Decomposition methods

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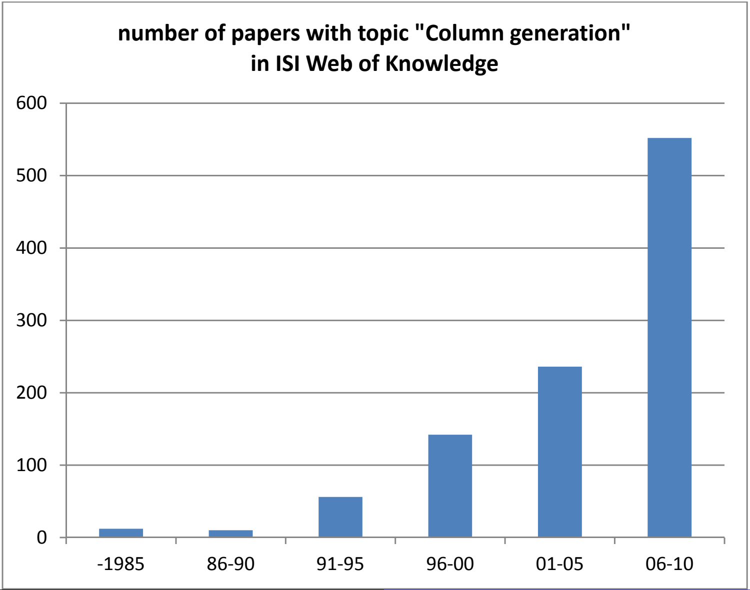
Global outline

- Part I - Decomposition methods
- Part II - Applications
- Part III - Branch-and-price algorithms
- Part IV - Branch-and-price algorithms (cont.)
- Part V - Practical issues, stabilization, accelerating strategies and heuristics
- Bibliography


number of papers with topic "Column generation"
in ISI Web of Knowledge
600
500
400
300
200
100
0
-1985
86-90
91-95
96-00
01-05
06-10

Decomposition Methods

Outline

- Strength of models in integer programming (IP)
- Dantzig-Wolfe decomposition (DW)
- Comparative strength of IP, DW and LP
- Application Example

Integer programming: strength of models

Integer Programming Problems (IP):

$$\begin{aligned} z_{IP} = \min \quad & cx \\ subj.\ to \quad & Ax = b \\ & x \geq 0 \text{ and integer} \end{aligned}$$

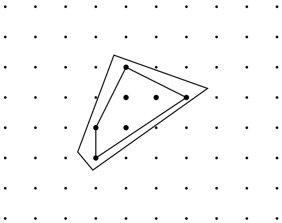
can be solved by branch-and-bound using the Linear Programming (LP) relaxation that results from relaxing the integrality conditions:

$$\begin{aligned} z_{LP} = \min \quad & cx \\ subj.\ to \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Crucial issue: some IP models are stronger, because their LP relaxations:

- provide closer description of convex hull of valid integer solutions.
- have LP optimal solution values closer to IP optimal solution values (smaller gap).

Motivation for branch-and-price



Some strong IP models have an exponential number of variables.

Solve them combining **column generation** and **branch-and-bound**.

May provide strong models (stronger than plain LP relaxation)...
... with an exponential number of variables.

$$
\begin{array}{ll}
\min & cx \\
suj. & Ax = b \\
 & x \in X \\
 & x \geq 0 \text{ and integer}
\end{array}
$$

Constraints decomposed in two sets:

- first set: general constraints → **Master Problem**.
- second set: constraints with special structure → **Subproblem**

Subproblem must be amenable for separate solution.

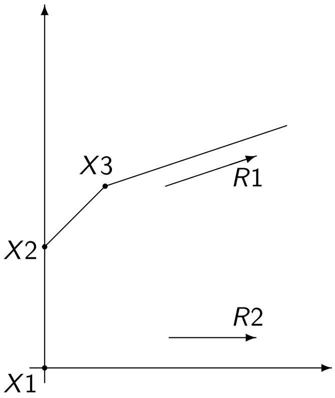
Dantzig-Wolfe decomposition: representation of a point

$$
\begin{array}{ll}
\min & cx \\
suj. & Ax = b \\
 & \boxed{x \in X} \\
 & x \geq 0
\end{array}
$$

- Polyhedron X has I extreme points, denoted as $X_1, X_2, \ldots, X_I$, and K extreme rays, denoted as $R_1, R_2, \ldots, R_K$.
- Any point $x \in X$ is expressed as a convex combination of the extreme points of X plus a non-negative combination of the extreme rays of X :

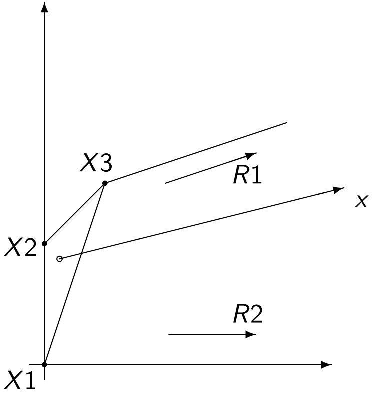
$$X = \left\{ x = \sum_{i=1}^{I} \lambda_i X_i + \sum_{k=1}^{K} \mu_k R_k, \sum_{i=1}^{I} \lambda_i = 1, \lambda_i \geq 0, \forall i, \mu_k \geq 0, \forall k \right\}$$

Dantzig-Wolfe decomposition: graphical representation



X_1,X_2 and X_3 are extreme points, and R_1 and R_2 are extreme rays.
Valid space is unbounded.

Dantzig-Wolfe decomposition: graphical representation



x is expressed as a convex combination of X_1, X_2 and X_3 plus a non-negative combination of R_1 and R_2.

Some rewriting work...

replacing x in $\min\{cx : Ax = b, x \in X, x \geq 0\}$, we obtain

$$
\begin{aligned}
\min \quad & c\Big(\sum_{i=1}^{I} \lambda_i X_i + \sum_{k=1}^{K} \mu_k R_k\Big) \\
suj. \quad & A\Big(\sum_{i=1}^{I} \lambda_i X_i + \sum_{k=1}^{K} \mu_k R_k\Big) = b \\
& \sum_{i=1}^{I} \lambda_i = 1 \\
& \lambda_i \geq 0, \forall i \\
& \mu_k \geq 0, \forall k
\end{aligned}
$$

Reformulation of the problem: master problem

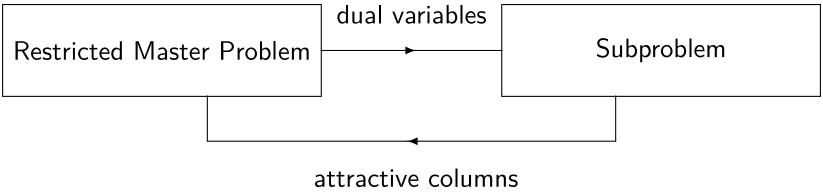
$$\begin{aligned} \min \quad & \sum_{i=1}^{I}(cX_i)\lambda_i + \sum_{k=1}^{K}(cR_k)\mu_k \\ \textit{subj. to} \quad & \sum_{i=1}^{I}(AX_i)\lambda_i + \sum_{k=1}^{K}(AR_k)\mu_k = b \\ & \sum_{i=1}^{I}\lambda_i = 1 \\ & \lambda_i \geq 0, \forall i \\ & \mu_k \geq 0, \forall k \end{aligned}$$

Decision variables: λ_i and μ_k.
Reformulated model is equivalent to original model.
Number of extreme points and extreme rays can be exponentially large.
Use column generation!

Column generation

Solve linear programming relaxation using column generation:

Choose an initial restricted set of columns
While (there is a column with negative reduced cost) do
  add column to restricted problem
  reoptimize
End While



[Dantzig, Wolfe, 1960; Ford, Fulkerson, 1958]

Integrality property

> If X does not have the integrality property, the reformulated model is stronger than the linear programming relaxation.

Instead of searching extreme points and extreme rays in:

$$x \in Conv\{x \in X\},$$

search in:

$$x \in Conv\{x \in X \text{ and integer}\}.$$

That may not be too hard: in the Cutting Stock Problem, we have to find an integer solution of the subproblem (knapsack problem).

Comparative strengths of 3 different models:

- Integer programming model (IP)
- Linear programming relaxation model (LP)
- Dantzig-Wolfe decomposition model (DW)

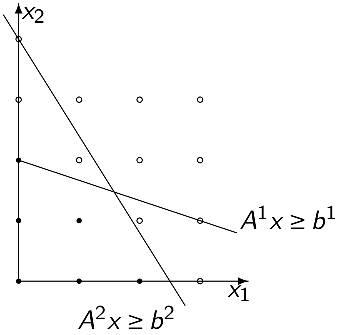
Three different models: IP, LP, DW

$$
\begin{aligned}
z_{IP} = \min \quad & cx \\
\textit{subj. to} \quad & Ax = b \\
& x \in X \\
& x \geq 0 \text{ and integer}
\end{aligned}
$$

$$
\begin{aligned}
z_{LP} = \min \quad & cx \\
\textit{subj. to} \quad & Ax = b \\
& x \in X \\
& x \geq 0
\end{aligned}
$$

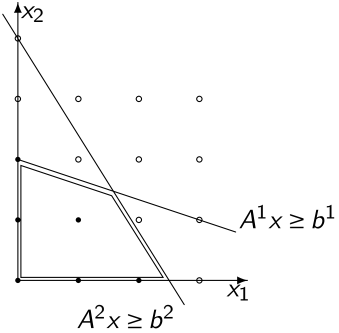
$$
\begin{aligned}
z_{DW} = \min \quad & cx \\
\textit{subj. to} \quad & Ax = b \\
& x \in Conv\{x \in X \text{ and integer}\} \\
& x \geq 0
\end{aligned}
$$

Integer Problem: domain is a finite set of points



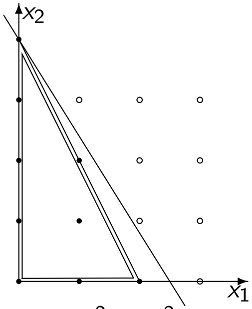
$$
\begin{aligned}
z_{IP} \quad = \quad & \min cx \\
\textit{subj. to} \quad & A^1x \geq b^1 \\
& A^2x \geq b^2 \\
& x \geq 0 \text{ and integer}
\end{aligned}
$$

Linear programming relaxation



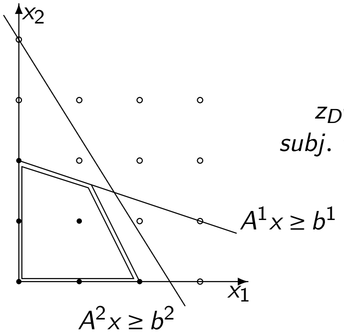
$$
\begin{aligned}
z_{LP} &= \min cx \\
\textit{subj. to} \quad & A^1x \geq b^1 \\
& A^2x \geq b^2 \\
& x \geq 0
\end{aligned}
$$

$A^2x \geq b^2$ does not have the integrality property



$x \in Conv\{A^2x \geq b^2 \text{ and integer}\}$

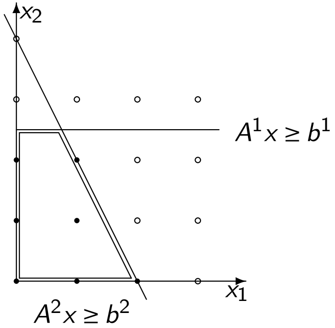
Reformulated model



$$\begin{array}{rcl} z_{DW} & = & \min cx \\ \textit{subj. to} & & A^1x \geq b^1 \\ & & x \in Conv\{A^2x \geq b^2 \text{ and integer}\} \\ & & x \geq 0 \end{array}$$

Reformulated model is stronger than LP relaxation.

If X is an integer polytope $\Rightarrow$ same bound as LP



$z_{LP} = z_{DW}$, because $X = Conv\{x \in X$ and integer$\}$ (compare models presented before).

Example

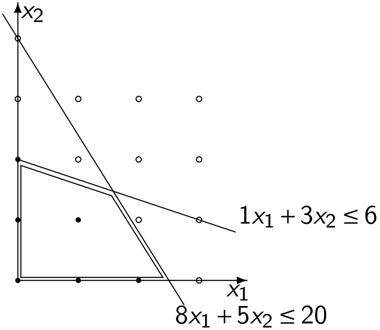
Integer programming:

- $\max\{x_1 + x_2 : 8x_1 + 5x_2 \leq 20, x_1 + 3x_2 \leq 6, x1, x2 \geq 0$ and integer$\}$.
- Several alternative integer optimal solutions: $(x_1, x_2) = (2,0), (0,2)$ and $(1,1)$ have objective function value $z_{IP} = 2$.

LP relaxation:

- LP relaxation optimal solution is $(x_1, x_2) = (30/19, 28/19)$, with objective function value $z_{LP} = 58/19 = 3.053$.
- Integrality gap $z_{LP} - z_{IP}$ equal to 1.053.

Integer and linear programming relaxation domains



$$
\begin{aligned}
z_{IP} \quad &= \quad \max 1x_1 + 1x_2 \\
\textit{subj. to} \quad & \quad 1x_1 + 3x_2 \leq 6 \\
& \quad 8x_1 + 5x_2 \leq 20 \\
& \quad x_1, x_2 \geq 0, \text{ integer}
\end{aligned}
$$

Dantzig-Wolfe decomposition: $Conv\{x \in X$ and integer$\}$

$Conv\{x \in X$ and integer$\} = Conv\{(x_1, x_2) : 8x_1 + 5x_2 \leq 20, x1, x2 \geq 0$ and integer$\}$

The integer extreme points of set X are:

$$X_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, X_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

and the corresponding polytope is:

$$Conv\{x \in X \text{ and integer}\} = \{x \in \mathrm{IR}^2 : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda_1 \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 4 \end{pmatrix},$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1, \lambda_1, \lambda_2, \lambda_3 \geq 0\}.$$

Dantzig-Wolfe decomposition: reformulation

$$
\begin{array}{llllll}
\max & 1x_1 & +1x_2 & & & \\
\textit{subj.to} & 1x_1 & +3x_2 & \leq & 6 & \text{(master problem)} \\
 & 8x_1 & +5x_2 & \leq & 20 & \text{(subproblem)} \\
 & \multicolumn{5}{l}{x_1,\ x_2 \geq 0 \text{ and integer}}
\end{array}
$$

$c = (1,1)$, $A^1 = (1,3)$, $A^2 = (8,5)$, $b^1 = 6$ and $b^2 = 20$, for the extreme points, we get:

$$
cX_1 = \boxed{\begin{array}{cc} 1 & 1 \end{array}} * \boxed{\begin{array}{c} 0 \\ 0 \end{array}} = 0 \qquad A^1X_1 = \boxed{\begin{array}{cc} 1 & 3 \end{array}} * \boxed{\begin{array}{c} 0 \\ 0 \end{array}} = 0
$$

$$
cX_2 = \boxed{\begin{array}{cc} 1 & 1 \end{array}} * \boxed{\begin{array}{c} 2 \\ 0 \end{array}} = 2 \qquad A^1X_2 = \boxed{\begin{array}{cc} 1 & 3 \end{array}} * \boxed{\begin{array}{c} 2 \\ 0 \end{array}} = 2
$$

$$
cX_3 = \boxed{\begin{array}{cc} 1 & 1 \end{array}} * \boxed{\begin{array}{c} 0 \\ 4 \end{array}} = 4 \qquad A^1X_3 = \boxed{\begin{array}{cc} 1 & 3 \end{array}} * \boxed{\begin{array}{c} 0 \\ 4 \end{array}} = 12
$$

Reformulated model (bounded case):

$$
\begin{array}{ll}
\max & \sum_{i=1}^{I} (cX_i)\lambda_i \\
\textit{subj. to} & \sum_{i=1}^{I} (AX_i)\lambda_i \leq b \\
& \sum_{i=1}^{I} \lambda_i = 1 \\
& \lambda_i \geq 0, \forall i
\end{array}
$$

$$
\begin{array}{lrrrcl}
\max z_{DW} = & 0\lambda_1 & +2\lambda_2 & +4\lambda_3 & & \\
\textit{subj. to} & 0\lambda_1 & +2\lambda_2 & +12\lambda_3 & \leq & 6 \\
& \lambda_1 & +\lambda_2 & +\lambda_3 & = & 1 \\
& \multicolumn{5}{l}{\lambda_1,\ \lambda_2,\ \lambda_3 \geq 0}
\end{array}
$$

Solution of complete model with simplex method

Notice that the order of the columns was changed, in order to have the identity matrix in the last two columns.

	z_{DW}	λ_2	λ_3	s_1	λ_1	
s_1	0	2	12	1	0	6
λ_1	0	1	1	0	1	1
	1	-2	-4	0	0	0

	z_{DW}	λ_2	λ_3	s_1	λ_1	
λ_3	0	1/6	1	1/12	0	1/2
λ_1	0	5/6	0	-1/12	1	1/2
	1	-4/3	0	1/3	0	2

	z_{DW}	λ_2	λ_3	s_1	λ_1	
λ_3	0	0	1	1/10	-1/5	2/5
λ_2	0	1	0	-1/10	6/5	3/5
	1	0	0	1/5	8/5	14/5

Optimal solution

- $\lambda_2 = 3/5$ and $\lambda_3 = 2/5$, with objective function value $14/5$.
- Optimal solution in the original space is a convex combination of the extreme points X_2 e X_3 with weights λ_2 e λ_3 :

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^* = 0\begin{pmatrix} 0 \\ 0 \end{pmatrix} + 3/5\begin{pmatrix} 2 \\ 0 \end{pmatrix} + 2/5\begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 6/5 \\ 8/5 \end{pmatrix}$$

- Integrality gap $z_{DW} - z_{IP}$ reduced to $14/5 - 2 = 0.8$ (stronger model).

Linear Programming (primal) maximization problem:

$$\begin{array}{ll} \max & cx \\ s.to & Ax \le b \\ & x \ge 0 \end{array}$$

where $x \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}$.
In matrix form:

A	I	b
$-c$	0	0

A basic solution (basis) is characterized by a set of m linearly independent columns, which form matrix $B \in \mathbb{R}^{m \times m}$.
$c_B \in \mathbb{R}^m$: vector of cost coefficients of the columns of B.

Exchange of Basis

Pre-multiplying B and c_B, the "matrix operator" performs an exchange of basis:

$$\left[\begin{array}{c|c} B^{-1} & \mathbf{0} \\ \hline c_B B^{-1} & 1 \end{array}\right] * \left[\begin{array}{c} B \\ \hline -c_B \end{array}\right] = \left[\begin{array}{c} I \\ \hline \mathbf{0} \end{array}\right]$$

For the entire tableau:

$$\left[\begin{array}{c|c} B^{-1} & \mathbf{0} \\ \hline c_B B^{-1} & 1 \end{array}\right] * \left[\begin{array}{c|c|c} A & I & b \\ \hline -c & \mathbf{0} & 0 \end{array}\right] =$$

$$= \left[\begin{array}{c|c|c} B^{-1}A & B^{-1} & B^{-1}b \\ \hline c_B B^{-1}A - c & c_B B^{-1} & c_B B^{-1} b \end{array}\right]$$

Dual Problem and Optimality Conditions

$\min\{yb : yA \geq c, y \geq 0\}$ is the dual problem of $\max\{cx : Ax \leq b, x \geq 0\}$.

$y = c_B B^{-1}$ is a dual solution, which is valid when dual constraints are obeyed, *i.e.*:

- $yA \geq c \quad \equiv \quad c_B B^{-1} A - c \geq 0$
- $y \geq 0 \quad \equiv \quad c_B B^{-1} \geq 0$

Optimality conditions: if

- solution is dual valid (as shown above),
- solution is primal valid: $x = B^{-1} b \geq 0$, and
- complementary slackness conditions are obeyed, then
- finite optimal solution has value $c_B B^{-1} b$.

Example

	z_{DW}	λ_2	λ_3	s_1	λ_1	
s_1	0	2	12	1	0	6
λ_1	0	1	1	0	1	1
z_{DW}	1	-2	-4	0	0	0

Matrix B and corresponding cost vector c_B :

$$B = \begin{array}{cc} \lambda_3 & \lambda_1 \\ \hline 12 & 0 \\ 1 & 1 \end{array}$$

$$-c_B = \boxed{\begin{array}{cc} -4 & 0 \end{array}}$$

Exchange of basis:

$$\left[\begin{array}{cc|c} 1/12 & 0 & 0 \\ -1/12 & 1 & 0 \\ \hline 1/3 & 0 & 1 \end{array}\right] * \left[\begin{array}{cc} 12 & 0 \\ 1 & 1 \\ \hline -4 & 0 \end{array}\right] = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \end{array}\right]$$

Initial and final basis

	z_{DW}	λ_2	λ_3	s_1	λ_1	
s_1	0	2	12	1	0	6
λ_1	0	1	1	0	1	1
	1	-2	-4	0	0	0

	z_{DW}	λ_2	λ_3	s_1	λ_1	
λ_3	0	1/6	1	1/12	0	1/2
λ_1	0	5/6	0	-1/12	1	1/2
	1	-4/3	0	1/3	0	2

$$\begin{array}{|c|c|} \hline B^{-1} & 0 \\ \hline c_B B^{-1} & 1 \\ \hline \end{array} * \begin{array}{|c|c|c|} \hline A & I & b \\ \hline -c & 0 & 0 \\ \hline \end{array} =$$

$$= \begin{array}{|c|c|c|} \hline B^{-1}A & B^{-1} & B^{-1}b \\ \hline c_B B^{-1} A - c & c_B B^{-1} & c_B B^{-1} b \\ \hline \end{array}$$

Examples: $B^{-1}b$, $B^{-1}A_{X_2}$ and $c_B B^{-1} A_{X_2} - c_{X_2}$

$$B^{-1}b = \boxed{\begin{array}{rr} 1/12 & 0 \\ -1/12 & 1 \end{array}} * \boxed{\begin{array}{c} 6 \\ 1 \end{array}} = \boxed{\begin{array}{c} 1/2 \\ 1/2 \end{array}}$$

$$B^{-1}A_{X_2} = \boxed{\begin{array}{rr} 1/12 & 0 \\ -1/12 & 1 \end{array}} * \boxed{\begin{array}{c} 2 \\ 1 \end{array}} = \boxed{\begin{array}{c} 1/6 \\ 5/6 \end{array}}$$

$$c_B B^{-1} A_{X_2} - c_{X_2} = \boxed{\begin{array}{cc} 1/3 & 0 \end{array}} * \boxed{\begin{array}{c} 2 \\ 1 \end{array}} - 2 = -4/3$$

Solution of model with column generation

In this small example, it is easy to describe $Conv\{x \in X$ and integer$\}$ using only linear inequalities, and to obtain a polytope with integer extreme points:

$$\begin{aligned} Conv\{x \in X \text{ and integer}\} &= Conv\{(x_1, x_2) : 8x_1 + 5x_2 \leq 20, x1, x2 \geq 0 \text{ and intege} \\ &= \{(x_1, x_2) : 2x_1 + 1x_2 \leq 4, x1, x2 \geq 0\} \end{aligned}$$

Unfortunately, in general, for integer programming problems, that may not be easy (exponential number of constraints needed).

Subproblem: pricing extreme points of $Conv\{X\}$ out of the restricted master problem:

Reduced cost of $X_j : c_B B^{-1} A_j - c_j$ (maximization problem)
Column is attractive if its reduced cost < 0

Expression of reduced cost - I

$$
\begin{array}{llllllll}
\max z_{DW} = & 0\lambda_1 & +2\lambda_2 & +4\lambda_3 & & & \text{(dual variable)} \\
\textit{subj. to} & 0\lambda_1 & +2\lambda_2 & +12\lambda_3 & \leq & 6 & (\pi_1) \\
 & \lambda_1 & +\lambda_2 & +\lambda_3 & = & 1 & (u) \\
 & \lambda_1,\ \lambda_2,\ \lambda_3 \geq 0 & & & & &
\end{array}
$$

Dual values of reformulated model: $c_B B^{-1} = \boxed{\pi_1 \quad u}$

Structure of columns in reformulated model (main body of tableau):

- upper part: results from changing variables in DW decomposition.
- lower part: convexity constraint ($\sum \lambda = 1$)

$$A_j = \boxed{\begin{array}{c} A^1 X_j \\ 1 \end{array}}$$

Structure of columns in reformulated model (objective function):

$$c_j = cX_j$$

Expression of reduced cost - II

By substitution, we obtain:

Reduced cost of column of the reformulated model expressed in terms of original variables:

$$
\begin{aligned}
c_B B^{-1} A_j - c_j &= \pi_1 A^1 X_j + u * 1 - c X_j = \\
&= (\pi_1 A^1 - c) X_j + u
\end{aligned}
$$

Transferral of dual information to subproblem:

- we are now able to calculate the value of each point in $Conv\{X\}$
- look for the most attractive point (should be an extreme point)
- optimize the subproblem.

Insight: in the subproblem we have an objective function with:

- cost coefficients of the original variables x_1 and x_2 given by elements of the vector $(\pi_1 A^1 - c)$
- a constant term u

Solution 1 (starting solution)

	z_{DW}	s_1	λ_1	
s_1	0	1	0	6
λ_1	0	0	1	1
	1	0	0	0

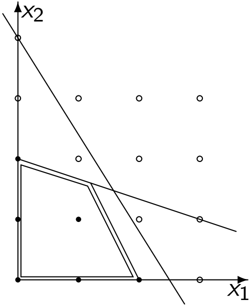
Dual solution: $\pi_1 = 0, u = 0$

Reduced cost:

$$\begin{aligned}(\pi_1 A^1 - c)X_j + u &= \left(0 * \boxed{1 \quad 3} - \boxed{1 \quad 1} \right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 0 = \\ &= -1\ x_1 - 1\ x_2\end{aligned}$$

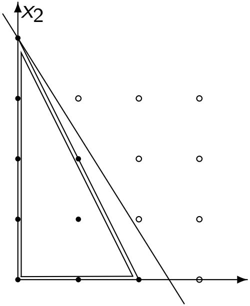
and Subproblem 1 is equivalent to :

$$\begin{array}{ll} \max & x_1 + x_2 \\ \text{s. to} & x \in Conv\{x \in X \text{ and integer}\} \end{array}$$



To which solution in the space of the original variables x_1, x_2 does the current solution $\lambda_1 = 1, \lambda_2 = \lambda_3 = 0$ correspond ?

Graphical solution of subproblem 1



Which is the solution $x \in Conv\{x \in X$ and integer$\}$ that maximizes $x_1 + x_2$?

Insert column in restrict master problem and re-optimize

Solution of Subproblem 1

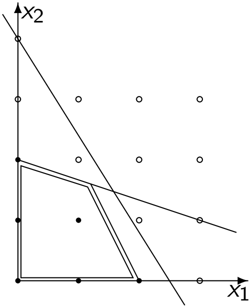
Optimal solution of subproblem 1 is $X_j^* = X_3 = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$ with objective solution value 4.

	z_{DW}	λ_3	s_1	λ_1	
s_1	0	12	1	0	6
λ_1	0	1	0	1	1
	1	-4	0	0	0

	z_{DW}	λ_3	s_1	λ_1	
λ_3	0	1	1/12	0	1/2
λ_1	0	0	-1/12	1	1/2
	1	0	1/3	0	2

Dual solution: $\pi_1 = 1/3, u = 0$

Solution 2 in the space of original variables



To which solution in the space of the original variables x_1, x_2 does the current solution $\lambda_1 = \lambda_3 = 1/2, \lambda_2 = 0$ correspond ?

Second subproblem:

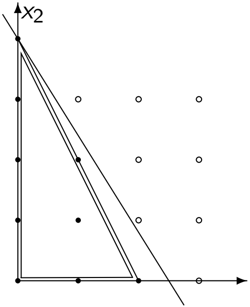
Reduced cost:

$$
\begin{aligned}
(\pi_1 A^1 - c)X_j + u &= \left(\quad 1/3 \quad * \boxed{1 \quad 3} - \boxed{1 \quad 1} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 0 = \\
&= -2/3 \; x_1 - 0 \; x_2
\end{aligned}
$$

and Subproblem 2 is equivalent to :

$$
\begin{array}{ll}
\max & 2/3x_1 + \; 0x_2 \\
\text{s. to} & x \in Conv\{x \in X \text{ and integer}\}
\end{array}
$$

Graphical solution of subproblem 2



Which is the solution $x \in Conv\{x \in X$ and integer$\}$ that maximizes $2/3x_1 + 0x_2$?

Re-expressing the column

Optimal solution of subproblem 2 is $X_j^* = X_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ with objective solution value $4/3$.

To insert the new column $A_{X_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ in the simplex tableau, we need to express it in terms of the current base:

$$B^{-1} A_{X_2} = \boxed{\begin{matrix} 1/12 & 0 \\ -1/12 & 1 \end{matrix}} * \boxed{\begin{matrix} 2 \\ 1 \end{matrix}} = \boxed{\begin{matrix} 1/6 \\ 5/6 \end{matrix}}$$

$$c_B B^{-1} A_{X_2} - c_{X_2} = \boxed{\begin{matrix} 1/3 & 0 \end{matrix}} * \boxed{\begin{matrix} 2 \\ 1 \end{matrix}} - 2 = -4/3$$

This step was skipped in the last iteration, because $B^{-1} = I$.

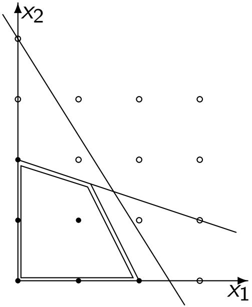
Second iteration

	z_{DW}	λ_2	λ_3	s_1	λ_1	
λ_3	0	1/6	1	1/12	0	1/2
λ_1	0	5/6	0	-1/12	1	1/2
	1	-4/3	0	1/3	0	2

	z_{DW}	λ_2	λ_3	s_1	λ_1	
λ_3	0	0	1	1/10	-1/5	2/5
λ_2	0	1	0	-1/10	6/5	3/5
	1	0	0	1/5	8/5	14/5

Dual solution: $\pi_1 = 1/5, u = 8/5$

Solution 3 in the space of original variables



To which solution in the space of the original variables x_1, x_2 does the current solution $\lambda_2 = 3/5,\ \lambda_3 = 2/5,\ \lambda_1 = 0$ correspond ?

Subproblem 3

Reduced cost:

$$
\begin{aligned}
(\pi_1 A^1 - c)X_j + u &= \left(\quad 1/5 \quad * \boxed{1 \quad 3} - \boxed{1 \quad 1} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 8/5 = \\
&= -4/5\ x_1 - 2/5\ x_2 + 8/5
\end{aligned}
$$

and Subproblem 3 is equivalent to :

$$
\begin{array}{ll}
\max & 4/5x_1 + \ 2/5x_2 - 8/5 \\
\text{s. to} & x \in Conv\{x \in X \text{ and integer}\}
\end{array}
$$

Solution of subproblem:

- Optimal solution has objective function value equal to 0.
- There are no attractive solutions.

Solution of reformulated problem is optimal.

Concluding remarks

- decomposition is a general tool to convert a difficult problem into a sequence of manageable problems.
- models from decomposition may be stronger.
- column generation has profited from developments in LP software.