Bias of group generators in the solvable case

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ISCHIA GROUP THEORY 2014
April, 1st - 5th

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Let $Q_{G,n}$ be the probability distribution on G of the first components of n-tuples chosen uniformly from $\Phi_n(G)$.

EXAMPLE: $G = \mathrm{Sym}(3)$

$$\Phi_2(G) = \left\{ \begin{matrix} ((1,2),(1,2,3)), & ((1,3),(1,2,3)), & ((2,3),(1,2,3)) \\ ((1,2,3),(1,2)), & ((1,2,3),(1,3)), & ((1,2,3),(2,3)) \\ ((1,2),(1,3,2)), & ((1,3),(1,3,2)), & ((2,3),(1,3,2)) \\ ((1,3,2),(1,2)), & ((1,3,2),(1,3)), & ((1,3,2),(2,3)) \\ ((1,2),(1,3)), & ((1,2),(2,3)), & ((1,3),(2,3)) \\ ((1,3),(1,2)), & ((2,3),(1,2)), & ((2,3),(1,3)) \end{matrix} \right\}$$

$$Q_{G,2}(g) = \begin{cases} 0 & \text{if } g = 1 \\ \frac{4}{18} & \text{if } g = (i,j) \\ \frac{3}{18} & \text{if } g = (i,j,k) \end{cases}$$

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The product replacement algorithm (PRA) is a practical algorithm to construct random elements of a finite group. It was designed by Leedham-Green and Soicher (1995) to generate efficiently nearly uniform group elements.

Given a generating k-tuple, a move to another such k-tuple is defined by first uniformly selecting a pair (i, j) with $1 \leq i \neq j \leq k$ and then applying one of the following operations with equal probability:

$$R_{i,j}^{\pm} : (g_1, \ldots, g_i, \ldots, g_k) \mapsto (g_1, \ldots, g_i \cdot g_j^{\pm 1}, \ldots, g_k),$$
$$L_{i,j}^{\pm} : (g_1, \ldots, g_i, \ldots, g_k) \mapsto (g_1, \ldots, g_j^{\pm 1} \cdot g_i, \ldots, g_k).$$

To produce a random element in G, start with some generating k-tuple, apply the above moves several times, and finally return a random element of the generating k-tuple that was reached.

The moves in the PRA can be conveniently encoded by the PRA graph $\Gamma_k(G)$ whose vertices are the tuples in $\Phi_k(G)$, with edges corresponding to the moves $R^{\pm}_{i,j}, L^{\pm}_{i,j}$.

If k is large enough, then the graph $\Gamma_k(G)$ is connected.

The algorithm consists of running a nearest neighbor random walk on this graph and returning a random component.

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For the product replacement algorithm to generate "random" group elements, it is necessary that $Q_{G,k}$ be close to U_G, the uniform distribution on G. Even if the graph $\Gamma_k(G)$ is connected, even if the product replacement random walk mixes rapidly, the resulting distribution of the output can still be biased.

We want to estimate the bias of the distribution $Q_{G,t}$ considering the variation distance between $Q_{G,t}$ and U_G.

$$\|Q_{G,t} - U_G\|_{\text{tv}} = \max_{B \subseteq G} |Q_{G,t}(B) - U_G(B)| = \frac{1}{2} \sum_{g \in G} \left| Q_{G,t}(g) - \frac{1}{|G|} \right|.$$

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$0 \leq \|Q_{G,t} - U_G\|_{\mathrm{tv}} \leq 1$ and $\|Q_{G,t} - U_G\|_{\mathrm{tv}} = 0$ if and only if $Q_{G,t} = U_G$.

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$$= \frac{1}{2} \left(\left| 0 - \frac{1}{6} \right| + 3 \left| \frac{4}{18} - \frac{1}{6} \right| + 2 \left| \frac{3}{18} - \frac{1}{6} \right| \right) = \frac{1}{6}.$$

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Theorem (Babai and Pak, 2000)

Let $G = \mathrm{Alt}(n)^{n!/8}$: if $n \geq 5$, then G is 2-generated but, for $t \geq 4$, the variation distance $\left\|Q_{G,t} - U_G\right\|_{tv}$ tends to 1 as $n \to \infty$.

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For every $N \in \mathcal{N}$ two probability distributions $Q_{G/N,t}$ and $U_{G/N}$ are defined on the quotient group G/N: this allows us to consider G as a measure space obtained as an inverse system of finite probability spaces in two different ways.

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One of the two measures obtained in this way is the usual normalized Haar measure μ_G. The other measure $\kappa_{G,t}$ has the property that $\kappa_{G,t}(X) = \inf_{N \in \mathcal{N}} Q_{G/N,t}(XN/N)$ for every closed subset X of G.

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We estimate the bias of the measure $\kappa_{G,t}$ considering

$$\|\kappa_{G,t} - \mu_G\|_{\text{tv}} = \sup_{B \in \mathcal{B}(G)} |\kappa_{G,t}(B) - \mu_G(B)|$$

where $\mathcal{B}(G)$ is the set of the measurable subsets of G.

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Pak proposed the following open problem: can one exhibit the bias for a sequence of finite solvable groups?

In other words can we produce a sequence (B_n, H_n) where H_n is a t-generated finite solvable group and B_n is a subset of H_n, such that $|B_n|/|H_n| \to 1$ and $|Q_{H_n,t}(B_n)| \to 0$ as $n \to \infty$?

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Equivalently does there exist a t-generated prosolvable group G with $\|\kappa_{G,t} - \mu_G\|_{\mathrm{tv}} = 1$?

It is not difficult to give an affirmative answer in the particular case when $t = d(G)$.

If $G = \hat{\mathbb{Z}}$, then $d(G) = 1$ but the probability of generating G with 1 element is 0.

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THEOREM (E. CRESTANI, A. L. 2013)

There exists a 2-generated metabelian profinite group G with the property that

$$\mu_G(\{x \in G \mid \langle x, y \rangle = G \text{ for some } y \in G\}) = 0.$$

In particular $\|\kappa_{G,2} - \mu_G\|_{tv} = 1$.

Sketch of the proof.

Let $\{p_n\}_{n\in\mathbb{N}}$ be the sequence of the odd primes in increasing order and let $A = \{1, y_1, y_2, y_3\}$ be an elementary abelian group of order 4.

$$H_m = (\langle x_1\rangle \times \langle x_2\rangle \times \langle x_3\rangle) \rtimes A$$

where $\langle x_1\rangle, \langle x_2\rangle, \langle x_3\rangle$ are cyclic groups of order $p_1 \cdots p_m$ and

$$x_i^{y_j} = x_i^{-1} \quad \text{if } i \neq j \quad \text{and} \quad [x_i, y_i] = 1 \quad \text{otherwise.}$$

$(x_1^{n_1}, x_2^{n_2}, x_3^{n_3})y_j$ belongs to a generating pair $\iff (n_j, p_1 \cdots p_m) = 1$.

Let π_m be the probability that an element of H_m appears in a generating pair:

$$\pi_m = \frac{3\left(\prod_{1\leq i\leq m}(p_i - 1)p_i^2\right)}{4\left(\prod_{1\leq i\leq m} p_i^3\right)} = \frac{3}{4}\left(\prod_{1\leq i\leq m}\left(1 - \frac{1}{p_i}\right)\right).$$

□

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If G is a t-generated profinite group then $\|\kappa_{G,t} - \mu_G\|_{\text{tv}} \leq 1 - P_G(t)$ being $P_G(t)$ the probability that t randomly chosen elements in G generate G.

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If we consider arbitrary profinite groups, this does not represent a serious obstacle: for example if G is the free profinite group or rank $d \geq 2$ then $P_G(t) = 0$ for every $t \geq d$.

The situation is different in the case of finitely generated prosolvable groups: $P_G(t) > 0$ whenever G is a finitely generated prosolvable group and $t \geq c(d(G) - 1) + 1$, with $c \backsimeq 3.243$, the Pàlfy-Wolf constant.

So if G is a t-generated prosolvable group and $\|\kappa_{G,t} - \mu_G\|_{\text{tv}} = 1$, then the difference $t - d(G)$ cannot be arbitrarily large.

However we can construct examples of prosolvable t-generated groups G with $\|\kappa_{G,t} - \mu_G\|_{\mathrm{tv}} = 1$ and where the difference $t - d(G)$ tends to infinity as $d(G) \to \infty$.

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Let $d \in \mathbb{N}$ with $d \geq 3$. There exists a finitely generated prosolvable group G such that $d(G) = d$ and $\|\kappa_{G,d(G)+k} - \mu_G\|_{tv} = 1$ for every k such that $2k \leq d(G) - 3$.

Sketch of the proof

$\mathrm{Sym}(4^{m+1})$ contains a 2-generated transitive subgroup

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- $V = \mathrm{soc}(T_m)$ is an absolutely irreducible T_m-modulo of order $q = 4^{4^m}$ and has a complement X_m in T_m.
- $T_m \leq \mathrm{Sym}(4) \wr T_{m-1}$ and $V \cong (C_2 \times C_2)^{4^m} \leq (\mathrm{Sym}(4))^{4^m}$.

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Roughly speaking, T_m is as large as possible compatibly with the property of being 2-generated. In particular $|X_m| > |V|^2$.

Let $d \geq 3$ and let F be the free group of rank d.

There exists $\Phi \subseteq \mathrm{Epi}(F, X_m)$ such that

- $|\Phi| > q^{2(d-2)}$;
- different elements of Φ have different kernels;
- $x \notin \bigcap_{\phi \in \Phi} \ker(\phi) \Rightarrow C_V(x^{\phi}) \neq 0$ for at least $|\Phi|/4$ choices of ϕ.

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$$H_m := \frac{F}{\bigcap_{\phi \in \Phi} \ker(\phi)}$$

To every $\phi \in \Phi$ there corresponds an absolutely irreducible H_m-module V_ϕ of cardinality q (we identify V_ϕ with V and we set $v \cdot h := v^{h^{\phi}}$).

$$G_m := \left(\prod_{\phi \in \Phi} V_\phi^n \right) \rtimes H_m \quad \text{with } n = 2 \cdot 4^m (d-1).$$

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It turns out that $d(G_m) = d$.

- Let $W_\phi = V_\phi^n$, $\quad W = \prod_\phi W_\phi \cong \mathrm{soc}(G_m)$, $\quad G_m = \prod_\phi W_\phi \rtimes H$.
- Let h be a fixed element in H_m : $(v_1, \ldots, v_n) \in W_\phi$ is h-positive if $\langle [V_\phi, h], v_1, \ldots, v_n \rangle = V_\phi$, h-negative otherwise.
- For $a \leq \nu = 2 \cdot 4^m$, let $\Sigma_a := \{\phi \mid \dim C_{V_\phi}(h) = a\}$, $\sigma_a := |\Sigma_a|$.
- Let $U_a = \prod_{\phi \in \Sigma_a} W_\phi$ and for any $u = (w_1, \ldots, w_{\sigma_a}) \in U_a$, let $\gamma_a(u)$ be the number of $i \in \{1, \ldots, \sigma_a\}$ such that w_i is h-negative.

Take $w \in W$ and write $w = (u_0, \ldots, u_\nu)$ with $u_a \in U_a$:

$$\frac{Q_{G_m,d+k}(wh)}{Q_{H_m,d+k}(h)} \leq \frac{\prod_{a \neq 0} \rho_a}{|W|} \quad \textit{with} \quad \rho_a = \frac{\left(1 - \frac{1}{q^k}\right)^{\gamma_a(u_a)}}{\prod_{0 \leq i \leq a-1} \left(1 - \frac{2^i}{q^{d+k-1}}\right)^{\sigma_a}}.$$

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The complete discussion requires standard probability estimates for large deviations in Bernoulli trials: there exists $a \neq 0$ such that σ_a is large and ρ_a is small for almost all $u_a \in U_a$.

Let G be a finitely generated pronilpotent group and let $t \geq d(G) = d$: $\|\kappa_{G,t} - \mu_G\|_{\mathrm{tv}} = 1$ if and only if $t = d = 1$ and $\sum_{p||G|} \frac{1}{p} = \infty$.

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Let G be a finitely generated pro-p-group: if $t \geq d(G) = d$ then

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Does there exist a finitely generated prosupersolvable group G with $\|\kappa_{G,t} - \mu_G\|_{\text{tv}} = 1$ for some $t > d(G)$?

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