Comparing the subgroup and the probabilistic zeta function

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G a finitely generated profinite group

$a_n(G)$ the number of subgroups of index n in G

$b_n(G) := \sum_{|G:H|=n} \mu(H, G)$

μ is the Möbius function of the subgroup lattice of G :

$$\mu(H, G) = \begin{cases} 1 & \text{if } H = G \\ -\sum_{H<K\leq G} \mu(K, G) & \text{otherwise} \end{cases}$$

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Subgroup zeta function

$$\zeta_G(s) = \sum_{n \in \mathbb{N}} \frac{a_n(G)}{n^s}$$

Probabilistic zeta function

$$p_G(s) = \sum_{n \in \mathbb{N}} \frac{b_n(G)}{n^s}$$

If $G = \hat{\mathbb{Z}}$, then

- $\zeta_G(s) = \sum_n \frac{1}{n^s} = \zeta(s)$ the Riemann zeta function
- $p_G(s) = \sum_n \frac{\mu(n)}{n^s} = (\zeta(s))^{-1}$

Remark

$\mu(H, G) \neq 0 \Rightarrow$ H is an intersection of maximal subgroups of G.

The probabilistic zeta function $p_G(s)$ depends only on the sublattice generated by the maximal subgroups.

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The probabilistic zeta function encodes information about the lattice generated by the maximal subgroups of G, just as the Riemann zeta function encodes information about the primes.

Question

$\zeta_G(s)$ and $p_G(s)$ can be considered as formal Dirichlet series. Do they converge in some part of the complex plane?

$\zeta_G(s)$ converges in a suitable half plane if and only if G has Polynomial Subgroup Growth.

The question whether $p_G(s)$ converges in a half plane seems to be related with the probabilistic meaning of $p_G(s)$ and with the behaviour of the maximal subgroup growth of G.

Let ν be the normalized Haar measure on G or on some direct power G^t; we define the probability that t random elements generate G as:

$$\mathrm{Prob}_G(t) = \nu\left(\left\{ (g_1, \dots, g_t) \in G^t \,\middle|\, \overline{\langle g_1, \dots, g_t \rangle} = G \right\}\right)$$

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If G is finite and $t \in \mathbb{N}$, then $p_G(t) = \mathrm{Prob}_G(t)$.

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Definition

G is Positively Finitely Generated when $\mathrm{Prob}_G(t) > 0$ *for some* $t \in \mathbb{N}$.

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Theorem (Mann - Shalev)

G is PFG if and only if G has polynomial maximal subgroup growth.

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Conjectures

- *$p_G(s)$ converges in a suitable half plane iff G is PFG.*
- *If G is PFG, then $p_G(t) = \mathrm{Prob}_G(t)$ for $t \in \mathbb{N}$ large enough.*

Finitely generated (virtually) prosolvable groups are PFG and satisfy the previous conjectures.

If $G = \hat{\mathbb{Z}}$, then $\zeta_G(s)p_G(s) = 1$.

PROBLEM

To study the finitely generated profinite groups G satisfying the condition

$$\zeta_G(s)p_G(s) = 1$$

DEFINITION

Just for this talk, we will say that G is ζ-reversible if $\zeta_G(s)p_G(s) = 1$.

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If G is a finite group and $t \in \mathbb{N}$ (and more in general if G has PSG and t is large enough) then

$$\sum_{H \leq_o G} \frac{\mathrm{Prob}_H(t)}{|G : H|^t} = 1.$$

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$$\sum_{H \leq_o G} \frac{\mathrm{Prob}_H(t)}{|G:H|^t} = 1.$$

Independently of the convergency properties of the series $p_H(s)$ and their probabilistic meaning, the following formal identity holds:

$$\sum_{H \leq_o G} \frac{p_H(s)}{|G:H|^s} = 1.$$

$$\zeta_G(s)p_G(s) = 1$$

$$\Updownarrow$$

$$\sum_{H \leq_o G} \frac{p_G(s)}{|G:H|^s} = 1 = \sum_{H \leq_o G} \frac{p_H(s)}{|G:H|^s}$$

$$\Updownarrow$$

$$\sum_{H \leq_o G} \frac{p_G(s) - p_H(s)}{|G:H|^s} = 0$$

Corollary

$p_G(s) = p_H(s)$ for each open subgroup H of $G \Rightarrow G$ is ζ-reversible.

EXAMPLES

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- If $H \cong G$ for each $H \leq_o G$, then G is ζ-reversible. This occurs if and only if G is abelian and torsion free.

  For example if $G = \widehat{\mathbb{Z}}^r$ then

$$\zeta_G(s) = \zeta(s)\zeta(s-1)\cdots\zeta(s-(r-1)) = (p_G(s))^{-1}$$

EXAMPLES

- If $H \cong G$ for each $H \leq_o G$, then G is ζ-reversible. This occurs if and only if G is abelian and torsion free.
- Let $d(G)$ be the smallest cardinality of a generating set of G. If G is a pro-p group then

$$p_G(s) = \prod_{0 \leq i \leq d(G)-1} \left(1 - \frac{p^i}{p^s}\right)$$

depends only on $d(G)$. If G is a pro-p group with $d(G) = d(H)$ for each $H \leq_o G$, then G is ζ-reversible.

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Non abelian examples. The pro-p group G with the presentation

$$\langle x_1, \ldots, x_r, y \mid [x_i, x_j] = 1, [x_i, y] = x_i^{p^t} \rangle$$

satisfies $d(H) = r + 1 \ \forall \ H \leq_o G$.

Non pronilpotent examples

For any $m \in \mathbb{Z}$, $m \neq 0$, let G_m be the profinite completion of the Baumslag-Solitar group $B_m = \langle a, b \mid a^{-1}ba = b^m \rangle$.

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- $p_{G_m}(s) = \prod_p \left(1 - \frac{1}{p^s}\right) \prod_{(p,m)=1} \left(1 - \frac{p}{p^s}\right)$
- $H \leq_o G_m \Rightarrow H \cong G_{m^u}$ for some $u \in \mathbb{N} \Rightarrow p_H(s) = p_{G_m}(s)$.

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$$\zeta_{G_m}(s) = (p_{G_m}(s))^{-1} = \zeta(s)\zeta(s-1) \prod_{p|m} \left(1 - \frac{p}{p^s}\right)$$

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$$\zeta_{G_m}(s) = (p_{G_m}(s))^{-1} = \zeta(s)\zeta(s-1)\prod_{p|m}\left(1 - \frac{p}{p^s}\right)$$

A direct computation of $\zeta_{G_m}(s)$ is due to Gelman (2005).

G_m is virtually pronilpotent if and only if $m = \pm 1$.

How can we compute $p_{G_m}(s)$?

Let $\{N_i\}_i$ be a chain of open normal subgroups of G with $\bigcap_i N_i = 1$ and N_i/N_{i+1} a chief factor of G/N_{i+1} for each i (a chief series of G).

For each i consider the finite Dirichlet series

$$p_i(s) = \sum_n \frac{b_i(n)}{n^s} \quad \text{with} \quad b_i(n) = \sum_{\substack{N_{i+1} \leq H \\ HN_i = G \\ |G:H| = n}} \mu(H, G)$$

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$p_G(s)$ can be written as an infinite formal product of these finite series:

$$p_G(s) = \prod_i p_i(s)$$

HOW CAN WE COMPUTE $p_{G_m}(s)$?

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If G is prosolvable, then N_i/N_{i+1} is abelian and $p_i(s) = 1 - c_i/q_i^s$ with $q_i = |N_i/N_{i+1}|$, c_i the number of complements of N_i/N_{i+1} in G/N_{i+1}.

Let G_m be the profinite completion of the Baumslag-Solitar group B_m.

- H a finite epimorphic image of $G_m \Rightarrow \exists K \trianglelefteq H$ such that
  - K and H/K are cyclic,
  - K is complemented in H,
  - $(|K|, m) = 1$.
- p divides $m \Rightarrow$ in a chief series of G_m there is only one complemented p-factor, it is central and the corresponding finite Dirichlet series is $1 - 1/p^s$.
- p does not divide $m \Rightarrow$ in a chief series of G there are 2 complemented p-factors, both are cyclic of order p, one has only 1 complement, the other has p complements, the product of the the corresponding finite Dirichlet series is $(1 - 1/p^s)(1 - p/p^s)$.
- $p_{G_m}(s) = \prod_p \left(1 - \frac{1}{p^s}\right) \prod_{(p,m)=1} \left(1 - \frac{p}{p^s}\right)$

A final remark about our examples

Let G be the group with the following profinite presentation:

$$G = \langle x_1, \dots, x_r, y \mid [x_i, x_j] = 1, x_i^y = x_i^m \rangle.$$

- G is ζ-reversible
- $\zeta_G(s) = \zeta(s)\zeta(s-1)\cdots\zeta(s-r)\prod_{p|m}\left(1 - \frac{p^r}{p^s}\right)$
- If $A = \widehat{\mathbb{Z}}^r \times \prod_{(p,m)=1} \mathbb{Z}_p$, then $\zeta_G(s) = \zeta_A(s)$ and $p_G(s) = p_A(s)$.

All the examples that we have presented can be obtained as epimorphic images of G, for suitable r and m. Are there different examples?

Question

Do there exist ζ-reversible groups that are not prosolvable?

- G ζ-reversible $\Rightarrow$ the coefficients of $(p_G(s))^{-1}$ are non negative.
- G prosolvable $\Rightarrow$ $p_G(s) = \prod_i (1 - \frac{c_i}{q_i^s})$ $\Rightarrow$

$$(p_G(s))^{-1} = \prod_i \left(1 + \frac{c_i}{q_i^s} + \frac{c_i^2}{q_i^{2s}} + \dots\right)$$

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Prosolvable groups

If G is a finitely generated prosolvable group then the series $p_G(s) = \sum_n b_n(G)/n^s$ satisfies the following properties (which are preserved under inversion):

- the sequence $\{b_n(G)\}_n$ has polynomial growth.
- $p_G(s)$ has an Euler factorization over the prime numbers:

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This is equivalent to say that the sequence $\{b_n(G)\}_{n\in\mathbb{N}}$ is multiplicative, i.e. $b_{rs}(G) = b_r(G)b_s(G)$ whenever $(r, s) = 1$.

PROSOLVABLE GROUPS

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CONSEQUENCES

If G is a ζ-reversible prosolvable group, then

- G has Polynomial Subgroup Growth, hence it has finite rank.
- $\zeta_G(s)$ has an Euler factorization over the prime numbers.

Remark

- (Detomi, AL 2004) The probabilistic zeta function $p_G(s)$ has an Euler factorization if and only if G is prosolvable.
- It is still open the problem of characterizing the finitely generated profinite groups G whose subgroup zeta function $\zeta_G(s)$ has an Euler factorization.
- If G is pronilpotent, then $\zeta_G(s)$ has an Euler factorization. The only other known examples of groups whose subgroup zeta function has an Euler factorization come from the profinite completions of the Baumslag-Solitar groups described before.

Proposition

Let G be a ζ-reversible prosolvable group and let π be the set of the prime divisors of the order of G. For each π-number n, G contains an open subgroup of index n.

Proof

- It suffices to prove: $a_{p^m}(G) \neq 0 \ \forall p \in \pi, \ \forall \ m \in \mathbb{N}$.

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- Fix p: G contains an open subgroup of index pu, with $(u, p) = 1$.
- $a_{pu}(G) \neq 0$ and $p_G(s)\zeta_G(s) = 1 \Rightarrow b_{pv}(G) \neq 0 \, \exists \; v$ with $(v, p) = 1$.

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- $p_G(s) = \prod_i (1 - c_i/q_i^s)$ with $c_i \geq 0$ and q_i prime-powers. Since $b_{pv}(G) \neq 0$ it must be $q_i = p$ for some i.

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- $$(p_G(s))^{-1} = (1 - c_i/p^s)^{-1}\left(\prod_{j \neq i}(1 - c_j/q_j^s)\right)^{-1}$$
  $$= (1 + c_i/p^s + c_i^2/p^{2s} + \dots)\left(\sum_m d_m/m^s\right) \text{ with } d_m \geq 0.$$

Proposition

Assume that G is a ζ-reversible prosolvable group of rank 2. Then

- *G is prosupersolvable;*
- *for each prime divisor p of $|G|$, G contains a normal subgroup of index p;*
- $p_G(s) = p_H(s)$ *for each* $H \leq_o G$.

Pro-p-groups

Assume that G is a ζ-reversible pro-p-group (G must be p-adic analytic):

$$\sum_{H \leq_o G} \frac{\prod_{0 \leq i \leq d(G)-1}(1 - \frac{p^i}{p^s}) - \prod_{0 \leq i \leq d(H)-1}(1 - \frac{p^i}{p^s})}{|G : H|^s} = 0.$$

Does this imply $d(G) = d(H)$ for each $H \leq_o G$?

Partial answers

If $d(H) \neq d(G)$ for some $H \leq_o G$, then

- $d(H) \neq d(G)$ for infinitely many open subgroups H of G.
- If r is minimal with respect to the property that there exists H with $d(H) \neq d(G)$ and $|G : H| = p^r$, then there exist H_1 and H_2 with $|G : H_1| = |G : H_2| = p^r$ and $d(H_1) < d(G) < d(H_2)$.
- G does not contain pro-cyclic open subgroups.
- $d(G) > 2$.

If X acts on G, then we may consider the lattice of the open X-subgroups of G and the Möbius function μ^X in this lattice.

We may define:

- $a_G^X(n)$: the number of X-subgroups of G with index n.
- $b_G^X(n) := \sum_{H \leq_X G, |G:H|=n} \mu^X(H, G)$

and the corresponding zeta functions: $\zeta_G^X(s)$ and $p_G^X(s)$.

"Normal" variation

If $X = G$ then

- $\zeta_G^X(s) = \zeta_G^{\triangleleft}(s)$ the normal subgroup zeta function
- $p_G^X(s) = p_G^{\triangleleft}(s)$ the normal probabilistic zeta function

If $\sum_{H \trianglelefteq_o G} \mu^{\triangleleft}(H, G)|G : H|^{-s}$ is absolutely convergent and $k \in \mathbb{N}$ is large enough, then $p_G^{\triangleleft}(k)$ gives the probability that the smallest closed normal subgroup containing k random elements is G.

Normal ζ-reversible groups

What can we say about a profinite group G with the property

$$\zeta_G^{\triangleleft}(s)p_G^{\triangleleft}(s) = 1?$$

Normal ζ-reversible groups

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$$\zeta_G^{\triangleleft}(s)p_G^{\triangleleft}(s) = 1 \Leftrightarrow \sum_{H \trianglelefteq_o G} \frac{p_G^G(s) - p_H^G(s)}{|G:H|^s} = 0$$

For any finite simple group T, let $m(T)$ be the largest m such that T^m is an epimorphic image of G :

$$p^{\lhd}_G(s) = p^{\lhd}_{G,ab}(s)p^{\lhd}_{G,nonab}(s) \quad \text{with}$$

$$p^{\lhd}_{G,ab}(s) = \prod_p \left(\prod_{0 \le i \le m(C_p)-1} \left(1 - \frac{p^i}{p^s} \right) \right) = p_{G/G'}(s)$$

$$p^{\lhd}_{G,nonab}(s) = \prod_{T \text{ non abelian}} \left(1 - \frac{1}{|T|^s} \right)^{m(T)}$$

Proposition

If $\zeta_G^{\triangleleft}(s)p_G^{\triangleleft}(s) = 1$ and no nonabelian simple group appears as an epimorphic image of G, then G is pronilpotent.

Proof

- No nonabelian simple group is an epimorphic image of G

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Proof

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$$\Downarrow$$

- $p_G^{\triangleleft}(s) = p_{G,ab}^{\triangleleft}(s)$ has an Euler factorization over the primes

$$\Downarrow$$

PROPOSITION

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PROOF

- No nonabelian simple group is an epimorphic image of G

$$\Downarrow$$

- $p_G^{\triangleleft}(s) = p_{G,ab}^{\triangleleft}(s)$ has an Euler factorization over the primes

$$\Downarrow$$

- $\zeta_G^{\triangleleft}(s)$ has an Euler factorization over the primes

$$\Downarrow$$

Proposition

If $\zeta_G^{\triangleleft}(s)p_G^{\triangleleft}(s) = 1$ and no nonabelian simple group appears as an epimorphic image of G, then G is pronilpotent.

Proof

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- the normal subgroup growth of G is multiplicative

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Proposition

If $\zeta_G^{\lhd}(s)p_G^{\lhd}(s) = 1$ and no nonabelian simple group appears as an epimorphic image of G, then G is pronilpotent.

Proof

- No nonabelian simple group is an epimorphic image of G

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- $p_G^{\lhd}(s) = p_{G,ab}^{\lhd}(s)$ has an Euler factorization over the primes

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$$\Downarrow$$

- the normal subgroup growth of G is multiplicative

$$\Downarrow$$

- (J.C. Puchta 2001) G is pronilpotent

A more general result

Let $(p^{\triangleleft}_{G,ab}(s))^{-1} = \sum_n \gamma_n/n^s$. If $\zeta^{\triangleleft}_G(s)p^{\triangleleft}_G(s) = 1$, then γ_n is the number of normal subgroups N of G with G/N a nilpotent group of order n.

Question

Do there exist non pronilpotent groups G satisfying $\zeta^{\triangleleft}_G(s)p^{\triangleleft}_G(s) = 1$?

For any set π of prime numbers let G_π be the largest epimorphic image G which is a π group.

Remark

$\zeta_G^{\triangleleft}(s)p_G^{\triangleleft}(s) = 1 \Leftrightarrow \zeta_{G_\pi}^{\triangleleft}(s)p_{G_\pi}^{\triangleleft}(s) = 1$ *for each finite set* π *of primes.*

- When we study the groups G with $\zeta_G^{\triangleleft}(s)p_G^{\triangleleft}(s) = 1$, it is not restrictive to assume that $|G|$ is divisible only by finitely many primes.
- This implies that $p_G^{\triangleleft}(s)$ is a finite Dirichlet series and consequently that G has polynomial normal subgroup growth.

Proposition

Assume that all the nonabelian composition factors of G are alternating groups. If $\zeta_G^{\triangleleft}(s)p_G^{\triangleleft}(s) = 1$, then G is pronilpotent.

Proposition

Assume that G is a perfect profinite group. If $\zeta_G^{\triangleleft}(s)p_G^{\triangleleft}(s) = 1$, then there exists two simple groups S and T and an irreducible T-module V such that

- *S and $V \rtimes T$ are epimorphic images of G.*
- $|S| < |T| < |S|^2$.
- $|S|^2 = |T||V|$.

There are only finitely many possibilities for (S, T, V).

$$\zeta_G^{\lhd}(s)p_G^{\lhd}(s) = 1 \Leftrightarrow \sum_{H \unlhd_o G} \frac{p_G^G(s) - p_H^G(s)}{|G:H|^s} = 0$$

Proposition

Assume that the following stronger property holds:

$$\sum_{|G/H|=n} \frac{p_G^G(s) - p_H^G(s)}{|G:H|^s} = 0 \text{ for each } n \in \mathbb{N}.$$

Then G is pronilpotent.

$$\sum_{H\leq_o G} \frac{p_H(s)}{|G:H|^s} = \sum_{H\leq_o G} \frac{\left(\sum_{K\leq_o H} \frac{\mu(K,H)}{|H:K|^s}\right)}{|G:H|^s}$$
$$= \sum_{K\leq_o H\leq_o G} \frac{\mu(K,H)}{|G:K|^s}$$
$$= \sum_{K\leq_o G} \frac{\sum_{K\leq_o H} \mu(K,H)}{|G:K|^s}$$
$$= \sum_{K\leq_o G} \frac{\delta_{K,G}}{|G:K|^s} = 1.$$