-\bar{u}-\bar{v}) - a\bar{u}}. \end{aligned}$$

Now, since $\bar{v} \in [0, v_s^1)$, we have that $\rho\bar{v}(1-\bar{u}-\bar{v}) - a\bar{u} > 0$ (recall (2.7) and notice that $(\bar{u}, \bar{v}) \in \mathcal{A}_4$). This and the fact that $c_2 > c_1$ give that

$$\frac{df_{c_1}}{dv}(\bar{v}) > \frac{df_{c_2}}{dv}(\bar{v}),$$

which is in contradiction with (3.9), thus establishing (3.8).

Now we prove that

$$\text{the claim in (3.4) is true in the interval } (v_s^1, \min\{u_{\mathcal{M}}^{c_1}, u_{\mathcal{M}}^{c_2}\}]. \tag{3.10}$$

Indeed, if $ac_1 < 1$, we argue towards a contradiction, supposing that there exists $v > v_s^1$ such that $f_{c_1}(v) \geqslant f_{c_2}(v)$. Hence, we can define

$$\widehat{v} := \inf\left\{v > v_s^1 \ \text{ s.t. } \ f_{c_1}(v) \geqslant f_{c_2}(v)\right\},$$

and we deduce from (3.6) that $\widehat{v} > v_s^1$. By continuity, we see that $f_{c_1}(\widehat{v}) = f_{c_2}(\widehat{v})$. Therefore, since $f_{c_1}(v) < f_{c_2}(v)$ for any $v < \widehat{v}$, we conclude that

$$\frac{df_{c_1}}{dv}(\widehat{v}) > \frac{df_{c_2}}{dv}(\widehat{v}). \tag{3.11}$$

On the other hand, setting $\widehat{u} := f_{c_1}(\widehat{v})$ and exploiting (1.1), we get that

$$\frac{df_{c_1}}{dv}(\widehat{v}) = \frac{\dot{u}}{\dot{v}}(\widehat{v}) = \frac{\widehat{(1-\widehat{u}-\widehat{v})} - ac_1\widehat{u}}{\rho\widehat{v}(1-\widehat{u}-\widehat{v}) - a\widehat{u}}$$

$$\text{and} \qquad \frac{df_{c_2}}{dv}(\widehat{v}) = \frac{\widehat{u}}{\widehat{v}}(\widehat{v}) = \frac{\rho\widehat{u}(1-\widehat{u}-\widehat{v}) - ac_2\widehat{u}}{\widehat{v}(1-\widehat{u}-\widehat{v}) - a\widehat{u}}.$$

Moreover, recalling (2.5) and (2.7), we have that $(f_{c_1}(\widehat{v}), \widehat{v})$ and $(f_{c_2}(\widehat{v}), \widehat{v})$ belong to the interior of $\mathcal{A}_2$, and therefore $\rho\widehat{v}(1-\widehat{u}-\widehat{v}) - a\widehat{u} < 0$. This ad the fact that $c_2 > c_1$ give that

$$\frac{df_{c_1}}{dv}(\widehat{v}) < \frac{df_{c_2}}{dv}(\widehat{v}),$$

which is in contradiction with (3.11). This establishes (3.10) in this case.

If instead $ac_1 \geqslant 1$, then also $ac_2 \geqslant 1$, and therefore we have that $(u_s^2, v_s^2) = (u_s^1, v_s^1) = (0,0)$. In this setting, we use Propositions 2.1 and 2.7 to say that at $v = 0$ the function f_{c_1} is tangent to the line $(\rho - 1 + ac_1)v - au = 0$, while f_{c_2} is tangent to $(\rho - 1 + ac_2)v - au = 0$. Now, since

$$\frac{\rho-1}{a} + c_1 < \frac{\rho-1}{a} + c_2,$$

we have that for positive v the second line is above the first one. Also, thanks to the fact that f_{c_1} and f_{c_2} are tangent to these lines, we conclude that there exists $\varepsilon > 0$ such that

$$f_{c_1}(v) < f_{c_2}(v) \quad \text{for any} \quad v < \varepsilon. \tag{3.12}$$

Now, we suppose by contradiction that there exists some $v > 0$ such that $f_{c_1}(v) \geqslant f_{c_2}(v)$. Hence, we can define

$$\tilde{v} := \inf\left\{v > 0 \ \text{ s.t. } \ f_{c_1}(v) \geqslant f_{c_2}(v)\right\}.$$

In light of (3.12), we have that $\tilde{v} \geqslant \varepsilon > 0$. Moreover, by continuity, we see that $f_{c_1}(\tilde{v}) = f_{c_2}(\tilde{v})$. Accordingly, since $f_{c_1}(v) < f_{c_2}(v)$ for any $v < \tilde{v}$, then it must be

$$\frac{df_{c_1}}{dv}(\tilde{v}) > \frac{df_{c_2}}{dv}(\tilde{v}). \tag{3.13}$$

On the other hand, setting $\tilde{u} := f_{c_1}(\tilde{v})$ and exploiting (1.1), we see that

$$\frac{df_{c_1}}{dv}(\tilde{v}) = \frac{\dot{u}}{\dot{v}}(\tilde{v}) = \frac{\tilde{u}(1-\tilde{u}-\tilde{v}) - ac_1\tilde{u}}{\rho\tilde{v}(1-\tilde{u}-\tilde{v}) - a\tilde{u}}$$

$$\text{and} \qquad \frac{df_{c_2}}{dv}(\tilde{v}) = \frac{\tilde{u}}{\tilde{v}}(\tilde{v}) = \frac{\rho\tilde{u}(1-\tilde{u}-\tilde{v}) - ac_2\tilde{u}}{\tilde{v}(1-\tilde{u}-\tilde{v}) - a\tilde{u}}.$$

Now, thanks to (2.10) and (2.11), we have that $(f_{c_1}(\tilde{v}), \tilde{v})$ and $(f_{c_2}(\tilde{v}), \tilde{v})$ belong to the interior of $\mathcal{A}_2$, and therefore $\rho\tilde{v}(1-\tilde{u}-\tilde{v}) - a\tilde{u} < 0$. This ad the fact that $c_2 > c_1$ give that

$$\frac{df_{c_1}}{dv}(\tilde{v}) < \frac{df_{c_2}}{dv}(\tilde{v}),$$

which is in contradiction with (3.13). This completes the proof of (3.10).

Gathering together (3.5), (3.8) and (3.10), we obtain (3.4), as desired.

(ii) We first show that for all $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that for all $c \geqslant c_\varepsilon$ it holds that

$$\mathcal{E}(c) \subset \big\{(u,v) \in [0,1] \times [0,1] \text{ s.t. } v < \varepsilon u\big\}. \tag{3.14}$$

The inclusion in (3.14) is also equivalent to

$$\big\{(u,v) \in [0,1] \times [0,1] \text{ s.t. } v > \varepsilon u\big\} \subset \mathcal{B}(c), \tag{3.15}$$

and the strict inequality is justified by the fact that $\mathcal{E}(c)$ and $\mathcal{B}(c)$ are separated by $\mathcal{M}$, according to Proposition 2.9. We now establish the inclusion in (3.15). For this, let

$$\mathcal{T}_\varepsilon := \big\{(u,v) \in [0,1] \times [0,1] \text{ s.t. } v > \varepsilon u\big\}. \tag{3.16}$$

Now, we can choose c large enough such that the condition $ac \geqslant 1$ is fulfilled. In this way, thanks to (ii) and (iii) of Theorem 1.2, the only equilibria are the points $(0,0)$ and $(0,1)$.

Now, the component of the velocity in the inward normal direction to $\mathcal{T}_\varepsilon$ on the side $\{v = \varepsilon u\}$ is given by

$$\begin{aligned}
(\dot{u}, \dot{v}) \cdot \frac{(-\varepsilon, 1)}{\sqrt{1+\varepsilon^2}} &= \frac{\dot{v} - \varepsilon\dot{u}}{\sqrt{1+\varepsilon^2}} \\
&= \frac{1}{\sqrt{1+\varepsilon^2}}\big(\rho v(1-u-v) - au - \varepsilon u(1-u-v) + \varepsilon acu\big) \\
&= \frac{1}{\sqrt{1+\varepsilon^2}}\big[(\rho v - \varepsilon u)(1-u-v) + (\varepsilon c - 1)au\big] \\
&= \frac{1}{\sqrt{1+\varepsilon^2}}\big[(\rho\varepsilon u - \varepsilon u)(1-u-\varepsilon u) + (\varepsilon c - 1)au\big],
\end{aligned}$$

that is positive for

$$c > c_\varepsilon := \frac{2\varepsilon(1+\rho)+a}{\varepsilon a}. \tag{3.17}$$

This says that no trajectory in $\mathcal{T}_\varepsilon$ can exit $\mathcal{T}_\varepsilon$ from the side $\{v = \varepsilon u\}$.

The other parts of $\partial\mathcal{T}_\varepsilon$ belong to $\partial((0,1)\times(0,1))$ but not to $[0,1]\times\{0\}$. As a consequence, by Proposition 1.1,

$$\text{every trajectory in } \mathcal{T}_\varepsilon \text{ belongs to } \mathcal{T}_\varepsilon \text{ for all } t \geqslant 0. \tag{3.18}$$

From this, (2.48) and the Poincaré-Bendixson Theorem (see e.g. [20]), we conclude that the ω-limit of any trajectory starting in $\mathcal{T}_\varepsilon$ can be either an equilibrium or a union of (finitely many) equilibria and non-closed orbits connecting these equilibria.

Now, we claim that, possibly taking c larger in (3.17),

$$\mathcal{M} \subset \big([0,1]\times[0,1]\big) \setminus \mathcal{T}_\varepsilon. \tag{3.19}$$

Indeed, suppose by contradiction that there exists $(\tilde u, \tilde v) \in \mathcal{M} \cap \mathcal{T}_\varepsilon$. Then, in light of (3.18), a trajectory passing through $(\tilde u, \tilde v)$ and converging to $(0,0)$ has to be entirely contained in $\mathcal{T}_\varepsilon$.

On the other hand, by Propositions 2.1 and 2.7, we know that at $u = 0$ the manifold $\mathcal{M}$ is tangent to the line $(\rho - 1 + ac)v - au = 0$. Hence, if we choose c large enough such that

$$\frac{a}{\rho - 1 + ac} < \varepsilon,$$

we obtain that this line is below the line $v = \varepsilon u$, thus reaching a contradiction. This establishes (3.19).

From (3.19), we deduce that, given $(\tilde u, \tilde v) \in \mathcal{T}_\varepsilon$, and denoting $\omega_{(\tilde u, \tilde v)}$ the ω-limit of $(\tilde u, \tilde v)$,

$$\omega_{(\tilde u, \tilde v)} \neq \{(0,0)\}, \tag{3.20}$$

provided that c is taken large enough.

Furthermore, $\omega_{(\tilde u, \tilde v)}$ cannot consist of the two equilibria $(0,0)$ and $(0,1)$ and non-closed orbits connecting these equilibria, since $(0,1)$ is a sink. As a consequence of this and (3.20), we obtain that $\omega_{(\tilde u, \tilde v)} = \{(0,1)\}$ for any $(\tilde u, \tilde v) \in \mathcal{T}_\varepsilon$, provided that c is large enough.

Thus, recalling (1.9) and (3.16), this proves (3.15), and therefore (3.14).

Now, using (3.14), we see that for every $\varepsilon > 0$,

$$\bigcap_{c>0} \mathcal{E}(c) \subseteq \mathcal{E}(c_\varepsilon) \subseteq \big\{(u,v) \in [0,1]\times[0,1] \ \text{ s.t. } \ v < \varepsilon u\big\}.$$

Accordingly,

$$\bigcap_{c>0} \mathcal{E}(c) \subseteq \bigcap_{\varepsilon>0} \big\{(u,v) \in [0,1]\times[0,1] \ \text{ s.t. } \ v < \varepsilon u\big\} = (0,1]\times\{0\},$$

which implies (3.1), as desired. □

3.2 Dependence of the dynamics on the parameter ρ

Now we analyze the dependence of the dynamics on the parameter ρ, that is the fitness of the second population v with respect to the fitness of the first one u.

In the following proposition, we will make it explicit the dependence on ρ by writing $\mathcal{E}(\rho)$ and $\mathcal{B}(\rho)$.

Proposition 3.2 (Dependence of the dynamics on ρ). *With the notation in* (1.9) *and* (1.10), *we have that*

(i) When $\rho = 0$, for any $v \in [0,1]$ the point $(0,v)$ is an equilibrium. If $v \in (1-ac, 1]$, then it corresponds to a strictly negative eigenvalue and a null one. If instead $v \in [0, 1-ac)$, then it corresponds to a strictly positive eigenvalue and a null one

Moreover,

$$\mathcal{B}(0) = \varnothing, \tag{3.21}$$

and for any $\varepsilon < ac/2$ and any $\delta < \varepsilon c/2$ we have that

$$[0,1] \times [0, 1-ac) \subseteq \mathcal{E}(0) \subseteq \mathcal{T}_{\varepsilon,\delta}, \tag{3.22}$$

where

$$\mathcal{T}_{\varepsilon,\delta} := \big\{(u,v) \in [0,1] \times [0,1] \;\; s.t. \;\; \delta v - \varepsilon u \leqslant \delta(1-\varepsilon)\big\}. \tag{3.23}$$

(ii) For any $\varepsilon < ac/3$ and any $\delta < \varepsilon c/2$ it holds that

$$\bigcap_{a>0} \bigcup_{0<\rho<a/3} \mathcal{E}(\rho) \subseteq \mathcal{T}_{\varepsilon,\delta},$$

where $\mathcal{T}_{\varepsilon,\delta}$ is defined in (3.23).

(iii) It holds that

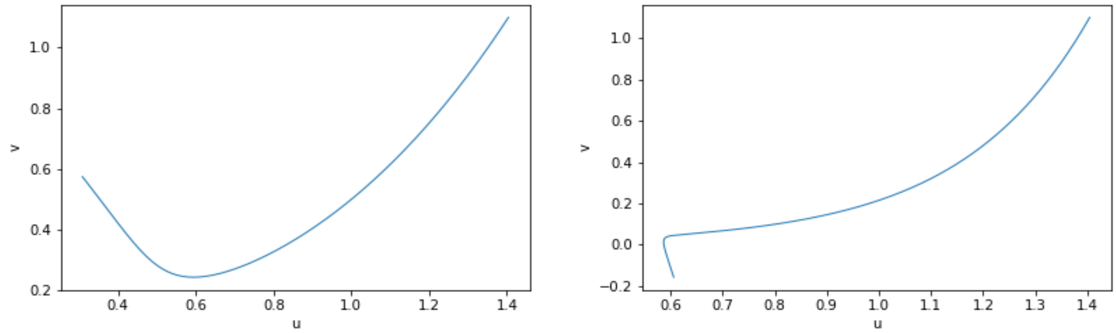
$$\bigcap_{\omega>0} \bigcup_{\rho>\omega} \mathcal{E}(\rho) = (0,1] \times \{0\}. \tag{3.24}$$

We point out that the case $\rho = 0$ is not comprehended in Theorem 1.2. As a matter of fact, the dynamics of this case is qualitatively very different from all the other cases. Indeed, for $\rho = 0$ the domain $[0,1] \times [0,1]$ is not divided into $\mathcal{E}$ and $\mathcal{B}$, since more attractive equilibria appear on the line $\{0\} \times (0,1)$. Thus, even if the second population cannot grow, it still has some chance of victory.

As soon as ρ is positive, on the line $u = 0$ only the equilibrium $(0,1)$ survives, and it attracts all the points that were going to the line $\{0\} \times (0,1)$ for $\rho = 0$.

When $\rho \to +\infty$, the basin of attraction of $(0,1)$ tends to invade the domain, thus the first population tends to have almost no chance of victory and the second population tends to win. However, the dependence on the parameter ρ is not monotone as one could think, at least not in $[0,+\infty) \times [0,+\infty)$.

Indeed, by performing some simulation, one could find some values ρ_1 and ρ_2, with $0 < \rho_1 < \rho_2$, and a point $(u^*, v^*) \in [0,+\infty) \times [0,+\infty)$ such that $(u^*, v^*) \notin \mathcal{E}(\rho_1)$ and $(u^*, v^*) \in \mathcal{E}(\rho_2)$, see Figure 5.



(a) $a = 0.2$, $c = 0.1$, and $\rho = 3$ (b) $a = 0.2$, $c = 0.1$, and $\rho = 7$

Figure 5: *Figure (a) and Figure (b) show the trajectory starting from the point* $(u_0, v_0) = (1.4045, 1.1)$ *for* $\rho = 3$ *and* $\rho = 7$ *respectively. For* $\rho = 3$ *the trajectory leads to the equilibrium* $(0, 1)$*, so* $(u_0, v_0) \notin \mathcal{E}(\rho = 3)$*, while for* $\rho = 7$ *the second population goes to extinction in finite time, so* $(u_0, v_0) \in \mathcal{E}(\rho = 7)$*.*

This means that, sometimes, a big value of fitness for the second population may lead to extinction while a small value brings to victory. This is counterintuitive, but can be easily explained: the parameter ρ is multiplied by the term $1 - u - v$, that is negative past the counterdiagonal of the square $[0, 1] \times [0, 1]$. So in the model (1.1), as well as in any model of Lotka-Volterra type, the population that grows faster is also the one that suffers more the consequences of overpopulation. Moreover, the usual dynamics of Lotka-Volterra models is altered by the presence of the term $-au$, and this leads to the lack of monotonicity that we observe.

We now give the proof of Proposition 3.2:

Proof of Proposition 3.2. (i) For $\rho = 0$, the equation $\dot{v} = 0$ collapses to $u = 0$. Since for $u = 0$ also the equation $\dot{u} = 0$ is satisfied, each point on the line $u = 0$ is an equilibrium.

Calculating the eigenvalues for the points $(0, \tilde{v})$, with $\tilde{v} \in [0, 1]$, using the Jacobian matrix in (2.3), one gets the values 0 and $1 - ac - \tilde{v}$. Accordingly, this entail that, if $\tilde{v} < 1 - ac$, the point $(0, \tilde{v})$ corresponds to a strictly negative eigenvalue and a null one, while if $\tilde{v} > 1 - ac$ then $(0, \tilde{v})$ corresponds to a strictly negative eigenvalue and a null one. These considerations proves the first statement in (i).

We notice also that in the whole square $(0, 1] \times [0, 1]$ we have $\dot{v} = -au < 0$, hence there is no trajectory that can go to $(0, 1)$, and there is no cycle. In particular this implies (3.21).

Now, we observe that on the side $[0, 1] \times \{1\}$ the inward normal derivative is given by $-\dot{v} = au$, which is nonnegative, and therefore a trajectory cannot exit the square on this side. Similarly, along the side $\{1\} \times [0, 1]$ the inward normal derivative is given by $-\dot{u} = v + ac$, which is positive, hence a trajectory cannot exit the square on this side either.

The side $\{0\} \times [0, 1]$ is made of equilibrium points at which the first population u is extinct, while on the side $(0, 1] \times \{0\}$ we have extinction of the population v.

Thus a trajectory either converges to one of the equilibria on the side $\{0\} \times [0, 1]$, or exits $[0, 1] \times [0, 1]$ through the side $(0, 1] \times \{0\}$.

In particular, since $\{0\} \times [0, 1 - ac)$ consists of repulsive equilibria, we have that

$$[0, 1] \times [0, 1 - ac) \subseteq \mathcal{E}(0),$$

that is, trajectories starting in $[0, 1] \times [0, 1 - ac)$ go to the extinction of v. This proves the first inclusion in (3.22).

To prove the second inclusion in (3.22), we first show that

$$\text{points in } \big([0, 1] \times [0, 1]\big) \setminus \mathcal{T}_{\varepsilon,\delta} \text{ are mapped into } \big([0, 1] \times [0, 1]\big) \setminus \mathcal{T}_{\varepsilon,\delta} \text{ itself.} \tag{3.25}$$

Indeed, on the line $\{\delta v - \varepsilon u = \delta(1 - \varepsilon)\}$ we have that the inward-pointing normal derivative is given by

$$\begin{aligned}
(\dot u, \dot v) \cdot \frac{(-\varepsilon, \delta)}{\sqrt{\varepsilon^2 + \delta^2}} &= \frac{1}{\sqrt{\varepsilon^2 + \delta^2}} \big(\delta \dot v - \varepsilon \dot u\big) \\
&= \frac{1}{\sqrt{\varepsilon^2 + \delta^2}} \big(- \delta a u - \varepsilon u(1 - u - v) + \varepsilon a c u\big) \\
&= \frac{u}{\sqrt{\varepsilon^2 + \delta^2}} \left[\varepsilon \left(-1 + ac + u + \frac{\varepsilon}{\delta} u + 1 - \varepsilon\right) - \delta a\right] \\
&= \frac{1}{\sqrt{\varepsilon^2 + \delta^2}} \left[u^2 \left(1 + \frac{\varepsilon}{\delta}\right) + u(\varepsilon a c - \delta a - \varepsilon^2)\right].
\end{aligned} \tag{3.26}$$

The first term is always positive; the second one is positive for the choice

$$\delta < \frac{\varepsilon c}{2} \quad \text{and} \quad \varepsilon < \frac{ac}{2}.$$

Hence, under the assumption in (i), on the line $\{\delta v - \varepsilon u = \delta(1 - \varepsilon)\}$ the inward-pointing normal derivative is positive, which implies that no trajectories in $\big([0, 1] \times [0, 1]\big) \setminus \mathcal{T}_{\varepsilon,\delta}$ can exit from $\big([0, 1] \times [0, 1]\big) \setminus \mathcal{T}_{\varepsilon,\delta}$. This establishes (3.25).

As a consequence of (3.25), we obtain also the second inclusion (3.22), as desired.

(ii) We claim that

$$\big([0, 1] \times [0, 1]\big) \setminus \mathcal{T}_{\varepsilon,\delta} \subseteq \mathcal{B}(\rho), \tag{3.27}$$

for all $0 < \rho < a/3$. To this end, we observe that, in order to determine the sign of the inward pointing normal derivative on the side $\{\delta v - \varepsilon u = \delta(1 - \varepsilon)\}$, by (3.26) we have to check that $\delta \dot v - \varepsilon \dot u \geqslant 0$. In order to simplify the calculation, we use the change of coordinates $x := u$ and $y := 1 - v$. In this way, one needs to verify that $\delta \dot y + \varepsilon \dot x \leqslant 0$ on the line $\{\delta y + \varepsilon x = \delta \varepsilon\}$. For this, we compute

$$\begin{aligned}
\delta \dot y + \varepsilon \dot x &= \delta \rho (y - 1)(y - x) + \delta a x + \varepsilon x (y - x) - \varepsilon a c x, \\
&= -\delta \rho (1 - y) y + x\big(\delta \rho (1 - y) + \delta a + \varepsilon (y - x) - \varepsilon a c\big), \\
&= -\delta \rho (1 - y) y + x\big(\delta \rho - \delta \rho y + \delta a + \varepsilon y - \varepsilon x - \varepsilon a c\big) \\
&\leqslant x\big(\delta \rho - \delta \rho y + \delta a + \varepsilon y - \varepsilon x - \varepsilon a c\big).
\end{aligned} \tag{3.28}$$

Now we choose $\delta < \varepsilon c/2$ and we recall that $\rho < a/3$. Moreover, we notice that

$$y = \varepsilon - \frac{\varepsilon}{\delta} x \leqslant \varepsilon,$$

and therefore $\varepsilon y \leqslant \varepsilon^2$. Thus, we have that

$$-\delta\rho y + \delta\rho + \delta a + \varepsilon y - \varepsilon x - \varepsilon ac \leqslant \frac{\varepsilon ac}{6} + \frac{\varepsilon ac}{2} + \varepsilon^2 - \varepsilon ac = \varepsilon \left(\frac{2}{3} ac + \varepsilon - ac \right)$$

that is negative for $\varepsilon < ac/3$. Plugging this information into (3.28), we obtain that $\delta \dot{y} + \varepsilon \dot{x} \leqslant 0$, as desired.

This proves that trajectories in $\big([0,1]\times[0,1]\big)\setminus\mathcal{T}_{\varepsilon,\delta}$ cannot exit $\big([0,1]\times[0,1]\big)\setminus\mathcal{T}_{\varepsilon,\delta}$. This, the fact that there are no cycles in $[0,1] \times [0,1]$ and the Poincaré-Bendixson Theorem (see e.g. [20]) give that trajectories in $\big([0,1]\times[0,1]\big)\setminus\mathcal{T}_{\varepsilon,\delta}$ converge to $(0,1)$, that is the only equilibrium in $\big([0,1] \times [0,1]\big) \setminus \mathcal{T}_{\varepsilon,\delta}$. Hence, (3.27) is established.

From (3.27) we deduce that

$$\mathcal{E}(\rho) \subseteq \mathcal{T}_{\varepsilon,\delta}$$

for all $0 < \rho < a/3$, which implies the desired result in (ii).

(iii) We consider $\varepsilon_1 > \varepsilon_2 > 0$ to be taken sufficiently small in what follows, and we show that there exists $R > 0$, depending on ε_1 and ε_2, such that for all $\rho \geqslant R$ it holds that

$$\mathcal{R}_{\varepsilon_1,\varepsilon_2} := [0, 1 - \varepsilon_1] \times [\varepsilon_2, 1] \subseteq \mathcal{B}(\rho). \tag{3.29}$$

For this, we first observe that

$$\text{no trajectory starting in } \mathcal{R}_{\varepsilon_1,\varepsilon_2} \text{ can exit the set.} \tag{3.30}$$

Indeed, looking at the velocity fields on the sides $\{0\} \times [\varepsilon_2, 1]$ and $[0, 1 - \varepsilon_1] \times \{1\}$, one sees that no trajectory in $\mathcal{R}_{\varepsilon_1,\varepsilon_2}$ can exit from these sides.

Moreover, on the side $\{1 - \varepsilon_1\} \times [\varepsilon_2, 1]$, the normal inward derivative is

$$-\dot{u} = -[u(1 - u - v) - acu] = -(1 - \varepsilon_1)(\varepsilon_1 - v - ac),$$

and this is positive for $\varepsilon_1 \leqslant ac$ (which is fixed from now on). In addition, on the side $[0, 1 - \varepsilon_1] \times \{\varepsilon_2\}$, the inward normal derivative is

$$\begin{aligned} \dot{v} = [\rho v(1 - u - v) - au] &= \rho\varepsilon_2(1 - u - \varepsilon_2) - au \\ &\geqslant \rho\varepsilon_2(\varepsilon_1 - \varepsilon_2) - a(1 - \varepsilon_1), \end{aligned}$$

and this is positive for

$$\rho > \frac{a(1 - \varepsilon_1)}{\varepsilon_2(\varepsilon_1 - \varepsilon_2)} =: R. \tag{3.31}$$

These observations complete the proof of (3.30).

From (2.48), (3.30) and the Poincaré-Bendixson Theorem (see e.g. [20]), we have that all the trajectories in the interior of $\mathcal{R}_{\varepsilon_1,\varepsilon_2}$ must converge to either an equilibrium or a union of (finitely many) equilibria and non-closed orbits connecting these equilibria.

In addition, we claim that, if $0 < ac < 1$, recalling (1.11) and possibly enlarging ρ in (3.31),

$$(u_s, v_s) \notin \mathcal{R}_{\varepsilon_1,\varepsilon_2}. \tag{3.32}$$

Indeed, we have that $u_s \to 1 - ac$ and $v_s \to 0$, as $\rho \to +\infty$. Hence, we can choose ρ large enough such that the statement in (3.32) is satisfied.

As a consequence of (3.32), we get that all the trajectories in the interior of $\mathcal{R}_{\varepsilon_1,\varepsilon_2}$ must converge to the equilibrium $(0, 1)$, and this establishes (3.29).

Accordingly, (3.29) entails that, for $\varepsilon_1 > \varepsilon_2 > 0$ sufficiently small, there exists $R > 0$, depending on ε_1 and ε_2, such that for all $\rho \geqslant R$

$$\mathcal{E}(\rho) \subset \big((0, 1] \times [0, \varepsilon_2)\big) \cup \big((1 - \varepsilon_1, 1] \times (\varepsilon_2, 1]\big).$$

This implies (3.24), as desired. □

3.3 Dependence of the dynamics on the parameter a

The consequences of the lack of variational structure become even more extreme when we observe the dependence of the dynamics on the parameter a, that is the aggressiveness of the first population towards the other. Throughout this section, we take $\rho > 0$ and $c > 0$, and we perform our analysis taking into account the limit cases $a \to 0$ and $a \to +\infty$. We start analyzing the dynamics of (1.1) in the case $a = 0$.

Proposition 3.3 (Dynamics of (1.1) when $a = 0$)**.** *For $a = 0$ the system (1.1) has the following features:*

i) The system has the equilibrium $(0, 0)$, which is a source, and a straight line of equilibria $(u, 1 - u)$, for all $u \in [0, 1]$, which correspond to a strictly negative eigenvalue and a null one.

ii) Given any $(u(0), v(0)) \in (0, 1) \times (0, 1)$ we have that

$$(u(t), v(t)) \to (\bar{u}, 1 - \bar{u}) \qquad \textit{as } t \to +\infty, \tag{3.33}$$

where $\bar{u}$ satisfies

$$\frac{v(0)}{u^\rho(0)}\bar{u}^\rho + \bar{u} - 1 = 0. \tag{3.34}$$

iii) The equilibrium (u_s^0, v_s^0) given in (1.14) has a stable manifold, which can be written as the graph of an increasing smooth function $\gamma_0 : [0, u_{\mathcal{M}}^0] \to [0, v_{\mathcal{M}}^0]$, for some $(u_{\mathcal{M}}^0, v_{\mathcal{M}}^0) \in \big(\{1\}\times[0, 1]\big)\cup\big((0, 1]\times\{1\}\big)$, such that $\gamma_0(0) = 0$, $\gamma_0(u_{\mathcal{M}}^0) = v_{\mathcal{M}}^0$.

More precisely,

$$\gamma_0(u) := \frac{v_s^0}{(u_s^0)^\rho}u^\rho \qquad \textit{and} \qquad u_{\mathcal{M}}^0 := \min\left\{1, \frac{u_s^0}{(v_s^0)^{\frac{1}{\rho}}}\right\}, \tag{3.35}$$

being (u_s^0, v_s^0) defined in (1.14).

We point out that formula (3.33) says that for $a = 0$ every point in the interior of $[0,1] \times [0,1]$ tends to a coexistence equilibrium. The shape of the trajectories depends on ρ, being convex in the case $\rho > 1$, a straight line in the case $\rho = 1$, and concave in the case $\rho < 1$. This means that if the second population v is alive at the beginning, then it does not get extinct in finite time.

Proof of Proposition 3.3. (i) For $a = 0$, we look for the equilibria of the system (1.1) by studying when $\dot{u} = 0$ and $\dot{v} = 0$. It is easy to see that the point $(0,0)$ and all the points on the line $u + v = 1$ are the only equilibria.

The Jacobian of the system (see (2.3), with $a = 0$) at the point $(0,0)$ has two positive eigenvalues, 1 and ρ , and thereofore $(0,0)$ is a source.

Furthermore, the characteristic polynomial at a point $(\tilde{u}, \tilde{v})$ on the line $u + v = 1$ is given by

$$(\lambda + \tilde{u})(\lambda + \rho\tilde{v}) - \rho\tilde{u}\tilde{v} = \lambda(\lambda + \tilde{u} + \rho\tilde{v}),$$

and therefore, the eigenvalues are 0 and $-\tilde{u} - \rho\tilde{v} < 0$.

(ii) We point out that when $a = 0$

$$\mu(t) := v(t)/u^\rho(t) \text{ is a prime integral for the system.} \tag{3.36}$$

Indeed,

$$\mu' = \frac{\dot{v}u^\rho - \rho u^{\rho-1}\dot{u}v}{u^{2\rho}} = u^{\rho-1}\frac{\rho uv(1-u-v) - \rho uv(1-u-v)}{u^{2\rho}} = 0.$$

As a result, the trajectory starting at a point $(u(0), v(0)) \in (0,1) \times (0,1)$ lies on the curve

$$v(t) = \frac{v(0)}{u^\rho(0)}\, u^\rho(t). \tag{3.37}$$

Moreover, the trajectory starting at $(u(0), v(0))$ is asymptotic as $t \to +\infty$ to an equilibrium on this curve. Since $(0,0)$ is a source, the only possibility is that the trajectory starting at $(u(0), v(0))$ converges to an equlibrium $(\bar{u}, \bar{v})$ such that $\bar{v} = 1 - \bar{u}$. This entails that

$$1 - \bar{u} = \bar{v} = (v(0)/u^\rho(0))\bar{u}^\rho,$$

which is exactly equation (3.34).

(iii) We observe that the point (u_s^0, v_s^0) given in (1.14) lies on the straight line $u + v = 1$, and therefore, thanks to (i) here, it is an equilibrium of the system (1.1), which corresponds to a strictly negative eigenvalue $-u_s^0 - \rho v_s^0$ and a null one.

Hence, by the Center Manifold Theorem (see e.g. Theorem 1 on page 16 of [3]), the point (u_s^0, v_s^0) has a stable manifold, which has dimension 1 and is tangent to the eigenvector of the linearized system associated to the strictly negative eigenvalue $-u_s^0 - \rho v_s^0$.

Also, the graphicality and the monotonicity properties follow from the strict sign of $\dot{u}$ and $\dot{u}$. The smoothnes of the graphs follows from the smoothness of the center manifold. The fact that $\gamma_0(0) = 0$ is a consequence of the monotonicity property of u and v, which ensures that the limit at $t \to -\infty$ exists, and the fact that this limit has to lie on the prime integral in (3.37). The fact that $\gamma_0(u_{\mathcal{M}}^0) = v_{\mathcal{M}}^0$ follows from formula (3.33) and the monotonicity property. Formula (3.35) follows from the fact that any trajectory has to lie on the prime integral in (3.37). □

To state our next result concerning the dependence of the basin of attraction $\mathcal{E}$ defined in (1.10) on the parameter a, we give some notation. We will make it explicit the dependence of the sets $\mathcal{E}$ and $\mathcal{B}$ on the parameter a, by writing explicitly $\mathcal{E}(a)$ and $\mathcal{B}(a)$, and we will call

$$\mathcal{E}_0 := \bigcap_{a'>0} \bigcup_{a'>a>0} \mathcal{E}(a)$$

and

$$\mathcal{E}_\infty := \bigcap_{a'>0} \bigcup_{a>a'} \mathcal{E}(a). \tag{3.38}$$

In this setting, we have the following statements:

Proposition 3.4 (Dependence of the dynamics on a)**.**

(i) We have that

$$\begin{aligned} &\big\{(u,v) \in [0,1]\times[0,1] \;\; s.t. \;\; v < \gamma_0(u) \;\; if \;\; u \in [0, u^0_{\mathcal{M}}] \\ &\qquad\qquad and \;\; v \leqslant 1 \;\; if \;\; u \in (u^0_{\mathcal{M}}, 1]\big\} \\ &\subseteq \mathcal{E}_0 \subseteq \\ &\big\{(u,v) \in [0,1]\times[0,1] \;\; s.t. \;\; v \leqslant \gamma_0(u) \;\; if \;\; u \in [0, u^0_{\mathcal{M}}] \\ &\qquad\qquad and \;\; v \leqslant 1 \;\; if \;\; u \in (u^0_{\mathcal{M}}, 1]\big\}, \end{aligned} \tag{3.39}$$

where γ_0 and $u^0_{\mathcal{M}}$ are given in (3.35).

(ii) It holds that

$$\mathcal{S}_c \subseteq \mathcal{E}_\infty \subseteq \overline{\mathcal{S}_c}, \tag{3.40}$$

where

$$\mathcal{S}_c := \left\{(u,v) \in [0,1]\times[0,1] \;\; s.t. \;\; v - \frac{u}{c} < 0\right\}. \tag{3.41}$$

We point out that the set $\mathcal{E}_0$ in (3.39) does not coincide with the basin of attraction for the system (1.1) when $a = 0$. Indeed, as already mentioned, formula (3.33) in Proposition 3.3 says that for $a = 0$ every point in the interior of $[0,1]\times[0,1]$ tends to a coexistence equilibrium and thus if $v(0) \neq 0$ then $v(t)$ does not get extinct in finite time.

Also, as $a \to +\infty$, we have that the set $\mathcal{E}_\infty$ is determined by $\mathcal{S}_c$, defined in (3.41), that depends only on the parameter c.

The statement in (i) of Proposition 3.4 will be a direct consequence of the following result. Recalling the function γ introduced in Propositions 2.1 and 2.7, we express here the dependence on the parameter a by writing γ_a, u_a, v_a, u^a_s, $u^a_{\mathcal{M}}$. We will also denote by $\mathcal{M}^a$ the stable manifold of the point (u_s, v_s) in (1.11), and by $\mathcal{M}^0$ the stable manifold of the point (u^0_s, v^0_s) in (1.14). The key lemma is the following:

Lemma 3.5. *For all $u \in [0,1]$, we have that $\gamma_a(u) \to \gamma_0(u)$ uniformly as $a \to 0$, where $\gamma_0(u)$ is the function defined in (3.35).*

Proof. Since we are dealing with the limit as a goes to zero, throughout this proof we will always assume that we are in the case $ac < 1$.

Also, we denote by $\phi^a_p(t)$ the flow at time t of the point $p \in [0,1] \times [0,1]$ associated with (1.1), and similarly by $\phi^{(0)}_p(t)$ the flow at time t of the point p associated with (1.1) when $a = 0$. With a slight abuse of notation, we will also write $\phi^a_p(t) = (u_a(t), v_a(t))$, with $p = (u_a(0), v_a(0))$.

Let us start by proving that

$$\mathcal{M}^a \cap \big([0, u^0_s] \times [0, v^0_s]\big) \to \mathcal{M}^0 \cap \big([0, u^0_s] \times [0, v^0_s]\big) \quad \text{as } a \to 0. \tag{3.42}$$

For this, we claim that, for every $\varepsilon > 0$, if

$$(u_a(0))^2 + (v_a(0))^2 \geqslant \frac{\varepsilon^2}{4} \tag{3.43}$$

and

$$\big|(u_a(t), v_a(t)) - (u^a_s, v^a_s)\big| > \frac{\varepsilon}{2}, \tag{3.44}$$

then

$$|\dot{u}_a(t)|^2 + |\dot{v}_a(t)|^2 > \frac{\varepsilon^4}{C_0}, \tag{3.45}$$

for some $C_0 > 0$, depending only on ρ and c.

Indeed, by (v) of Theorem 1.2 and (3.44), the trajectory $(u_a(t), v_a(t))$ belongs to the set $[0, u^a_s] \times [0, v^a_s] \setminus B_{\frac{\varepsilon}{2}}(u^a_s, v^a_s)$.

Moreover, we claim that

$$1 - ac - u_a(t) - v_a(t) \geqslant \frac{\varepsilon\sqrt{2}}{4}, \tag{3.46}$$

for any $t > 0$ such that (3.44) is satisfied. To prove this, we recall that (u^a_s, v^a_s) lies on the straight line ℓ given by $v = -u + 1 - ac$ when $0 < ac < 1$ (see (2.1)). Clearly, there is no point of the set $[0, u^a_s] \times [0, v^a_s] \setminus B_{\frac{\varepsilon}{2}}(u^a_s, v^a_s)$ lying on ℓ, and we notice that the points in the set $[0, u^a_s] \times [0, v^a_s] \setminus B_{\frac{\varepsilon}{2}}(u^a_s, v^a_s)$ with minimal distance from ℓ are given by $p := (u^a_s - \varepsilon/2, v^a_s)$ and $q := (u^a_s, v^a_s - \varepsilon/2)$. Also, the distance of the point p from the straight line ℓ is given by $\frac{\varepsilon}{2} \cdot \tan \frac{\pi}{4} = \frac{\varepsilon\sqrt{2}}{4}$. Thus, the distance between $(u_a(t), v_a(t))$ and the line ℓ is greater than $\frac{\varepsilon\sqrt{2}}{4}$, and this implies (3.46).

As a consequence of (3.46), we obtain that

$$(\dot{u}_a(t))^2 = \big(u_a(t)(1 - ac - u_a(t) - v_a(t))\big)^2 > (u_a(t))^2 \left(\frac{\varepsilon\sqrt{2}}{4}\right)^2 \tag{3.47}$$

and that

$$\begin{aligned} (\dot{v}_a(t))^2 &= \big(\rho v_a(t)(1 - u_a(t) - v_a(t)) - a u_a(t)\big)^2 \\ &\geqslant \left(\rho v_a(t)\left(ac + \frac{\varepsilon\sqrt{2}}{4}\right) - a u_a(t)\right)^2. \end{aligned} \tag{3.48}$$

Now, if $u_a(t) \geqslant \rho c v_a(t)$, then from (3.47) and (3.43) we obtain that

$$\begin{aligned}(\dot{u}_a(t))^2 + (\dot{v}_a(t))^2 &\geqslant (\dot{u}_a(t))^2 > (u_a(t))^2 \left(\frac{\varepsilon\sqrt{2}}{4}\right)^2 \\ &\geqslant \frac{(u_a(t))^2}{2}\left(\frac{\varepsilon\sqrt{2}}{4}\right)^2 + \frac{(\rho c v_a(t))^2}{2}\left(\frac{\varepsilon\sqrt{2}}{4}\right)^2 \\ &\geqslant \min\{1, \rho^2 c^2\}\frac{\varepsilon^2}{16}\big((u_a(t))^2 + (v_a(t))^2\big) \\ &\geqslant \min\{1, \rho^2 c^2\}\frac{\varepsilon^2}{16}\big((u_a(0))^2 + (v_a(0))^2\big) \\ &\geqslant \min\{1, \rho^2 c^2\}\frac{\varepsilon^4}{64},\end{aligned}$$

which proves (3.45) in this case.

If instead $u_a(t) < \rho c v_a(t)$, we use (3.48) to see that

$$\begin{aligned}(\dot{u}_a(t))^2 + (\dot{v}_a(t))^2 \geqslant (\dot{v}_a(t))^2 &\geqslant \left(\rho v_a(t)\left(ac + \frac{\varepsilon\sqrt{2}}{4}\right) - a u_a(t)\right)^2 \\ &= \left(\frac{\varepsilon\sqrt{2}\rho v_a(t)}{4} + a\big(\rho c v_a(t) - u_a(t)\big)\right)^2 \geqslant \left(\frac{\varepsilon\sqrt{2}\rho v_a(t)}{4}\right)^2 \\ &\geqslant \frac{1}{2}\left(\frac{\varepsilon\sqrt{2}\rho v_a(t)}{4}\right)^2 + \frac{1}{2}\left(\frac{\varepsilon\sqrt{2}u_a(t)}{4c}\right)^2 \\ &\geqslant \min\left\{\rho^2, \frac{1}{c^2}\right\}\frac{\varepsilon^2}{16}\big((u_a(t))^2 + (v_a(t))^2\big) \\ &\geqslant \min\left\{\rho^2, \frac{1}{c^2}\right\}\frac{\varepsilon^2}{16}\big((u_a(0))^2 + (v_a(0))^2\big) \\ &\geqslant \min\left\{\rho^2, \frac{1}{c^2}\right\}\frac{\varepsilon^4}{64},\end{aligned}$$

which completes the proof of (3.45).

Now, for any $\eta > 0$, we define

$$\mathcal{P}_\eta := \left\{(u, v) \in [0, 1] \times [0, 1] \;\text{ s.t. }\; v = \frac{v_s^0 - \eta'}{(u_s^0 + \eta')^\rho} u^\rho \;\text{ with } |\eta'| \leqslant \eta\right\}.$$

Given $\varepsilon > 0$, we define

$$\eta(\varepsilon) \text{ to be the smallest } \eta \text{ for which } \mathcal{P}_\eta \supset B_\varepsilon(u_s^0, v_s^0). \tag{3.49}$$

We remark that

$$\lim_{\varepsilon\to 0} \eta(\varepsilon) = 0. \tag{3.50}$$

Also, given $\delta > 0$, we define a tubular neighborhood $\mathcal{U}_\delta$ of $\mathcal{M}^0$ as

$$\mathcal{U}_\delta := \bigcup_{q\in\mathcal{M}^0} B_\delta(q).$$

Furthermore, we define

$$\delta(\varepsilon) \text{ the smallest } \delta \text{ such that } \mathcal{U}_\delta \supset \mathcal{P}_{\eta(\varepsilon)}. \tag{3.51}$$

Recalling (3.50), we have that

$$\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0. \tag{3.52}$$

We remark that, as $a \to 0$, the point (u_s^a, v_s^a) in (1.11), which is a saddle point for the dynamics of (1.1) when $ac < 1$ (recall Theorem 1.2), tends to the point (u_s^0, v_s^0) in (1.14), that belongs to the line $v + u = 1$, which is an equilibrium point for the dynamics of (1.1) when $a = 0$, according to Proposition 3.3.

As a consequence, for every $\varepsilon > 0$, there exists $a_\varepsilon > 0$ such that if $a \in (0, a_\varepsilon)$,

$$|(u_s^a, v_s^a) - (u_s^0, v_s^0)| \leqslant \frac{\varepsilon}{8}. \tag{3.53}$$

This gives that the intersection of $\mathcal{M}^a$ with $B_{\varepsilon/2}(u_s^0, v_s^0)$ is nonempty.

Furthermore, since $\gamma_a(0) = 0$, in light of Proposition 2.1, we have that the intersection of $\mathcal{M}^a$ with $B_{\varepsilon/2}$ is nonempty. Hence, there exists $p_{\varepsilon,a} \in \mathcal{M}^a \cap \partial B_{\varepsilon/2}$.

We also notice that

$$\mathcal{M}^a = \phi^a_{p_{\varepsilon,a}}(\mathbb{R}). \tag{3.54}$$

In addition,

$$\phi^a_{p_{\varepsilon,a}}\big((-\infty, 0]\big) \subset B_{\varepsilon/2}. \tag{3.55}$$

Also, since the origin belongs to $\mathcal{M}^0$, we have that $B_{\varepsilon/2} \subset \mathcal{U}_\varepsilon$. From this and (3.55), we deduce that

$$\phi^a_{p_{\varepsilon,a}}\big((-\infty, 0]\big) \subset \mathcal{U}_\varepsilon. \tag{3.56}$$

Now, we let C_0 be as in (3.45) and we claim that there exists $t_{\varepsilon,a} \in (0, 3\sqrt{C_0}\varepsilon^{-2})$ such that

$$\phi^a_{p_{\varepsilon,a}}(t_{\varepsilon,a}) \in \partial B_{3\varepsilon/4}(u_s^0, v_s^0). \tag{3.57}$$

To check this, we argue by contradiction and we suppose that

$$\phi^a_{p_{\varepsilon,a}}\big((0, 3\sqrt{C_0}\varepsilon^{-2})\big) \cap B_{3\varepsilon/4}(u_s^0, v_s^0) = \varnothing.$$

Then, for every $t \in (0, 3\sqrt{C_0}\varepsilon^{-2})$, recalling also (3.53),

$$\left|\phi^a_{p_{\varepsilon,a}}(t) - (u_s^a, v_s^a)\right| \geqslant \left|\phi^a_{p_{\varepsilon,a}}(t) - (u_s^0, v_s^0)\right| - \left|(u_s^a, v_s^a) - (u_s^0, v_s^0)\right| \geqslant \frac{3\varepsilon}{4} - \frac{\varepsilon}{8} > \frac{\varepsilon}{2},$$

and consequently (3.44) is satisfied for every $t \in (0, 3\sqrt{C_0}\varepsilon^{-2})$.

Moreover, we observe that $p_{\varepsilon,a}$ satisfies (3.43), and therefore, by (3.45),

$$|\dot{u}_a(t)|^2 + |\dot{v}_a(t)|^2 > \frac{\varepsilon^4}{C_0},$$

for all $t \in (0, 3\sqrt{C_0}\varepsilon^{-2})$, where we used the notation $\phi^a_{p_{\varepsilon,a}}(t) = (u_a(t), v_a(t))$, being $p_{\varepsilon,a} = (u_a(0), v_a(0))$. As a result,

$$\big(\dot{u}_a(t) + \dot{v}_a(t)\big)^2 > \frac{\varepsilon^4}{C_0},$$

and thus

$$\dot{u}_a(t) + \dot{v}_a(t) > \frac{\varepsilon^2}{\sqrt{C_0}}.$$

This leads to

$$\begin{aligned} u_a\left(\frac{3\sqrt{C_0}}{\varepsilon^2}\right) + v_a\left(\frac{3\sqrt{C_0}}{\varepsilon^2}\right) &= u_a(0) + v_a(0) + \int_0^{\frac{3\sqrt{C_0}}{\varepsilon^2}} \big(\dot{u}_a(t) + \dot{v}_a(t)\big)\,dt \\ &\geqslant u_a(0) + v_a(0) + \int_0^{\frac{3\sqrt{C_0}}{\varepsilon^2}} \frac{\varepsilon^2}{\sqrt{C_0}}\,dt = u_a(0) + v_a(0) + 3 \geqslant 3, \end{aligned}$$

which forces the trajectory to exit the region $[0,1]\times[0,1]$. This is against the assumption that $p_{\varepsilon,a} \in \mathcal{M}^a$, and therefore the proof of (3.57) is complete.

In light of (3.57), we can set $q_{\varepsilon,a} := \phi^a_{p_{\varepsilon,a}}(t_{\varepsilon,a})$, and we deduce from (3.49) that $q_{\varepsilon,a} \in \mathcal{P}_{\eta(\varepsilon)}$. We also observe that the set $\mathcal{P}_\eta$ is invariant for the flow with $a = 0$, thanks to (3.36). These observations give that $\phi^0_{q_{\varepsilon,a}}(t) \in \mathcal{P}_{\eta(\varepsilon)}$ for all $t \in \mathbb{R}$.

As a result, using (3.51), we conclude that

$$\phi^0_{q_{\varepsilon,a}}(t) \in \mathcal{U}_{\delta(\varepsilon)} \qquad \text{for all } t \in \mathbb{R}. \tag{3.58}$$

In addition, by the continuous dependence of the flow on the parameter a (see e.g. Section 2.4 in [8], or Theorem 2.4.2 in [10]),

$$\left|\phi^0_{q_{\varepsilon,a}}(t) - \phi^a_{q_{\varepsilon,a}}(t)\right| < \varepsilon,$$

for all $t \in [-3\sqrt{C_0}\varepsilon^{-2}, 0]$, provided that a is sufficiently small, possibly in dependence of ε. This fact and (3.58) entail that

$$\phi^a_{q_{\varepsilon,a}}(t) \in \mathcal{U}_{\delta(\varepsilon)+\varepsilon} \qquad \text{for all } t \in [-3\sqrt{C_0}\varepsilon^{-2}, 0].$$

In particular, for all $t \in [0, t_{\varepsilon,a}]$,

$$\phi^a_{p_{\varepsilon,a}}(t) = \phi^a_{q_{\varepsilon,a}}(t - t_{\varepsilon,a}) \in \mathcal{U}_{\delta(\varepsilon)+\varepsilon}. \tag{3.59}$$

We now claim that for all $t \geqslant t_{\varepsilon,a}$,

$$\phi^a_{p_{\varepsilon,a}}(t) \subset B_\varepsilon(u^a_s, v^a_s). \tag{3.60}$$

Indeed, this is true when $t = t_{\varepsilon,a}$ thanks to (3.53) and (3.57). Hence, since the trajectory $\phi^a_{p_{\varepsilon,a}}(t)$ is contained in the domain where $\dot{u} \geqslant 0$ and $\dot{v} \geqslant 0$, thanks to (2.7), we deduce that (3.60) holds true.

From (3.53) and (3.60), we conclude that

$$\phi^a_{p_{\varepsilon,a}}(t) \subset B_{2\varepsilon}(u^0_s, v^0_s),$$

for all $t \geqslant t_{\varepsilon,a}$.

Using this, (3.56) and (3.59), we obtain that

$$\phi^a_{p_{\varepsilon,a}}(\mathbb{R}) \subset \mathcal{U}_{\delta(\varepsilon)+2\varepsilon}.$$

This and (3.52) give that (3.42) is satisfied, as desired.

One can also show that

$$\mathcal{M}^a \cap \big([u_s^0, u_{\mathcal{M}}^0] \times [v_s^0, v_{\mathcal{M}}^0]\big) \to \mathcal{M}^0 \cap \big([u_s^0, u_{\mathcal{M}}^0] \times [v_s^0, v_{\mathcal{M}}^0]\big) \quad \text{as } a \to 0. \tag{3.61}$$

The proof of (3.61) is similar to that of (3.42), just replacing $p_{\varepsilon,a}$ with $(u_{\mathcal{M}}^a, v_{\mathcal{M}}^a)$ (in this case the analysis near the origin is simply omitted since the trajectory has only one limit point).

With (3.42) and (3.61) the proof of Lemma 3.5 is thereby complete. □

Now we are ready to give the proof of Proposition 3.4:

Proof of Proposition 3.4. (i) We call $\mathcal{G}$ the right-hand-side of (3.39), that is

$$\begin{aligned} \mathcal{G} \;:=\; \big\{(u,v) \in [0,1]\times[0,1] \ \text{ s.t. } \ v < \gamma_0(u) \ \text{ if } \ u \in [0, u_{\mathcal{M}}^0] \\ \text{and } \ v \leqslant 1 \ \text{ if } \ u \in (u_{\mathcal{M}}^0, 1]\big\}, \end{aligned}$$

and we aim at proving that $\mathcal{G} \subseteq \mathcal{E}_0 \subseteq \overline{\mathcal{G}}$.

For this, we observe that, by Lemma 3.5, $\gamma_a(u)$ converges to $\gamma_0(u)$ pointwise as $a \to 0$. In particular, $u_{\mathcal{M}}^a \to u_{\mathcal{M}}^0$ as $a \to 0$.

Also, recalling (3.35), we notice that if $u_{\mathcal{M}}^0 = u_s^0/(v_s^0)^{\frac{1}{\rho}} < 1$, then $\gamma_0(u_{\mathcal{M}}^0) = 1$, otherwise if $u_{\mathcal{M}}^0 = 1$ then $\gamma_0(u_{\mathcal{M}}^0) < 1$, being $\gamma_0(u)$ strictly monotone increasing.

Furthermore, thanks to Proposition 2.9, we know that he set $\mathcal{E}(a)$ is bounded from above by the graph of the function $\gamma_a(u)$ for $u \in [0, u_{\mathcal{M}}^a]$ and from the straight line $v = 1$ for $u \in (u_{\mathcal{M}}^a, 1]$ (that is non empty for $u_{\mathcal{M}}^a < 1$).

Now we claim that, for all $a' > 0$,

$$\mathcal{G} \subseteq \bigcup_{0<a<a'} \mathcal{E}(a). \tag{3.62}$$

To show this, we take a point $(u,v) \in \mathcal{G}$. Hence, in light of the considerations above, we have that $(u,v) \in \mathcal{E}(a)$ for any a sufficiently small, which proves (3.62).

From (3.62), we deduce that

$$\mathcal{G} \subseteq \bigcap_{a'>0} \bigcup_{0<a<a'} \mathcal{E}(a). \tag{3.63}$$

Now we show that

$$\bigcap_{a'>0} \bigcup_{0<a<a'} \mathcal{E}(a) \subseteq \overline{\mathcal{G}}. \tag{3.64}$$

For this, we take

$$(\hat u, \hat v) \in \bigcap_{a'>0} \bigcup_{0<a<a'} \mathcal{E}(a),$$

then it must hold that for every $a' > 0$ there exists $a < a'$ such that $(\hat u, \hat v) \in \mathcal{E}(a)$, namely $\hat v < \gamma_a(\hat u)$ if $\hat u \in [0, u_{\mathcal{M}}^a]$ and $\hat v \leqslant 1$ if $\hat u \in (u_{\mathcal{M}}^a, 1]$. Thus, by the pointwise convergence, we have that $\hat v \leqslant \gamma_0(\hat u)$ if $\hat u \in [0, u_{\mathcal{M}}^0]$ and $\hat v \leqslant 1$ if $\hat u \in (u_{\mathcal{M}}^0, 1]$, which proves (3.64).

From (3.63) and (3.64), we conclude that

$$\mathcal{G} \subseteq \bigcap_{a'>0} \bigcup_{0<a<a'} \mathcal{E}(a) = \mathcal{E}_0 \subseteq \overline{\mathcal{G}},$$

as desired.

(ii) Since we deal with the limit case as $a \to +\infty$, from now on we suppose from now on that $ac > 1$. We fix $\varepsilon > 0$ and we consider the set

$$\mathcal{S}_{\varepsilon^+} := \left\{ (u, v) \in [0, 1] \times [0, 1] \text{ s.t. } v > u \left(\frac{1}{c} + \varepsilon \right) \right\}.$$

We claim that

$$\mathcal{S}_{\varepsilon^+} \subseteq \mathcal{B}(a) \tag{3.65}$$

for a big enough, possibly in dependence of ε. For this, we first analyze the component of the velocity in the inward normal directions along the boundary of $\mathcal{S}_{\varepsilon^+}$. On the side $\{0\} \times [0, 1]$, the trajectories cannot cross the boundary thanks to Proposition 1.1, and the same happens for the sides $[0, 1] \times \{1\}$ and $\{1\} \times [\varepsilon + 1/c, 1]$.

Hence, it remains to check the sign of the normal derivative along the side given by the straight line $v - u(\varepsilon + 1/c) = 0$. We compute

$$\begin{aligned} (\dot{u}, \dot{v}) \cdot \left(-\left(\varepsilon + \frac{1}{c} \right), 1 \right) &= \dot{v} - \dot{u} \left(\varepsilon + \frac{1}{c} \right) \\ &= \rho v(1 - u - v) - au - \left(\varepsilon + \frac{1}{c} \right) u(1 - u - v) + \left(\varepsilon + \frac{1}{c} \right) acu \\ &= \left[\rho v - \left(\varepsilon + \frac{1}{c} \right) u \right] (1 - u - v) + \varepsilon acu. \end{aligned}$$

Thus, by using that $v - u(\varepsilon + 1/c) = 0$, we obtain that

$$(\dot{u}, \dot{v}) \cdot \left(-\left(\varepsilon + \frac{1}{c} \right), 1 \right) \geqslant u \left[a\varepsilon c + (\rho - 1)(1 - u - v) \left(\varepsilon + \frac{1}{c} \right) \right].$$

Notice that $u \leqslant 1$ and $|1 - u - v| \leqslant 2$, and therefore

$$(\dot{u}, \dot{v}) \cdot \left(-\left(\varepsilon + \frac{1}{c} \right), 1 \right) \geqslant u \left[a\varepsilon c - 2(\rho + 1) \left(\varepsilon + \frac{1}{c} \right) \right].$$

Accordingly, the normal velocity is positive for $a \geqslant a_1$, where

$$a_1 := 2(\rho + 1) \left(\varepsilon + \frac{1}{c} \right) \frac{1}{\varepsilon c}.$$

These considerations, together with the fact that there are no cycles in $[0, 1] \times [0, 1]$ and the Poincaré-Bendixson Theorem (see e.g. [20]) give that the ω-limit set of any trajectory starting in the interior of $\mathcal{S}_{\varepsilon^+}$ can be either an equilibrium or a union of (finitely many) equilibria and non-closed orbits connecting these equilibria.

We remark that

$$\text{the } \omega\text{-limit set of any trajectory cannot be the equilibrium } (0,0). \tag{3.66}$$

Indeed, if the ω-limit of a trajectory were $(0,0)$, then this trajectory must lie on the stable manifold of $(0,0)$, and moreover it must be contained in $\mathcal{S}_{\varepsilon^+}$, since no trajectory can exit $\mathcal{S}_{\varepsilon^+}$. On the other hand, by Proposition 2.1, we have that at $u=0$ the stable manifold is tangent to the line

$$v = \frac{a}{\rho - 1 + ac} u = \frac{1}{\frac{\rho-1}{a} + c} u.$$

Now, if we take a sufficiently large, this line lies below the line $v = u(1/c + \varepsilon)$, thus providing a contradiction. Hence, the proof of (3.66) is complete.

Accordingly, since $(0,1)$ is a sink, the only possibility is that the ω-limit set of any trajectory starting in the interior of $\mathcal{S}_{\varepsilon^+}$ is the equilibrium $(0,1)$. Namely, we have established (3.65).

As a consequence of (3.65), we deduce that for every $\varepsilon > 0$ there exists $a_\varepsilon > 0$ such that

$$\bigcup_{a \geqslant a_\varepsilon} \mathcal{E}(a) \subseteq \left\{ (u,v) \in [0,1] \times [0,1] \text{ s.t. } v \leqslant u \left(\frac{1}{c} + \varepsilon \right) \right\}. \tag{3.67}$$

In addition,

$$\begin{aligned} &\bigcap_{\varepsilon>0} \left\{ (u,v) \in [0,1] \times [0,1] \text{ s.t. } v \leqslant u \left(\frac{1}{c} + \varepsilon \right) \right\} \\ &\qquad = \left\{ (u,v) \in [0,1] \times [0,1] \text{ s.t. } v \leqslant \frac{u}{c} \right\} = \overline{\mathcal{S}_c}. \end{aligned}$$

This and (3.67) entail that

$$\bigcap_{a'>0} \bigcup_{a>a'} \mathcal{E}(a) \subseteq \overline{\mathcal{S}_c},$$

which implies the second inclusion in (3.40).

Now, to show the first inclusion in (3.40), for every $\varepsilon \in (0, 1/c)$ we consider the set

$$\mathcal{S}_{\varepsilon^-} := \left\{ (u,v) \in [0,1] \times [0,1] \text{ s.t. } v < u \left(\frac{1}{c} - \varepsilon \right) \right\}.$$

We claim that, for all $\varepsilon \in (0, 1/c)$,

$$\mathcal{S}_{\varepsilon^-} \subseteq \mathcal{E}_\infty. \tag{3.68}$$

For this, we first show that if a is sufficiently large, possibly in dependence of ε,

$$\begin{aligned} &\text{every trajectory starting in the interior of } \mathcal{S}_{\varepsilon^-} \\ &\text{can exit } \mathcal{S}_{\varepsilon^-} \text{ from the side } [0,1] \times \{0\}. \end{aligned} \tag{3.69}$$

Indeed, on the side $\{1\} \times [0,1]$ the trajectory cannot exit the set, thanks to Proposition 1.1. On the side given by $v - (-\varepsilon + 1/c)u = 0$, the component of the velocity in the direction of the outward normal is

$$\begin{aligned}(\dot{u}, \dot{v}) \cdot \left(-\left(\frac{1}{c} - \varepsilon\right), 1\right) &= \dot{v} - \dot{u}\left(\frac{1}{c} - \varepsilon\right) \\ &= \rho v(1-u-v) - au - \left(\frac{1}{c} - \varepsilon\right) u(1-u-v) + \left(\frac{1}{c} - \varepsilon\right) acu \\ &= u\left[\left(\frac{1}{c} - \varepsilon\right)(\rho - 1)(1-u-v) - \varepsilon ac\right] \\ &\leqslant u\left[2\left(\frac{1}{c} - \varepsilon\right)(\rho + 1) - \varepsilon ac\right],\end{aligned}$$

which is negative if $a \geqslant a_2$, with

$$a_2 := 2\left(\frac{1}{c} - \varepsilon\right)(\rho + 1)\frac{1}{\varepsilon c}.$$

Hence, if $(u(0), v(0)) \in \mathcal{S}_{\varepsilon^-}$, then either $T_s(u(0), v(0)) < \infty$ or $(u(t), v(t)) \in \mathcal{S}_{\varepsilon^-}$ for all $t \geqslant 0$, where the notation in (1.8) has been used. We also notice that, for $a > 1/c$, the points $(0,1)$ and $(0,0)$ are the only equilibria of the system, and there are no cycles. We have that $(0,1) \notin \overline{\mathcal{S}_{\varepsilon^-}}$ and $(0,0) \in \overline{\mathcal{S}_{\varepsilon^-}}$, thus if

$$(u(t), v(t)) \in \mathcal{S}_{\varepsilon^-} \text{ for all } t \geqslant 0 \tag{3.70}$$

then

$$(u(t), v(t)) \to (0,0). \tag{3.71}$$

On the other hand, by Proposition 2.1, we have that at $u = 0$ the stable manifold is tangent to the line

$$v = \frac{a}{\rho - 1 + ac} u = \frac{1}{\frac{\rho-1}{a} + c} u,$$

and, if we take a large enough, this line lies above the line $v = u(1/c - \varepsilon)$. This says that, for sufficiently large t, the trajectory must lie outside $\mathcal{S}_{\varepsilon^-}$, and this is in contradiction with (3.70).

As a result of these considerations, we conclude that if $(u(0), v(0)) \in \mathcal{S}_{\varepsilon^-}$ then $T_s(u(0), v(0)) < \infty$, which implies (3.69).

As a consequence of (3.69), we obtain that for every $\varepsilon \in (0, 1/c)$ there exists $a_\varepsilon > 0$ such that

$$\mathcal{S}_{\varepsilon^-} \subseteq \bigcap_{a \geqslant a_\varepsilon} \mathcal{E}(a).$$

In particular for all $\varepsilon \in (0, 1/c)$ it holds that

$$\mathcal{S}_{\varepsilon^-} \subseteq \bigcap_{a'>0} \bigcup_{a>a'} \mathcal{E}(a) = \mathcal{E}_\infty,$$

which proves (3.68), as desired.

Then, the first inclusion in (3.40) plainly follows from (3.68). $\square$

4 Analysis of the strategies for the first population

The main theorems on the winning strategy have been stated in Subsection 1.4. In particular, Theorem 1.3 gives the characterization of the set of points that have a winning strategy $\mathcal{V}_{\mathcal{A}}$ in (1.13), and Theorem 1.4 establishes the non equivalence of constant and non-constant strategies when $\rho \neq 1$ (and their equivalence when $\rho = 1$). Nonetheless, in Theorem 1.5 we state that Heaviside functions are enough to construct a winning strategy for every point in $\mathcal{V}_{\mathcal{A}}$.

In the following subsections we will give the proofs of these results.

4.1 Construction of winning non-constant strategies

We want to put in light the construction of non-constant winning strategies for the points for which constant strategies fail.

For this, we recall the notation introduced in (1.14), (1.18) and (3.35), and we have the following statement:

Proposition 4.1. *Let $M > 1$. Then we have:*

1. *For $\rho < 1$, let (u_0, v_0) be a point of the set*

$$\mathcal{P} := \left\{ (u,v) \in [0,1] \times [0,1] \quad s.t. \quad u \in [u_s^0, 1], \ \gamma_0(u) \leqslant v < \frac{u}{c} + \frac{1-\rho}{1+\rho c} \right\}. \tag{4.1}$$

   Then there exist $a^* > M$, $a_* < \frac{1}{M}$, and $T \geqslant 0$, depending on (u_0, v_0), c, and ρ, such that the Heaviside strategy defined by

$$a(t) = \begin{cases} a^*, & \text{if } t < T, \\ a_*, & \text{if } t \geqslant T, \end{cases} \tag{4.2}$$

   belongs to $\mathcal{V}_{\mathcal{A}}$.

2. *For $\rho > 1$, let (u_0, v_0) be a point of the set*

$$\mathcal{Q} := \left\{ (u,v) \in [0,1] \times [0,1] \quad s.t. \quad u \in [u_\infty, 1], \ \frac{u}{c} \leqslant v < \zeta(u) \right\}. \tag{4.3}$$

   Then there exist $a^* > M$, $a_* < \frac{1}{M}$, and $T \geqslant 0$, depending on (u_0, v_0), c, and ρ, such that the Heaviside strategy defined by

$$a(t) = \begin{cases} a_*, & \text{if } t < T, \\ a^*, & \text{if } t \geqslant T, \end{cases}$$

   belongs to $\mathcal{V}_{\mathcal{A}}$.

Proof. We start by proving the first claim in Proposition 4.1. To this aim, we take $(\bar{u}, \bar{v}) \in \mathcal{P}$, and we observe that

$$\bar{v} - \frac{\bar{u}}{c} < \frac{1-\rho}{1+\rho c} = v_s^0 - \frac{u_s^0}{c}.$$

Therefore, there exists $\xi > 0$ such that

$$\xi < \frac{v_s^0 - \bar{v} - \frac{1}{c}(u_s^0 - \bar{u})}{\bar{u} - u_s^0}.$$

Hence, setting

$$v_S := \left(\frac{1}{c} - \xi\right)(u_s^0 - \bar{u}) + \bar{v}, \tag{4.4}$$

we see that

$$v_S < v_s^0. \tag{4.5}$$

Now, we want to show that there exists $a^* > 0$ such that, for any $a > a^*$ and $u > u_s^0$, we have that

$$\frac{\dot{v}}{\dot{u}} > \frac{1}{c} - \xi. \tag{4.6}$$

To prove this, we first notice that

$$\text{if } a > \frac{2}{c}, \text{ then } \dot{u} \leqslant -u < 0. \tag{4.7}$$

Moreover, we set

$$a_1 := \frac{1 + \rho c}{4c},$$

and we claim that,

$$\text{if } a > a_1 \text{ and } u > u_s^0, \text{ then } \dot{v} < 0. \tag{4.8}$$

Indeed, we recall that the function σ defined in (2.6) represents the points in $[0,1] \times [0,1]$ where $\dot{v} = 0$ and separates the points where $\dot{v} > 0$, which lie on the left of the curve described by σ, from the points where $\dot{v} < 0$, which lie on the right of the curve described by σ.

Therefore, in order to show (4.8), it is sufficient to prove that the curve described by σ is contained in $\{u \leqslant u_s^0\}$ whenever $a > a_1$. For this, one computes that, if $u = \sigma(v)$ and $a > a_1$, then

$$\begin{aligned}
u - u_s^0 &= \sigma(v) - \frac{\rho c}{1 + \rho c} = 1 - \frac{\rho v^2 + a}{\rho v + a} - \frac{\rho c}{1 + \rho c} \\
&= \frac{\rho v - \rho v^2}{\rho v + a} - \frac{\rho c}{1 + \rho c} = \frac{\rho v(1 - v)}{\rho v + a} - \frac{\rho c}{1 + \rho c} \\
&\leqslant \frac{\rho}{4(\rho v + a)} - \frac{\rho c}{1 + \rho c} \leqslant \frac{\rho}{4a} - \frac{\rho c}{1 + \rho c} \\
&\leqslant \frac{\rho}{4a_1} - \frac{\rho c}{1 + \rho c} \leqslant 0.
\end{aligned}$$

This completes the proof of (4.8).

Now we define

$$a_2 := \left(\rho + \frac{1}{c} + \xi\right)\frac{2}{u_s^0 c \xi}.$$

and we claim that

$$\text{if } a > a_2 \text{ and } u > u_s^0, \text{ then } \dot{v} < \left(\frac{1}{c} - \xi\right)\dot{u}. \tag{4.9}$$

Indeed, under the assumptions of (4.9), we deduce that

$$\begin{aligned}\dot{v}-\left(\frac{1}{c}-\xi\right)\dot{u}&=\rho v(1-u-v)-au-\left(\frac{1}{c}-\xi\right)\Big(u(1-u-v)-acu\Big)\\&=(1-u-v)\left(\rho v-\left(\frac{1}{c}-\xi\right)u\right)-ac\xi u\leqslant 2\left(\rho v+\frac{u}{c}+\xi u\right)-ac\xi u\\&<2\left(\rho+\frac{1}{c}+\xi\right)-a_2\,c\xi u_s^0=0,\end{aligned}$$

and this establishes the claim in (4.9).

Then, choosing

$$a^*:=\max\left\{\frac{2}{c},a_1,a_2,M\right\},$$

we can exploit (4.7), (4.8) and (4.9) to deduce (4.6), as desired.

Now we claim that, for any $a>a^*$, there exists $T\geqslant 0$ such that the trajectory $(u(t),v(t))$ starting from $(\bar{u},\bar{v})$ satisfies

$$u(T)=u_s^0 \text{ and } v(T)<v_S. \tag{4.10}$$

Indeed, we define $T\geqslant 0$ to be the first time for which $u(T)=u_s^0$. This is a fair definition, since $u(0)=\bar{u}\geqslant u_s^0$ and $\dot{u}$ is negative, and bounded away from zero till $u\geqslant u_s^0$, thanks to (4.7). Then, we see that

$$\begin{aligned}v(T)&=\bar{v}+\int_0^T\dot{v}(t)\,dt<\bar{v}+\int_0^T\left(\frac{1}{c}-\xi\right)\dot{u}(t)\,dt=\bar{v}+\left(\frac{1}{c}-\xi\right)(u(T)-u(0))\\&=\bar{v}+\left(\frac{1}{c}-\xi\right)(u_s^0-\bar{u})=v_S,\end{aligned}$$

thanks to (4.4) and (4.6), and this establishes (4.10).

Now we observe that

$$v(T)<v_S<v_s^0=\gamma_0(u_s^0)=\gamma_0(u(T))$$

due to (4.5) and (4.10)

As a result, recalling Lemma 3.5, we can choose $a_*<1/M$ such that

$$v(T)<\gamma_{a_*}(u(T)).$$

Accordingly, by Proposition 2.9, we obtain that $(u(T),v(T))\in\mathcal{E}(a_*)$. Hence, applying the strategy in (4.2), we accomplish the desired result and complete the proof of the first claim in Proposition 4.1.

Now we focus on the proof of the second claim in Proposition 4.1. For this, let

$$(u_0,v_0)\in\mathcal{Q}, \tag{4.11}$$

and consider the trajectory $(u_0(t),v_0(t))$ starting from (u_0,v_0) for the strategy $a=0$. In light of formula (3.33) of Proposition 3.3, we have that

$$\begin{aligned}&\text{the trajectory } (u_0(t),v_0(t)) \text{ converges}\\&\text{to a point of the form } (u_F,1-u_F) \text{ as } t\to+\infty.\end{aligned} \tag{4.12}$$

We define

$$v_F := 1 - u_F, \quad v_\infty := 1 - u_\infty = \frac{1}{c+1}, \tag{4.13}$$

where the last equality can be checked starting from the value of u_∞ given in (1.18). Using the definition of ζ in (1.18) and the information in (3.36), we also notice that the curve given by $v = \zeta(u)$ is a trajectory for $a = 0$. Moreover

$$\zeta(u_\infty) = \frac{1}{c(u_\infty)^{\rho-1}} u_\infty^\rho = \frac{c}{c(c+1)} = v_\infty$$

and, recalling (4.13) and formula (3.33) of Proposition 3.3, we get that the graph of ζ is a trajectory for $a = 0$ that converges to $(u_\infty, 1 - u_\infty)$ as $t \to +\infty$.

Also, by (4.11), we have that $v_0 < \zeta(u_0)$. Thus, since by Cauchy's uniqueness result for ODEs, two orbits never intersect, we have that

$$\text{the orbit } (u_0(t), v_0(t)) \text{ must lie below the graph of } \zeta. \tag{4.14}$$

Since both (u_F, v_F) and (u_∞, v_∞) belong to the line given by $v = 1 - u$, from (4.14) we get that

$$u_\infty < u_F \tag{4.15}$$

and

$$v_\infty > v_F. \tag{4.16}$$

Thanks to (4.15) and (4.16) and recalling the values of u_∞ from (1.18) and of v_∞ from (4.13), we get that

$$v_F < v_\infty = \frac{u_\infty}{c} < \frac{u_F}{c}. \tag{4.17}$$

As a consequence, since the inequality in (4.17) is strict, we find that there exists $T' > 0$ such that

$$v_0(T') < \frac{u_0(T')}{c}. \tag{4.18}$$

Moreover, since $\dot{u} < 0$ for $v > 1 - u$ and $a = 0$, we get that $u_0(t)$ is decreasing in t, and therefore $u_F < u_0(T') < u_0$.

By the strict inequality in (4.18), and claim (ii) in Proposition 3.4, we have that $(u_0(T'), v_0(T')) \in \mathcal{E}_\infty$, where $\mathcal{E}_\infty$ is defined in (3.38). In particular, we have that $(u_0(T'), v_0(T')) \in \bigcup_{a>a'} \mathcal{E}(a)$, for every $a' > 0$. Consequently, there exists $a^* > M$ such that $(u_0(T'), v_0(T')) \in \mathcal{E}(a^*)$. Therefore, applying the strategy

$$a(t) = \begin{cases} 0, & t < T', \\ a^*, & t \geqslant T, \end{cases}$$

we reach the victory. □

4.2 Proof of Theorem 1.3

To avoid repeating passages in the proofs of Theorems 1.3 and 1.4, we first state and prove the following lemma:

Lemma 4.2. *If $\rho = 1$, then for all $a > 0$ we have $\mathcal{E}(a) = \mathcal{S}_c$, where $\mathcal{S}_c$ was defined in* (3.41).

Proof. Let $(u(t), v(t))$ be a trajectory starting at a point in $[0,1]\times[0,1]$. For any $a > 0$, we consider the function

$$\mu(t) := \frac{v\left(\frac{t}{a}\right)}{u\left(\frac{t}{a}\right)}.$$

Notice that

$$(u(0), v(0)) \in \mathcal{E}(a) \text{ if and only if there exists } T > 0 \text{ such that } \mu(T) = 0. \tag{4.19}$$

In addition, we observe that

$$\begin{aligned} \dot{\mu}(t) &= \frac{\dot{v}\left(\frac{t}{a}\right) u\left(\frac{t}{a}\right) - v\left(\frac{t}{a}\right) \dot{u}\left(\frac{t}{a}\right)}{au^2\left(\frac{t}{a}\right)} \\ &= \frac{-u^2\left(\frac{t}{a}\right) + cu\left(\frac{t}{a}\right) v\left(\frac{t}{a}\right)}{u^2\left(\frac{t}{a}\right)} \\ &= c\mu(t) - 1. \end{aligned} \tag{4.20}$$

The equation in (4.20) is integrable and leads to

$$\mu(t) = \frac{e^{ct}\left(c\mu(0) - 1\right) + 1}{c}.$$

From this and (4.19), we deduce that

$$(u(0), v(0)) \in \mathcal{E}(a) \text{ if and only if } c\mu(0) - 1 < 0.$$

This leads to

$$(u(0), v(0)) \in \mathcal{E}(a) \text{ if and only if } \frac{v(0)}{u(0)} < \frac{1}{c},$$

which, recalling the definition of $\mathcal{S}_c$ in (3.41), ends the proof. □

Now we provide the proof of Theorem 1.3, exploiting the result obtained in Section 4.1.

Proof of Theorem 1.3.
(i) Let $\rho = 1$. For the sake of simplicity, we suppose that $c \geqslant 1$, and therefore the second line in (1.15) is not present (the proof of (1.15) when $c < 1$ is similar, but one has to take into account also the set $(c, 1] \times [0, 1]$ and show that it is contained in $\mathcal{V}_{\mathcal{A}}$ by checking the sign of the component of the velocity field in the normal direction).

We claim that

$$\mathcal{V}_{\mathcal{A}} = \mathcal{S}_c, \tag{4.21}$$

where $\mathcal{S}_c$ was defined in (3.41) (incidentally, $\mathcal{S}_c$ is precisely the right-hand-side of equation (1.15)).

From Lemma 4.2 we have that for $\rho = 1$ and $a > 0$ it holds $\mathcal{S}_c = \mathcal{E}(a) \subset \mathcal{V}_{\mathcal{A}}$. Thus, to show (4.21) we just need to check that

$$\mathcal{V}_{\mathcal{A}} \subseteq \mathcal{S}_c, \tag{4.22}$$

which is equivalent to

$$\mathcal{S}_c^C \subseteq \mathcal{V}_{\mathcal{A}}^C, \tag{4.23}$$

where the superscript C denotes the complement of the set in the topology of $[0,1] \times [0,1]$.

First, by definition we have that

$$\mathcal{S}_c^C \cap ((0,1] \times \{0\}) = \varnothing. \tag{4.24}$$

Now, we analyze the behavior of the trajectories at $\partial \mathcal{S}_c^C$. By Proposition 1.1, no trajectory can exit $\mathcal{S}_c^C$ from a point on $\partial([0,1] \times [0,1]) \setminus ((0,1] \times \{0\})$. Moreover, $\partial \mathcal{S}_c^C \cap ((0,1] \times \{0\}) = \varnothing$ thanks to (4.24) and the fact that $\mathcal{S}_c^C$ is closed in the topology of $[0,1] \times [0,1]$. Hence,

$$\text{no trajectory can exit } \mathcal{S}_c^C \text{ from a point on } \partial([0,1] \times [0,1]). \tag{4.25}$$

Furthermore, it holds that

$$\partial \mathcal{S}_c^C \cap \big((0,1) \times (0,1)\big) = \left\{(u,v) \in (0,1) \times (0,1) \ \text{ s.t. } \ v = \frac{u}{c}\right\}.$$

The velocity of a trajectory starting on the line $v = \frac{u}{c}$ in the orthogonal direction pointing inward $\mathcal{S}_c^C$ is

$$(\dot{u}, \dot{v}) \cdot \frac{(-1,c)}{\sqrt{c^2+1}} = \frac{1}{\sqrt{c^2+1}}(cv - u)(1-u-v) = 0,$$

the last equality coming from the fact that $cv = u$ on $\partial \mathcal{S}_c^C \cap \big((0,1) \times (0,1)\big)$. This means that

$$\text{no trajectory can exit } \mathcal{S}_c^C \text{ from a point on the line } v = \tfrac{u}{c}. \tag{4.26}$$

From (4.25) and (4.26), we get that no trajectory exits $\mathcal{S}_c^C$. Then, by (4.24), no trajectory starting in $\mathcal{S}_c^C$ can reach the set $(0,1] \times \{0\}$, therefore $\mathcal{S}_c^C \cap \mathcal{V}_{\mathcal{A}} = \varnothing$ and this implies that (4.23) is true. As a result, the proof of (4.22) is established and the proof is completed for $\rho = 1$.

(ii) Let $\rho < 1$. For the sake of simplicity, we suppose that $\frac{\rho c(c+1)}{1+\rho c} \geqslant 1$. Let $\mathcal{Y}$ be the set in the right-hand-side of (1.16), and

$$\mathcal{F}_0 := \big\{(u,v) \in [0,1] \times [0,1] \ \text{ s.t. } \ v < \gamma_0(u) \ \text{ if } u \in [0,1]\big\}. \tag{4.27}$$

Notice that

$$\mathcal{Y} = \mathcal{F}_0 \cup \mathcal{P}, \tag{4.28}$$

being $\mathcal{P}$ the set defined in (4.1).

Moreover,

$$\mathcal{P} \subseteq \mathcal{V}_{\mathcal{A}}, \tag{4.29}$$

thanks to Proposition 4.1.

We also claim that

$$\mathcal{F}_0 \subseteq \mathcal{V}_{\mathcal{K}}, \tag{4.30}$$

where $\mathcal{K}$ is the set of constant functions. Indeed, if $(u,v) \in \mathcal{F}_0$, we have that $v < \gamma_0(u)$ and consequently $v < \gamma_a(u)$, as long as a is small enough, due to Lemma 3.5.

From this and Proposition 2.9, we deduce that (u,v) belongs to $\mathcal{E}(a)$, as long as a is small enough, and this proves (4.30).

From (4.30) and the fact that $\mathcal{K} \subseteq \mathcal{A}$, we obtain that

$$\mathcal{F}_0 \subseteq \mathcal{V}_{\mathcal{A}}. \tag{4.31}$$

Then, as a consequence of (4.28), (4.29) and (4.31), we get that $\mathcal{Y} \subseteq \mathcal{V}_{\mathcal{A}}$.

Hence, we are left with proving that

$$\mathcal{V}_{\mathcal{A}} \subseteq \mathcal{Y}. \tag{4.32}$$

For this, we show that

$$\text{on } \partial\mathcal{Y} \cap \big((0,1)\times(0,1)\big) \text{ the outward normal derivative is nonnegative.} \tag{4.33}$$

To prove this, we calculate the outward normal derivative on the part of $\partial\mathcal{Y}$ lying on the graph of $v = \gamma_0(u)$, that is

$$\dot v - \frac{u^{\rho-1}\dot u}{c(u_s^0)^{\rho-1}} = \rho v(1-u-v) - au - \frac{u^\rho(1-u-v-ac)}{c(u_s^0)^{\rho-1}}.$$

By substituting $v = \gamma_0(u) = \frac{u^\rho}{\rho c(u_s^0)^{\rho-1}}$ we get

$$\begin{aligned}\dot v - \frac{u^{\rho-1}\dot u}{c(u_s^0)^{\rho-1}} &= \frac{u^\rho}{c(u_s^0)^{\rho-1}}(1-u-v) - au - \frac{u^\rho(1-u-v-ac)}{c(u_s^0)^{\rho-1}} \\ &= -au + \frac{acu^\rho}{c(u_s^0)^{\rho-1}} = au^\rho\left(-u^{1-\rho} + \frac{1}{(u_s^0)^{\rho-1}}\right).\end{aligned}$$

As a result, since $\rho < 1$, we have

$$\dot v - \frac{u^{\rho-1}\dot u}{c(u_s^0)^{\rho-1}} \geqslant 0 \quad \text{for } u \leqslant u_s^0. \tag{4.34}$$

On the part of $\partial\mathcal{Y}$ contained on the line $v = \frac{u}{c} + \frac{1-\rho}{1+\rho c}$, the outward normal derivative is

$$\begin{aligned}\dot v - \frac{\dot u}{c} &= \rho v(1-u-v) - au - \frac{u(1-ac-u-v)}{c} = \left(\rho v - \frac{u}{c}\right)(1-u-v) \\ &= \left(\frac{\rho u}{c} + \frac{\rho(1-\rho)}{1+\rho c} - \frac{u}{c}\right)\left(1 - u - \frac{u}{c} - \frac{1-\rho}{1+\rho c}\right) \\ &= \left(\frac{(\rho-1)u}{c} + \frac{\rho(1-\rho)}{1+\rho c}\right)\left(-\frac{u(c+1)}{c} + \frac{\rho(1+c)}{1+\rho c}\right).\end{aligned} \tag{4.35}$$

We also observe that, when $u > u_s^0 = \frac{\rho c}{1+\rho c}$, the condition $\rho < 1$ gives that

$$\frac{(\rho-1)u}{c} + \frac{\rho(1-\rho)}{1+\rho c} < \frac{\rho(\rho-1)}{1+\rho c} + \frac{\rho(1-\rho)}{1+\rho c} = 0$$

and

$$-\frac{u(c+1)}{c} + \frac{\rho(1+c)}{1+\rho c} < -\frac{\rho(c+1)}{1+\rho c} + \frac{\rho(1+c)}{1+\rho c} = 0.$$

Therefore, when $u > u_s^0$, we deduce from (4.35) that

$$\dot{v} - \frac{\dot{u}}{c} > 0.$$

Combining this and (4.34), we obtain (4.33), as desired.

Now, by (4.33), we have that, for any value of a, no trajectory starting in $\big([0,1]\times[0,1]\big)\setminus\mathcal{Y}$ can enter in $\mathcal{Y}$, and in particular no trajectory starting in $\big([0,1]\times[0,1]\big)\setminus\mathcal{Y}$ can hit $\{v=0\}$, which ends the proof of (4.32).

(iii) Let $\rho > 1$. For the sake of simplicity, we suppose that $\frac{c}{(c+1)^\rho} \geqslant 1$. Let $\mathcal{X}$ be the right-hand-side of (1.17). We observe that

$$\mathcal{X} = \mathcal{S}_c \cup \mathcal{Q}, \tag{4.36}$$

where $\mathcal{S}_c$ was defined in (3.41) and $\mathcal{Q}$ in (4.3). Thanks to Proposition 3.4, one has that $\mathcal{S}_c \subseteq \bigcup_{a>a'} \mathcal{E}(a)$, for every $a' > 0$, and therefore $\mathcal{S}_c \subseteq \mathcal{V}_\mathcal{A}$. Moreover, by the second claim in Proposition 4.1, one also has that $\mathcal{Q} \subseteq \mathcal{V}_\mathcal{A}$. Hence,

$$\mathcal{X} \subseteq \mathcal{V}_\mathcal{A}. \tag{4.37}$$

Accordingly, to prove equality in (4.37) and thus complete the proof of (1.17), we need to show that $\mathcal{V}_\mathcal{A} \subseteq \mathcal{X}$. First, we prove that

$$(0,1] \times \{0\} \subseteq \mathcal{X}. \tag{4.38}$$

Indeed, for $u > 0$ we have $v = \frac{u}{c} > 0$, therefore $(u,0) \in \mathcal{X}$ for $u \in (0, u_\infty]$. Then, $\zeta(u)$ is increasing in u since it is a positive power function, therefore $v = \zeta(u) > 0$ for $u \in (u_\infty, 1]$, hence $(u,0) \in \mathcal{X}$ for $u \in (u_\infty, 1]$. These observations prove (4.38).

We now prove that the component of the velocity field in the outward normal direction with respect to $\mathcal{X}$ is nonnegative on

$$\partial\mathcal{X} \cap \partial(\mathcal{X}^C) = \\ \left\{(u,v) \in (0,u_\infty] \times (0,1) \ : \ v = \frac{u}{c}\right\} \cup \{(u,v) \in (u_\infty, 1) \times (0,1) \ \ v = \zeta(u)\}.$$

To this end. we observe that on the line $v = \frac{u}{c}$, the outward normal derivative is

$$\dot{v} - \frac{1}{c}\dot{u} = \rho v(1-u-v) - au - \frac{u}{c}(1-ac-u-v) = (\rho v - \frac{u}{c})(1-u-v). \tag{4.39}$$

The first term is positive because for $\rho > 1$ we have

$$\rho v > v = \frac{u}{c}.$$

Moreover, for $u \leqslant u_\infty$ we have that

$$1 - u - v \geqslant 1 - u_\infty - \frac{u_\infty}{c} = 0,$$

thanks to (1.18). Thus, the left hand side of (4.39) is nonnegative, which proves that the component of the velocity field in the outward normal direction is nonnegative on $\partial\mathcal{X} \cap \{v = \frac{u}{c}\}$.

On the part of $\partial\mathcal{X}$ lying in the graph of $v = \zeta(u)$, the component of the velocity field in the outward normal direction is given by

$$\dot{v} - \frac{\rho u^{\rho-1}\dot{u}}{\rho c(u_\infty)^{\rho-1}} = \rho v(1-u-v) - au - \frac{\rho u^\rho}{\rho c(u_\infty)^{\rho-1}}(1-u-v-ac). \tag{4.40}$$

Now we substitute $v = \zeta(u) = \frac{u^\rho}{\rho c(u_\infty)^{\rho-1}}$ in (4.40) and we get

$$\dot{v} - \frac{u^{\rho-1}\dot{u}}{c(u_\infty)^{\rho-1}} = au\left(-1 + \frac{u^{\rho-1}}{(u_\infty)^{\rho-1}}\right)$$

which leads to

$$\dot{v} - \frac{\rho u^{\rho-1}\dot{u}}{\rho c(u_\infty)^{\rho-1}} > 0 \quad \text{if } u > u_\infty,$$

as desired.

As a consequence of these considerations, we find that no trajectory starting in $\mathcal{X}^C$ can enter in $\mathcal{X}$ and therefore hit $\{v = 0\}$, by (4.38). Hence, we conclude that $\mathcal{V}_\mathcal{A} \subseteq \mathcal{X}$, which, together with (4.37), establishes (1.17). □

4.3 Proof of Theorem 1.4

In order to prove Theorem 1.4, we will establish a geometrical lemma in order to understand the reciprocal position of the function γ, as given by Propositions 2.1 and 2.7, and the straight line where the saddle equilibria lie. To emphasize the dependence of γ on the parameter a we will often use the notation $\gamma = \gamma_a$. Moreover, we recall the notation of the saddle points (u_s, v_s) defined in (1.11) and of the points $(u_\mathcal{M}, v_\mathcal{M})$ given by Propositions 2.1 and 2.7, with the convention that

$$(u_s, v_s) = (0, 0) \text{ if } ac \geqslant 1, \tag{4.41}$$

and we state the following result:

Lemma 4.3. *If $\rho < 1$, then*

$$\frac{u}{\rho c} \leqslant \gamma_a(u) \quad \textit{for } u \in [0, u_s] \tag{4.42}$$

and

$$\gamma_a(u) \leqslant \frac{u}{\rho c} \qquad \textit{for } u \in [u_s, u_{\mathcal{M}}]. \tag{4.43}$$

If instead $\rho > 1$*, then*

$$\gamma_a(u) \leqslant \frac{u}{\rho c} \qquad \textit{for } u \in [0, u_s] \tag{4.44}$$

and

$$\frac{u}{\rho c} \leqslant \gamma_a(u) \qquad \textit{for } u \in [u_s, u_{\mathcal{M}}]. \tag{4.45}$$

Moreover equality holds in (4.42) *and* (4.44) *if and only if either* $u = u_s$ *or* $u = 0$*. Also, strict inequality holds in* (4.43) *and* (4.45) *for* $u \in (u_s, u_{\mathcal{M}})$*.*

Proof. We focus here on the proof of (4.43), since the other inequalities are proven in a similar way. Moreover, we deal with the case $ac < 1$, being the case $ac \geqslant 1$ analogous with obvious modifications.

We suppose by contradiction that (4.43) does not hold true. Namely, we assume that there exists $\tilde{u} \in (u_s, u_{\mathcal{M}}]$ such that

$$\gamma_a(\tilde{u}) > \frac{\tilde{u}}{\rho c}.$$

Since γ_a is continuous thanks to Propositions 2.1, we have that

$$\gamma_a(u) > \frac{u}{\rho c} \qquad \text{in a neighborhood of } \tilde{u}.$$

Hence, we consider the largest open interval $(u_1, u_2) \subset (u_s, u_{\mathcal{M}}]$ containing $\tilde{u}$ and such that

$$\gamma_a(u) > \frac{u}{\rho c} \qquad \text{for all } u \in (u_1, u_2). \tag{4.46}$$

Moreover, in light of (1.11), we see that

$$\gamma_a(u_s) = v_s = \frac{1 - ac}{1 + \rho c} = \frac{u_s}{\rho c}. \tag{4.47}$$

Hence, by the continuity of γ_a, we have that $\gamma_a(u_1) = \frac{u_1}{\rho c}$ and

$$\text{either } \gamma_a(u_2) = \frac{u_2}{\rho c} \text{ or } u_2 = u_{\mathcal{M}}. \tag{4.48}$$

Now, we consider the set

$$\mathcal{T} := \left\{ (u, v) \in [u_1, u_2] \times [0, 1] \;\text{ s.t. }\; \frac{u}{\rho c} < v < \gamma_a(u) \right\},$$

that is non empty, thanks to (4.46). We claim that

$$\text{for all } (u(0), v(0)) \in \mathcal{T}, \text{ the } \omega\text{-limit of its trajectory is } (u_s, v_s). \tag{4.49}$$

To prove this, we analyze the normal derivative on

$$\begin{aligned} &\partial\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3, \\ \text{where} \quad &\mathcal{T}_1 := \big\{(u, \gamma_a(u)) \text{ with } u \in (u_1, u_2)\big\}, \\ &\mathcal{T}_2 := \left\{\left(u, \frac{u}{\rho c}\right) \text{ with } u \in (u_1, u_2)\right\} \\ \text{and} \quad &\mathcal{T}_3 := \left\{(u_2, v) \text{ with } v \in \left(\frac{u_2}{\rho c}, \min\{\gamma_a(u_2), 1\}\right)\right\}, \end{aligned}$$

with the convention that $\partial\mathcal{T}$ does contain $\mathcal{T}_3$ only if the second possibility in (4.48) occurs.

We notice that the set $\mathcal{T}_1$ is an orbit for the system, and thus the component of the velocity in the normal direction is null. On $\mathcal{T}_2$, we have that the sign of the component of the velocity in the inward normal direction is given by

$$\begin{aligned} (\dot{u}, \dot{v}) \cdot \left(-\frac{1}{\rho c}, 1\right) &= \dot{v} - \frac{1}{\rho c}\dot{u} = \rho v(1-u-v) - au - \frac{u}{\rho c}(1-u-v) + \frac{au}{\rho} \\ &= \frac{u}{c}\left(1 - u - \frac{u}{\rho c}\right)\left(1 - \frac{1}{\rho}\right) - au\left(1 - \frac{1}{\rho}\right) \\ &= \frac{u}{c}\left(1 - \frac{1}{\rho}\right)\left(1 - u - \frac{u}{\rho c} - ac\right). \end{aligned} \tag{4.50}$$

Notice that for $u \geqslant u_s$ we have that

$$1 - u - v - ac \leqslant 0, \tag{4.51}$$

thus the sign of last term in (4.50) depends only on the quantity $1 - \frac{1}{\rho}$. Consequently, since $\rho < 1$ the sign of the component of the velocity in the inward normal direction is positive.

Furthermore, in the case in which the second possibility in (4.48) occurs, we also check the sign of the component of the velocity in the inward normal direction along $\mathcal{T}_3$. In this case, if $\gamma_a(u_2) < 1$ then $u_2 = 1$, and therefore we find that

$$(\dot{u}, \dot{v}) \cdot (-1, 0) = -\dot{u} = -u(1-u-v) + acu = v + ac,$$

which is positive. If instead $\gamma_a(u_2) = 1$

$$(\dot{u}, \dot{v}) \cdot (-1, 0) = -\dot{u} = -u(1-u-v) + acu = -u(1 - ac - u - v),$$

which is positive, thanks to (4.51).

We also point out that there are no cycle in $\mathcal{T}$, since $\dot{u}$ has a sign. These considerations and the Poincaré-Bendixson Theorem (see e.g. [20]) give that the ω-limit set of $(u(0), v(0))$ can be either an equilibrium or a union of (finitely many) equilibria and non-closed orbits connecting these equilibria. Since $(0,0)$ and $(0,1)$ do not belong to the closure of $\mathcal{T}$, in this case the only possibility is that the ω-limit is the equilibrium (u_s, v_s). Consequently, we have that $u_1 = u_s$, and that (4.49) is satisfied.

Accordingly, in light of (4.49), we have that the set $\mathcal{T}$ is contained in the stable manifold of (u_s, v_s), which is in contradiction with the definition of $\mathcal{T}$. Hence, (4.43) is established, as desired.

Now we show that strict inequality holds true in (4.43) if $u \in (u_s, u_{\mathcal{M}})$. To this end, we suppose by contradiction that there exists $\bar{u} \in (u_s, u_{\mathcal{M}})$ such that

$$\gamma_a(\bar{u}) = \frac{\bar{u}}{\rho c}. \tag{4.52}$$

Now, since (4.43) holds true, we have that the line $v - \frac{u}{\rho c} = 0$ is tangent to the curve $v = \gamma_a(u)$ at $(\bar{u}, \gamma_a(\bar{u}))$, and therefore at this point the components of the velocity along the normal directions to the curve and to the line coincide. On the other hand, the normal derivative at a point on the line has a sign, as computed in (4.50), while the normal derivative to $v = \gamma_a(u)$ is 0 because the curve is an orbit.

This, together with (4.47), proves that equality in (4.43) holds true if $u = u_s$, but strict inequality holds true for all $u \in (u_s, u_{\mathcal{M}})$, and thus the proof of Lemma 4.3 is complete. □

For each $a > 0$, we define $(u_d^a, v_d^a) \in [0,1] \times [0,1]$ as the unique intersection of the graph of γ_a with the line $\{v = 1 - u\}$, that is the solution of the system

$$\begin{cases} v_d^a = \gamma_a(u_d^a), \\ v_d^a = 1 - u_d^a. \end{cases} \tag{4.53}$$

We recall that the above intersection is unique since the function γ_a is increasing. Also, by construction,

$$u_d^a \leqslant u_{\mathcal{M}}. \tag{4.54}$$

Now, recalling (1.11) and making explicit the dependence on a by writing u_s^a (with the convention in (4.41)), we give the following result:

Lemma 4.4. *We have that:*

1. *For $\rho < 1$, for all $a^* > 0$ it holds that*

$$\gamma_a(u) \leqslant \gamma_{a^*}(u) \quad \textit{for all } a > a^* \textit{ and for all } u \in [u_s^{a^*}, u_d^{a^*}]. \tag{4.55}$$

2. *For $\rho > 1$, for all $a^* > 0$ it holds that*

$$\gamma_a(u) \leqslant \gamma_{a^*}(u) \quad \textit{for all } a < a^* \textit{ and for all } u \in [u_s^{a^*}, u_d^{a^*}]. \tag{4.56}$$

Proof. We claim that

$$u_s^{a^*} < u_d^{a^*}. \tag{4.57}$$

Indeed, when $a^* c \geqslant 1$, we have that $u_s^{a^*} = 0 < u_d^{a^*}$ and thus (4.57) holds true. If instead $a^* c < 1$, by (1.11) and (4.53) we have that

$$\gamma_{a^*}(u_s^{a^*}) + u_s^{a^*} = 1 - a^* c < 1 = \gamma_{a^*}(u_d^{a^*}) + u_d^{a^*}. \tag{4.58}$$

Also, since γ_{a^*} is increasing, we have that the map $r \mapsto \gamma_{a^*}(r)+r$ is strictly increasing. Consequently, we deduce from (4.58) that (4.57) holds true in this case as well.

Now we suppose that $\rho < 1$ and we prove (4.55). For this, we claim that, for every $a^* > 0$ and every $a > a^*$,

$$\gamma_a(u_s^{a^*}) \leqslant \gamma_{a^*}(u_s^{a^*}) \quad \text{with strict inequality when } a^* \in \left(0, \frac{1}{c}\right). \tag{4.59}$$

To check this, we distinguish two cases. If $a^* \in \big(0, \frac{1}{c}\big)$, then for all $a > a^*$

$$u_s^a = \max\left\{0, \rho c \frac{1-ac}{1+\rho c}\right\} < \rho c \frac{1-a^*c}{1+\rho c} = u_s^{a^*}. \tag{4.60}$$

By (4.60) and formula (4.43) in Lemma 4.3, we have that

$$\gamma_a(u_s^{a^*}) < \frac{u_s^{a^*}}{\rho c} = \gamma_{a^*}(u_s^{a^*}) \quad \text{for all } a > a^*. \tag{4.61}$$

If instead $a^* \geqslant \frac{1}{c}$, then $u_s^{a^*} = 0$ and for all $a > a^*$ we have $u_s^a = 0$. As a consequence,

$$\gamma_{a^*}(u_s^{a^*}) = \gamma_a(u_s^{a^*}) \quad \text{for all } a > a^*. \tag{4.62}$$

The claim in (4.59) thus follows from (4.61) and (4.62).

Furthermore, by Propositions 2.1 and 2.7,

$$\gamma_a'(0) = \frac{a}{\rho + ac - 1} < \frac{a^*}{\rho + a^*c - 1} = \gamma_{a^*}'(0) \quad \text{for all } a > a^* \geqslant \frac{1}{c}. \tag{4.63}$$

Moreover, for all $a \geqslant a^*$ and $u > u_s^{a^*}$ it holds that, when $v = \gamma_{a^*}(u)$,

$$-\big(acu - u(1-u-v)\big) = u(1-u-\gamma_{a^*}(u) - ac) < u(1-u_s^{a^*} - v_s^{a^*} - ac) \leqslant 0. \tag{4.64}$$

Now, we establish that

$$u(\rho c v - u)(1-u-v)(a-a^*) < 0 \quad \text{for all } a > a^*,\ u \in (u_s^{a^*}, u_d^{a^*}),\ v = \gamma_{a^*}(u). \tag{4.65}$$

Indeed, for the values of a, u and v as in (4.65) we have that $v \leqslant \gamma_{a^*}(u_d^{a^*})$ and hence

$$(1-u-v) > (1-u_d^{a^*} - \gamma_{a^*}(u_d^{a^*})) = 0. \tag{4.66}$$

Moreover, by formula (4.43) in Lemma 4.3, for $u \in (u_s^{a^*}, u_d^{a^*})$ and $v = \gamma_{a^*}(u)$ and we have that

$$\rho c v - u = \rho c \gamma_{a^*}(u) - u < 0.$$

From this and (4.66), we see that (4.65) plainly follows, as desired.

As a consequence of (4.64) and (4.65), one deduces that, for all $a > a^*$, $u \in (u_s^{a^*}, u_d^{a^*})$ and $v = \gamma_{a^*}(u)$,

$$\begin{aligned}
&\frac{au - \rho v(1-u-v)}{acu - u(1-u-v)} - \frac{a^*u - \rho v(1-u-v)}{a^*cu - u(1-u-v)} \\
&= \frac{(a-a^*)c\rho uv(1-u-v) - (a-a^*)u^2(1-u-v)}{\big(acu - u(1-u-v)\big)\big(a^*cu - u(1-u-v)\big)} \\
&= \frac{(a-a^*)(1-u-v)u(c\rho v - u)}{\big(acu - u(1-u-v)\big)\big(a^*cu - u(1-u-v)\big)} \\
&\leqslant 0.
\end{aligned} \tag{4.67}$$

Now, we define

$$\mathcal{Z}(u) := \gamma_a(u) - \gamma_{a^*}(u) \tag{4.68}$$

and we claim that

$$\text{if } u_o \in (u_s^{a^*}, u_d^{a^*}) \text{ is such that } \mathcal{Z}(u_o) = 0, \text{ then } \mathcal{Z}'(u_o) < 0. \tag{4.69}$$

Indeed, since γ_a is a trajectory for (1.1), if $(u_a(t), v_a(t))$ is a solution of (1.1), we have that $v_a(t) = \gamma_a(u_a(t))$, whence

$$\begin{aligned}\rho v_a(t)(1 - u_a(t) - v_a(t)) - a u_a(t) &= \dot v_a(t) = \gamma_a'(u_a(t))\, \dot u_a(t) \\ &= \gamma_a'(u_a(t))\big(u_a(t)(1 - u_a(t) - v_a(t)) - acu_a(t)\big).\end{aligned} \tag{4.70}$$

Then, we let $v_o := \gamma_a(u_o)$ and we notice that v_o coincides also with $\gamma_{a^*}(u_o)$. Hence, we take trajectories of the system with parameter a and a^* starting at (u_o, v_o), and by (4.67) we obtain that

$$\begin{aligned} 0 &> \frac{au_o - \rho v(1-u_o-v_o)}{acu_o - u_o(1-u_o-v_o)} - \frac{a^*u_o - \rho v(1-u_o-v_o)}{a^*cu_o - u_o(1-u_o-v_o)} \\ &= \frac{au_a(0) - \rho v(1-u_a(0)-v_a(0))}{acu_a(0) - u(1-u_a(0)-v_a(0))} - \frac{a^*u_{a^*}(0) - \rho v(1-u_{a^*}(0)-v_a(0))}{a^*cu_{a^*}(0) - u(1-u_{a^*}(0)-v_{a^*}(0))} \\ &= \gamma_a'(u_a(0)) - \gamma_{a^*}'(u_{a^*}(0)) \\ &= \gamma_a'(u_o) - \gamma_{a^*}'(u_o), \end{aligned}$$

which establishes (4.69).

Now we claim that

$$\begin{aligned}&\text{there exists } \underline{u} \in [u_s^{a^*}, u_d^{a^*}] \text{ such that } \mathcal{Z}(\underline{u}) < 0 \\ &\text{and } \mathcal{Z}(u) \leqslant 0 \text{ for every } u \in [u_s^{a^*}, \underline{u}].\end{aligned} \tag{4.71}$$

Indeed, if $a^* \in \left(0, \frac{1}{c}\right)$, we deduce from (4.59) that $\mathcal{Z}(u_s^{a^*}) < 0$ and therefore (4.71) holds true with $\underline{u} := u_s^{a^*}$. If instead $a^* \geqslant \frac{1}{c}$, we have that $u_s^a = u_s^{a^*} = 0$ and we deduce from (4.59) and (4.63) that $\mathcal{Z}(u_s^{a^*}) = 0$ and $\mathcal{Z}'(u_s^{a^*}) < 0$, from which (4.71) follows by choosing $\underline{u} := u_s^{a^*} + \epsilon$ with $\epsilon > 0$ sufficiently small.

Now we claim that

$$\mathcal{Z}(u) \leqslant 0 \qquad \text{for every } u \in [u_s^{a^*}, u_d^{a^*}]. \tag{4.72}$$

To prove this, in light of (4.71), it suffices to check that $\mathcal{Z}(u) \leqslant 0$ for every $u \in (\underline{u}, u_d^{a^*}]$. Suppose not. Then there exists $u^\sharp \in (\underline{u}, u_d^{a^*}]$ such that $\mathcal{Z}(u) < 0$ for all $[\underline{u}, u^\sharp)$ and $\mathcal{Z}(u^\sharp) = 0$. This gives that $\mathcal{Z}'(u^\sharp) \geqslant 0$. But this inequality is in contradiction with (4.69) and therefore the proof of (4.72) is complete.

The desired claim in (4.55) follows easily from (4.72), hence we focus now on the proof of (4.56).

To this end, we take $\rho > 1$ and we claim that, for every $a^* > 0$ and every $a \in (0, a^*)$,

$$\gamma_a(u_s^{a^*}) \leqslant \gamma_{a^*}(u_s^{a^*}) \qquad \text{with strict inequality when } a^* \in \left(0, \frac{1}{c}\right). \tag{4.73}$$

To prove this, we first notice that, if $a < a^* < \frac{1}{c}$, then

$$u_s^{a^*} = \rho c \frac{1 - a^* c}{1 + \rho c} < \rho c \frac{1 - ac}{1 + \rho c} = u_s^a.$$

Hence by (4.44) in Lemma 4.3 we have

$$\gamma_a(u_s^{a^*}) < \frac{u_s^{a^*}}{\rho c} = \gamma_{a^*}(u_s^{a^*}) \quad \text{for } a < a^* < \frac{1}{c},$$

and this establishes (4.73) when $a^* \in \left(0, \frac{1}{c}\right)$. Thus, we now focus on the case $a^* \geqslant \frac{1}{c}$. In this situation, we have that $u_s^{a^*} = 0$ and accordingly $\gamma_a(u_s^{a^*}) = \gamma_a(0) = \gamma_{a^*}(0) = \gamma_{a^*}(u_s^{a^*})$, that completes the proof of (4.73).

In addition, by Propositions 2.1 and 2.7 we have that

$$\gamma_a'(0) = \frac{a}{\rho - 1 + ac} \leqslant \frac{a^*}{\rho - 1 + a^* c} = \gamma_{a^*}'(0) \quad \text{for } a \in \left[\frac{1}{c}, a^*\right]. \tag{4.74}$$

Moreover, for $u > u_s^a$, if $v = \gamma_a(u)$ we have that $v > \gamma_a(u_s^a) = v_s^a$, thanks to the monotonicity of γ_a, and, as a result,

$$u(1 - u - v - ac) < u(1 - u_s^a - v_s^a - ac) = 0. \tag{4.75}$$

Now we claim that, for all $a < a^*$, $u \in (u_s^{a^*}, u_d^{a^*})$ and $v = \gamma_{a^*}(u)$, we have

$$u(1 - u - v)(a^* - a)(u - \rho c v) < 0. \tag{4.76}$$

Indeed, by the monotonicity of γ_{a^*}, in this situation we have that $v \leqslant \gamma_{a^*}(u_d^{a^*})$, and therefore, by (4.53),

$$1 - u - v > 1 - u_d^{a^*} - \gamma_{a^*}(u_d^{a^*}) = 1 - u_d^{a^*} - 1 + u_d^{a^*} = 0. \tag{4.77}$$

Moreover, by (4.45) in Lemma (4.3), we have that $\gamma_{a^*}(u) > \frac{u}{\rho c}$, and hence $u - \rho c v > 0$. Combining this inequality with (4.77), we obtain (4.76), as desired.

Now, by (4.75), for all $a < a^*$, $u \in (u_s^a, u_d^{a^*})$ and $v = \gamma_{a^*}(u)$,

$$0 < -u(1 - u - v - ac) = acu - u(1 - u - v) < a^* cu - u(1 - u - v)$$

and then, by (4.76),

$$\begin{aligned} &\frac{au - \rho v(1 - u - v)}{acu - u(1 - u - v)} - \frac{a^* u - \rho v(1 - u - v)}{a^* cu - u(1 - u - v)} \\ &= \frac{u(1 - u - v)(a^* - a)(u - \rho c v)}{\big(acu - u(1 - u - v)\big)\big(a^* cu - u(1 - u - v)\big)} \\ &< 0. \end{aligned} \tag{4.78}$$

Now we recall the definition of $\mathcal{Z}$ in (4.68) and we claim that

$$\text{if } u_o \in (u_s^{a^*}, u_d^{a^*}) \text{ is such that } \mathcal{Z}(u_o) = 0, \text{ then } \mathcal{Z}'(u_o) < 0. \tag{4.79}$$

To prove this, we let $v_o := \gamma_a(u_o)$, we notice that $v_o = \gamma_{a^*}(u_o)$, we recall (4.70) and apply it to a trajectory starting at (u_o, v_o), thus finding that

$$\rho v_o(1-u_o-v_a(t)) - au_o = \gamma_a'(u_o)\big(u_o(1-u_o-v_o)-acu_o\big).$$

This and (4.78) yield that

$$0 > \frac{au-\rho v(1-u-v)}{acu-u(1-u-v)} - \frac{a^*u-\rho v(1-u-v)}{a^*cu-u(1-u-v)} = \gamma_a'(u_o) - \gamma_{a^*}'(u_o) = \mathcal{Z}'(u_o),$$

which proves the desired claim in (4.79).

We now point out that

$$\begin{aligned}&\text{there exists } \underline{u} \in [u_s^{a^*}, u_d^{a^*}] \text{ such that } \mathcal{Z}(\underline{u}) < 0\\ &\text{and } \mathcal{Z}(u) \leqslant 0 \text{ for every } u \in [u_s^{a^*}, \underline{u}].\end{aligned} \tag{4.80}$$

Indeed, if $a^* \in \big(0, \frac{1}{c}\big)$, this claim follows directly from (4.59) by choosing $\underline{u} := u_s^{a^*}$, while if $a^* \geqslant \frac{1}{c}$, the claim follows from (4.59) and (4.69) by choosing $\underline{u} := u_s^{a^*} + \epsilon$ with $\epsilon > 0$ sufficiently small.

Now we claim that

$$\mathcal{Z}(u) \leqslant 0 \qquad \text{for every } u \in [u_s^{a^*}, u_d^{a^*}]. \tag{4.81}$$

Indeed, by (4.80), we know that the claim is true for all $u \in [u_s^{a^*}, \underline{u}]$. Then, the claim for $u \in (\underline{u}, u_d^{a^*}]$ can be proved by contradiction, supposing that there exists $u^\sharp \in (\underline{u}, u_d^{a^*}]$ such that $\mathcal{Z}(u) < 0$ for all $[\underline{u}, u^\sharp)$ and $\mathcal{Z}(u^\sharp) = 0$. This gives that $\mathcal{Z}'(u^\sharp) \geqslant 0$, which is in contradiction with (4.69).

Having completed the proof of (4.81), one can use it to obtain the desired claim in (4.56). □

Now we perform the proof of Theorem 1.4, analyzing separately the cases $\rho = 1$, $\rho < 1$ and $\rho > 1$.

Proof of Theorem 1.4, case $\rho = 1$. We notice that

$$\mathcal{V}_{\mathcal{K}} \subseteq \mathcal{V}_{\mathcal{A}}, \tag{4.82}$$

since $\mathcal{K} \subset \mathcal{A}$.

Also, from Theorem 1.3, part (i), we get that $\mathcal{V}_{\mathcal{A}} = \mathcal{S}_c$, where $\mathcal{S}_c$ was defined in (3.41). On the other hand, by Lemma 4.2, we know that for $\rho = 1$ and for all $a > 0$ we have $\mathcal{E}(a) = \mathcal{S}_c$. But since every constant a belongs to the set $\mathcal{K}$, we have $\mathcal{E}(a) \subseteq \mathcal{V}_{\mathcal{K}}$. This shows that $\mathcal{V}_{\mathcal{A}} = \mathcal{E}(a) \subseteq \mathcal{V}_{\mathcal{K}}$, and together with (4.82) concludes the proof. □

Proof of Theorem 1.4, case $\rho < 1$. We notice that

$$\mathcal{V}_{\mathcal{K}} \subseteq \mathcal{V}_{\mathcal{A}}, \tag{4.83}$$

since $\mathcal{K} \subset \mathcal{A}$. To prove that the inclusion is strict, we aim to find a point $(\bar{u}, \bar{v}) \in \mathcal{V}_{\mathcal{A}} \setminus \mathcal{V}_{\mathcal{K}}$. Namely, we have to prove that there exists $(\bar{u}, \bar{v}) \in \mathcal{V}_{\mathcal{A}}$ such that, for all constant strategies $a > 0$, we have that $(\bar{u},\bar{v}) \notin \mathcal{E}(a)$, that is, by the characterization in Proposition 2.9, it must hold true that $\bar{v} \geqslant \gamma_a(\bar{u})$ and $\bar{u} \leqslant u^a_{\mathcal{M}}$.

To do this, we define

$$f(u) := \frac{u}{c} + \frac{1-\rho}{1+\rho c} \quad \text{and} \quad m := \min\left\{\frac{\rho c(c+1)}{1+\rho c}, 1\right\}. \tag{4.84}$$

By inspection, one can see that $(u, f(u)) \in [0,1] \times [0,1]$ if and only if $u \in [0,m]$. We point out that, by (ii) of Theorem 1.3, for $\rho < 1$ and $u \in [u^0_s, m]$, a point (u,v) belongs to $\mathcal{V}_{\mathcal{A}}$ if and only if $v < f(u)$. Here u^0_s is defined in (1.14). We underline that the interval $[u^0_s, m]$ is non empty since

$$u^0_s = \frac{\rho c}{1+\rho c} < \min\left\{\frac{\rho c(c+1)}{1+\rho c}, 1\right\} = m. \tag{4.85}$$

Now we point out that

$$m \leqslant u^a_{\mathcal{M}}. \tag{4.86}$$

Indeed, by (4.84) we already know that $m \leqslant 1$, thus if $u^a_{\mathcal{M}} = 1$ the inequality in (4.86) is true. On the other hand, when $u^a_{\mathcal{M}} < 1$ we have that $(u^a_{\mathcal{M}}, 1) \times (0,1) \subseteq \mathcal{E}(a)$. This and (4.83) give that $(u^a_{\mathcal{M}}, 1) \times (0,1) \subseteq \mathcal{V}_{\mathcal{K}} \subseteq \mathcal{V}_{\mathcal{A}}$. Hence, in view of (1.16), we deduce that $\frac{\rho c(c+1)}{1+\rho c} \leqslant u^a_{\mathcal{M}}$. In particular, we find that $m \leqslant u^a_{\mathcal{M}}$, and therefore (4.86) is true also in this case.

With this notation, we claim the existence of a value $\bar{v} \in (0,1]$ such that for all $a > 0$ we have $\gamma_a(m) \leqslant \bar{v} < f(m)$. That is, we prove now that there exists $\theta > 0$ such that

$$\gamma_a(m) + \theta < f(m) \qquad \text{for all } a > 0. \tag{4.87}$$

The strategy is to study two cases separately, namely we prove (4.87) for sufficiently small values of a and then for the other values of a.

To prove (4.87) for small values of a, we start by looking at the limit function γ_0 defined in (3.35). One observes that

$$\gamma_0(u^0_s) = v^0_s = \frac{1}{1+\rho c} = \frac{\rho c}{c(1+\rho c)} + \frac{1-\rho}{1+\rho c} = f(u^0_s). \tag{4.88}$$

Moreover, for all $u \in (u^0_s, m]$, we have that

$$\gamma_0'(u) = \frac{v^0_s}{(u^0_s)^\rho}\,\rho u^{\rho-1} < \frac{v^0_s}{(u^0_s)^\rho}\,\rho (u^0_s)^{\rho-1} = \frac{\rho v^0_s}{u^0_s} = \frac{1}{c} = f'(u).$$

Hence, using the fundamental theorem of calculus on the continuous functions $\gamma_0(u)$ and $f(u)$, we get

$$\gamma_0(m) = \gamma_0(u^0_s) + \int_{u^0_s}^m \gamma_0'(u)\,du < f(u^0_s) + \int_{u^0_s}^m f'(u)\,du = f(m).$$

Then, the quantity

$$\theta_1 := \frac{f(m) - \gamma_0(m)}{4}$$

is positive and we have

$$\gamma_0(m) + 2\theta_1 < f(m). \tag{4.89}$$

Now, by the uniform convergence of γ_a to γ_0 given by Lemma 3.5, we know that there exists $\varepsilon \in \left(0, \frac{1}{c}\right)$ such that, if $a \in (0, \varepsilon]$,

$$\sup_{u \in [u_s^0, m]} |\gamma_a(u) - \gamma_0(u)| < \theta_1. \tag{4.90}$$

By this and (4.89), we obtain that

$$\gamma_a(m) + \theta_1 < f(m) \quad \text{ for all } a \in (0, \varepsilon]. \tag{4.91}$$

We remark that formula (4.91) will give the desired claim in (4.87) for conveniently small values of a.

We are now left with considering the case $a > \varepsilon$. To this end, recalling (1.11), (4.53), by the first statement in Lemma 4.4, used here with $a^* := \varepsilon$, we get

$$\gamma_a(u) \leqslant \gamma_\varepsilon(u) \quad \text{ for all } a > \varepsilon \text{ and for all } u \in [u_s^\varepsilon, u_d^\varepsilon]. \tag{4.92}$$

Now we observe that

$$u_d^a \geqslant u_s^\varepsilon. \tag{4.93}$$

Indeed, suppose not, namely

$$u_d^a < u_s^\varepsilon. \tag{4.94}$$

Then, by the monotonicity of γ_a, we have that $\gamma_a(u_d^a) \leqslant \gamma_a(u_s^\varepsilon)$. This and (4.92) yield that $\gamma_a(u_d^a) \leqslant \gamma_\varepsilon(u_s^\varepsilon)$. Hence, the monotonicity of γ_ε gives that $\gamma_a(u_d^a) \leqslant \gamma_\varepsilon(u_d^\varepsilon)$. This and (4.53) lead to $1 - u_d^a \leqslant 1 - u_d^\varepsilon$, that is $u_d^\varepsilon \leqslant u_d^a$. From this inequality, using again (4.94), we deduce that $u_d^\varepsilon < u_s^\varepsilon$. This is in contradiction with (4.57) and thus the proof of (4.93) is complete.

We also notice that

$$u_d^a \geqslant u_d^\varepsilon. \tag{4.95}$$

Indeed, suppose not, say

$$u_d^a < u_d^\varepsilon. \tag{4.96}$$

Then, by (4.93), we have that $u_d^a \in [u_s^\varepsilon, u_d^\varepsilon]$ and therefore we can apply (4.92) to say that $\gamma_a(u_d^a) \leqslant \gamma_\varepsilon(u_d^a)$. Also, by the monotonicity of γ_ε, we have that $\gamma_\varepsilon(u_d^a) \leqslant \gamma_\varepsilon(u_d^\varepsilon)$.

With these items of information and (4.53), we find that

$$1 - u_d^a = \gamma_a(u_d^a) \leqslant \gamma_\varepsilon(u_d^\varepsilon) = 1 - u_d^\varepsilon,$$

and accordingly $u_d^a \geqslant u_d^\varepsilon$. This is in contradiction with (4.96) and establishes (4.95).

Moreover, by (1.11) and (1.14), we know that $u_s^0 > u_s^{a^*}$, for every $a^* > 0$. Therefore, setting $\tilde{u}_d^{a^*} := \min\{u_d^{a^*}, u_s^0\}$, we have that $\tilde{u}_d^{a^*} \in [u_s^{a^*}, u_d^{a^*}]$. Thus, we are in the position of using the first statement in Lemma 4.4 with $a := \varepsilon$ and deduce that

$$\gamma_\varepsilon(\tilde{u}_d^{a^*}) \leqslant \gamma_{a^*}(\tilde{u}_d^{a^*}) \qquad \text{for all} \quad a^* < \varepsilon. \tag{4.97}$$

We also remark that

$$u_d^{a^*} \to u_s^0 \qquad \text{as} \quad a^* \to 0. \tag{4.98}$$

Indeed, up to a subsequence we can assume that $u_d^{a^*} \to \tilde{u}$ as $a^* \to 0$, for some $\tilde{u} \in [0,1]$. Also, by (4.53),

$$\gamma_{a^*}(u_d^{a^*}) = 1 - u_d^{a^*},$$

and then the uniform convergence of γ_{a^*} in Lemma 3.5 yields that

$$\gamma_0(\tilde{u}) = 1 - \tilde{u}.$$

This and (4.53) lead to $\tilde{u} = u_d^0$. Since

$$u_d^0 = u_s^0 \tag{4.99}$$

in virtue of (1.14), we thus conclude that $\tilde{u} = u_s^0$ and the proof of (4.98) is thereby complete.

As a consequence of (4.98), we have that $\tilde{u}_d^{a^*} \to u_s^0$ as $a^* \to 0$. Hence, using again the uniform convergence of γ_{a^*} in Lemma 3.5, we obtain that $\gamma_{a^*}(\tilde{u}_d^{a^*}) \to \gamma_0(u_s^0)$. From this and (4.97), we conclude that

$$\gamma_\varepsilon(u_s^0) \leqslant \gamma_0(u_s^0). \tag{4.100}$$

Now we claim that

$$u_d^\varepsilon > u_s^0. \tag{4.101}$$

Indeed, suppose, by contradiction, that

$$u_d^\varepsilon \leqslant u_s^0. \tag{4.102}$$

Then, the monotonicity of γ_ε, together with (4.99) and (4.100), gives that

$$1 - u_d^\varepsilon = \gamma_\varepsilon(u_d^\varepsilon) \leqslant \gamma_\varepsilon(u_s^0) = 1 - u_s^0.$$

From this and (4.102) we deduce that $u_d^\varepsilon = u_s^0$. In particular, we have that $u_s^0 \in (u_s^\varepsilon, u_{\mathcal{M}}^\varepsilon)$. Accordingly, by (4.43),

$$1 - u_s^0 = 1 - u_d^\varepsilon = \gamma_\varepsilon(u_d^\varepsilon) = \gamma_\varepsilon(u_s^0) < \frac{u_s^0}{\rho c}.$$

As a consequence,

$$u_s^0 > \frac{\rho c}{1 + \rho c},$$

and this is in contradiction with (1.14). The proof of (4.101) is thereby complete.

As a byproduct of (4.99) and (4.101), we have that

$$v_d^\varepsilon = \gamma_\varepsilon(u_d^\varepsilon) = 1 - u_d^\varepsilon < 1 - u_s^0 = 1 - u_d^0 = \gamma_0(u_d^0) = \gamma_0(u_s^0) = v_s^0. \tag{4.103}$$

Similarly, by means of (4.95),

$$v_d^a = \gamma_a(u_d^a) = 1 - u_d^a \leqslant 1 - u_d^\varepsilon = \gamma_\varepsilon(u_d^\varepsilon) = v_d^\varepsilon. \tag{4.104}$$

In light of (4.95), (4.101), (4.103) and (4.104), we can write that

$$1 > u_d^a \geqslant u_d^\varepsilon > u_s^0 > 0 \quad \text{and} \quad 1 > v_s^0 > v_d^\varepsilon \geqslant v_d^a > 0. \tag{4.105}$$

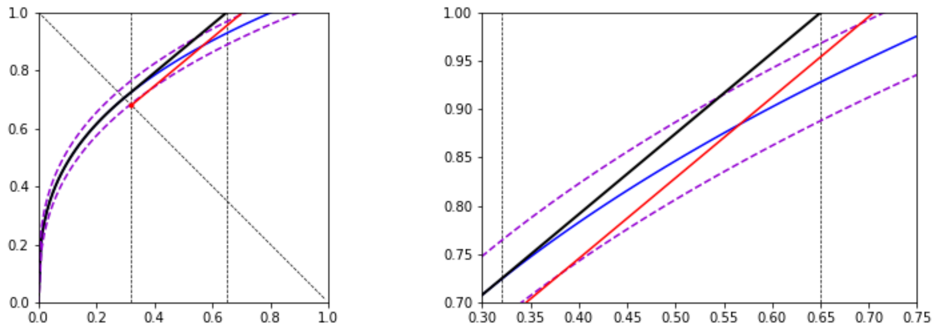


Figure 6: *The figures illustrate the functions involved in the proof of Theorem 1.4 for the case* $\rho < 1$. *The two vertical lines correspond to the values* u_d^ε *and* m. *The thick black line represents the boundary of* $\mathcal{V}_\mathcal{A}$*; the blue line is the graph of* $\gamma_0(u)$*; the dark violet lines delimit the area where* $\gamma_a(u)$ *for* $a \leqslant \varepsilon$ *might be; the red line is the upper limit of* $\gamma_a(u)$ *for* $a > \varepsilon$. *The image was realized using a simulation in Python for the values* $\rho = 0.35$ *and* $c = 1.2$.

Now, to complete the proof of (4.87) when $a > \varepsilon$, we consider two cases depending on the order of m and u_d^ε. If $u_d^\varepsilon \geqslant m$, by (4.105) we have that $m < 1$ and $f(m) = 1$. Then,

$$\gamma_a(m) \leqslant \gamma_a(u_d^\varepsilon) \leqslant \gamma_\varepsilon(u_d^\varepsilon) = v_d^\varepsilon < 1 = f(m), \tag{4.106}$$

thanks to the monotonicity of γ_a, (4.92) and (4.105). We define

$$\theta_2 := \frac{1 - v_d^\varepsilon}{2},$$

which is positive thanks to (4.105). From (4.106), we get that

$$\gamma_a(m) + \theta_2 \leqslant v_d^\varepsilon + \theta_2 < 1 = f(m). \tag{4.107}$$

This formula proves the claim in (4.87) for $a > \varepsilon$ and $u_d^\varepsilon \geqslant m$.

If instead $u_d^\varepsilon < m$, then we proceed as follows. By (4.105) we have

$$\gamma_a(u_d^a) = v_d^a \leqslant v_d^\varepsilon < v_s^0 = f(u_s^0). \tag{4.108}$$

Now we set

$$\theta_3 := \frac{f(u_d^\varepsilon) - f(u_s^0)}{2}.$$

Using the definition of f in (4.84), we see that

$$\theta_3 = \frac{u_d^\varepsilon - u_s^0}{2c},$$

and accordingly θ_3 is positive, due to (4.105).

From (4.108) we have

$$\gamma_a(u_d^a) + \theta_3 < f(u_s^0) + \theta_3 < f(u_d^\varepsilon). \tag{4.109}$$

Now we show that, on any trajectory $(u(t), v(t))$ lying on the graph of γ_a, it holds that

$$\dot{v}(t) > \frac{\dot{u}(t)}{c} \quad \text{provided that } u(t) \in (u_d^a, u_{\mathcal{M}}^a). \tag{4.110}$$

To prove this, we first observe that $u(t) > u_d^a > u_s^a$, thanks to (4.57). Hence, we can exploit formula (4.43) of Lemma 4.3 and get that

$$\gamma_a(u(t)) - \frac{u(t)}{\rho c} < 0. \tag{4.111}$$

Also, by the monotonicity of γ_a and (4.53),

$$\gamma_a(u(t)) \geqslant \gamma_a(u_d^a) = 1 - u_d^a > 1 - u(t).$$

From this and (4.111) it follows that

$$\left(\dot{v}(t) - \frac{\dot{u}(t)}{c}\right) = \rho \left(\gamma_a(u(t)) - \frac{u(t)}{\rho c}\right) (1 - u(t) - \gamma_a(u(t))) > 0$$

provided that $u(t) \in (u_d^a, u_{\mathcal{M}}^a)$, and this proves (4.110).

In addition, for such a trajectory $(u(t), v(t))$ we have that

$$\begin{aligned}\dot{u}(t) &= u(t)\,(1 - u(t) - \gamma_a(u(t)) - ac) \\ &< u(t)\,(1 - u(t) - \gamma_a(u_d^a)) = u(t)\,(1 - u(t) - 1 + u_d^a) < 0,\end{aligned}$$

provided that $u(t) \in (u_d^a, u_{\mathcal{M}}^a)$.

From this and (4.110), we get

$$\gamma_a'(u(t)) = \frac{\dot{v}(t)}{\dot{u}(t)} < \frac{1}{c} = f'(u(t)),$$

provided that $u(t) \in (u_d^a, u_{\mathcal{M}}^a)$.

Consequently, taking as initial datum of the trajectory an arbitrary point $(u, \gamma_a(u))$ with $u \in (u_d^a, u_{\mathcal{M}}^a)$, we can write that, for all $u \in (u_d^a, u_{\mathcal{M}}^a)$,

$$\gamma_a'(u) < f'(u).$$

As a result, integrating and using (4.92), for all $u \in (u_d^a, u_{\mathcal{M}}^a)$, we have

$$\gamma_a(u) = \gamma_a(u_d^a) + \int_{u_d^a}^{u} \gamma_a'(u)\,du < \gamma_a(u_d^a) + \int_{u_d^a}^{u} f'(u)\,du = \gamma_a(u_d^a) + f(u) - f(u_d^a).$$

Then, making use (4.109), for $u \in (u_d^a, u_{\mathcal{M}}^a)$,

$$\gamma_a(u) + \theta_3 < \gamma_a(u_d^a) + f(u) - f(u_d^a) + \theta_3 \leqslant f(u) - f(u_d^a) + f(u_d^\varepsilon). \tag{4.112}$$

Also, recalling (4.105) and the monotonicity of f, we see that $f(u_d^\varepsilon) \leqslant f(u_d^a)$. Combining this and (4.112), we deduce that

$$\gamma_a(u) + \theta_3 < f(u) \qquad \text{for all } u \in (u_d^a, u_{\mathcal{M}}^a). \tag{4.113}$$

We also observe that if $u \in (u_d^\varepsilon, u_d^a]$, then the monotonicity of γ_a yields that $\gamma_a(u) \leqslant \gamma_a(u_d^a)$. It follows from this and (4.109) that $\gamma_a(u) + \theta_3 < f(u_d^\varepsilon)$. This and the monotonicity of f give that

$$\gamma_a(u) + \theta_3 < f(u) \qquad \text{for all } u \in (u_d^\varepsilon, u_d^a].$$

Comparing this with (4.113), we obtain

$$\gamma_a(u) + \theta_3 < f(u) \qquad \text{for all } u \in (u_d^\varepsilon, u_{\mathcal{M}}^a)$$

and therefore

$$\gamma_a(u) + \theta_3 \leqslant f(u) \qquad \text{for all } u \in [u_d^\varepsilon, u_{\mathcal{M}}^a]. \tag{4.114}$$

Now, in view of (4.86), we have that $m \in [u_d^\varepsilon, u_{\mathcal{M}}^a]$. Consequently, we can utilize (4.114) with $u := m$ and find that

$$\gamma_a(m) + \theta_3 \leqslant f(m) \tag{4.115}$$

which gives (4.87) in the case $a > \varepsilon$ and $u_d^\varepsilon \leqslant m$ (say, in this case with $\theta \leqslant \theta_3/2$).

That is, by (4.91), (4.107) and (4.115) we obtain that (4.87) holds true for

$$\theta := \frac{1}{2} \min\{\theta_1, \ \theta_2, \ \theta_3\}.$$

If we choose $\bar{v} := f(m) - \frac{\theta}{2}$ we have that

$$0 < \gamma_a(m) \leqslant \bar{v} < f(m) \leqslant 1. \tag{4.116}$$

This completes the proof of Theorem 1.4 when $\rho < 1$, in light of the characterizations of $\mathcal{E}(a)$ and $\mathcal{V}_{\mathcal{A}}$ from Proposition 2.9 and Theorem 1.3, respectively. □

Now we focus on the case $\rho > 1$.

Proof of Theorem 1.4, case $\rho > 1$. As before, the inclusion $\mathcal{V}_{\mathcal{K}} \subseteq \mathcal{V}_{\mathcal{A}}$ is trivial since $\mathcal{K} \subset \mathcal{A}$. To prove that it is strict, we aim to find a point $(\bar{u}, \bar{v}) \in \mathcal{V}_{\mathcal{A}}$ such that $(\bar{u}, \bar{v}) \notin \mathcal{V}_{\mathcal{K}}$. Thus, we have to prove that there exists $(\bar{u}, \bar{v}) \in \mathcal{V}_{\mathcal{A}}$ such that, for all constant strategies $a > 0$, we have that $(\bar{u}, \bar{v}) \notin \mathcal{E}(a)$.

To this end, using the characterizations given in Proposition 2.9 and Theorem 1.3, we claim that

$$\begin{aligned}&\text{there exists a point } (\bar{u}, \bar{v}) \in [0,1] \times [0,1] \\ &\text{satisfying } u_\infty \leqslant \bar{u} \leqslant u_{\mathcal{M}}^a \text{ and } \gamma_a(\bar{u}) \leqslant \bar{v} < \zeta(\bar{u}) \text{ for all } a > 0.\end{aligned} \tag{4.117}$$

For this, we let

$$m := \min\left\{1, \frac{c}{(c+1)^{\frac{\rho-1}{\rho}}}\right\}.$$

By (1.18) one sees that

$$u_\infty < m. \tag{4.118}$$

In addition, we point out that

$$m \leqslant u_{\mathcal{M}}^a. \tag{4.119}$$

Indeed, since $m \leqslant 1$, if $u^a_{\mathcal{M}} = 1$ the desired inequality is obvious. If instead $u^a_{\mathcal{M}} < 1$ we have that $(u^a_{\mathcal{M}}, 1) \times (0,1) \subseteq \mathcal{E}(a) \subseteq \mathcal{V}_{\mathcal{K}} \subseteq \mathcal{V}_{\mathcal{A}}$. Hence, by (1.17), it follows that $\frac{c}{(c+1)^{\frac{\rho-1}{\rho}}} \leqslant u^a_{\mathcal{M}}$, which leads to (4.119), as desired.

Now we claim that there exists $\theta > 0$ such that

$$\gamma_a(m) + \theta < \zeta(m) \quad \text{for all} \quad a > 0. \tag{4.120}$$

We first show some preliminary facts for $\gamma_a(u)$. For all $a > 0$, we have that $\mathcal{E}(a) \subseteq \mathcal{V}_{\mathcal{A}}$. Owing to the characterization of $\mathcal{E}(a)$ from Proposition 2.9 and of $\mathcal{V}_{\mathcal{A}}$ from Theorem 1.3 (which can be used here, thanks to (4.118) and (4.119)), we get that

$$\gamma_a(u) \leqslant \frac{u}{c} \quad \text{for all} \quad u \in (0, u_\infty] \quad \text{and} \quad a > 0. \tag{4.121}$$

This is true in particular for $u = u_\infty$.

We choose

$$\delta \in \left(0, \frac{\rho - 1}{c}\right) \qquad \text{and} \qquad M := \max\left\{\frac{1}{c}, \frac{\rho + \frac{1}{c} + \delta}{\delta c u_\infty}\right\}, \tag{4.122}$$

and we prove (4.120) by treating separately the cases $a > M$ and $a \in (0, M]$.

We first consider the case $a > M$. We let $(u(t), v(t))$ be a trajectory for (1.1) lying on γ_a and we show that

$$\dot{v}(t) - \left(\frac{1}{c} + \delta\right)\dot{u}(t) > 0 \quad \text{provided that} \quad u(t) > u_\infty \quad \text{and} \quad a > M. \tag{4.123}$$

To check this, we observe that

$$\begin{aligned}\dot{v}(t) - \left(\frac{1}{c} + \delta\right)\dot{u}(t) &= \left[\rho\gamma_a(u(t)) - \left(\frac{1}{c} + \delta\right) u(t)\right] (1 - u(t) - \gamma_a(u(t))) + \delta a c u(t) \\ &\geqslant - \left|\rho + \frac{1}{c} + \delta\right| + \delta a c u_\infty > 0,\end{aligned}$$

where the last inequality is true thanks to the hypothesis $a > M$ and the definition of M in (4.122). This proves (4.123).

Moreover, for $a > M \geqslant \frac{1}{c}$ we have $\dot{u} < 0$. From this, (4.123) and the invariance of γ_a for the flow, we get

$$\gamma_a'(u(t)) = \frac{\dot{v}(t)}{\dot{u}(t)} < \frac{1}{c} + \delta, \tag{4.124}$$

provided that $u(t) > u_\infty$ and $a > M$.

For this reason and (4.121), we get

$$\gamma_a(u(t)) = \gamma_a(u_\infty) + \int_{u_\infty}^{u(t)} \gamma_a'(\tau)\, d\tau \leqslant \frac{u_\infty}{c} + \left(\frac{1}{c} + \delta\right)(u(t) - u_\infty) \tag{4.125}$$

provided that $u(t) > u_\infty$ and $a > M$.

Furthermore, thanks to the choice of δ in (4.122), we have

$$\zeta'(u) = \frac{\rho u^{\rho-1}}{c u_\infty^{\rho-1}} > \frac{\rho}{c} > \frac{1}{c} + \delta \quad \text{for all} \quad u > u_\infty.$$

Since also $\zeta(u_\infty) = \frac{u_\infty}{c}$, by (4.125) we deduce that

$$\gamma_a(u(t)) \leqslant \frac{u_\infty}{c} + \left(\frac{1}{c} + \delta\right)(u(t) - u_\infty) < \zeta(u_\infty) + \int_{u_\infty}^{u(t)} \zeta'(\tau)\, d\tau = \zeta(u(t)), \quad (4.126)$$

provided that $u(t) > u_\infty$ and $a > M$.

In particular, given any $u > u_\infty$, we can take a trajectory starting at $(u, \gamma_a(u))$ and deduce from (4.126) that

$$\gamma_a(u) \leqslant \frac{u_\infty}{c} + \left(\frac{1}{c} + \delta\right)(u - u_\infty) < \zeta(u_\infty) + \int_{u_\infty}^{u} \zeta'(\tau)\, d\tau = \zeta(u),$$

whenever $a > M$. We stress that, in light of (4.118), we can take $u := m$ in the above chain of inequalities, concluding that

$$\gamma_a(m) \leqslant \frac{u_\infty}{c} + \left(\frac{1}{c} + \delta\right)(m - u_\infty) < \zeta(m).$$

We rewrite this in the form

$$\gamma_a(m) \leqslant \left(\frac{1}{c} + \delta\right) m - \delta u_\infty < \zeta(m). \quad (4.127)$$

We define

$$\theta_1 := \frac{1}{2}\left[\zeta(m) - \left(\frac{1}{c} + \delta\right) m + \delta u_\infty\right], \quad (4.128)$$

that is positive thanks to the last inequality in (4.127). Then by the first inequality in (4.127) we have

$$\gamma_a(m) + \theta_1 \leqslant \left(\frac{1}{c} + \delta\right) m - \delta u_\infty + \theta_1 = \frac{1}{2}\left[\left(\frac{1}{c} + \delta\right) m - \delta u_\infty\right] + \frac{\zeta(m)}{2}.$$

Hence, using again the last inequality in (4.127), we obtain that

$$\gamma_a(m) + \theta_1 < \zeta(m), \quad (4.129)$$

which gives the claim in (4.120) for the case $a > M$.

Now we treat the case $a \in (0, M]$. We claim that

$$u_d^M > u_\infty. \quad (4.130)$$

Here, we are using the notation u_d^M to denote the point u_d^a when $a := M$. To prove (4.130) we argue as follows. Since $M \geqslant \frac{1}{c}$, by Propositions 2.1 and 2.7 we have

$$\gamma_M'(0) = \frac{M}{\rho - 1 + Mc} < \frac{1}{c}. \quad (4.131)$$

Moreover, since the graph of $\gamma_M(u)$ is a parametrization of a trajectory for (1.1) with $a = M$, we have that $\dot{v}(t) = \gamma'_M(u(t))\dot{u}(t)$. Hence, at all points $(\bar{u}, \bar{v})$ with $\bar{u} \in (0, u_\infty)$ and $\bar{v} = \gamma_M(\bar{u})$ we have

$$\gamma'_M(\bar{u}) = \frac{M\bar{u} - \rho\bar{v}(1 - \bar{u} - \bar{v})}{Mc\bar{u} - \bar{u}(1 - \bar{u} - \bar{v})}. \tag{4.132}$$

We stress that the denominator in the right hand side of (4.132) is strictly positive, since $M \geqslant \frac{1}{c}$ and $\bar{u} > 0$.

In addition, we have that

$$\frac{1}{c} - \frac{M\bar{u} - \rho\bar{v}(1 - \bar{u} - \bar{v})}{Mc\bar{u} - \bar{u}(1 - \bar{u} - \bar{v})} = \frac{(\rho c\bar{v} - \bar{u})(1 - \bar{u} - \bar{v})}{Mc^2\bar{u} - c\bar{u}(1 - \bar{u} - \bar{v})}. \tag{4.133}$$

Also,

$$u_s^M = 0 < \bar{u} < u_\infty < m \leqslant u_{\mathcal{M}}^M,$$

thanks to (4.118) and (4.119). Hence, we can exploit formula (4.45) in Lemma 4.3 with the strict inequality, thus obtaining that

$$\rho c\bar{v} - \bar{u} = \rho c\gamma_M(\bar{u}) - \bar{u} > 0. \tag{4.134}$$

Moreover, by (4.121),

$$1 - \bar{u} - \bar{v} = 1 - \bar{u} - \gamma_M(\bar{u}) \geqslant 1 - \bar{u} - \frac{\bar{u}}{c} > 1 - u_\infty - \frac{u_\infty}{c} = 0.$$

Therefore, using the latter estimate and (4.134) into (4.133), we get that

$$\frac{1}{c} - \frac{M\bar{u} - \rho\bar{v}(1 - \bar{u} - \bar{v})}{Mc\bar{u} - \bar{u}(1 - \bar{u} - \bar{v})} > 0.$$

From this and (4.132), we have that

$$\gamma'_M(u) < \frac{1}{c} \quad \text{for all } u \in (0, u_\infty).$$

This, together with (4.131) and the fact that $\gamma_M(0) = 0$, gives

$$\gamma_M(u) = \gamma_M(u) - \gamma_M(0) = \int_0^u \gamma'_M(\tau)\, d\tau < \frac{u}{c}$$

for all $u \in (0, u_\infty]$. This inequality yields that

$$\gamma_M(u_\infty) < \frac{u_\infty}{c} = 1 - u_\infty. \tag{4.135}$$

Now, to complete the proof of (4.130) we argue by contradiction and suppose that the claim in (4.130) is false, hence

$$u_d^M \leqslant u_\infty. \tag{4.136}$$

Thus, by (4.135), the monotonicity of $\gamma_M(u)$ and the definition of u_d^M given in (4.53), we get

$$1 - u_d^M = \gamma_M(u_d^M) \leqslant \gamma_M(u_\infty) < 1 - u_\infty$$

which is in contraddiction with (4.136). Hence, (4.130) holds true, as desired.

Also, by the second statement in Lemma 4.4, used here with $a^* := M$,

$$\gamma_a(u) \leqslant \gamma_M(u) \quad \text{for all} \quad u \in [0, u_d^M]. \tag{4.137}$$

We claim that

$$u_d^M \leqslant u_d^a. \tag{4.138}$$

Indeed, suppose, by contradiction, that

$$u_d^M > u_d^a. \tag{4.139}$$

Then, by the monotonicity of γ_a and (4.137), used here with $u := u_d^M$, we find that

$$1 - u_d^a = \gamma_a(u_d^a) \leqslant \gamma_a(u_d^M) \leqslant \gamma_M(u_d^M) = 1 - u_d^M.$$

This entails that $u_d^a \geqslant u_d^M$, which is in contradiction with (4.139), and thus establishes (4.138).

We note in addition that

$$v_d^M = \gamma_M(u_d^M) = 1 - u_d^M < 1 - u_\infty, \tag{4.140}$$

thanks to the definition of (u_d^M, v_d^M) and (4.130).

Similarly, by (4.138),

$$v_d^a = \gamma_a(u_d^a) = 1 - u_d^a \leqslant 1 - u_d^M = \gamma_M(u_d^M) = v_d^M. \tag{4.141}$$

Collecting the pieces of information in (4.130), (4.138), (4.140) and (4.141), we thereby conclude that, for all $a \in (0, M]$,

$$0 < u_\infty < u_d^M \leqslant u_d^a < 1 \qquad \text{and} \qquad 0 < v_d^a \leqslant v_d^M < 1 - u_\infty =: v_\infty < 1. \tag{4.142}$$

Now we consider two cases depending on the order of m and u_d^M. If $u_d^M \geqslant m$, by (4.142) we have $m < 1$ and $\zeta(m) = 1$. Accordingly, for $a \in (0, M]$, by (4.142) and (4.137) we have

$$\gamma_a(m) \leqslant \gamma_a(u_d^M) \leqslant \gamma_M(u_d^M) = v_d^M < 1 = \zeta(m).$$

Hence, we can define

$$\theta_2 := \frac{1 - v_d^M}{2},$$

and observe that θ_2 is positive by (4.142), thus obtaining that

$$\gamma_a(m) + \theta_2 < \zeta(m). \tag{4.143}$$

This is the desired claim in (4.120) for $a \in (0, M]$ and $u^* \geqslant m$.

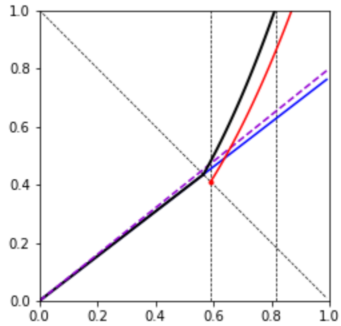


Figure 7: *The figure illustrates the functions involved in the proof of Theorem 1.4 for the case $\rho > 1$. The two vertical lines correspond to the values u_d^M and m. The thick black line represents the boundary of $\mathcal{V}_{\mathcal{A}}$; the blue line is the graph of the line $v = \frac{u}{c}$; the dark violet line is the upper bound for $\gamma_a(u)$ for $a > M$; the red line is $\phi(u)$. The image was realized using a simulation in Python for the values $\rho = 2.3$ and $c = 1.3$.*

If instead $u_d^M < m$, we consider the function

$$\phi(u) := v_d^M \left(\frac{u}{u_d^M}\right)^\rho, \qquad \text{for } u \in [u_d^M, m]$$

and we claim that

$$\gamma_a(u) \leqslant \phi(u) \quad \text{for all} \quad a \in (0, M] \quad \text{and} \quad u \in [u_d^M, m]. \tag{4.144}$$

To prove this, we recall (4.142) and the fact that γ_a is an increasing function to see that

$$\gamma_a(u_d^M) \leqslant \gamma_a(u_d^a) = v_d^a \leqslant v_d^M = \phi(u_d^M). \tag{4.145}$$

Now we remark that

$$\gamma_M(u_d^M) + u_d^M = 1 > 1 - Mc = \gamma_M(u_s^M) + u_s^M,$$

and therefore $u_d^M > u_s^M$. Notice also that $u_d^M < m \leqslant u_{\mathcal{M}}^M$, thanks to (4.119). As a result, we find that $\rho c \gamma_M(u_d^M) > u_d^M$ by inequality (4.45) in Lemma 4.3. Therefore, if $u \geqslant u_d^M$ and $v = \phi(u)$, then

$$\begin{aligned} au\left(1 - \rho c \frac{v_d^M}{(u_d^M)^\rho} u^{\rho-1}\right) &= au\left(1 - \frac{\rho c \gamma_M(u_d^M)}{(u_d^M)^\rho} u^{\rho-1}\right) \\ &< au\left(1 - \left(\frac{u}{u_d^M}\right)^{\rho-1}\right) \leqslant 0 = \rho\left(v - \frac{v_d^M}{(u_d^M)^\rho} u^\rho\right)(1 - u - v). \end{aligned}$$

Using this and (4.75), we deduce that, if $a \in [0, M]$, $u \in [u_d^M, m]$ and $v = \phi(u)$,

$$\begin{aligned}
&\frac{au - \rho v(1-u-v)}{acu - u(1-u-v)} - \frac{v_d^M}{(u_d^M)^\rho}\rho u^{\rho-1} \\
&= \frac{au - \rho v(1-u-v) - \big(acu - u(1-u-v)\big)\frac{v_d^M}{(u_d^M)^\rho}\rho u^{\rho-1}}{acu - u(1-u-v)} \\
&= \frac{au\left(1 - \rho c \frac{v_d^M}{(u_d^M)^\rho} u^{\rho-1}\right) - \rho(1-u-v)\left(v - \frac{v_d^M}{(u_d^M)^\rho}u^\rho\right)}{acu - u(1-u-v)} \\
&< 0.
\end{aligned} \tag{4.146}$$

Now we take $a \in (0, M]$, $u \in [u_d^M, m]$ and suppose that $v = \phi(u) = \gamma_a(u)$, we consider an orbit $(u(t), v(t))$ lying on γ_a with $(u(0), v(0)) = (u, v)$, and we notice that, by (4.75) and (4.146),

$$\begin{aligned}
\gamma_a'(u) = \gamma_a'(u(0)) = \frac{\dot v(0)}{\dot u(0)} &= \frac{au(0) - \rho v(0)\,(1-u(0)-v(0))}{acu(0) - u(0)(1-u(0)-v(0))} \\
&= \frac{au - \rho v\,(1-u-v)}{acu - u(1-u-v)} < \frac{v_d^M}{(u_d^M)^\rho}\rho u^{\rho-1} = \phi'(u).
\end{aligned} \tag{4.147}$$

To complete the proof of (4.144), we define

$$\mathcal{H}(u) := \gamma_a(u) - \phi(u)$$

and we claim that for every $a \in (0, M]$ there exists $\underline{u} \in [u_d^M, m]$ such that

$$\mathcal{H}(\underline{u}) < 0 \text{ and } \mathcal{H}(u) \leqslant 0 \text{ for every } u \in [u_d^M, \underline{u}]. \tag{4.148}$$

Indeed, by (4.145), we know that $\mathcal{H}(u_d^M) \leqslant 0$. Thus, if $\mathcal{H}(u_d^M) < 0$ then we can choose $\underline{u} := u_d^M$ and obtain (4.148). If instead $\mathcal{H}(u_d^M) = 0$, we have that $\gamma_a(u_d^M) = \phi(u_d^M)$ and thus we can exploit (4.147) and find that $\mathcal{H}'(u_d^M) < 0$, from which we obtain (4.148).

Now we claim that, for every $a \in (0, M]$ and $u \in [u_d^M, m]$,

$$\mathcal{H}(u) \leqslant 0. \tag{4.149}$$

For this, given $a \in (0, M]$, we define

$$\mathcal{L} := \{u_* \in [u_d^M, m] \text{ s.t. } \mathcal{H}(u) \leqslant 0 \text{ for every } u \in [u_d^M, u_*]\} \qquad \text{and} \qquad \overline{u} := \sup \mathcal{L}.$$

We remark that $\underline{u} \in \mathcal{L}$, thanks to (4.148) and therefore $\overline{u}$ is well defined. We have that

$$\overline{u} = m, \tag{4.150}$$

otherwise we would have that $\mathcal{H}(\overline{u}) = 0$ and thus $\mathcal{H}'(\overline{u}) < 0$, thanks to (4.147), which would contradict the maximality of $\overline{u}$. Now, the claim in (4.149) plainly follows from (4.150).

We notice that by the inequalities in (4.142) we have

$$\zeta(u) = \frac{v_\infty}{(u_\infty)^\rho} u^\rho > \frac{v_d^M}{(u_d^M)^\rho} u^\rho = \phi(u). \tag{4.151}$$

Then, we define

$$\theta_3 := \frac{\zeta(m) - \phi(m)}{2}, \tag{4.152}$$

that is positive thanks to (4.151). We get that

$$\phi(m) + \theta_3 < \zeta(m). \tag{4.153}$$

From this and (4.144), we conclude that

$$\gamma_a(m) + \theta_3 \leqslant \phi(m) + \theta_3 < \zeta(m) \quad \text{ for } a \in (0, M]. \tag{4.154}$$

By (4.129), (4.143) and (4.154) we have that (4.120) is true for $\theta = \min\{\theta_1,\ \theta_2,\ \theta_3\}$. This also establishes the claim in (4.117), and the proof is completed. □

4.4 Proof of Theorem 1.5

Now, we can complete the proof of Theorem 1.5 by building on the previous work.

Proof of Theorem 1.5. Since the class of Heaviside functions $\mathcal{H}$ is contained in the class of piecewise continuous functions $\mathcal{A}$, we have that

$$\mathcal{V}_\mathcal{H} \subseteq \mathcal{V}_\mathcal{A}, \tag{4.155}$$

hence we are left with proving the converse inclusion. We treat separately the cases $\rho = 1$, $\rho < 1$ and $\rho > 0$.

If $\rho = 1$, the desired claim follows from Theorem 1.4, part (i).

If $\rho < 1$, we deduce from (1.16) and (4.28) that

$$\mathcal{V}_\mathcal{A} = \mathcal{F}_0 \cup \mathcal{P}, \tag{4.156}$$

where $\mathcal{P}$ has been defined in (4.1) and $\mathcal{F}_0$ in (4.27).

Moreover, by (4.30), we have that

$$\mathcal{F}_0 \subseteq \mathcal{V}_\mathcal{K} \subseteq \mathcal{V}_\mathcal{H}. \tag{4.157}$$

Also, in Proposition 4.1 we construct a Heaviside winning strategy for every point in $\mathcal{P}$. Accordingly, it follows that $\mathcal{P} \subseteq \mathcal{V}_\mathcal{H}$. This, (4.156) and (4.157) entail that $\mathcal{V}_\mathcal{A} \subseteq \mathcal{V}_\mathcal{H}$, which completes the proof of Theorem 1.5 when $\rho < 1$.

Hence, we now focus on the case $\rho > 1$. By (1.17) and (4.36),

$$\mathcal{V}_\mathcal{A} = \mathcal{S}_c \cup \mathcal{Q}, \tag{4.158}$$

where $\mathcal{S}_c$ was defined in (3.41) and $\mathcal{Q}$ in (4.3).

For every point $(u_0, v_0) \in \mathcal{S}_c$ there exists $\bar{a}$ that is a constant winning strategy for (u_0, v_0), thanks to Proposition 3.4, therefore $\mathcal{S}_c \subseteq \mathcal{V}_\mathcal{H}$. Moreover, in Proposition 4.1 for every point $(u_0, v_0) \in \mathcal{Q}$ we constructed a Heaviside winning strategy, whence $\mathcal{Q} \subseteq \mathcal{V}_\mathcal{H}$. In light of these observations and (4.158), we see that also in this case $\mathcal{V}_\mathcal{A} \subseteq \mathcal{V}_\mathcal{H}$ and the proof is complete. □

4.5 Bounds on winning initial positions under pointwise constraints for the possible strategies

This subsection is dedicated to the analysis of $\mathcal{V}_{\mathcal{A}}$ when we put some constraints on $a(t)$. In particular, we consider $M \geqslant m \geqslant 0$ with $M > 0$ and the set $\mathcal{A}_{m,M}$ of the functions $a(t) \in \mathcal{A}$ with $m \leqslant a(t) \leqslant M$ for all $t > 0$. We will prove Theorem 1.6 via a technical proposition giving informative bounds on $\mathcal{V}_{m,M}$.

For this, we denote by (u_s^m, v_s^m) the point (u_s, v_s) introduced in (1.11) when $a(t) = m$ for all $t > 0$ (this when $mc < 1$, and we use the convention that $(u_s^m, v_s^m) = (0, 0)$ when $mc \geqslant 1$). In this setting, we have the following result obtaining explicit bounds on the favorable set $\mathcal{V}_{m,M}$:

Proposition 4.5. *Let $M \geqslant m \geqslant 0$ with $M > 0$ and*

$$\varepsilon \in \left(0, \min\left\{\frac{M(c+1)}{M+1}, 1\right\}\right). \tag{4.159}$$

Then

(i) If $\rho < 1$, we have

$$\mathcal{V}_{m,M} \subseteq \Big\{(u,v) \in [0,1] \times [0,1] \quad s.t. \quad v < f_\varepsilon(u)\Big\} \tag{4.160}$$

where $f_\varepsilon : [0, u_{\mathcal{M}}] \to [0,1]$ is the continuous function given by

$$f_\varepsilon(u) = \begin{cases} \dfrac{(u_s^m)^{1-\rho} u^\rho}{\rho c} & \text{if } u \in [0, u_s^m), \\ \dfrac{u}{\rho c} & \text{if } u \in [u_s^m, u_s^0), \\ \dfrac{u}{c} + \dfrac{1-\rho}{1+\rho c} & \text{if } u \in [u_s^0, u_1), \\ hu + p & \text{if } u \in [u_1, 1], \end{cases}$$

with the convention that the first interval is empty if $m \geqslant \frac{1}{c}$, the second interval is empty if $m = 0$, and h, u_1 and p take the following values:

$$\begin{aligned} h &:= \frac{1}{c}\left(1 - \frac{\varepsilon^2(1-\rho)}{M(1+\rho c)(c+1-\varepsilon)^2 + \varepsilon(\rho c + \rho + \varepsilon - \varepsilon\rho)}\right), \\ u_1 &:= \frac{c(\rho c + \rho + \varepsilon - \varepsilon\rho)}{(1+\rho c)(c+1-\varepsilon)}, \\ p &:= \frac{c + 1 - hc(\rho c + \rho + \varepsilon - \varepsilon\rho)}{(1+\rho c)(c+1-\varepsilon)}. \end{aligned}$$

(ii) If $\rho > 1$, we have

$$\mathcal{V}_{m,M} \subseteq \Big\{(u,v) \in [0,1] \times [0,1] \quad s.t. \quad v < g_\varepsilon(u)\Big\}$$

where $g_\varepsilon : [0, u_{\mathcal{M}}] \to [0,1]$ is the continuous function given by

$$g_\varepsilon(u) = \begin{cases} k\,u & \text{if } u \in [0, u_2), \\ \dfrac{u}{c} + q & \text{if } u \in [u_2, u_3), \\ \dfrac{(1-u_3)u^\rho}{(u_3)^\rho} & \text{if } u \in [u_3, 1] \end{cases}$$

for the following values:

$$k := \frac{(c+1-\varepsilon)M}{(\rho-1)\varepsilon c + (c+1-\varepsilon)Mc}, \qquad q := \frac{(kc-1)(1-\varepsilon)}{c(k-k\varepsilon+1)},$$
$$u_2 := \frac{1-\varepsilon}{k-k\varepsilon+1} \qquad \textit{and} \qquad u_3 := \frac{c+1-\varepsilon}{(c+1)(k-k\varepsilon+1)}.$$

We observe that it might be that for some $u \in [0,1]$ we have $f_\varepsilon(u) > 1$ or $g_\varepsilon(u) > 1$. In this case, the above proposition would produce the trivial result that $\mathcal{V}_{m,M} \cap (\{u\} \times [0,1]) \subseteq \{u\} \times [0,1]$. On the other hand, a suitable choice of ε would lead to nontrivial consequences entailing, in particular, the proof of Theorem 1.6.

Proof of Proposition 4.5. We start by proving the claim in (i). For this, we will show that

$$\mathcal{D} := \Big\{(u,v) \in [0,1] \times [0,1] \;\text{ s.t. }\; v \geqslant f_\varepsilon(u)\Big\} \subseteq \mathcal{V}^C_{m,M}. \tag{4.161}$$

where $\mathcal{V}^C_{m,M}$ is the complement of $\mathcal{V}_{m,M}$ in the topology of $[0,1] \times [0,1]$. We remark that once (4.161) is established, then the desired claim in (4.160) plainly follows by taking the complement sets.

To prove (4.161) we first show that

$$0 \leqslant u_s^m < u_s^0 < u_1 < 1. \tag{4.162}$$

Notice, as a byproduct, that the above inequalities also give that f_ε is well defined. To prove (4.162) we notice that, by (1.11), (1.14) and (4.41),

$$0 \leqslant u_s^m = \max\left\{0, \frac{1-mc}{1+\rho c}\rho c\right\} < \frac{\rho c}{1+\rho c} = u_s^0$$

(and actually the first inequality is strict if $m < \frac{1}{c}$). Next, one can check that, since $\varepsilon > 0$,

$$u_s^0 - u_1 = \frac{\rho c}{1+\rho c} - \frac{c(\rho c + \rho + \varepsilon - \varepsilon\rho)}{(1+\rho c)(c+1-\varepsilon)} = -\frac{c\varepsilon}{(1+\rho c)(c+1-\varepsilon)} < 0.$$

Furthermore, since $\varepsilon < 1$,

$$u_1 - 1 = \frac{c(\rho c + \rho + \varepsilon - \varepsilon\rho)}{(1+\rho c)(c+1-\varepsilon)} - 1 = \frac{(\varepsilon-1)(c+1)}{(1+\rho c)(c+1-\varepsilon)} < 0.$$

These observations prove (4.162), as desired.

Now we point out that

$$f_\varepsilon \text{ is a continuous function.} \tag{4.163}$$

Indeed,

$$\frac{(u_s^m)^{1-\rho}}{\rho c}(u_s^m)^\rho = \frac{u_s^m}{\rho c} \qquad \text{and} \qquad \frac{u_s^0}{\rho c} = \frac{u_s^0}{c} + \frac{1-\rho}{1+\rho c}. \tag{4.164}$$

Furthermore, by the definitions of p and u_1 we see that

$$\begin{aligned} p &= \frac{c+1}{(1+\rho c)(c+1-\varepsilon)} - \frac{hc(\rho c+\rho+\varepsilon-\varepsilon\rho)}{(1+\rho c)(c+1-\varepsilon)} \\ &= \frac{c+1}{(1+\rho c)(c+1-\varepsilon)} - hu_1. \end{aligned} \tag{4.165}$$

Moreover, from the definition of u_1,

$$\frac{u_1}{c} + \frac{1-\rho}{1+\rho c} = \frac{c+1}{(1+\rho c)(c+1-\varepsilon)}.$$

Combining this and (4.165), we deduce that

$$\frac{u_1}{c} + \frac{1-\rho}{1+\rho c} = hu_1 + p. \tag{4.166}$$

This observation and (4.164) entail the desired claim in (4.163).

Next, we show that

$$f_\varepsilon(u) > 0 \quad \text{for } u > 0. \tag{4.167}$$

To prove this, we note that for $u \in (0, u_s^m)$ the function is an exponential times the positive constant $\frac{(u_s^m)^{1-\rho}}{\rho c}$, hence is positive. If $u \in [u_s^m, u_s^0)$ then $f_\varepsilon(u)$ is a linear function and it is positive since $\rho c > 0$. On $[u_s^0, u_1)$, $f_\varepsilon(u)$ coincide with a linear function with positive angular coefficient, hence we have

$$f_\varepsilon(u) \geqslant \min_{u\in[u_s^0,u_1)} f_\varepsilon(u) = f_\varepsilon(u_s^0) = \frac{u_s^0}{\rho c} > 0.$$

By inspection one can check that $h > 0$. Hence, in the interval $[u_1, 1]$ we have

$$f_\varepsilon(u) \geqslant \min_{u\in[u_1,1]} f_\varepsilon(u) = f_\varepsilon(u_1) \geqslant \frac{u_s^0}{\rho c} > 0.$$

This completes the proof of (4.167).

Let us notice that, as a consequence of (4.167),

$$\mathcal{D} \cap \big((0,1] \times \{0\}\big) = \varnothing. \tag{4.168}$$

Now we show that

$$\text{for any strategy } a \in \mathcal{A}_{m,M}, \text{ no trajectory starting in } \mathcal{D} \text{ leaves } \mathcal{D}. \tag{4.169}$$

To this end, we notice that, since $\partial\mathcal{D} \cap \{v = 0\} = \{(0,0)\}$, and the origin is an equilibrium, we already have that no trajectory can exit $\mathcal{D}$ by passing through the points in $\partial\mathcal{D} \cap \partial([0,1] \times [0,1])$. Hence, we are left with considering the possibility of leaving $\mathcal{D}$ through $\partial\mathcal{D} \cap ((0,1) \times (0,1))$. To exclude this possibility, we compute the velocity of a trajectory in the inward normal direction at $\partial\mathcal{D} \cap ((0,1) \times (0,1))$.

For every $u \in [0, u_s^m)$ we have that this normal velocity is

$$\begin{aligned} &\dot{v} - \frac{(u_s^m)^{1-\rho}\rho(u)^{\rho-1}\dot{u}}{\rho c} \\ &\quad = \rho\left(v - \frac{(u_s^m)^{1-\rho}\,u^\rho}{\rho c}\right)(1-u-v) - au\left(1 - \frac{(u_s^m)^{1-\rho}}{u^{1-\rho}}\right). \end{aligned} \tag{4.170}$$

Notice that the term $v - \frac{(u_s^m)^{1-\rho}\,u^\rho}{\rho c}$ vanishes on $\partial\mathcal{D} \cap ((0,1) \times (0,1))$ when $u \in [0, u_s^m)$. Also, for all $u \in [0, u_s^m)$ we have

$$1 - \frac{(u_s^m)^{1-\rho}}{u^{1-\rho}} < 0,$$

thus the left hand side in (4.170) is positive. This observation rules out the possibility of leaving $\mathcal{D}$ through $\partial\mathcal{D} \cap ((0,1) \times (0,1))$ at points where $u \in [0, u_s^m)$.

It remains to exclude egresses at points of $\partial\mathcal{D} \cap ((0,1) \times (0,1))$ with $u \in [u_s^m, 1)$. We first consider this type of points when (u_s^m, u_s^0). At these points, we have that the velocity in the inward normal direction on $\{v = \frac{u}{\rho c}\}$ is

$$\dot{v} - \frac{\dot{u}}{\rho c} = \left(\rho v - \frac{u}{\rho c}\right)(1-u-v) + au\left(\frac{1}{\rho} - 1\right)$$

Expressing u with respect to v on $\partial\mathcal{D} \cap ((0,1) \times (0,1))$ with $u \in (u_s^m, u_s^0)$, we have

$$\begin{aligned} \dot{v} - \frac{\dot{u}}{\rho c} &= v\,(\rho - 1)\,(1 - \rho c v - v) + a\rho c v\,\frac{1-\rho}{\rho} \\ &= v(1-\rho)(\rho c v + v - 1 + ac). \end{aligned} \tag{4.171}$$

We also remark that, for these points,

$$v > v_s^m = \frac{1 - mc}{1 + \rho c} \geqslant \frac{1 - ac}{1 + \rho c},$$

thanks to (1.11). This gives that the quantity in (4.171) is strictly positive and, as a consequence, we have excluded the possibility of exiting $\mathcal{D}$ at points of $\partial\mathcal{D} \cap ((0,1) \times (0,1))$ with $u \in (u_s^m, u_s^0)$.

It remains to consider the case $u \in \{u_s^m\} \cup [u_s^0, 1)$. We first focus on the range $u \in (u_s^0, u_1)$. In this interval, the velocity of a trajectory starting at a point $(u,v) \in \partial\mathcal{D} \cap ((0,1) \times (0,1))$ lying on the line $v = \frac{u}{c} + \frac{1-\rho}{1+\rho c}$ in the inward normal direction with respect to $\partial\mathcal{D}$ is given by

$$\dot{v} - \frac{1}{c}\dot{u} = \left(\rho v - \frac{u}{c}\right)(1-u-v). \tag{4.172}$$

We also observe that, in light of (1.14),

$$u > u_s^0 = \frac{\rho c}{1+\rho c},$$

and therefore, for any $u \in (u_s^0, u_1)$ lying on the above line,

$$1-u-v = 1-u-\frac{u}{c}-\frac{1-\rho}{1+\rho c} = (c+1)\left(\frac{\rho}{1+\rho c}-\frac{u}{c}\right) < 0$$

and

$$\rho v - \frac{u}{c} = \frac{\rho u}{c} + \frac{\rho(1-\rho)}{1+\rho c} - \frac{u}{c} = (1-\rho)\left(\frac{\rho}{1+\rho c}-\frac{u}{c}\right) < 0.$$

Using these pieces of information in (4.172), we conclude that the inward normal velocity of a trajectory starting at a point $(u,v) \in \partial\mathcal{D} \cap ((0,1)\times(0,1))$ with $u \in (u_s^0, u_1)$ is strictly positive. This gives that no trajectory can exit $\mathcal{D}$ at this type of points, and we need to exclude the case $u \in \{u_s^m, u_s^0\} \cup [u_1, 1)$.

We consider now the interval $[u_1, 1)$. In this interval, the component of the velocity of a trajectory at a point on the straight line given by $v = hu + p$ in the orthogonal inward pointing direction is

$$\begin{aligned}(\dot u, \dot v)\cdot \frac{(-h,1)}{\sqrt{1+h^2}} &= \frac{(\rho v - hu)(1-u-v) - au(1-hc)}{\sqrt{1+h^2}} \\ &= \frac{((1-\rho)hu - \rho p)(u+v-1) - au(1-hc)}{\sqrt{1+h^2}}\end{aligned} \tag{4.173}$$

We observe that, if $u \in [u_1, 1)$,

$$\begin{aligned}&(1-\rho)hu - \rho p \geqslant (1-\rho)hu_1 - \rho p = hu_1 - \rho(hu_1 + p) \\ &\quad = hu_1 - \rho\left(\frac{u_1}{c} + \frac{1-\rho}{1+\rho c}\right) = hu_1 - \rho\left(\frac{\rho c + \rho + \varepsilon - \varepsilon\rho}{(1+\rho c)(c+1-\varepsilon)} + \frac{1-\rho}{1+\rho c}\right) \\ &\quad = hu_1 - \frac{\rho(c+1)}{(1+\rho c)(c+1-\varepsilon)},\end{aligned} \tag{4.174}$$

thanks to (4.166).

We also remark that

$$\begin{aligned}hu_1 &= \left(1 - \frac{\varepsilon^2(1-\rho)}{M(1+\rho c)(c+1-\varepsilon)^2 + \varepsilon(\rho c + \rho + \varepsilon - \varepsilon\rho)}\right)\frac{\rho c + \rho + \varepsilon - \varepsilon\rho}{(1+\rho c)(c+1-\varepsilon)}, \\ &= \frac{\rho c + \rho + \varepsilon - \varepsilon\rho}{(1+\rho c)(c+1-\varepsilon)} \\ &\qquad - \frac{\varepsilon^2(1-\rho)\big(\rho c + \rho + \varepsilon - \varepsilon\rho\big)}{\big(M(1+\rho c)(c+1-\varepsilon)^2 + \varepsilon(\rho c + \rho + \varepsilon - \varepsilon\rho)\big)(1+\rho c)(c+1-\varepsilon)}.\end{aligned}$$

Accordingly,

$$\begin{aligned}
hu_1 - \frac{\rho(c+1)}{(1+\rho c)(c+1-\varepsilon)} &= \frac{\varepsilon(1-\rho)}{(1+\rho c)(c+1-\varepsilon)} \\
&\quad - \frac{\varepsilon^2(1-\rho)\big(\rho c+\rho+\varepsilon-\varepsilon\rho\big)}{\big(M(1+\rho c)(c+1-\varepsilon)^2+\varepsilon(\rho c+\rho+\varepsilon-\varepsilon\rho)\big)(1+\rho c)(c+1-\varepsilon)} \\
&= \frac{\varepsilon(1-\rho)}{(1+\rho c)(c+1-\varepsilon)}\left(1-\frac{\varepsilon\big(\rho c+\rho+\varepsilon-\varepsilon\rho\big)}{M(1+\rho c)(c+1-\varepsilon)^2+\varepsilon(\rho c+\rho+\varepsilon-\varepsilon\rho)}\right) \\
&= \frac{\varepsilon(1-\rho)}{(1+\rho c)(c+1-\varepsilon)}\cdot\frac{M(1+\rho c)(c+1-\varepsilon)^2}{M(1+\rho c)(c+1-\varepsilon)^2+\varepsilon(\rho c+\rho+\varepsilon-\varepsilon\rho)} \\
&= \frac{\varepsilon M(1-\rho)(c+1-\varepsilon)}{M(1+\rho c)(c+1-\varepsilon)^2+\varepsilon(\rho c+\rho+\varepsilon-\varepsilon\rho)}.
\end{aligned}$$

From this and (4.174), we gather that

$$(1-\rho)hu-\rho p \geqslant \frac{\varepsilon M(1-\rho)(c+1-\varepsilon)}{M(1+\rho c)(c+1-\varepsilon)^2+\varepsilon(\rho c+\rho+\varepsilon-\varepsilon\rho)}. \tag{4.175}$$

Furthermore, we point out that, when $[u_1,1)$ and $v=hu+p$,

$$\begin{aligned}
u+v-1 &\geqslant u_1+hu_1+p-1 = u_1+\frac{u_1}{c}+\frac{1-\rho}{1+\rho c}-1 \\
&= \frac{(c+1)(\rho c+\rho+\varepsilon-\varepsilon\rho)}{(1+\rho c)(c+1-\varepsilon)}-\frac{\rho(c+1)}{1+\rho c} = \frac{\varepsilon(c+1)}{(1+\rho c)(c+1-\varepsilon)} > \frac{\varepsilon}{c+1-\varepsilon},
\end{aligned}$$

thanks to (4.166).

Combining this inequality and (4.175), we deduce that

$$((1-\rho)hu-\rho p)(u+v-1) > \frac{\varepsilon^2 M(1-\rho)}{M(1+\rho c)(c+1-\varepsilon)^2+\varepsilon(\rho c+\rho+\varepsilon-\varepsilon\rho)}.$$

Therefore, noticing that $h<\frac{1}{c}$,

$$\begin{aligned}
&((1-\rho)hu-\rho p)(u+v-1)-au(1-hc) \\
>\ & \frac{\varepsilon^2 M(1-\rho)}{M(1+\rho c)(c+1-\varepsilon)^2+\varepsilon(\rho c+\rho+\varepsilon-\varepsilon\rho)} - Mu(1-hc) \\
=\ & \frac{\varepsilon^2 M(1-\rho)(1-u)}{M(1+\rho c)(c+1-\varepsilon)^2+\varepsilon(\rho c+\rho+\varepsilon-\varepsilon\rho)},
\end{aligned}$$

which is strictly positive.

Using this information in (4.173), we can thereby exclude the possibility of leaving $\mathcal{D}$ through $\partial\mathcal{D}\cap((0,1)\times(0,1))$ with $u\in[u_1,1)$. As a result, it only remains to exclude the possibility of an egress from $\mathcal{D}$ through $\partial\mathcal{D}\cap((0,1)\times(0,1))$ with $u\in\{u_s^m,u_s^0\}$.

For this, we perform a general argument of dynamics, as follows. We denote by P_s^m and P_s^0 the points on $\partial\mathcal{D}\cap((0,1)\times(0,1))$ with $u=u_s^m$ and $u=u_s^0$, respectively (these points may also coincide, as it happens when $m = 0$). We stress that we already know by the previous arguments that

$$\text{if a trajectory leaves } \mathcal{D} \text{ it must pass through } \{P_s^m, P_s^0\}. \tag{4.176}$$

Our goal is to show that no trajectory leaves $\mathcal{D}$ and for this we argue by contradiction, supposing that there exist $\bar{P} \in \mathcal{D}$ and $T > 0$ such that $\phi^T(\bar{P})$ lies in the complement of $\mathcal{D}$ in $[0,1] \times [0,1]$. Here, we have denoted by ϕ^T the flow associated to (1.1). We let $\bar{Q} := \phi^T(\bar{P})$ and, since the complement of $\mathcal{D}$ is open in $[0,1] \times [0,1]$, we can find $\rho > 0$ such that $B_\rho(\bar{Q}) \cap ([0,1] \times [0,1])$ is contained in the complement of $\mathcal{D}$.

Also, from (4.176), there exists $\bar{t} \in [0,T)$ such that $\phi^{\bar{t}}(\bar{P}) \in \{P_s^m, P_s^0\}$. We suppose that $\phi^{\bar{t}}(\bar{P}) = P_s^m$ (the case $\phi^{\bar{t}}(\bar{P}) = P_s^0$ being completely analogous). We let $\bar{T} := T - \bar{t}$ and we notice that $\phi^{\bar{T}}(P_s^m) = \phi^T(\bar{P}) = \bar{Q}$. Hence, by continuity with respect to the data, we can find $r > 0$ such that

$$\phi^{\bar{T}}\big(B_r(P_s^m) \cap ([0,1] \times [0,1])\big) \subseteq B_\rho(\bar{Q}) \cap ([0,1] \times [0,1]).$$

We define $\mathcal{U} := B_r(P_s^m) \cap \mathcal{D}$. We observe that

$$\mathcal{U} \text{ has strictly positive Lebesgue measure,} \tag{4.177}$$

since $P_s^m \in \partial\mathcal{D}$ and $\mathcal{D}$ has boundary of Hölder class. In addition,

$$\phi^{\bar{T}}(\mathcal{U}) \subseteq B_\rho(\bar{Q}) \cap ([0,1] \times [0,1]) \subseteq \big([0,1] \times [0,1]\big) \setminus \mathcal{D}.$$

This and (4.176) give that for every $P \in \mathcal{U}$ there exists $t_P \in [0, \bar{T}]$ such that $\phi^{t_P}(P) \in \{P_s^m, P_s^0\}$. In particular,

$$P \in \phi^{-t_P}\{P_s^m, P_s^0\} \subseteq \big\{\phi^t(P_s^m),\ t \in [-\bar{T}, 0]\big\} \cup \big\{\phi^t(P_s^0),\ t \in [-\bar{T}, 0]\big\}.$$

Since this is valid for every $P \in \mathcal{U}$, we conclude that

$$\mathcal{U} \subseteq \big\{\phi^t(P_s^m),\ t \in [-\bar{T}, 0]\big\} \cup \big\{\phi^t(P_s^0),\ t \in [-\bar{T}, 0]\big\}. \tag{4.178}$$

Now we remark that $\big\{\phi^t(P_s^m),\ t \in [-\bar{T}, 0]\big\}$ is an arc of a smooth curve, whence it has null Lebesgue measure, and a similar statement holds true for $\big\{\phi^t(P_s^0),\ t \in [-\bar{T}, 0]\big\}$. Consequently, we deduce from (4.178) that $\mathcal{U}$ has null Lebesgue measure, in contradiction with (4.177).

In this way, we have shown that no trajectory can leave $\mathcal{D}$ and the proof of (4.169) is complete.

By (4.168) and (4.169), no trajectory starting in $\mathcal{D}$ can arrive in $(0,1] \times [0,1]$ when the bound $m \leqslant a(t) \leqslant M$ holds, hence (4.161) is true. Therefore the statement (i) in Proposition 4.5 is true.

Now we establish the claim in (ii). To this end, we point out that claim (ii) is equivalent to

$$\mathcal{G} := \Big\{(u,v) \in [0,1] \times [0,1] \ \text{ s.t. } \ v \geqslant g_\varepsilon(u)\Big\} \subseteq \mathcal{V}_{m,M}^C, \tag{4.179}$$

for all $\varepsilon \in (0,1)$, where $\mathcal{V}_{m,M}^C$ is the complement of $\mathcal{V}_{m,M}$ in the topology of $[0,1] \times [0,1]$.

First, we point out that

$$g_\varepsilon \text{ is a well defined continuous function.} \tag{4.180}$$

Indeed, one can easily check for $\varepsilon \in (0,1)$ that

$$\begin{aligned} 0 < u_2 = \frac{1-\varepsilon}{k-k\varepsilon+1} - \frac{c+1-\varepsilon}{(c+1)(k-k\varepsilon+1)} + u_3 = -\frac{c\varepsilon}{(c+1)(k-k\varepsilon+1)} + u_3 \\ < u_3 < \frac{c+1}{(c+1)(k-k\varepsilon+1)} < 1. \end{aligned} \tag{4.181}$$

Then, one checks that

$$ku_2 = \frac{u_2}{c} + q,$$

hence g_ε is continuous at the point u_2. In addition, one can check that g_ε is continuous at the point u_3 by observing that

$$\begin{aligned} \frac{u_3}{c} + q - (1-u_3) &= \frac{(c+1)u_3}{c} + q - 1 \\ &= \frac{c+1-\varepsilon}{c(k-k\varepsilon+1)} + \frac{(kc-1)(1-\varepsilon)}{c(k-k\varepsilon+1)} - 1 \\ &= \frac{c+1-\varepsilon+(kc-1)(1-\varepsilon)-c(k-k\varepsilon+1)}{c(k-k\varepsilon+1)} = 0. \end{aligned} \tag{4.182}$$

This completes the proof of (4.180).

Now we show that

$$g_\varepsilon(u) > 0 \quad \text{for every } u \in (0,1]. \tag{4.183}$$

We have that $k > 0$ for every $\varepsilon < 1$, and therefore $g_\varepsilon(u) > 0$ for all $u \in (0, u_2)$. Also, since $g_\varepsilon(u_2) = ku_2 > 0$ and g_ε is linear in (u_2, u_3), we have that $g_\varepsilon(u) > 0$ for all $u \in (u_2, u_3)$. Moreover, in the interval $\in [u_3, 1]$ we have that g_ε is an exponential function multiplied by a positive constant, thanks to (4.181), hence it is positive. These considerations prove (4.183).

As a consequence of (4.183), we have that

$$\mathcal{G} \cap \big((0,1] \times \{0\}\big) = \varnothing. \tag{4.184}$$

Now we claim that

$$\text{for any strategy } a \in \mathcal{A}_{m,M}, \text{ no trajectory starting in } \mathcal{G} \text{ leaves } \mathcal{G}. \tag{4.185}$$

For this, we observe that, in light of (4.184), all the points on

$$\partial\mathcal{G} \setminus \{(u, g_\varepsilon(u)) \ \text{ with } \ u \in [0,1]\}$$

belong to $\partial([0,1] \times [0,1]) \setminus \{v = 0\}$, and these three sides of the square do not allow the flow to exit. Hence, to prove (4.185) it suffices to check that the trajectories starting on $\partial\mathcal{G} \cap \big((0,1)\times(0,1)\big)$ enter $\mathcal{G}$. We do this by showing that the inner pointing derivative of the trajectory is nonnegative, according to the computation below.

At a point on the line $v = ku$, the velocity of a trajectory in the direction that is orthogonal to $\partial\mathcal{G}$ for $u \in [0, u_2)$ and pointing inward is:

$$(\dot{u}, \dot{v}) \cdot \frac{(-k, 1)}{\sqrt{1+k^2}} = \frac{(\rho v - ku)(1-u-v) - au(1-kc)}{\sqrt{1+k^2}}. \tag{4.186}$$

We also note that

$$kc = \frac{(c+1-\varepsilon)M}{(\rho-1)\varepsilon + (c+1-\varepsilon)M} < 1, \tag{4.187}$$

and therefore, at a point on $v = ku$ with $u \in [0, u_2)$,

$$\begin{aligned} 1-u-v \geqslant 1-u_2-ku_2 &= 1 - \frac{(1+k)(1-\varepsilon)}{k-k\varepsilon+1} = \frac{\varepsilon}{k(1-\varepsilon)+1} \\ &= \frac{\varepsilon c}{kc(1-\varepsilon)+c} > \frac{\varepsilon c}{1+c-\varepsilon}. \end{aligned}$$

This inequality entails that

$$k = \frac{(1+c-\varepsilon)M}{(\rho-1)\varepsilon c + (1+c-\varepsilon)Mc} = \frac{M}{\frac{(\rho-1)\varepsilon c}{1+c-\varepsilon} + Mc} > \frac{M}{(\rho-1)(1-u-v) + Mc}.$$

Consequently,

$$(\rho-1)(1-u-v)k > M(1-kc).$$

From this and (4.186), one deduces that, for all $u \in (0, u_2)$, $a \leqslant M$, and $v = ku$,

$$\begin{aligned} (\dot{u}, \dot{v}) \cdot \frac{(-k,1)}{\sqrt{1+k^2}} &= \frac{ku(\rho-1)(1-u-v) - au(1-kc)}{\sqrt{1+k^2}} \\ &> \frac{Mu(1-kc) - au(1-kc)}{\sqrt{1+k^2}} \geqslant 0. \end{aligned}$$

This (and the fact that the origin is an equilibrium) rules out the possibility of exiting $\mathcal{G}$ from $\{u \in [0, u_2) \text{ and } v = ku\}$.

It remains to consider the portions of $\partial\mathcal{G} \cap ((0,1)\times(0,1))$ given by

$$\left\{ u \in [u_2, u_3) \text{ and } v = \frac{u}{c} + q \right\} \tag{4.188}$$

and by

$$\left\{ u \in [u_3, 1] \text{ and } v = \frac{(1-u_3)u^\rho}{(u_3)^\rho} \right\}. \tag{4.189}$$

Let us deal with the case in (4.188). In this case, the velocity of a trajectory in the direction orthogonal to $\partial\mathcal{G}$ for $u \in [u_2, u_3)$ and pointing inward is

$$(\dot{u}, \dot{v}) \cdot \frac{(-1, c)}{\sqrt{1+c^2}} = \frac{(\rho c v - u)(1-u-v)}{\sqrt{1+c^2}}. \tag{4.190}$$

Recalling (4.159), we also observe that

$$\begin{aligned} k-\frac{1}{\rho c} &= \frac{1}{c}\left(\frac{(c+1-\varepsilon)M}{(\rho-1)\varepsilon+(c+1-\varepsilon)M}-\frac{1}{\rho}\right) \\ &= \frac{(\rho-1)\big((c+1-\varepsilon)M-\varepsilon\big)}{\rho c\big((\rho-1)\varepsilon+(c+1-\varepsilon)M\big)} > 0. \end{aligned} \tag{4.191}$$

Thus, on the line given by $v=\frac{u}{c}+q$ we have that

$$\begin{aligned} \rho c v-u &= (\rho-1)u+\rho c q \geqslant (\rho-1)u_2+\rho c q \\ &= \frac{(\rho-1)(1-\varepsilon)}{k-k\varepsilon+1}+\frac{\rho(kc-1)(1-\varepsilon)}{k-k\varepsilon+1} \\ &= (1-\varepsilon)\frac{(\rho-1)+\rho(kc-1)}{k-k\varepsilon+1} = \frac{(1-\varepsilon)(\rho kc-1)}{k-k\varepsilon+1} > 0, \end{aligned} \tag{4.192}$$

where (4.191) has been used in the latter inequality.

In addition, recalling (4.182),

$$1-u-v > 1-u_3-\frac{u_3}{c}-q = 1-u_3-1+u_3 = 0.$$

From this and (4.192), we gather that the velocity calculated in (4.190) is positive in $[u_2, u_3)$ and this excludes the possibility of exiting $\mathcal{G}$ from the boundary given in (4.188).

Next, we focus on the portion of the boundary described in (4.189) by considering $u\in[u_3,1]$. That is, we now compute the component of the velocity at a point on $\partial\mathcal{G}$ for $u\in[u_3,1]$ in the direction that is orthogonal to $\partial\mathcal{G}$ and pointing inward, that is

$$\begin{aligned} &(\dot u,\dot v)\cdot\frac{\left(-\rho\frac{1-u_3}{(u_3)^\rho}u^{\rho-1},1\right)}{\sqrt{1+\rho^2\frac{(1-u_3)^2}{(u_3)^{2\rho}}u^{2\rho-2}}} \\ &= \frac{\rho(1-u-v)\left(v-\frac{1-u_3}{(u_3)^\rho}u^\rho\right)-au\left(1-\rho c\frac{1-u_3}{(u_3)^\rho}u^{\rho-1}\right)}{\sqrt{1+\rho^2\frac{(1-u_3)^2}{(u_3)^{2\rho}}u^{2\rho-2}}} \\ &= \frac{au\left(\rho c\frac{1-u_3}{(u_3)^\rho}u^{\rho-1}-1\right)}{\sqrt{1+\rho^2\frac{(1-u_3)^2}{(u_3)^{2\rho}}u^{2\rho-2}}} \\ &\geqslant \frac{au\left(\rho c\frac{1-u_3}{u_3}-1\right)}{\sqrt{1+\rho^2\frac{(1-u_3)^2}{(u_3)^{2\rho}}u^{2\rho-2}}}. \end{aligned} \tag{4.193}$$

Now we notice that

$$\rho c(1-u_3) = \rho c\left(\frac{u_3}{c}+q\right) = \rho u_3+\rho c q = \rho u_3+\frac{\rho(kc-1)(1-\varepsilon)(c+1)u_3}{c+1-\varepsilon},$$

thanks to (4.182).

As a result, using (4.191),

$$\begin{aligned}\rho c(1-u_3) &> \rho u_3 + \frac{(1-\rho)(1-\varepsilon)(c+1)u_3}{c+1-\varepsilon}\\ &= \frac{u_3}{c+1-\varepsilon}\Big(\rho(c+1-\varepsilon)+(1-\rho)(1-\varepsilon)(c+1)\Big)\\ &= \frac{u_3\big((1-\varepsilon)(c+1)+\varepsilon\rho c\big)}{c+1-\varepsilon}\\ &= u_3+\frac{\varepsilon c u_3(\rho-1)}{c+1-\varepsilon} > u_3.\end{aligned}$$

This gives that the quantity in (4.193) is positive. Hence, we have ruled out also the possibility of exiting $\mathcal{G}$ from the boundary given in (4.189), and this ends the proof of (4.185).

Since no trajectory can exit $\mathcal{G}$ for any a with $m \leqslant a \leqslant M$, we get that no point $(u,v) \in \mathcal{G}$ is mapped into $(0,1] \times \{0\}$ because of (4.184), thus (4.179) is true and the proof is complete. □

We end this section with the proof of Theorem 1.6.

Proof of Theorem 1.6. Since by definition $\mathcal{A}_{m,M} \subseteq \mathcal{A}$, we have that $\mathcal{V}_{m,M} \subseteq \mathcal{V}_{\mathcal{A}}$. Hence, we are left with proving that the latter inclusion is strict.

We start with the case $\rho < 1$. We choose

$$\varepsilon \in \left(0,\, \min\left\{\frac{\rho c(c+1)}{1+\rho c}, \frac{M(c+1)}{M+1}, 1\right\}\right). \tag{4.194}$$

We observe that this choice is compatible with the assumption on ε in (4.159). We note that

$$u_1 < \min\left\{\frac{\rho c(c+1)}{1+\rho c}, 1\right\}, \tag{4.195}$$

thanks to (4.194). Moreover, by (4.166) and the fact that $h < \frac{1}{c}$, it holds that

$$hu+p = h(u-u_1)+hu_1+p = h(u-u_1)+\frac{u_1}{c}+\frac{1-\rho}{1+\rho c} < \frac{u}{c}+\frac{1-\rho}{1+\rho c} \tag{4.196}$$

for all $u > u_1$.

Now we choose

$$\bar{u} \in \left(u_1, \min\left\{\frac{\rho c(c+1)}{1+\rho c}, 1\right\}\right),$$

which is possible thanks to (4.195), and

$$\bar{v} := \frac{1}{2}(h\bar{u}+p)+\frac{1}{2}\left(\frac{\bar{u}}{c}+\frac{1-\rho}{1+\rho c}\right). \tag{4.197}$$

By (4.196) we get that

$$h\bar{u}+p < \frac{1}{2}(h\bar{u}+p)+\frac{1}{2}\left(\frac{\bar{u}}{c}+\frac{1-\rho}{1+\rho c}\right) = \bar{v} < \frac{\bar{u}}{c}+\frac{1-\rho}{1+\rho c}. \tag{4.198}$$

Using Proposition 4.5 and (4.198), we deduce that $(\bar{u}, \bar{v}) \notin \mathcal{V}_{m,M}$. By Theorem 1.3 and (4.198) we obtain instead that $(\bar{u}, \bar{v}) \in \mathcal{V}_{\mathcal{A}}$. Hence, the set $\mathcal{V}_{m,M}$ is strictly included in $\mathcal{V}_{\mathcal{A}}$ when $\rho < 1$.

Now we consider the case $\rho > 1$, using again the notation of Proposition 4.5. We recall that $u_2 > 0$ and $u_\infty > 0$, due to (1.18) and (4.181), hence we can choose

$$\bar{u} \in (0, \min\{u_2, u_\infty\}).$$

We also define

$$\bar{v} := \frac{1}{2}\left(\frac{1}{c} + k\right)\bar{u}.$$

By (4.187), we get that

$$k\bar{u} < \frac{k\bar{u}}{2} + \frac{\bar{u}}{2c} = \bar{v} < \frac{\bar{u}}{c}. \tag{4.199}$$

Exploiting this and the characterization in Proposition 4.5, it holds that $(\bar{u}, \bar{v}) \notin \mathcal{V}_{m,M}$. On the other hand, by Theorem (1.3) and (4.199) we have instead that $(\bar{u}, \bar{v}) \in \mathcal{V}_{\mathcal{A}}$. As a consequence, the set $\mathcal{V}_{m,M}$ is strictly contained in $\mathcal{V}_{\mathcal{A}}$ for $\rho > 1$. This concludes the proof of Theorem 1.6. □

4.6 Minimization of war duration: proof of Theorem 1.7

We now deal with the strategies leading to the quickest possible victory of the first population.

Proof of Theorem 1.7. Our aim is to establish the existence of the strategy leading to the quickest possible victory and to determine its range. For this, we consider the following minimization problem under constraints for $x(t) := (u(t), v(t))$:

$$\begin{cases} \dot{x}(t) = f(x(t), a(t)), \\ x(0) = (u_0, v_0), \\ x(T_s) \in (0,1] \times \{0\}, \\ \displaystyle\min_{a(t)\in[m,M]} \int_0^{T_s} 1\, dt, \end{cases} \tag{4.200}$$

where

$$f(x,a) := \Big(u(1-u-v-ac),\ \rho v(1-u-v) - au\Big).$$

Here T_s corresponds to the exit time introduced in (1.8), in dependence of the strategy $a(\cdot)$.

Theorem 6.15 in [22] assures the existence of a minimizing solution $(\tilde{a}, \tilde{x})$ with $\tilde{a}(t) \in [m, M]$ for all $t \in [0, T]$, and $\tilde{x}(t) \in [0,1] \times [0,1]$ absolutely continuous, such that $\tilde{x}(T) = (\tilde{u}(T), 0)$ with $\tilde{u}(T) \in [0,1]$, where T is the exit time for $\tilde{a}$.

We now prove that

$$\tilde{u}(T) > 0. \tag{4.201}$$

Indeed, if this were false, then $(\tilde{u}(T), \tilde{v}(T)) = (0,0)$. Let us call $d(t) := \tilde{u}^2(t) + \tilde{v}^2(t)$. Then, we observe that the function $d(t)$ satisfies the following differential inequality:

$$-\dot{d}(t) \leqslant Cd, \qquad \text{for} \quad C := 4 + 4\rho + 2Mc + M. \tag{4.202}$$

To check this, we compute that

$$\begin{aligned}
-\dot{d} &= 2\left(-\tilde{u}^2(1-\tilde{u}-\tilde{v}-\tilde{a}c) - \tilde{v}^2\rho(1-\tilde{u}-\tilde{v}) + \tilde{u}\tilde{v}\tilde{a}\right)\\
&\leqslant 2\tilde{u}^2(2+Mc) + 4\rho\tilde{v}^2 + (\tilde{u}^2+\tilde{v}^2)M\\
&\leqslant C(\tilde{u}^2+\tilde{v}^2)\\
&= Cd,
\end{aligned}$$

which proves (4.202).

From (4.202), one has that

$$0 < (u_0^2+v_0^2)e^{-CT} \leqslant d(T) = \tilde{u}^2(T) + \tilde{v}^2(T) = \tilde{u}^2(T),$$

and this leads to (4.201), as desired. We remark that, in this way, we have found a trajectory $\tilde{a}$ which leads to the victory of the first population in the shortest possible time.

Theorem 6.15 in [22] assures that $\tilde{a}(t) \in L^1[0,T]$, so $\tilde{a}(t)$ is measurable. We have that the two vectorial functions F and G, defined by

$$F(u,v) := \begin{pmatrix} u(1-u-v) \\ \rho v(1-u-v) \end{pmatrix} \qquad \text{and} \qquad G(u,v) := \begin{pmatrix} -cu \\ -u \end{pmatrix},$$

and satisfying $f(x(t),a(t)) = F(x(t)) + a(t)G(x(t))$, are analytic. Moreover the set $\overline{\mathcal{V}}_{\mathcal{A}_{m,M}}$ is a subset of $\mathbb{R}^2$, therefore it can be seen as an analytic manifold with border which is also a compact set. For all $x_0 \in \mathcal{V}_{\mathcal{A}_{m,M}}$ and $t>0$ we have that the trajectory starting from x_0 satisfies $x(\tau) \in \overline{\mathcal{V}}_{\mathcal{A}_{m,M}}$ for all $\tau \in [0,t]$. Then, by Theorem 3.1 in [19], there exists a couple $(\tilde{a},\tilde{x})$ analytic a part from a finite number of points, such that $(\tilde{a},\tilde{x})$ solves (4.200).

Now, to study the range of $\tilde{a}$, we apply the Pontryagin Maximum Principle (see for example [22]). The Hamiltonian associated with system (4.200) is

$$H(x,p,p_0,a) := p \cdot f(x,a) + p_0$$

where $p = (p_u, p_v)$ is the adjoint to $x = (u,v)$ and p_0 is the adjoint to the cost function identically equal to 1. The Pontryagin Maximum Principle tells us that, since $\tilde{a}(t)$ and $\tilde{x}(t) = (\tilde{u}(t),\tilde{v}(t))$ give the optimal solution, there exist a vectorial function $\tilde{p} : [0,T] \to \mathbb{R}^2$ and a scalar $\tilde{p}_0 \in (-\infty, 0]$ such that

$$\begin{cases} \dfrac{d\tilde{x}}{dt}(t) = \dfrac{\partial H}{\partial p}(\tilde{x}(t),\tilde{p}(t),\tilde{p}_0,\tilde{a}(t)), & \text{for a.a. } t \in [0,T], \\[2ex] \dfrac{d\tilde{p}}{dt}(t) = -\dfrac{\partial H}{\partial x}(\tilde{x}(t),\tilde{p}(t),\tilde{p}_0,\tilde{a}(t)), & \text{for a.a. } t \in [0,T], \end{cases} \tag{4.203}$$

and

$$H(\tilde{x}(t),\tilde{p}(t),\tilde{p}_0,\tilde{a}(t)) = \max_{a(\cdot)\in[m,M]} H(\tilde{x}(t),\tilde{p}(t),\tilde{p}_0,a) \quad \text{for a.a. } t \in [0,T]. \tag{4.204}$$

Moreover, since the final time is free, we have

$$H(\tilde{x}(T),\tilde{p}(T),\tilde{p}_0,\tilde{a}(T)) = 0. \tag{4.205}$$

Also, since $H(x, p, p_0, a)$ does not depend on t, we get

$$H(\tilde{x}(t), \tilde{p}(t), \tilde{p}_0, \tilde{a}(t)) = \text{constant} = 0, \quad \text{for a.a. } t \in [0, T], \tag{4.206}$$

where the value of the constant is given by (4.205). By substituting the values of $f(x, a)$ in $H(x, p, p_0, a)$ and using (4.206), we get, for a.a. $t \in [0, T]$,

$$\tilde{p}_u \tilde{u}(1 - \tilde{u} - \tilde{v} - \tilde{a}c) + \tilde{p}_v \rho \tilde{v}(1 - \tilde{u} - \tilde{v}) - \tilde{p}_v \tilde{a}\tilde{u} + \tilde{p}_0 = 0,$$

where $\tilde{p} = (\tilde{p}_u, \tilde{p}_v)$.

Also, by (4.204) we get that

$$\max_{a \in [m,M]} H(\tilde{x}(t), \tilde{p}(t), \tilde{p}_0, a) = \max_{a \in [m,M]} \Big[-a\tilde{u}(c\tilde{p}_u + \tilde{p}_v) + \tilde{p}_u \tilde{u}(1 - \tilde{u} - \tilde{v}) + \tilde{p}_v \rho \tilde{v}(1 - \tilde{u} - \tilde{v}) + \tilde{p}_0 \Big]. \tag{4.207}$$

Thus, to maximize the term in the square brackets we must choose appropriately the value of $\tilde{a}$ depending on the sign of $\varphi(t) := c\tilde{p}_u(t) + \tilde{p}_v(t)$, that is we choose

$$\tilde{a}(t) := \begin{cases} m & \text{if } \varphi(t) > 0, \\ M & \text{if } \varphi(t) < 0. \end{cases} \tag{4.208}$$

When $\varphi(t) = 0$, we are for the moment free to choose $\tilde{a}(t) := a_s(t)$ for every $a_s(\cdot)$ with range in $[m, M]$, without affecting the maximization problem in (4.207).

Our next goal is to determine that $a_s(t)$ has the expression stated in (1.20) for a.a. $t \in [0, T] \cap \{\varphi = 0\}$.

To this end, we claim that

$$\dot{\varphi}(t) = 0 \text{ a.e. } t \in [0, T] \cap \{\varphi = 0\}. \tag{4.209}$$

Indeed, by (4.203), we know that $\tilde{p}$ is Lipschitz continuous in $[0, T]$, hence almost everywhere differentiable, and thus the same holds for φ. Hence, up to a set of null measure, given $t \in [0, T] \cap \{\varphi = 0\}$, we can suppose that t is not an isolated point in such a set, and that φ is differentiable at t. That is, there exists an infinitesimal sequence h_j for which $\varphi(t + h_j) = 0$ and

$$\dot{\varphi}(t) = \lim_{j \to +\infty} \frac{\varphi(t + h_j) - \varphi(t)}{h_j} = \lim_{j \to +\infty} \frac{0 - 0}{h_j} = 0,$$

and this establishes (4.209).

Consequently, in light of (4.209), a.a. $t \in [0, T] \cap \{\varphi = 0\}$ satisfies

$$\begin{aligned} 0 = \dot{\varphi}(t) &= c\frac{d\tilde{p}_u}{dt}(t) + \frac{d\tilde{p}_v}{dt}(t) \\ &= c\big[-\tilde{p}_u(t)(1 - 2\tilde{u}(t) - \tilde{v}(t) - ca_s(t)) + \tilde{p}_v(t)(\rho\tilde{v}(t) + a_s(t)) \big] \\ &\quad + \tilde{p}_u(t)\tilde{u}(t) - \tilde{p}_v(t)\rho(1 - \tilde{u}(t) - 2\tilde{v}(t)). \end{aligned}$$

Now, since $\varphi(t) = 0$, we have that $\tilde{p}_v(t) = -c\tilde{p}_u(t)$; inserting this information in the last equation, we get

$$0 = -\tilde{p}_u c(1 - 2\tilde{u} - \tilde{v} - a_s c) - \tilde{p}_u \rho c^2 \tilde{v} - \tilde{p}_u a_s c^2 + \tilde{p}_u \tilde{u} + \tilde{p}_u \rho c(1 - \tilde{u} - 2\tilde{v}). \tag{4.210}$$

Notice that if $\tilde{p}_u = 0$, then $\tilde{p}_v = -c\tilde{p}_u = 0$; moreover, by (4.206), one gets $\tilde{p}_0 = 0$. But by the Pontryagin Maximum Principle one cannot have $(\tilde{p}_u, \tilde{p}_v, \tilde{p}_0) = (0, 0, 0)$, therefore one obtains $\tilde{p}_u \neq 0$ in $\{\varphi = 0\}$. Hence, dividing (4.210) by $\tilde{p}_u$ and rearranging the terms, one gets

$$\tilde{u}(2c + 1 - \rho c) + c\tilde{v}(1 - \rho c - 2\rho) + c(\rho - 1) = 0. \tag{4.211}$$

Differentiating the expression in (4.211) with respect to time, we get

$$\tilde{u}(2c + 1 - \rho c)(1 - \tilde{u} - \tilde{v} - ac) + c(1 - \rho c - 2\rho)[\rho\tilde{v}(1 - \tilde{u} - \tilde{v}) - a\tilde{u}] = 0,$$

that yields

$$a_s = \frac{(1 - \tilde{u} - \tilde{v})(\tilde{u}(2c + 1 - \rho c) + \rho c)}{2c\tilde{u}(c + 1)}, \tag{4.212}$$

which is the desired expression. By a slight abuse of notation, we define the function $a_s(t) = a_s(\tilde{u}(t), \tilde{v}(t))$ for $t \in [0, T]$. Notice that since $\tilde{u}(t) > 0$ for $t \in [0, T]$, $a_s(t)$ is continuous for $t \in [0, T]$. □

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