Some questions arising from the study of the generating graph

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Many deep results about finite simple groups G can equivalently be stated as theorems about $\Gamma(G)$.

- (Guralnick and Kantor, 2000) There is no isolated vertex in $\Gamma(G)$.
- (Breuer, Guralnick and Kantor, 2008) The diameter of $\Gamma(G)$ is at most 2.

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Theorem (Breuer, Guralnick, AL , Maróti, Nagy (2010))

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Conjecture

Let G be a finite group with at least 4 elements. Then $\Gamma(G)$ contains a Hamiltonian cycle if and only if G/N is cyclic for all non-trivial normal subgroups N of G.

- The conjecture is true for finite soluble groups.
- For every sufficiently large symmetric group $\mathrm{Sym}(n)$, the graph $\Gamma(\mathrm{Sym}(n))$ contains a Hamiltonian cycle.
- For every sufficiently large non-abelian finite simple group S, the graph $\Gamma(S \wr C_m)$ contains a Hamiltonian cycle, where m denotes a prime power.

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- For every sufficiently large symmetric group $\mathrm{Sym}(n)$, the graph $\Gamma(\mathrm{Sym}(n))$ contains a Hamiltonian cycle.
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Moreover computer calculations show that the following groups contain Hamiltonian cycles.

- Non-abelian simple groups of orders at most 10^7,
- groups G containing a unique minimal normal subgroup N such that N has order at most 10^6, N is nonsolvable, and G/N is cyclic,
- alternating and symmetric groups on n points, with $5 \leq n \leq 13$,
- sporadic simple groups and automorphism groups of sporadic simple groups.

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A graph with m vertices and list of vertex degrees $d_1 \leq \ldots \leq d_m$ contains a Hamiltonian cycle if $d_k \geq k+1$ for all $k < m/2$.

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The *n*-closure of a graph Γ is the graph obtained from Γ by recursively joining pairs of nonadjacent vertices whose degree sum is at least n until no such pair remains.

Bondy, Chvátal

A graph with m vertices is Hamiltonian if and only if its m-closure is Hamiltonian.

Question

Let $m \in \mathbb{N}$ and let S be a nonabelian simple group and consider the wreath product $G = S \wr C_m$. Is the generating graph of G Hamiltonian?

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- Let $\pi : G = S \wr C_m \to C_m$ the projection to the top group C_m.
- If $\pi(g_1)$ and $\pi(g_2)$ generate C_m, then g_1 and g_2 are connected in $\Gamma(G)$ with high probability.
- $\deg(g, \Gamma(G)) \sim \deg(\pi(g), \Gamma(C_m))|S|^m$ if S is large.

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The conditions in Posa's criterion and in the Bondy-Chvátal Theorem are satisfied by $\Gamma(G)$ provided that the graph $\Gamma(C_m)$ satisfies similar conditions.

Definition

$\Lambda_m :=$ the $(m+1)$-closure of the generating graph $\Gamma(C_m)$.
We say that m is Hamiltonian if every $1 \neq x \in C_m$ generating a subgroup of C_m of odd index is connected in Λ_m to any other vertex.

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Theorem (E. Crestani and AL (2011))

Assume that m is Hamiltonian. There exists a positive integer τ such that if S is a nonabelian simple group with $|S| \geq \tau$, then the graph $\Gamma(S \wr C_m)$ contains a Hamiltonian cycle.

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Conjecture

Every positive integer is Hamiltonian.

Corollary

Let $m = \prod_{i=1}^{s} p_i^{\alpha_i}$*, where* $p_s < p_{s-1} < \cdots < p_1$ *are distinct primes and* $\alpha_i > 0$ *for every* $1 \leq i \leq s$*. Assume that one of the following holds:*

1. $s \leq 2$;
2. $\varphi(m)/m \geq (p_s - 1)/(2p_s - 1)$;
3. *m is odd and* $p_s \geq s + 1$;
4. *m is even and* $p_{s-1} \geq 2s - 1$.

There exists τ *such that if* $|S| \geq \tau$*, then the graph* $\Gamma(S \wr C_m)$ *contains a Hamiltonian cycle.*

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- $\delta(S) :=$ number of $(\mathrm{Aut}\, S)$-orbits on pairs of generators for S

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Nevertheless, the abundance of edges in the graph $\Gamma(S)$ reflects on $\Gamma(S^{\delta(S)})$.

Let $n \leq \delta(S) = \delta$. If n is large, it is no more true that $\Gamma(S^n)$ has no isolated vertices. We will concentrate our attention on the subgraph $\Gamma_n(S)$ obtained from $\Gamma(S^n)$ by removing the isolated vertices.

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- $\{(a_1, b_1), \dots, (a_\delta, b_\delta)\}$ a set of representatives for the Aut(S)-orbits on the ordered pairs of generators for S
- $C_1, \dots C_u$ the Aut(S)-orbits on $S \setminus \{1\}$
- $\delta_r = |\{i \mid a_i \in C_r\}|$

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$$S^n = \langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle \qquad \Longleftrightarrow$$

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$$x = (x_1, \dots, x_n) \text{ is a nonisolated vertex of } \Gamma(S^n)$$

$$\Leftrightarrow$$

$$|\{i \mid x_i \in C_r\}| \leq \delta_r \text{ for } 1 \leq r \leq u.$$

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$S = \mathrm{Alt}(5), \quad \delta(S) = 19, \quad u = 3$

$$(1,2,3,4,5) \in C_1 \Rightarrow \delta_1 = 10$$
$$(1,2,3) \in C_2 \Rightarrow \delta_2 = 6$$
$$(1,2)(3,4) \in C_3 \Rightarrow \delta_3 = 3$$

$$x = (x_1, \ldots, x_n)$$
is a non-isolated vertex in $\Gamma(S^n)$
if and only if x has
at most 10 entries of order 5,
at most 6 entries of order 3,
at most 3 entries of order 2.

- $\mathrm{Aut}(S^\delta) \cong \mathrm{Aut}(S) \wr \mathrm{Sym}(\delta) \leq \mathrm{Aut}(\Gamma_\delta(S))$.
- $S^\delta = \langle x, y \rangle = \langle \bar{x}, \bar{y} \rangle \Rightarrow (\bar{x}, \bar{y}) = (x^\alpha, y^\alpha)\ \exists \alpha \in \mathrm{Aut}(S^\delta)$.

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- $\Gamma_\delta(S)$ is vertex-transitive and edge-transitive.
- $\Gamma_\delta(S)$ has $\frac{|\mathrm{Aut}(S)|^\delta \delta!}{2}$ edges and $\frac{|\mathrm{Aut}(S)|^\delta \delta!}{\prod_{1 \leq i \leq u} \gamma_i^{\delta_i} \delta_i!}$ vertices, with $\gamma_i = |C_{\mathrm{Aut}(S)}(x_i)|$ for $x_i \in C_i$.

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For example $\Gamma_{19}(\mathrm{Alt}(5))$ is a graph with $2^{45} \cdot 3^{14} \cdot 5^9 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$ vertices, whose degree is equal to $2^{28} \cdot 3^{13} \cdot 5^{13} \cdot 7$.

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Let S be a finite non abelian simple group and let $\delta = \delta(S)$.

1. *If $n < \delta$, then the diameter of $\Gamma_n(S)$ is at most $2 + 4(n - 1)$.*
2. *The diameter of $\Gamma_\delta(S)$ is at most $4\delta + c$ for an absolute constant c.*
3. *The diameter of $\Gamma_\delta(S)$ is at most 4δ if $|S|$ is large enough.*

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Theorem (E. Crestani and Al (2011))

Let $S = \mathrm{SL}(2, 2^p)$ with p an odd prime. Then $\mathrm{diam}(\Gamma_\delta(S)) \geq 2^{p-2} - 1$ if p is large enough.

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If a graph has “many” edges, then by Turán’s Theorem, it should contain a “large” complete subgraph. Applying this result to $\Gamma(S)$ when S is a nonabelian simple group with large order, it follows:

Theorem (Liebeck and Shalev (1995))

There exists a positive constant c_1 such that $c_1 \cdot m(S) \leq \omega(\Gamma(S))$ for any finite simple group S where $m(S)$ denotes the minimal index of a proper subgroup in S.

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Theorem (A. Maroti and al (2009))

If $\delta = \delta(S)$, then $\omega(\Gamma(S^\delta)) = \omega(\Gamma_\delta(S))$ is at most $(1 + o(1))m(S)$, so $\omega(\Gamma_i(S))$ decreases drastically with i.

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- $G = \langle x, y \rangle \Rightarrow \{x, y, xy\}$ is a complete subgraph of $\Gamma(G)$.
- If G is 2-generated, then $\omega(\Gamma(G)) \geq 3$.

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Theorem

$$\omega(\Gamma_{19}(\mathrm{Alt}(5))) = 4.$$

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Assume that G is a 2-generated finite group and let $\sigma(G)$ denote the least number of proper subgroups of G whose union is G.

A set that generates G pairwise cannot contain two elements of any proper subgroup, hence $\omega(\Gamma(G)) \leq \sigma(G)$.

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A set that generates G pairwise cannot contain two elements of any proper subgroup, hence $\omega(\Gamma(G)) \leq \sigma(G)$.

In general $\omega(\Gamma(G)) \neq \sigma(G)$. $\omega(\Gamma(\mathrm{Alt}(5))) = 8$ and $\sigma(\mathrm{Alt}(5)) = 10$.

However no example is known of a finite 2-generated soluble non cyclic group G with $\omega(\Gamma(G)) \neq \sigma(G)$.

Theorem (E. Crestani and AL (2012))

If G is a finite, 2-generated, non cyclic, soluble group and $\mathcal{A}$ is the set of the chief factors G having more than one complement, then

$$\min_{A \in \mathcal{A}} (1 + |\operatorname{End}_G(A)|) \leq \omega(\Gamma(G)) \leq \sigma(G) = \min_{A \in \mathcal{A}} (1 + |A|).$$

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Corollary (A. Maroti and al (2009))

Let G be a finite soluble group with Fitting height at most 2. Then $\omega(\Gamma(G)) = \sigma(G)$.

Definition

$\mu_d(G)$:= the largest m for which there exists an m-tuple of elements of G such that any of its d entries generate G $\quad (\mu_2(G) = \omega(\Gamma(G)))$.

Theorem

Let G be a d-generated finite soluble group with $d \geq 2$ and let $\mathcal{A}$ be the set of the chief factors G having more than one complement. Assume that a positive integer t satisfies the following property:

1. $t \leq |A|$ *for each* $A \in \mathcal{A}$ *with* $C_G(A) = G$.
2. $\binom{t-1}{d-1} \leq |\mathrm{End}_G(A)|$ *for each* $A \in \mathcal{A}$ *with* $C_G(A) \neq G$.

Then $\mu_G(d) \geq t$.

Corollary

Let G be a d-generated finite group, with $d \geq 2$, and let p be the smallest prime divisor of the order of G. Then $\binom{\mu_d(G)}{d-1} > p$.

A question in linear algebra plays a crucial role in the study of the value of $\mu_d(G)$ when G is soluble.

$M_{r\times s}(F)$ the ring of the $r\times s$ matrices with coefficients over F.

Let $t\geq d$. Assume that $A_1,\dots,A_t\in M_{n\times n}(F)$ have the property that $\operatorname{rank}\begin{pmatrix}A_{i_1} & \cdots & A_{i_d}\end{pmatrix}=n$ whenever $1\leq i_1<i_2<\cdots<i_d\leq t$.

Can we find $B_1,\dots,B_t\in M_{n(d-1)\times n}(F)$ with the property that $\det\begin{pmatrix}A_{i_1} & \cdots & A_{i_d}\\ B_{i_1} & \cdots & B_{i_d}\end{pmatrix}\neq 0$ whenever $1\leq i_1<i_2<\cdots<i_d\leq t$?

Proof.

$$\begin{aligned} x &= (x_1, \dots, x_\delta) & \bar{x} &= (\bar{x}_1, \dots, \bar{x}_\delta) \\ y &= (y_1, \dots, y_\delta) & \bar{y} &= (\bar{y}_1, \dots, \bar{y}_\delta) \end{aligned}$$

$$\Downarrow$$

$$\begin{pmatrix} \bar{x}_{i\pi} \\ \bar{y}_{i\pi} \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix}^{a_i} \quad \exists \pi \in \mathrm{Sym}(\delta), (a_1, \dots, a_\delta) \in \mathrm{Aut}(S)^\delta$$

$$\Downarrow$$

$$(\bar{x}, \bar{y}) = (x^\alpha, y^\alpha) \text{ for } \alpha = (a_1, \dots, a_\delta)\pi \in \mathrm{Aut}(S^\delta).$$

□

TURÁN'S THEOREM

Let Γ be a graph with n vertices and e edges. If $\omega(\Gamma) \leq r$ then

$$e \leq \left(1 - \frac{1}{r}\right) \frac{n^2}{2}.$$

Assume that $\{g_1, \ldots, g_r\}$ is a complete subgraph of $\Gamma_\delta(S)$.

$S^\delta = \langle g_1, g_2 \rangle$ and for each $i \in \{3, \ldots, r\}$ there exists a word $w_i(x_1, x_2)$ such that $g_i = w_i(g_1, g_2)$.

$$S = \langle s_1, s_2 \rangle$$

$$\Downarrow$$

$\{s_1, s_2, w_3(s_1, s_2), \ldots, w_r(s_1, s_2)\}$ is a complete subgraph of $\Gamma(S)$