Rings as the unions of proper subrings

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Una giornata per Silvia
Padova, March 26 - 27, 2010

Exercize

No group is the union of two of its proper subgroups.

Scorza 1926

A group G is a union of three of its pairwise distinct proper subgroups A, B, C if and only if

- A, B, C have index 2 in G;
- $G/(A \cap B \cap C)$ is isomorphic to the Klein four group.

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Questions

Are there similar results for rings?

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- no non-trivial ideal of R is contained in $S_1 \cap S_2 \cap S_3$.

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Definition

A 4-tuple (R, S_1, S_2, S_3) of rings is good if S_1, S_2, S_3 are proper subrings of the ring R so that $R = S_1 \cup S_2 \cup S_3$ and that no non-trivial ideal of R is contained in $S_1 \cap S_2 \cap S_3$.

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THEOREM (ATTILA MAROTI - AL 2009)

All good 4-tuples of rings are completely described by the following ten Examples.

Example 1

$$R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \leq M_2(\mathbb{Z}/2\mathbb{Z}).$$

- A commutative ring of order 4 with a multiplicative identity.

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- R is the direct product of two fields of order 2.

Example 2

$$\left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\} \leq M_3(\mathbb{Z}/2\mathbb{Z})$$

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Example 5

$$R = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{Z}/2\mathbb{Z} \right\} \leq M_3(\mathbb{Z}/2\mathbb{Z})$$

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- $S_1 \cap S_2 \cap S_3 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \middle| a \in \mathbb{Z}/2\mathbb{Z} \right\}$ is a subring of R, but it is not an ideal.

EXAMPLE 6

$$R = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & a & 0 \\ c & 0 & a \end{pmatrix} \middle| a, b, c \in \mathbb{Z}/2\mathbb{Z} \right\} \leq M_3(\mathbb{Z}/2\mathbb{Z})$$

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- The subset of R obtained by imposing the restriction $a = 0$ is isomorphic to the ring R of Example 2.
- R can be obtained from Example 2 by adding a multiplicative identity 1 and imposing the relation $1 + 1 = 0$.

Example 7

$$R := \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \middle| a, b, c \in \mathbb{Z}/2\mathbb{Z} \right\} \leq M_2(\mathbb{Z}/2\mathbb{Z})$$

- A non-commutative ring of order 8 containing a multiplicative identity.

Example 7

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- R can be obtained from Examples 3 or 4 by adding a multiplicative identity 1 and imposing the relation $1 + 1 = 0$.

Examples 8 and 9

$$R = \left\{ \begin{pmatrix} 0 & b & c & d \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix} \middle| b, c, d \in \mathbb{Z}/2\mathbb{Z} \right\} \leq M_4(\mathbb{Z}/2\mathbb{Z}).$$

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- $(R, +) = \langle 1, x, y, z \rangle \cong (\mathbb{Z}/2\mathbb{Z})^4$ and
  - $z^2 = z$,
  - $x^2 = y^2 = xy = yx = 0$,
  - $xz = x, yz = y, zx = 0, zy = 0.$
- $S_1 = \langle 1, x, z \rangle, \quad S_2 = \langle 1, y, z \rangle, \quad S_3 = \langle 1, x + y, z \rangle$

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But this is not the case of Example 6.

Theorem

A ring R is the union of three of its proper subrings if and only if there exists a factor ring (of order 4 or 8) of R which is isomorphic to ring of examples 1, 2, 3, 4 or 6.

How can we prove our result?

Let (R, S_1, S_2, S_3) be a good 4-tuple of rings and let $S = S_1 \cap S_2 \cap S_3$.

- Scorza's Theorem $\Rightarrow |R : S_i| = 2$ for each $i \in \{1, 2, 3\}$ and $S_1 \cap S_2 = S_1 \cap S_3 = S_2 \cap S_3 = S$.
- $2R \subseteq S_1 \cap S_2 \cap S_3 \;\Rightarrow\; 2R = 0$.

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- S is a subring of R with $|R : S| = 4$.
- [J. Lewin (1967)] Suppose that a ring R contains a subring S of finite index. Then there exists an ideal I of R, contained in S and of finite index in R.
- R is finite ($|R| \leq 5^{25}$).

A useful remark

- Let M be a ring (with or without a multiplicative identity) with $2M = 0$.
- Consider the abelian group $M^* = M \oplus \langle u \rangle$ with $u + u = 0$ and define a multiplication on M^* by setting u to be the identity on M^*.
- M^* becomes a ring with a multiplicative identity.

A USEFUL REMARK

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- Consider the abelian group $M^* = M \oplus \langle u \rangle$ with $u + u = 0$ and define a multiplication on M^* by setting u to be the identity on M^*.
- M^* becomes a ring with a multiplicative identity.

Let (R, S_1, S_2, S_3) be a good 4-tuple . Suppose that R has no multiplicative identity. Then $(R^*, S_1^*, S_2^*, S_3^*)$ is also a good 4-tuple where a unique multiplicative identity was added to the four rings R, S_1, S_2, and S_3.

We may restrict our attention to rings with a multiplicative identity.

Proposition

Let R be a good ring. Then $|R| = 4$, 8, or 16.

Proof

- There exists x and y such that $R = S \oplus \{x, y, x + y, 0\}$, $S_1 = S \oplus \{x, 0\}$, $S_2 = S \oplus \{y, 0\}$, $S_3 = S \oplus \{x + y, 0\}$.

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$$sx \in S,\ sy \notin S$$

$$\Downarrow$$

$$sx = s_1 \ \exists\ s_1 \in S,\ sy = s_2 + y \ \exists\ s_2 \in S$$

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$$s(x+y) = s_1 + s_2 + y \notin S_3$$

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- S_R and S_T are subgroups of S with index at most 2, so T is a subgroup of S with $|S : T| \in \{1, 2, 4\}$.
- $t \in T \Rightarrow xty, ytx, xtx, yty \in S$

$$\begin{cases} xt \in S \Rightarrow xty \in S_2 \\ ty \in S \Rightarrow xty \in S_1 \end{cases} \Rightarrow \quad xty \in S_1 \cap S_2 = S$$

$$\begin{cases} xtx = s_1 + bx \\ xty = s_2 \end{cases} \Rightarrow xt(x+y) = s_1 + s_2 + bx \in S_3 \quad b = 0$$

Proposition

Let R be a good ring. Then $|R| = 4$, 8, or 16.

Proof

- There exists x and y such that $R = S \oplus \{x, y, x+y, 0\}$, $S_1 = S \oplus \{x, 0\}$, $S_2 = S \oplus \{y, 0\}$, $S_3 = S \oplus \{x+y, 0\}$.
- $\forall s \in S$ we have: $sx \in S \Leftrightarrow sy \in S$, $xs \in S \Leftrightarrow ys \in S$,
- $S_R := \{s \in S \mid sx \in S\}$, $S_L := \{s \in S \mid xs \in S\}$, $T := S_L \cap S_R$.
- S_R and S_T are subgroups of S with index at most 2, so T is a subgroup of S with $|S : T| \in \{1, 2, 4\}$.
- $t \in T \Rightarrow xty, ytx, xtx, yty \in S$
- $RTR \subseteq S \Rightarrow RTR = 0 \Rightarrow T = 0 \Rightarrow |S| \in \{1, 2, 4\}$.

Proposition

Let (R, S_1, S_2, S_3) be a good 4-tuple of rings. If R contains a multiplicative identity 1, then either $|R| = 4$ or $1 \in S_1 \cap S_2 \cap S_3$.

Proposition

Let (R, S_1, S_2, S_3) be a good 4-tuple of rings. If R contains a multiplicative identity 1, then either $|R| = 4$ or $1 \in S_1 \cap S_2 \cap S_3$.

Proof

$$1 \notin S$$

$$\Downarrow$$

$$\text{we may assume} \quad 1 \notin S_1,\ 1 \notin S_2$$

$$\Downarrow$$

$$S_1 \text{ and } S_2 \text{ are ideals of } R$$

$$\Downarrow$$

$$S = S_1 \cap S_2 \text{ is an ideal of } R$$

$$\Downarrow$$

$$S = \{0\}$$

Thanks

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thanks to Silvia, for the time she spent with us