On the (non) contractibility of the simplicial complex associated to the coset poset of a classical group

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The simplicial complex associated to the coset poset

**Definition (S. Bouc - K. Brown)**

The coset poset \( C(G) \) associated to a finite group \( G \) is the poset consisting of the proper (right) cosets \( Hx \), with \( H \leq G \) and \( x \in G \), ordered by inclusion.

\[
Hx \subseteq Ky \iff H \leq K \text{ and } Ky = Kx.
\]

We can apply topological concepts to this poset by using the simplicial complex \( \Delta = \Delta(C(G)) \) whose simplices are the finite chains \( H_1g_1 < H_2g_2 < ... < H_ng_n \) of elements of \( C(G) \).
Motivations and Results
Outline of the strategy of the proof

Introduction
A conjecture of Kennet Brown
The Main Result: a recent result by Massimiliano Patassini

Euler characteristic and contractibility of $\Delta(C(G))$

In particular, we can speak of the Euler characteristic of $C(G)$

$$\chi(C(G)) := \sum_{m} (-1)^{m} \alpha_{m}$$

where $\alpha_{m}$ is the number of chains in $C(G)$ of length $m$

and of the reduced Euler characteristic of $C(G)$

$$\tilde{\chi}(C(G)) := \chi(C(G)) - 1.$$

$$\tilde{\chi}(C(G)) \neq 0 \implies \text{the simplicial complex } \Delta(C(G)) \text{ is not contractible.}$$
In 2000, Kennet Brown pointed out a connection between $C(G)$ and the Dirichlet polynomial $P_G(s)$ associated to $G$, given by

$$P_G(s) = \sum_{k=1}^{\infty} \frac{a_k(G)}{k^s},$$
where

$$a_k(G) = \sum_{H \leq G, |G:H|=k} \mu_G(H),$$

and $\mu_G$ is the Möbius function of the subgroup lattice of $G$, defined by $\mu_G(G) = 1$, $\mu_G(H) = -\sum_{K > H} \mu_G(K)$ if $H < G$.

Remark (Bouc, Brown)

$$\tilde{\chi}(C(G)) = -P_G(-1).$$
Conjecture of Brown

**Conjecture (Brown, 2000)**

Let $G$ be a finite group. Then $P_G(-1) \neq 0$, hence the simplicial complex associated to the coset poset of $G$ is non-contractible.
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Evidences for the conjectures

1. $P_G(-1) \neq 0$ for all the finite groups for which $P_G(-1)$ have been computed by GAP.
3. (M. Patassini - 2008) True for $PSL(2, q)$, $^2B_2(q)$, $^2G_2(q)$. 
Theorem (Brown, 2000)

If $G$ is a finite soluble group, then $P_G(-1) \neq 1$. 
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Proof

Let $1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_t = G$ be a chief series of $G$. Define

1. $c_i :=$ the numbers of complements of $N_i/N_{i-1}$ in $G/N_{i-1}$,
2. $d_i :=$ the order of $N_i/N_{i-1}$.

Then

$$P_G(s) = \prod_{1 \leq i \leq t} \left(1 - \frac{c_i}{d_i^s}\right)$$

hence

$$P_G(-1) = \prod_{1 \leq i \leq t} \left(1 - \frac{c_i d_i}{1}\right) \neq 0.$$
Let $\mathcal{G}$ be the set of groups $X$ satisfying the following:

- $X$ is nearly simple, i.e. $X/Z(X)$ is almost simple with socle $G$.
- $G$ is a classical simple group (i.e. $\text{PSL}_n(q)$, $\text{PSU}_n(q)$, $\text{PSp}_n(q)$, $\text{P}Ω_n^ε(q)$, where $ε \in \{o, +, −\}$).
- $X$ does not contain a non-trivial graph automorphism.
- $G$ is not isomorphic to $\text{PSL}_2(49)$ or to $\text{PSp}_6(2)$.

**Theorem (M. Patassini 2009)**

*If $X \in \mathcal{G}$, then $P_X(−1) \neq 0$.***
For the sake of simplicity, we present an outline of the proof in the case $X = G$, a classical simple group.

**Definition**

Let $p$ be the characteristic of $G$. The $p$-Dirichlet polynomial of $G$ is

$$P_{G}^{(p)}(s) = \sum_{(p,n)=1} \frac{a_n(G)}{n^s}.$$
General Strategy

It suffices to prove:

- $P_G(p)(-1) \neq 0$;
- $p$ divides $|G : H|$ and $\mu_G(H) \neq 0 \Rightarrow |P_G(p)(-1)|_p < |G : H|^2_p.$
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General strategy
The \( p \)-Dirichlet polynomial of \( G \)
Two types of non-parabolic subgroups
\( H \) contained in a non-parabolic maximal
\( H \) non-parabolic, \( H \) intersection of parabolic maximal
So, we are done!

General Strategy

It suffices to prove:

- \( P^{(p)}_G(-1) \neq 0 \);
- \( p \) divides \( |G : H| \) and \( \mu_G(H) \neq 0 \) \( \Rightarrow |P^{(p)}_G(-1)|_p < |G : H|^2 \).

\( P_G(-1) = P^{(p)}_G(-1) + \sum_{H \in \mathcal{M}} \mu_G(H)|G : H| \) \text{ with } \mathcal{M} := \{ H \leq G \mid p \text{ divides } |G : H| \text{ and } \mu_G(H) \neq 0 \}

(\text{Isaacs, Hawkes, Özaydin}) \ G \ text{ perfect } \Rightarrow |G : H| \text{ divides } \mu_G(H)
General Strategy

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- \( P_G(-1) = P^{(p)}_G(-1) + \sum_{H \in \mathcal{M}} \mu_G(H)|G : H| \) with
  \( \mathcal{M} := \{ H \leq G \mid p \text{ divides } |G : H| \text{ and } \mu_G(H) \neq 0 \} \)
- (Isaacs, Hawkes, Özaydin) \( G \) perfect \( \Rightarrow |G : H| \) divides \( \mu_G(H) \)

\[
|P^{(p)}_G(-1)|_p < |G : H|^2_p \quad \forall H \in \mathcal{M} \Rightarrow \\
|P^{(p)}_G(-1)|_p < |\mu_G(H)|G : H|_p \quad \forall H \in \mathcal{M} \Rightarrow \\
|P_G(-1)|_p = |P^{(p)}_G(-1)|_p \neq 0 \Rightarrow P_G(-1) \neq 0.
\]
The $p$-part of $P_G^{(p)}(-1)$

$$P_G^{(p)}(s) = \sum_{H \geq B} \frac{\mu_G(H)}{|G:H|^{s-1}}$$
where $B$ is a Borel subgroup.

**Theorem**

Let $G$ be an untwisted simple group of Lie type of characteristic $p$ over $\mathbb{F}_q$. Then

$$|P_G^{(p)}(-1)|_p = (p, 2)q^l$$

with $l$ the number of nodes of the Dynkin diagram associated to $G$.

A similar result holds for twisted groups: in that case $l$ is the number of the $\rho$-orbits on the nodes, where $\rho$ is a suitable symmetry of the Dynkin diagram.
Two types of subgroups

Strategy

Prove that if $\mu_G(H) \neq 0$ and $p$ divides $|G : H|$, then


If $H$ is such a subgroup, then $|H|_p < |G|_p$ and $H$ is intersection of maximal subgroups of $G$.

We have two types of such subgroups $H$:

(A) $H$ is contained in a non parabolic maximal subgroup $M$ of $G$;

(B) $H$ is not of type (A), $H$ is an intersection of parabolic maximal subgroups of $G$ and $|H|_p < |G|_p$. 

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Subgroups of type \((A)\)

\((A)\) \(H\) is contained in a non parabolic maximal subgroup \(M\) of \(G\).

From what it is known about the maximal subgroups of classical groups, it follows

**Proposition**

\[ |G : H|^2_p > |P_G^{(p)}(-1)|_p, \text{ with few exceptions (e.g. } \text{PSL}_2(p^2)). \]
(B) \( H \) is an intersection of parabolic maximal subgroups of \( G \) and \(|H|_p < |G|_p\). Moreover \( H \) is not of type (A).

- To the classical group \( G \) it is associated a certain form \( \kappa \) defined on a vector space \( V \) of dimension \( n \) over a field \( \mathbb{F}_q \) of characteristic \( p \).
- So we can speak about totally singular subspace of \( V \) (w.r.t. the form \( \kappa \)).
Totally singular subspaces of $V$ and parabolic maximal subgroups

We have the following 1-1 correspondence:

$$\text{Stab}_G : \begin{cases} 
\text{non-trivial totally singular subspaces of } V \\
\text{s.t. } \text{Stab}_G(W) \geq H 
\end{cases} \rightarrow \begin{cases} 
\text{parabolic maximal subgroups of } G 
\end{cases}$$

Given a subgroup $H$ of type (B):

$$\text{Stab}_G : L_H = \begin{cases} 
W \text{ non-trivial totally singular subspaces of } V \\
s.t. \text{Stab}_G(W) \geq H 
\end{cases} \rightarrow \begin{cases} 
\text{parabolic maximal subgroups of } G \text{ containing } H 
\end{cases}$$
The property $\mathcal{P}$

**Definition**

We say that $\mathcal{L}_H$ fulfills the property $\mathcal{P}$ if there exists $W \in \mathcal{L}_H$ such that for each $U \in \mathcal{L}_H$, we have $W \leq U$ or $W \geq U$.

**Theorem**

Let $H$ be a subgroup of type $(B)$.

- If $\mathcal{L}_H$ fulfills the property $\mathcal{P}$, then $\mu_G(H) = 0$.
- If $\mathcal{L}_H$ does not fulfill the property $\mathcal{P}$, then $|G : H|_p^2 > \left| P_G^{(p)}(-1) \right|_p$. 

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Example: $\mathcal{L}_H$ fulfills the property $\mathcal{P}$

- Let $G = \text{PSL}_4(q)$, $q$ odd, let $V = \langle e_1, \ldots, e_4 \rangle$.
- $W_1 = \langle e_1 \rangle$, $W_2 = \langle e_2 \rangle$.
- $H = M_1 \cap M_2$, $M_i = \text{Stab}_G(W_i)$ for $i = 1, 2$.

Then

- $H$ is of type (B).
- $|G : H|_p = q$ and $|P_G^{(p)}(-1)|_p = q^3$, so $|G : H|_p^2 \ncong |P_G^{(p)}(-1)|_p$
- $\mathcal{L}_H = \{ W_1, W_2, W_1 + W_2 \}$;
- $\mathcal{L}_H$ fulfills the property $\mathcal{P}$. 
Example: $\mathcal{L}_H$ fulfills the property $\mathcal{P}$

- Let $G = \text{PSL}_4(q)$, $q$ odd, let $V = \langle e_1, ..., e_4 \rangle$.
- $W_1 = \langle e_1 \rangle$, $W_2 = \langle e_2 \rangle$.
- $H = M_1 \cap M_2$, $M_i = \text{Stab}_G(W_i)$ for $i = 1, 2$.

Let $M_3 = \text{Stab}_G(W_1 + W_2)$. The lattice generated by the maximal subgroups over $H$ is

$$
\begin{array}{c}
G \\
/ \ \\
M_1 \ \\
/ \ \\
M_1 \cap M_3 \\
/ \ \\
H
\end{array}
$$
Example: $\mathcal{L}_H$ fulfills the property $\mathcal{P}$

- Let $G = \text{PSL}_4(q)$, $q$ odd, let $V = \langle e_1, \ldots, e_4 \rangle$.
- $W_1 = \langle e_1 \rangle$, $W_2 = \langle e_2 \rangle$.
- $H = M_1 \cap M_2$, $M_i = \text{Stab}_G(W_i)$ for $i = 1, 2$.

Let $M_3 = \text{Stab}_G(W_1 + W_2)$. The corresponding values of the Möbius function are:

$$
\begin{array}{c}
1 \\
\downarrow & \downarrow & \downarrow \\
-1 & -1 & -1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 1 & 1 \\
\downarrow & \downarrow & \downarrow \\
0
\end{array}
$$
In general, the idea is: find a "redundant" element $W$ in $\mathcal{L}_H$, i.e for each $M \subseteq \mathcal{L}_H$ s.t. $W \in M$,

$$\bigcap_{U \in M} \text{Stab}_G(U) = H \Rightarrow \bigcap_{U \in M - \{W\}} \text{Stab}_G(U) = H.$$  

In the previous case, the subspace $W_1 + W_2$ is redundant.
Summarizing

(A). $H$ is contained in a maximal subgroup $M$ of $G$ and $|M|_p < |G|_p$

(B). $H$ is not of type (A) and $H$ is an intersection of parabolic maximal subgroups of $G$ and $|H|_p < |G|_p$. 

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Summarizing

(A). $H$ is contained in a maximal subgroup $M$ of $G$ and $|M|_p < |G|_p \Rightarrow |G:H|_p^2 > |P_G^{(p)}(-1)|_p$

(B). $H$ is not of type (A) and $H$ is an intersection of parabolic maximal subgroups of $G$ and $|H|_p < |G|_p$. 
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$H$ contained in a non-parabolic maximal $H$
$H$ non-parabolic, $H$ intersection of parabolic maximal $H$
So, we are done!

Summarizing

(A). $H$ is contained in a maximal subgroup $M$ of $G$ and
$|M|_p < |G|_p \Rightarrow |G : H|_p^2 > |P_G^{(p)}(-1)|_p$

(B). $H$ is not of type (A) and $H$ is an intersection of parabolic maximal subgroups of $G$ and $|H|_p < |G|_p$. \( \Rightarrow \) if $\mu_G(H) \neq 0$, then $|G : H|_p^2 > |P_G^{(p)}(-1)|_p$. 

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Summarizing

(A). $H$ is contained in a maximal subgroup $M$ of $G$ and

$$|M|_p < |G|_p \Rightarrow |G : H|_p^2 > |P_G^{(p)}(-1)|_p$$

(B). $H$ is not of type (A) and $H$ is an intersection of parabolic maximal subgroups of $G$ and $|H|_p < |G|_p$. \(\Rightarrow\) if $\mu_G(H) \neq 0$, then


Thus we conclude that $P_G(-1) \neq 0$. 
**Corollary**

If \( X = G^n \), with \( G \) a classical simple group, then \( P_X(-1) \neq 0 \).

**Proof**

\[
P_X(s) = \prod_{1 \leq i \leq n-1} \left( P_G(s) - \frac{i|\text{Aut}(G)|}{|G|^s} \right)
\]

\[
P_X(-1) = \prod_{1 \leq i \leq n-1} \left( P_G(-1) - i|\text{Aut}(G)||G| \right)
\]

\[
\downarrow \quad \left( \text{since } |P_G^{(p)}(-1)|_p < |G|^2 \right)
\]

\[
|P_X^{(p)}(-1)|_p = \prod_{1 \leq i \leq n-1} |P_G^{(p)}(-1)|_p \neq 0
\]