Four Italian friends [from *La Settimana Enigmistica*]

Andrea, Bruno, Carlo and Dario share an apartment and read four newspapers: "La Repubblica", "Il Messaggero", "La Stampa" and "La Gazzetta dello Sport" before going out. Each of them wants to read all newspapers in a specific order. Andrea starts with "La Repubblica" for one hour, then he reads "La Stampa" for 30 minutes, "Il Messaggero" for two minutes and then "La Gazzetta dello Sport" for 5 minutes. Bruno prefers to start with "La Stampa" for 75 minutes; he then has a look at "Il Messaggero" for three minutes, then he reads "La Repubblica" for 25 minutes and finally "La Gazzetta dello Sport" for 10 minutes. Carlo starts with "Il Messaggero" for 5 minutes, then he reads "La Stampa" for 15 minutes, "La Repubblica" for 10 minutes and "La Gazzetta dello Sport" for 30 minutes. Finally, Dario starts with "La Gazzetta dello Sport" for 90 minutes and then he dedicates just one minute to each of "La Repubblica", "La Stampa" and "Il Messaggero" in this order. The preferred order is so important that each is willing to wait and read nothing until the newspaper that he wants becomes available. Moreover, none of them would stop reading a newspaper and resume later. By taking into account that Andrea gets up at 8:30, Bruno and Carlo at 8:45 and Dario at 9:30, and that they can wash, get dressed and have breakfast while reading the newspapers, what is the earliest time they can leave home together?

Four Italian friends: a Job-Shop Scheduling Problem (JSP)

- **Jobs**: **A**ndrea, **B**runo, **C**arlo, **D**ario [set I]
- **Machines**: "La **R**epubblica", "Il **M**essaggero", "La **S**tampa" and "La **G**azzetta dello Sport" [set K]
- **Processing times and order**:

  A: R (60) → S (30) → M (2) → G (5);

  B: S (75) → M (3) → R (25) → G (10);

  C: M (5) → S (15) → R (10) → G (30);

  A: G (90) → R (1) → S (1) → M (1);

  [param: D_{ik}, processing times]

  [param: $\sigma[i,l] \in K$, newspaper read by i in position l)]
- **Release time**: A 8:30 – B 8:45 – C 8:45 – D 9:30. [param R_i]
- Objective: Minimize the **Makespan** (job-completion time)
- **No pre-emption**

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- **Release time**: A 8:30 – B 8:45 – C 8:45 – D 9:30. [param R_i]
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LP model for JSP

- h_{ik}: start time (in minutes after 8:30) of $i \in I$ on $k \in K$;
- y: completion time (in minutes after 8:30);
- x_{ijk}: binary, 1 if $i \in I$ precedes $j \in I$ on $k \in K$, 0 otherwise.

$$
\begin{array}{lll}
\min & y & \\
\text{s.t.} & y \geq h_{i\,\sigma[i,|K|]} + D_{i\,\sigma[i,|K|]} & \forall\, i \in I \\
& h_{i\,\sigma[i,l]} \geq h_{i\,\sigma[i,l-1]} + D_{i\,\sigma[i,l-1]} & \forall\, i \in I, l = 2 ... |K| \\
& h_{i\,\sigma[i,1]} \geq R_i & \forall\, i \in I \\
& h_{ik} \geq h_{jk} + D_{jk} - M\, x_{ijk} & \forall\, k \in K, i \in I, j \in I : i \neq j \\
& h_{jk} \geq h_{ik} + D_{ik} - M\,(1 - x_{ijk}) & \forall\, k \in K, i \in I, j \in I : i \neq j \\
& y \in \mathbb{R}_+ & \\
& h_{ik} \in \mathbb{R}_+ & \forall\, k \in K, i \in I \\
& x_{ijk} \in \{0,1\} & \forall\, k \in K, i \in I, j \in I : i \neq j
\end{array}
$$

Project scheduling in the boatyard industry

Constructing a boat requires the completion of the following operations :

Operations	Duration	Precedences
A	2	none
B	4	A
C	2	A
D	5	A
E	3	B,C
F	3	E
G	2	E
H	7	D,E,G
I	4	F,G

Some of the operations are alternative to each other. In particular, only one of B and C needs to be executed, and only one of F and G needs to be executed. Furthermore, if both C and G are executed, the duration of I increases by 2 days. The table also shows the precedences for each operation (i.e., operations that must be completed before the beginning of the new operation). For instance, H can start only after the completion of E, D and G (if G will be executed). Write a linear programming model that can be used to decide which operations should be executed in order to minimize the total duration of the construction of the boat.

Project scheduling in the boatyard industry: hints

$$
\begin{array}{lll}
\min & z & \\
s.t. & z \geq t_i \qquad \forall i \in A...I & t_H \geq t_D + d_H \\
& t_A \geq d_A & t_H \geq t_E + d_H \\
& t_B \geq t_A + d_B - M(1-y_B) & t_H \geq t_G + d_H \\
& t_C \geq t_A + d_C - M(1-y_C) & t_I \geq t_F + d_I + 2y_{CG} \\
& t_D \geq t_A + d_D & t_I \geq t_G + d_I + 2y_{CG} \\
& t_E \geq t_B + d_E & y_B + y_C = 1 \\
& t_E \geq t_C + d_E & y_F + y_G = 1 \\
& t_F \geq t_E + d_F - M(1-y_F) & y_C + y_G <= 1 + y_{CG} \\
& t_G \geq t_A + d_G - M(1-y_G) & z, t_i \geq 0 \qquad \forall i \in \{A...I\} \\
& & y. \in \{0,1\}
\end{array}
$$

where

- t_i completion time of operation $i \in \{A, B, C, D, E, F, G, H, I\}$;
- y_i 1 if operation $i \in \{B, C, F, G\}$ is executed, 0 otherwise;
- y_{CG} 1 if both C and G are executed, 0 otherwise;
- z completion time of the last operation;
- d_i parameter indicating the duration of operation i;
- M sufficiently large constant.

Exercise: write a more general model for generic sets of operations and precedence.

A (shift) covering problem

The pharmacy federation wants to organize the opening shifts on public holidays all over the region. The number of shifts is already decided, and the number of pharmacies open on the same day has to be as balanced as possible. Furthermore, every pharmacy is part of one shift only. For instance, if there are 12 pharmacies and the number of shifts is 3, every shift will consist of 4 pharmacies. Pharmacies and users are thought as concentrated in centroids (for instance, villages). For every centroid, the number of users and pharmacies are known. The distance between every ordered pair of centroids is also known. For the sake of simplicity, we ignore congestion problems and we assume that every user will go to the closest open pharmacy. The target is to determine the sifts so that the total distance covered by the users is minimized.

A (shift) covering problem: model 1

- y_{jk}: 1 if pharmacy $j \in P$ takes part in shift $k = 1 \dots K$, 0 otherwise;
- z_{ijk}: 1 if centroid $i \in C$ uses pharmacy $j \in P$ during shift $k = 1 \dots K$, 0 otherwise (notice: by optimality, z selects the nearest open pharmacy)

$$\min \sum_{k=1}^{K} \sum_{i \in C} \sum_{j \in P} D_{ij} z_{ijk} \qquad \text{(parameter } D_{ij}\text{: distance from } i \text{ to } j\text{)}$$

$$\text{s.t.} \sum_{k=1}^{K} y_{jk} = 1 \qquad \forall\, j \in P$$

$$\sum_{j \in P} z_{ijk} = 1 \qquad \forall\, i \in C, k = 1 \dots K$$

$$x_{ijk} \leq y_{jk} \qquad \forall\, i \in C, j \in P, k = 1 \dots K$$

$$(\lfloor |P|/K \rfloor \leq \sum_{j \in P} y_{jk} \leq \lceil |P|/K \rceil \qquad \forall\, k = 1 \dots K)$$

$$z_{ijk}, y_{jk} \in \{0, 1\} \qquad \forall\, i \in C, j \in P, k = 1 \dots K$$

Notice: the model has a polynomial number of variables and constraints but suffers from symmetries, that is, the same "real" solution can be represented in many different ways, by giving different names (i.e. value of k) to the same shifts.

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A (shift) covering problem: model 2

$\mathcal{P}$: set of all possible subsets of P (with balanced cardinality for balancing constraint)
$D(J)$: total distance covered by all users in C to reach the nearest pharmacy in $J \in \mathcal{P}$

- x_J: 1 if the set $J \in \mathcal{P}$ is selected as a shift, 0 otherwise;

$$\begin{aligned} \min & \sum_{J \in \mathcal{P}} D_J x_J & \\ \text{s.t.} & \sum_{J \in \mathcal{P}} x_J = K & \\ & \sum_{J \in \mathcal{P} : j \in J} x_J = 1 & \forall\, j \in P \\ & x_J \in \{0,1\} & \forall J \in \mathcal{P} \end{aligned}$$

Notice: the model does not suffer from symmetries (a shift is directly determined by the defining subset), but has an exponential number of variables [we will see how to face this issue].

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An energy flow problem

A company distributing electric energy has several power plants and distributing stations connected by wires. Each station i can:

- produce p_i kW of energy ($p_i = 0$ if the station cannot produce energy);
- distribute energy on a sub-network whose users have a total demand of d_i kW ($d_i = 0$ if the station serves no users);
- carry energy from/to different stations.

The wires connecting station i to station j have a maximum capacity of u_{ij} kW and a cost of c_{ij} euros for each kW carried by the wires. The company wants to determine the minimum cost distribution plan, under the assumption that the total amount of energy produced equals the total amount of energy required by all sub-networks.

Network flows models: single commodity

Parameters: u_{ij}, c_{ij} and

$G = (N, A)$, $N =$ power/distribution stations, $A =$ connections between stations

$b_v = d_v - p_v$, $v \in N$ [demand ($b_v > 0$)/supply (< 0)/transshipment ($= 0$) node]

Variables:

x_{ij} amount of energy to flow on arc $(i, j) \in A$

$$\begin{aligned} \min \quad & \sum_{(i,j)\in A} c_{ij}x_{ij} \\ s.t. \quad & \sum_{(i,v)\in A} x_{iv} - \sum_{(v,j)\in A} x_{vj} = b_v \quad \forall\, v \in N \\ & x_{ij} \leq u_{ij} \quad \forall\, (i,j) \in A \\ & x_{ij} \in \mathbb{R}_+ \end{aligned}$$

Minimum Cost Network Flow Problem

Network flows models: single commodity

Parameters: u_{ij}, c_{ij} and

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Minimum Cost Network Flow Problem

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Minimum Cost Network Flow Problem

An multi-type energy flow problem

A company distributing electric energy has several power and distributing stations connected by wires. Each station produces/distributes different kinds of energy. Each station i can:

- produce p_i^k kW of energy of type k (it may be $p_i^k = 0$);
- distribute energy of type k on a sub-network whose users have a total demand of d_i^k kW (it may be $d_i^k = 0$);
- carry energy from/to different stations.

Note that every station can produce and/or distribute different types of energy. The wires connecting station i to station j have a maximum capacity of u_{ij} kW, independently of the type of energy carried. The transportation cost depends both on the pair of stations (i, j) and the type of energy k, and is equal to c_{ij}^k euros for each kW. The company wants to determine the minimum cost distribution plan, under the assumption that, for each type of energy, the total amount produced equals the total amount of energy of the same type required by all sub-networks.

Network flows models: multi-commodity

Parameters: u_{ij}, c_{ij}^k, K (set of energy types or **commodities**) and

$G = (N, A)$, $N =$ power/distribution stations, $A =$ connections between stations

$b_v^k = d_v^k - p_v^k$, $v \in N$ [demand ($b_v^k > 0$)/supply (< 0)/transshipment ($= 0$) node]

Variables:

x_{ij}^k amount of energy of type k to flow on arc $(i, j) \in A$

$$\min \sum_{k \in K} \sum_{(i,j) \in A} c_{ij}^k x_{ij}^k$$

$$\text{s.t.} \sum_{(i,v) \in A} x_{iv}^k - \sum_{(v,j) \in A} x_{vj}^k = b_v^k \quad \forall\, v \in N, \forall\, k \in K$$

$$\sum_{k \in K} x_{ij}^k \leq u_{ij} \quad \forall\, (i,j) \in A$$

$$x_{ij}^k \in \mathbb{R}_+ \quad \forall\, (i,j) \in A, \forall\, k \in K$$

Minimum Cost Network Multi-commodity Flow Problem

Network flows models: multi-commodity

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Minimum Cost Network Multi-commodity Flow Problem

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Minimum Cost Network Multi-commodity Flow Problem

Lab exercise - Part I

Problem description

A company produces boards with holes used to build electric frames. Boards are positioned over a machines and a drill moves over the board, stops at the desired positions and makes the holes. Once a board is drilled, a new board is positioned and the process is iterated many times. Given the position of the holes on the board, the company asks us to determine the hole sequence that minimizes the total drilling time, taking into account that the time needed for making an hole is the same and constant for all the holes.

Lab exercise - Part I: graph model

- Can be seen as a Traveling Salesman Problem (TSP)
  - graph $G = (N, A)$: $N =$ holes, $A =$ trajectories from i to j, $i, j \in N$
  - find a minimum weight (c_{ij}) hamiltonian cycle on G
- Can be modeled as a network flow problem
  - $b_0 = -|N|, \quad (0 \in N$ is an arbitrary node)
  - $b_i = 1, \forall\, i \in N \setminus \{0\}$
  - the cost is related to the use of an arc (fixed cost)
  - at most one incoming and one outgoing arc, for each node
- Decision variables:

  x_{ij} amount of the flow shipped from i to j, $\forall\, (i,j) \in A$;
  y_{ij} 1 if arc (i,j) ships some flow, 0 otherwise, $\forall\, (i,j) \in A$.

Lab exercise - Part I: ILP model

$$
\begin{array}{llll}
\min & \sum_{i,j:(i,j)\in A} c_{ij} y_{ij} & & \\
s.t. & \sum_{j:(0,j)\in A} x_{0j} = |N| & & \\
 & \sum_{i:(i,k)\in A} x_{ik} - \sum_{j:(k,j)\in A} x_{kj} = 1 & & \forall\, k \in N \setminus \{0\} \\
 & \sum_{j:(i,j)\in A} y_{ij} = 1 & & \forall\, i \in N \\
 & \sum_{i:(i,j)\in A} y_{ij} = 1 & & \forall\, j \in N \\
 & x_{ij} \leq |N|\, y_{ij} & & \forall\, (i,j) \in A \\
 & x_{ij} \in \mathbb{Z}_+ & & \forall\, (i,j) \in A \\
 & y_{ij} \in \{0,1\} & & \forall\, (i,j) \in A
\end{array}
$$