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Decomposition Methods for Integer Programming **(PDMA, PDEIS)**

Exercise GAP

1. In the generalized assignment problem (GAP), we want to minimize the cost of assigning a set of items to bins with variable capacity $W_i, \forall i$. There is a weight associated to each item j, w_{ij}, which depends on the bin i where it is placed.

   This problem arises in machine scheduling: items correspond to jobs that are assigned to machines, which are the objects. The coefficient w_{ij} denotes the processing time of job j in machine i, W_i represents the time available in machine i, and we want to minimize the total processing cost, which is a function of the c_{ij}.

   Let us consider the decision variables x_{ij}, defined as follows:

$$x_{ij} = \begin{cases} 1 & \text{, if item } j \text{ is placed in bin } i \\ 0 & \text{, otherwise} \end{cases}$$

   One model for the GAP is as follows:

$$\min z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \tag{1}$$

$$\text{subj. to} \quad \sum_{j=1}^{n} w_{ij} x_{ij} \leq W_i, \ i = 1, \ldots, m \tag{2}$$

$$\sum_{i=1}^{m} x_{ij} = 1, \ j = 1, \ldots, n \tag{3}$$

$$x_{ij} = 0 \text{ ou } 1, \ \forall i, j \tag{4}$$

   There are two different Dantzig-Wolfe decompositions for this problem depending on the set of constraints that are placed in the subproblem, either set (2) or set (3).

   a) For each of the two decompositions, describe the resulting Master Problem and Subproblem.

   b) Discuss the quality of the bounds that are obtained from solving the linear programming (LP) relaxation of their column generation models, and compare them with the bound that is obtained by solving the LP model above.

   c) For each of the two decompositions, build an initial solution for the restricted master problem (RMP) (you may choose whatever method you want to build it).

2. Consider the data given in the following tables for a GAP with 2 machines and 4 jobs. The values of the processing costs $C = [c_{ij}]$ are as follows:

$$C = \begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ \hline 1 & 15 & 12 & 10 & 9 \\ 2 & 8 & 4 & 8 & 6 \\ \hline \end{array}$$

The values of the processing times $P = [p_{ij}]$ are as follows:

$$P = \begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ \hline 1 & 3 & 4 & 5 & 6 \\ 2 & 4 & 5 & 8 & 4 \\ \hline \end{array}$$

The capacities of the machines are 10 and 12, respectively.

Using the reformulated model with the knapsack constraints in the subproblem:

a) Solve the linear relaxation of the problem.

b) Describe a branching rule for the GAP.

c) Use branch-and-price to solve the integer problem.

Note: you should use an LP software package to solve this problem. In each iteration, you should indicate which are the RMP and the SubProblem you are going to solve, and then obtain their solutions (primal and dual) with the LP software package.

3. The optimal solution to the previous exercise, expressed in terms of the original variables, is $x_{11} = x_{13} = x_{24} = x_{24} = 1$, and all other variables equal to 0, meaning that jobs 1 and 3 are processed in machine 1 and jobs 2 and 4 are processed in machine 2. The optimal cost is 35.

   a) Suppose that you wanted to enforce the assignment of job 1 to machine 2, a constraint that can be stated, in terms of original variables, as $x_{21} = 1$. Modify the reformulated model so as to enforce this constraint in the reformulated model. Which is the new optimal solution?

   a) Suppose that you wanted to enforce the assignment of job 1 to a machine other than machine 1, a constraint that can be stated, in terms of original variables, as $x_{11} = 0$. Modify the reformulated model so as to enforce this constraint in the reformulated model. Which is the new optimal solution?