On the (non) contractibility of the simplicial complex associated to the coset poset of a classical group

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The simplicial complex associated to the coset poset

Definition (S. Bouc - K. Brown)

The coset poset $\mathcal{C}(G)$ associated to a finite group G is the poset consisting of the proper (right) cosets Hx, with $H < G$ and $x \in G$, ordered by inclusion.

$$Hx \subseteq Ky \Longleftrightarrow H \leq K \text{ and } Ky = Kx.$$

We can apply topological concepts to this poset by using the simplicial complex $\Delta = \Delta(\mathcal{C}(G))$ whose simplices are the finite chains $H_1g_1 < H_2g_2 < \ldots < H_ng_n$ of elements of $\mathcal{C}(G)$.

Euler characteristic and contractibility of $\Delta(\mathcal{C}(G))$

In particular, we can speak of the Euler characteristic of $\mathcal{C}(G)$

$$\chi(\mathcal{C}(G)) := \sum_m (-1)^m \alpha_m$$

where α_m is the number of chains in $\mathcal{C}(G)$ of length m

and of the reduced Euler characteristic of $\mathcal{C}(G)$

$$\tilde{\chi}(\mathcal{C}(G)) := \chi(\mathcal{C}(G)) - 1.$$

$\tilde{\chi}(\mathcal{C}(G)) \neq 0 \Rightarrow$ the simplicial complex $\Delta(\mathcal{C}(G))$ is not contractible.

The Dirichlet polynomial $P_G(s)$ of a group G

In 2000, Kennet Brown pointed out a connection between $\mathcal{C}(G)$ and the Dirichlet polynomial $P_G(s)$ associated to G, given by

$$P_G(s) = \sum_{k=1}^{\infty} \frac{a_k(G)}{k^s}, \quad \text{where} \quad a_k(G) = \sum_{H \leq G, |G:H|=k} \mu_G(H),$$

and μ_G is the Möbius function of the subgroup lattice of G, defined by $\mu_G(G) = 1$, $\mu_G(H) = -\sum_{K>H} \mu_G(K)$ if $H < G$.

Remark (Bouc, Brown)

$$\tilde{\chi}(\mathcal{C}(G)) = -P_G(-1).$$

Conjecture of Brown

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Evidences for the conjectures

1. $P_G(-1) \neq 0$ for all the finite groups for which $P_G(-1)$ have been computed by GAP.
2. (K. Brown - 2000) True for solvable groups.
3. (M. Patassini - 2008) True for $PSL(2, q)$, ${}^2B_2(q)$, ${}^2G_2(q)$.

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Proof

Let $1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_t = G$ be a chief series of G. Define

1. $c_i :=$ the numbers of complements of N_i/N_{i-1} in G/N_{i-1},
2. $d_i :=$ the order of N_i/N_{i-1}.

Then

$$P_G(s) = \prod_{1 \leq i \leq t} (1 - c_i/d_i^s)$$

hence

$$P_G(-1) = \prod_{1 \leq i \leq t} (1 - c_i d_i) \neq 0.$$

The Main Result: a recent result by Massimiliano Patassini

Let $\mathcal{G}$ be the set of groups X satisfying the following:

- X is nearly simple, i.e. $X/Z(X)$ is almost simple with socle G.
- G is a classical simple group (i.e. $\mathrm{PSL}_n(q)$, $\mathrm{PSU}_n(q)$, $\mathrm{PSp}_n(q)$, $\mathrm{P\Omega}^{\epsilon}_n(q)$, where $\epsilon \in \{o, +, -\}$).
- X does not contain a non-trivial graph automorphism.
- G is not isomorphic to $\mathrm{PSL}_2(49)$ or to $\mathrm{PSp}_6(2)$.

Theorem (M. Patassini 2009)

If $X \in \mathcal{G}$, then $P_X(-1) \neq 0$.

Assumptions and useful definitions

For the sake of simplicity, we present an outline of the proof in the case $X = G$, a classical simple group.

Definition

Let p be the characteristic of G. The p-Dirichlet polynomial of G is

$$P_G^{(p)}(s) = \sum_{(p,n)=1} \frac{a_n(G)}{n^s}.$$

General Strategy

It suffices to prove:

- $P_G^{(p)}(-1) \neq 0$;
- p divides $|G : H|$ and $\mu_G(H) \neq 0 \Rightarrow |P_G^{(p)}(-1)|_p < |G : H|_p^2$.

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- $P_G(-1) = P_G^{(p)}(-1) + \sum_{H \in \mathcal{M}} \mu_G(H)|G:H|$ with $\mathcal{M} := \{H \leq G \mid p \text{ divides } |G:H| \text{ and } \mu_G(H) \neq 0\}$
- (Isaacs, Hawkes, Özaydin) G perfect $\Rightarrow |G:H|$ divides $\mu_G(H)$

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$$|P_G^{(p)}(-1)|_p < |G:H|_p^2 \ \forall H \in \mathcal{M} \Rightarrow$$

$$|P_G^{(p)}(-1)|_p < \big|\mu_G(H)|G:H|\big|_p \ \forall H \in \mathcal{M} \Rightarrow$$

$$|P_G(-1)|_p = |P_G^{(p)}(-1)|_p \neq 0 \quad \Rightarrow \quad P_G(-1) \neq 0.$$

The p-part of $P_G^{(p)}(-1)$

$$P_G^{(p)}(s) = \sum_{H \geq B} \frac{\mu_G(H)}{|G:H|^{s-1}}$$ where B is a Borel subgroup.

Theorem

Let G be an untwisted simple group of Lie type of characteristic p over $\mathbb{F}_q$. Then

$$|P_G^{(p)}(-1)|_p = (p,2)q^l$$

with l the number of nodes of the Dynkin diagram associated to G.

A similar result holds for twisted groups: in that case l is the number of the ρ-orbits on the nodes, where ρ is a suitable symmetry of the Dynkin diagram.

Two types of subgroups

Strategy

Prove that if $\mu_G(H) \neq 0$ and p divides $|G : H|$, then

$$|P_G^{(p)}(-1)|_p < |G : H|_p^2.$$

If H is such a subgroup, then $|H|_p < |G|_p$ and H is intersection of maximal subgroups of G.

We have two **types** of such subgroups H:

(A) H is contained in a non parabolic maximal subgroup M of G;

(B) H is not of type (A), H is an intersection of parabolic maximal subgroups of G and $|H|_p < |G|_p$.

Subgroups of type (A)

(A) H is contained in a non parabolic maximal subgroup M of G.

From what it is known about the maximal subgroups of classical groups, it follows

Proposition

$|G:H|_p^2 > |P_G^{(p)}(-1)|_p$, *with few exceptions (e.g.* $\mathrm{PSL}_2(p^2)$*).*

Subgroups of type (B)

(B) H is an intersection of parabolic maximal subgroups of G and $|H|_p < |G|_p$. Moreover H is not of type (A).

- To the classical group G it is associated a certain form κ defined on a vector space V of dimension n over a field $\mathbb{F}_q$ of characteristic p.
- So we can speak about totally singular subspace of V (w.r.t. the form κ).

Totally singular subspaces of V and parabolic maximal subgroups

We have the following 1-1 correspondence:

$$\mathrm{Stab}_G : \left\{ \begin{array}{c} \text{non-trivial} \\ \text{totally singular} \\ \text{subspaces of } V \end{array} \right\} \to \left\{ \begin{array}{c} \text{parabolic maximal} \\ \text{subgroups of } G \end{array} \right\}$$

Given a subgroup H of type (B):

$$\mathrm{Stab}_G : \mathcal{L}_H = \left\{ \begin{array}{c} W \text{ non-trivial} \\ \text{totally singular} \\ \text{s.t. } \mathrm{Stab}_G(W) \geq H \end{array} \right\} \to \left\{ \begin{array}{c} \text{parabolic maximal} \\ \text{subgroups of } G \\ \text{containing } H \end{array} \right\}$$

The property $\mathcal{P}$

Definition

We say that $\mathcal{L}_H$ fulfills the property $\mathcal{P}$ if there exists $W \in \mathcal{L}_H$ such that for each $U \in \mathcal{L}_H$, we have $W \leq U$ or $W \geq U$.

Theorem

Let H be a subgroup of type (B).

- *If $\mathcal{L}_H$ fulfills the property $\mathcal{P}$, then $\mu_G(H) = 0$.*
- *If $\mathcal{L}_H$ does not fulfill the property $\mathcal{P}$, then $|G : H|_p^2 > |P_G^{(p)}(-1)|_p$.*

Example: $\mathcal{L}_H$ fulfills the property $\mathcal{P}$

- Let $G = \mathrm{PSL}_4(q)$, q odd, let $V = \langle e_1, \ldots, e_4 \rangle$.
- $W_1 = \langle e_1 \rangle$, $W_2 = \langle e_2 \rangle$.
- $H = M_1 \cap M_2$, $M_i = \mathrm{Stab}_G(W_i)$ for $i = 1, 2$.

Then

- H is of type (B).
- $|G : H|_p = q$ and $|P_G^{(p)}(-1)|_p = q^3$, so $|G : H|_p^2 \not> |P_G^{(p)}(-1)|_p$
- $\mathcal{L}_H = \{W_1, W_2, W_1 + W_2\}$;
- $\mathcal{L}_H$ fulfills the property $\mathcal{P}$.

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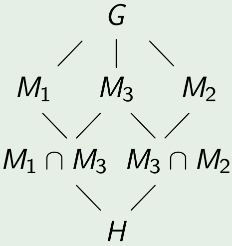
- $H = M_1 \cap M_2$, $M_i = \mathrm{Stab}_G(W_i)$ for $i = 1, 2$.

Let $M_3 = \mathrm{Stab}_G(W_1 + W_2)$. The lattice generated by the maximal subgroups over H is



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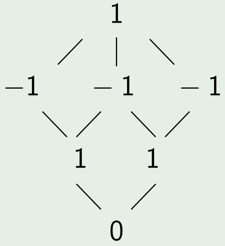

$\mathcal{L}_H$ fulfills the property $\mathcal{P}$

In general, the idea is: find a "redundant" element W in $\mathcal{L}_H$, i.e for each $M \subseteq \mathcal{L}_H$ s.t. $W \in M$,

$$\bigcap_{U \in M} \mathrm{Stab}_G(U) = H \Rightarrow \bigcap_{U \in M - \{W\}} \mathrm{Stab}_G(U) = H.$$

In the previous case, the subspace $W_1 + W_2$ is redundant.

Summarizing

(A). H is contained in a maximal subgroup M of G and $|M|_p < |G|_p$

(B). H is not of type (A) and H is an intersection of parabolic maximal subgroups of G and $|H|_p < |G|_p$.

Summarizing

(A). H is contained in a maximal subgroup M of G and $|M|_p < |G|_p \Rightarrow |G:H|_p^2 > |P_G^{(p)}(-1)|_p$

(B). H is not of type (A) and H is an intersection of parabolic maximal subgroups of G and $|H|_p < |G|_p$.

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(B). H is not of type (A) and H is an intersection of parabolic maximal subgroups of G and $|H|_p < |G|_p$. $\Rightarrow$ if $\mu_G(H) \neq 0$, then $|G:H|_p^2 > |P_G^{(p)}(-1)|_p$.

Summarizing

(A). H is contained in a maximal subgroup M of G and $|M|_p < |G|_p \Rightarrow |G:H|_p^2 > |P_G^{(p)}(-1)|_p$

(B). H is not of type (A) and H is an intersection of parabolic maximal subgroups of G and $|H|_p < |G|_p$. $\Rightarrow$ if $\mu_G(H) \neq 0$, then $|G:H|_p^2 > |P_G^{(p)}(-1)|_p$.

Thus we conclude that $P_G(-1) \neq 0$.

Corollary

If $X = G^n$, with G a classical simple group, then $P_X(-1) \neq 0$.

Proof

$$P_X(s) = \prod_{1\le i\le n-1}\left(P_G(s) - \frac{i|\mathrm{Aut}(G)|}{|G|^s}\right)$$

$$\Downarrow$$

$$P_X(-1) = \prod_{1\le i\le n-1}\left(P_G(-1) - i|\mathrm{Aut}(G)||G|\right)$$

$$\Downarrow \qquad (\text{ since } |P_G^{(p)}(-1)|_p < |G|_p^2)$$

$$|P_X^{(p)}(-1)|_p = \prod_{1\le i\le n-1}|P_G^{(p)}(-1)|_p \neq 0$$