

Linear Programming and the Simplex method

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Mathematical Programming models

$$\begin{aligned} \min(\max) \quad & f(x) \\ \text{s.t.} \quad & g_i(x) = b_i \quad (i = 1 \dots k) \\ & g_i(x) \leq b_i \quad (i = k + 1 \dots k') \\ & g_i(x) \geq b_i \quad (i = k' + 1 \dots m) \\ & x \in \mathbb{R}^n \end{aligned}$$

- $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a vector (column) of n **REAL** variables;
- f e g_i are functions $\mathbb{R}^n \rightarrow \mathbb{R}$
- $b_i \in \mathbb{R}$

Linear Programming (LP) models

f e g_i are **linear** functions of x

$$\begin{array}{ll} \min(\max) & c_1x_1 + c_2x_2 + \dots + c_nx_n \\ \text{s.t.} & a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i \quad (i = 1 \dots k) \\ & a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \leq b_i \quad (i = k + 1 \dots k') \\ & a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \geq b_i \quad (i = k' + 1 \dots m) \\ & x_j \in \mathbb{R} \quad (i = 1 \dots n) \end{array}$$

Notice: for the moment, just **CONTINUOUS variables are considered!!!**

We need different methods for models with integer or binary variables.

Resolution of an LP model

- *Feasible solution*: $x \in \mathbb{R}^n$ satisfying all the constraints
- *Feasible region*: set of all the feasible solutions x
- *Optimal solution* x^* [min]: $c^T x^* \leq c^T x, \forall x \in \mathbb{R}^n, x$ feasible.

Solving a LP model is determining if it:

- is unfeasible
- is unlimited
- has a (finite) optimal solution

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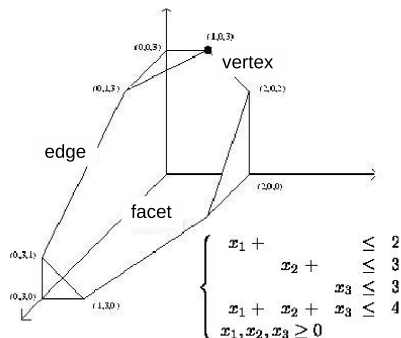
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Geometry of LP

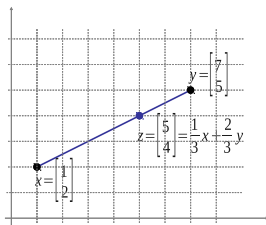
The feasible region is a **polyhedron** (intersection of a finite number of closed half-spaces and hyperplanes in \mathbb{R}^n)



LP problem: $\min(\max)\{c^T x : x \in P\}$, P is a polyhedron in \mathbb{R}^n .

Vertex of a polyhedron: definition

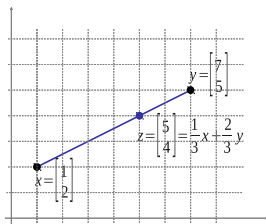
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$$z = \lambda x + (1 - \lambda)y$$



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- $v \in P$ is **vertex of a polyhedron** P if it is **not a strict convex combination** of two *distinct* points of P :
 $\nexists x, y \in P, \lambda \in (0, 1) : x \neq y, v = \lambda x + (1 - \lambda)y$

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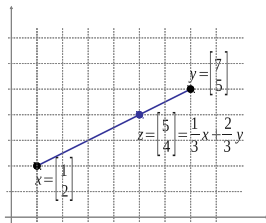
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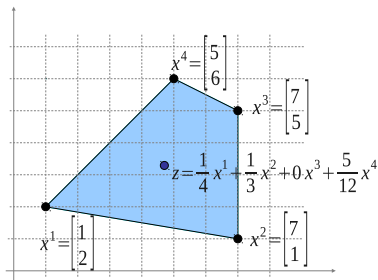


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Representation of polyhedra

$z \in \mathbb{R}^n$ is **convex combination** of $x^1, x^2 \dots x^k$ if $\exists \lambda_1, \lambda_2 \dots \lambda_k \geq 0$:

$$\sum_{i=1}^k \lambda_i = 1 \text{ and } z = \sum_{i=1}^k \lambda_i x^i$$



Theorem: representation of polyhedra [Minkowski-Weyl] - case limited

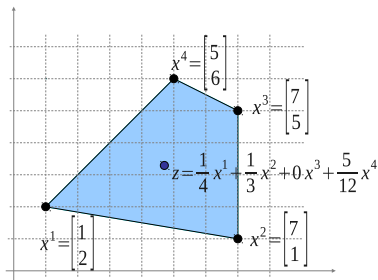
Polydron *limited* $P \subseteq \mathbb{R}^n$, v^1, v^2, \dots, v^k ($v^i \in \mathbb{R}^n$) vertices of P

if $x \in P$ then $x = \sum_{i=1}^k \lambda_i v^i$ with $\lambda_i \geq 0, \forall i = 1..k$ and $\sum_{i=1}^k \lambda_i = 1$
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Optimal vertex: from graphical intuition to proof

Theorem: optimal vertex (fix *min* objective function)

LP problem $\min\{c^T x : x \in P\}$, P non empty and limited

- LP ha optimal solution
- **one of the optimal solution of LP is a vertex of P**

Proof:

$$V = \{v^1, v^2 \dots v^k\} \quad v^* = \arg \min_{v \in V} c^T v$$

$$c^T x = c^T \sum_{i=1}^k \lambda_i v^i = \sum_{i=1}^k \lambda_i c^T v^i \geq \sum_{i=1}^k \lambda_i c^T v^* = c^T v^* \sum_{i=1}^k \lambda_i = c^T v^*$$

Summarizing: $\forall x \in P, c^T v^* \leq c^T x$ ■

We can limit the search of an optimal solution to the vertices of P !

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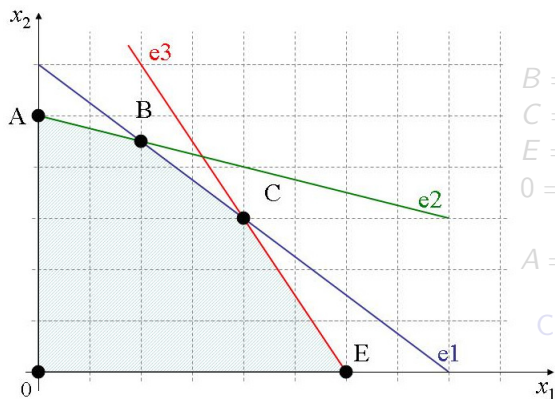
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Vertex comes from intersection of generating hyperplanes

$$\begin{aligned} \max \quad & 13x_1 + 10x_2 \\ \text{s.t.} \quad & 3x_1 + 4x_2 \leq 24 \quad (\text{e1}) \\ & x_1 + 4x_2 \leq 20 \quad (\text{e2}) \\ & 3x_1 + 2x_2 \leq 18 \quad (\text{e3}) \\ & x_1, x_2 \geq 0 \end{aligned}$$

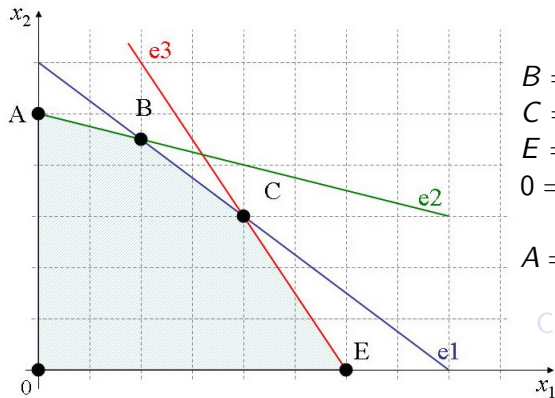


$B = e1 \cap e2$	$(2, 9/2)$	71
$C = e1 \cap e3$	$(4, 3)$	82
$E = e3 \cap (x_2 = 0)$	$(6, 0)$	78
$0 = (x_1 = 0) \cap (x_2 = 0)$	$(0, 0)$	0
$A = e2 \cap (x_1 = 0)$	$(0, 5)$	50

C optimum!

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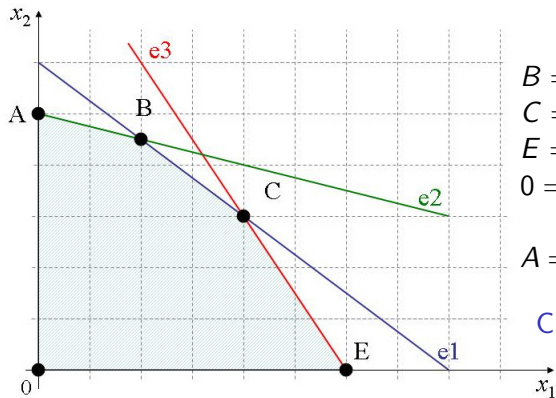


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Algebraic representation of vertices

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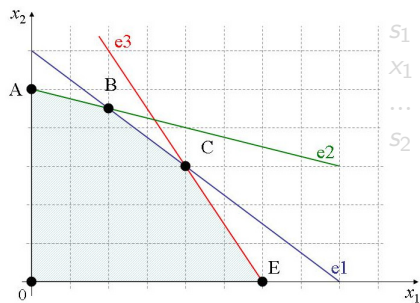
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5 - 3 = 2 degrees of freedom:

we can set (any) two variables to 0 and obtain a unique solution!



$$s_1 = s_2 = 0 \quad (2, 9/2, 0, 0, 3) \quad B$$

$$x_1 = s_2 = 0 \quad (0, 5, 4, 0, 8) \quad A$$

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$$s_2 = s_3 = 0 \quad (3.2, 4.2, -2.4, 0, 0)$$

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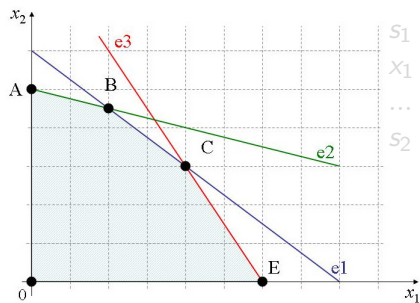
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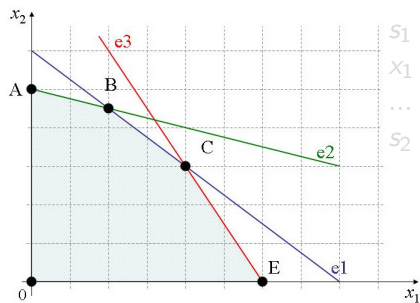
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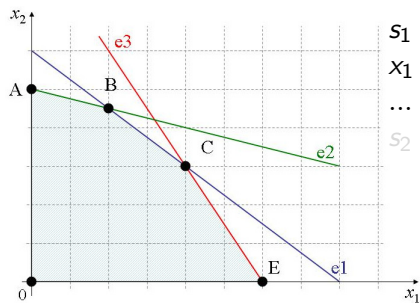
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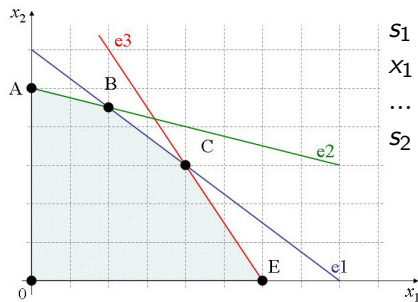
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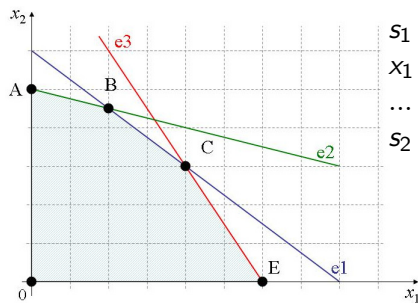
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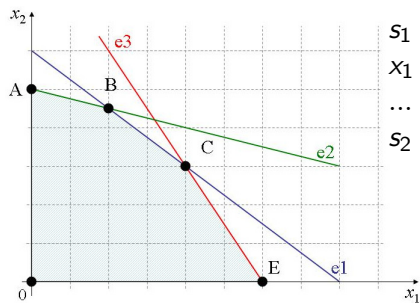
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Standard form for LP problems

$$\begin{array}{ll} \min & c_1x_1 + c_2x_2 + \dots + c_nx_n \\ \text{s.t.} & a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i \quad (i = 1 \dots m) \\ & x_i \in \mathbb{R}_+ \quad (i = 1 \dots n) \end{array}$$

- **minimizing** objective function (if not, multiply by -1);
- variables ≥ 0 ; (if not, substitution)
- all constraints are equalities; (+/- slack/surplus variables)
- $b_i \geq 0$. (if not, multiply by -1)

Standard form: example

$$\begin{aligned} \max \quad & 5(-3x_1 + 5x_2 - 7x_3) + 34 \\ \text{s.t.} \quad & -2x_1 + 7x_2 + 6x_3 - 2x_1 \leq 5 \\ & -3x_1 + x_3 + 12 \geq 13 \\ & x_1 + x_2 \leq -2 \\ & x_1 \leq 0 \\ & x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \hat{x}_1 &= -x_1 & (\hat{x}_1 \geq 0) \\ x_3 &= x_3^+ - x_3^- & (x_3^+ \geq 0, x_3^- \geq 0) \end{aligned}$$

$$\begin{aligned} \min \quad & -3\hat{x}_1 - 5x_2 + 7x_3^+ - 7x_3^- \\ \text{s.t.} \quad & 4\hat{x}_1 + 7x_2 + 6x_3^+ - 6x_3^- + s_1 = 5 \\ & 3\hat{x}_1 + x_3^+ - x_3^- - s_2 = 1 \\ & \hat{x}_1 - x_2 - s_3 = 2 \\ & \hat{x}_1 \geq 0, x_2 \geq 0, x_3^+ \geq 0, x_3^- \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0. \end{aligned}$$

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Linear algebra: definitions

- column vector $v \in \mathbb{R}^{n \times 1}$: $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$
- row vector $v^T \in \mathbb{R}^{1 \times n}$: $v^T = [v_1, v_2, \dots, v_n]$
- matrix $A \in \mathbb{R}^{m \times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$
- $v, w \in \mathbb{R}^n$, scalar product $v \cdot w = \sum_{i=1}^n v_i w_i = v^T w = v w^T$
- Rank of $A \in \mathbb{R}^{m \times n}$, $\rho(A)$, max linearly independent rows/columns
- $B \in \mathbb{R}^{m \times m}$ invertible $\iff \rho(B) = m \iff \det(B) \neq 0$

Systems of linear equations

- *Systems of equations in matrix form*: a system of m equations in n variables can be written as:

$$Ax = b, \text{ where } A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \text{ e } x \in \mathbb{R}^n.$$

- *Theorem of Rouché-Capelli*:

$$Ax = b \text{ has solutions } \iff \rho(A) = \rho(A|b) = r \text{ (}\infty^{n-r} \text{ solutions)}.$$

- *Elementary row operations*:

- ▶ swap row i and row j ;
- ▶ multiply row i by a non-zero scalar;
- ▶ substitute row i by row i plus α times row j ($\alpha \in \mathbb{R}$).

Elementary operations on (augmented) matrix $[A|b]$ leave the same solutions as $Ax = b$.

- *Gauss-Jordan method* for solving $Ax = b$: make elementary row operations on $[A|b]$ so that A contains an identity matrix of dimension $\rho(A) = \rho(A|b)$.

Basic solutions

- **Assumptions:** system $Ax = b$, $A \in \mathbb{R}^{m \times n}$, $\rho(A) = m$, $m < n$
- **Basis of A :** square submatrix with maximum rank, $B \in \mathbb{R}^{m \times m}$
- $A = [B|N]$ $B \in \mathbb{R}^{m \times m}$, $\det(B) \neq 0$
 $x = \begin{bmatrix} x_B \\ x_N \end{bmatrix}$, $x_B \in \mathbb{R}^m$, $x_N \in \mathbb{R}^{n-m}$
- $Ax = b \implies [B|N] \begin{bmatrix} x_B \\ x_N \end{bmatrix} = Bx_B + Nx_N = b$
- $x_B = B^{-1}b - B^{-1}Nx_N$
- imposing $x_N = 0$, we obtain a so called **basic solution**:
 $x = \begin{bmatrix} x_B \\ x_N \end{bmatrix} = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix}$
- many different basic solutions by choosing a **different basis** of A
- **variables equal to 0** are $n - m$ (or more: *degenerate* basic solutions)

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- $Ax = b \implies [B|N] \begin{bmatrix} x_B \\ x_N \end{bmatrix} = Bx_B + Nx_N = b$

- $x_B = B^{-1}b - B^{-1}Nx_N$

- imposing $x_N = 0$, we obtain a so called **basic solution**:

$$x = \begin{bmatrix} x_B \\ x_N \end{bmatrix} = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix}$$

- many different basic solutions by choosing a **different basis** of A
- **variables equal to 0** are $n - m$ (or more: *degenerate* basic solutions)

Basic solutions

- **Assumptions:** system $Ax = b$, $A \in \mathbb{R}^{m \times n}$, $\rho(A) = m$, $m < n$
- **Basis of A :** square submatrix with maximum rank, $B \in \mathbb{R}^{m \times m}$
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Basic solutions and LP in standard form

$$\min \quad c_1x_1 + c_2x_2 + \dots + c_nx_n$$

$$\text{s.t.} \quad a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i \quad (i = 1 \dots m)$$

$$x_j \in \mathbb{R}_+$$

$$\min \quad c^T x$$

$$\text{s.t.} \quad Ax = b$$

$$x \geq 0$$

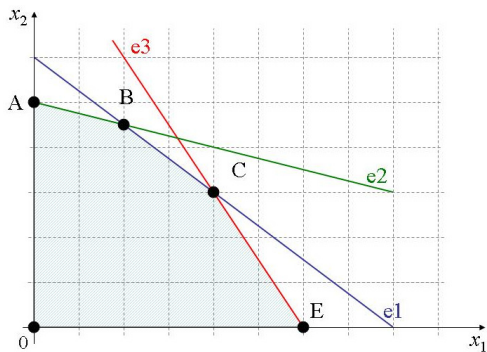
- basis B gives a **feasible basic solution** if $x_B = B^{-1}b \geq 0$

$$3x_1 + 4x_2 + s_1 = 24$$

$$x_1 + 4x_2 + s_2 = 20$$

$$3x_1 + 2x_2 + s_3 = 18$$

$$A = \begin{bmatrix} 3 & 4 & 1 & 0 & 0 \\ 1 & 4 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 24 \\ 20 \\ 18 \end{bmatrix}$$



Basic solutions and LP in standard form

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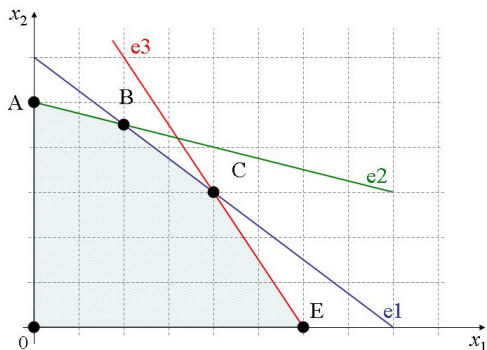
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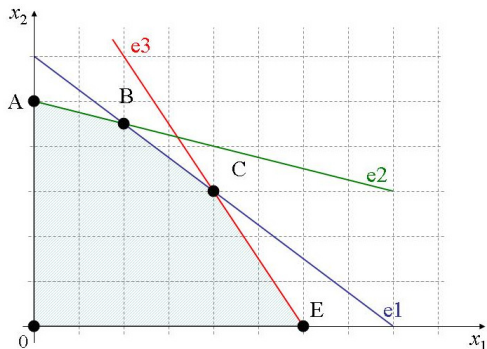
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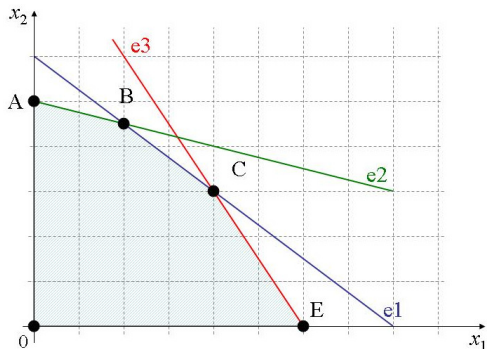
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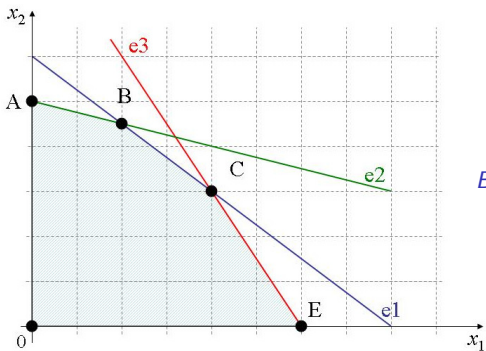
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$$B_1 = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

Basic solutions and LP in standard form

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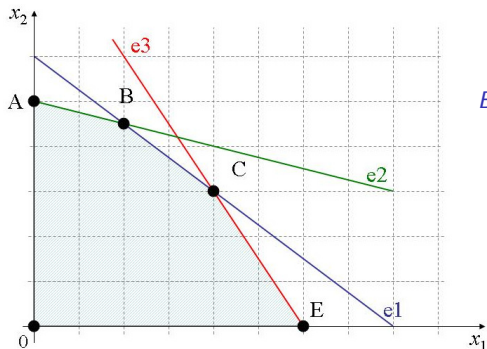
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$$B_1 = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$x_B = \begin{bmatrix} x_1 \\ x_2 \\ s_3 \end{bmatrix} = B_1^{-1}b = \begin{bmatrix} 2 \\ 4,5 \\ 3 \end{bmatrix}$$

$$x_N = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Basic solutions and LP in standard form

$$\min \quad c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

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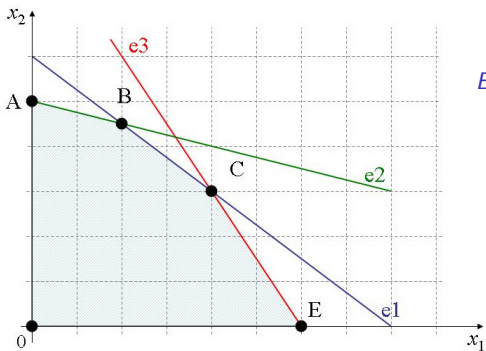
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$$x^T = (2 \ 9/2 \ 0 \ 0 \ 3) \quad \rightarrow \text{vertex B}$$

Basic solutions and LP in standard form

$$\min \quad c_1x_1 + c_2x_2 + \dots + c_nx_n$$

$$\text{s.t.} \quad a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i \quad (i = 1 \dots m)$$

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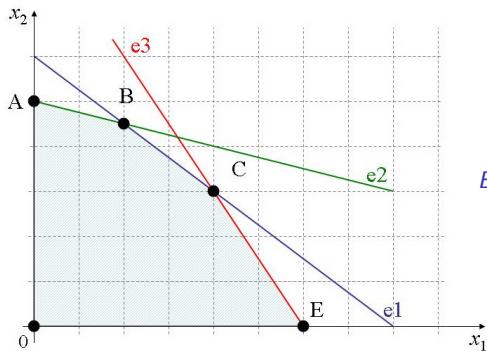
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$$B_2 = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 4 & 1 \\ 3 & 2 & 0 \end{bmatrix}$$

Basic solutions and LP in standard form

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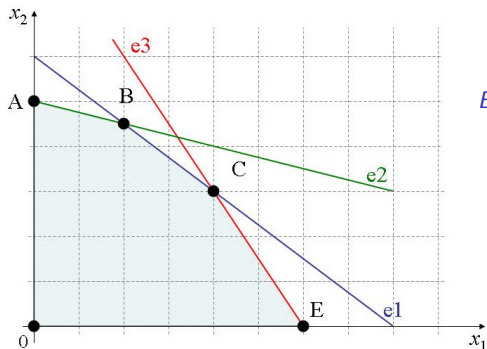
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$$x_B = \begin{bmatrix} x_1 \\ x_2 \\ s_2 \end{bmatrix} = B_2^{-1}b = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

$$x_N = \begin{bmatrix} s_1 \\ s_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Basic solutions and LP in standard form

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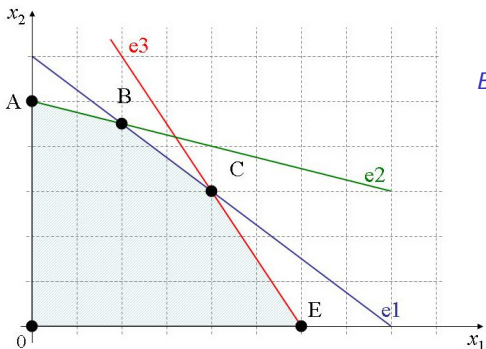
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$$x_N = \begin{bmatrix} s_1 \\ s_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x^T = (4 \ 3 \ 0 \ 2 \ 0) \rightarrow \text{vertex C}$$

Basic solutions and LP in standard form

$$\min \quad c_1x_1 + c_2x_2 + \dots + c_nx_n$$

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$$(i = 1 \dots n)$$

$$\min \quad c^T x$$

$$\text{s.t.} \quad Ax = b$$

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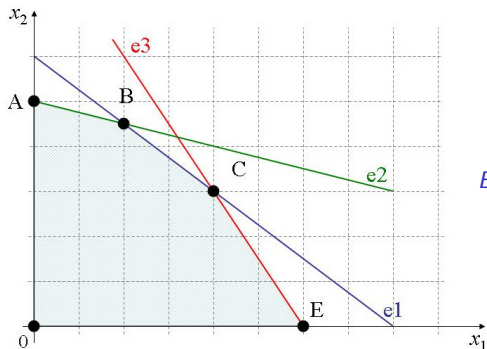
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$$B_3 = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix}$$

Basic solutions and LP in standard form

$$\min \quad c_1x_1 + c_2x_2 + \dots + c_nx_n$$

$$\text{s.t.} \quad a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i \quad (i = 1 \dots m)$$

$$x_j \in \mathbb{R}_+$$

$$(i = 1 \dots n)$$

$$\min \quad c^T x$$

$$\text{s.t.} \quad Ax = b$$

$$x \geq 0$$

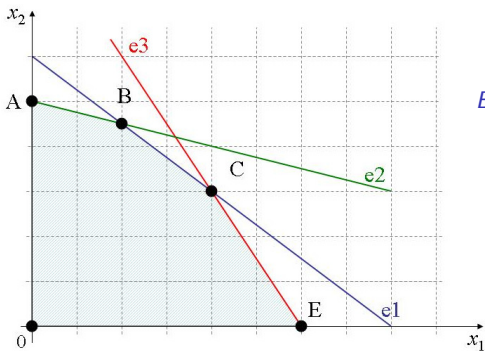
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$$B_3 = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix}$$

$$x_B = \begin{bmatrix} x_1 \\ s_1 \\ s_2 \end{bmatrix} = B_3^{-1}b = \begin{bmatrix} 6 \\ 6 \\ 14 \end{bmatrix}$$

$$x_N = \begin{bmatrix} x_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Basic solutions and LP in standard form

$$\min \quad c_1x_1 + c_2x_2 + \dots + c_nx_n$$

$$\text{s.t.} \quad a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i \quad (i = 1 \dots m)$$

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$$\text{s.t.} \quad Ax = b$$

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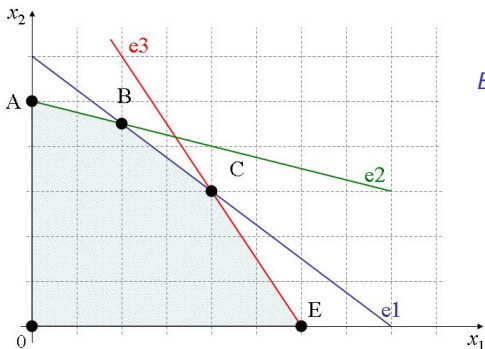
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$$x^T = (6 \ 0 \ 6 \ 14 \ 0) \rightarrow \text{vertex E}$$

Basic solutions and LP in standard form

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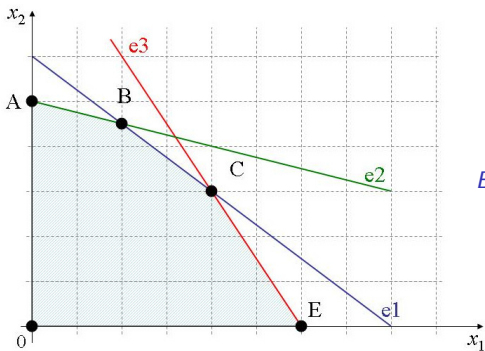
- basis B gives a **feasible basic solution** if $x_B = B^{-1}b \geq 0$

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Basic solutions and LP in standard form

$$\min \quad c_1x_1 + c_2x_2 + \dots + c_nx_n$$

$$\text{s.t.} \quad a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i \quad (i = 1 \dots m)$$

$$x_j \in \mathbb{R}_+$$

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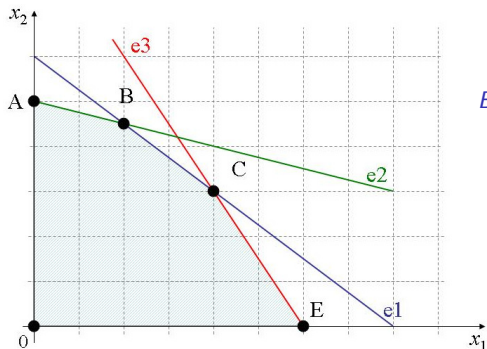
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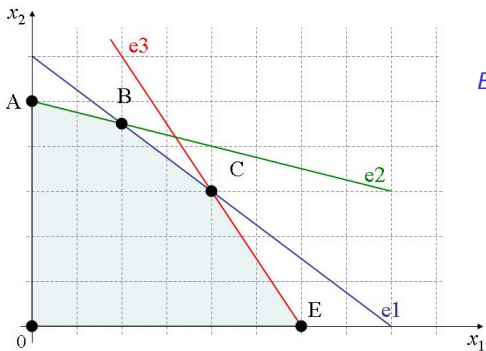
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Vertices and basic solution

Feasible basic solution \rightsquigarrow $n - m$ variables are 0 \rightsquigarrow
intersection of the right number of hyperplanes \rightsquigarrow vertex!

$$\text{PL } \min\{c^T x : Ax = b, x \geq 0\} \quad P = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$$

Theorem: vertices correspond to feasible basic solutions
(algebraic characterization of the vertices of a polyhedron)

x feasible basic solution of $Ax = b$ \iff x is a vertex of P

Corollary: optimal basic solution

If P non empty and limited, then **there exists at least an optimal solution which is a basic feasible solution**

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Algorithm for LP (case limited): sketch

Consider **all** the feasible basic solutions:

- 1 put the LP in standard form $\min\{c^T x : Ax = b, x \geq 0\}$
- 2 *incumbent* = $+\infty$
- 3 **repeat**
- 4 generate a combination of m columns of A
- 5 let B be the corresponding submatrix of A
- 6 **if** $\det(B) == 0$ **then continue** **else** compute $x_B = B^{-1}b$
- 7 **if** $x_B \geq 0$ **and** $c^T x_B < \textit{incumbent}$ **then** update *incumbent*
- 8 **until**(no other column combinations)

Complexity: up to $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ basic solution!!!

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Example

LP problem in **standard form**:

$$\begin{array}{ll} \min & z = -13x_1 - 10x_2 \\ \text{s.t.} & 3x_1 + 4x_2 + s_1 = 24 \\ & x_1 + 4x_2 + s_2 = 20 \\ & 3x_1 + 2x_2 + s_3 = 18 \\ & x_1, x_2, s_1, s_2, s_3 \geq 0 \end{array}$$

an initial **basic feasible solution** (vertex B):

- $B = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$
- $x_B = \begin{bmatrix} x_1 \\ x_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 9/2 \\ 3 \end{bmatrix} \quad x_N = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- $z_B = c^T x = c_B^T x_B + c_N^T x_N = -71$

Example

Change basis: **New basic solution** \Rightarrow one non-basic variable increases
affecting x_B and z_B

$$x_B = B^{-1}b - B^{-1}N x_N$$

$$\begin{aligned} z &= c^T x = c_B^T x_B + c_N^T x_N = c_B^T (B^{-1}b - B^{-1}N x_N) + c_N^T x_N \\ &= c_B^T B^{-1}b + (c_N^T - c_B^T B^{-1}N) x_N \end{aligned}$$

Write x_B and z as functions of only **non-basic** variables

For the sake of manual computation, use **Gauss-Jordan**:

$$Ax = b \quad \rightsquigarrow \quad [B \ N \mid b] \quad \rightsquigarrow \quad [B^{-1}B = I \ B^{-1}N = \bar{N} \mid B^{-1}b = \bar{b}]$$

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$(R_1 - 1/2 R_2)$	1	0	0	1/2	-1/2	2
$(3/8 R_2)$	0	1	0	-1/8	3/8	9/2
$(R_3 + 3/4 R_2)$	0	0	1	-5/4	3/4	3

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- while preserving non-negativity:

$$x_1 \geq 0 \Rightarrow 2 + 1/2 s_2 \geq 0 \Rightarrow s_2 \geq -4 \text{ always!}$$

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New basic solution! s_2 (now > 0) takes the place of s_3 (now $= 0$):

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LP in *canonical* form

PL $\min\{z = c^T x : Ax = b, x \geq 0\}$ is in **canonical form with respect to basis B** if all basic variables and the objective are explicitly written as functions of **non-basic variables only**:

$$\begin{aligned} z &= \bar{z}_B + \bar{c}_{N_1} x_{N_1} + \bar{c}_{N_2} x_{N_2} + \dots + \bar{c}_{N_{(n-m)}} x_{N_{(n-m)}} \\ x_{B_i} &= \bar{b}_i - \bar{a}_{iN_1} x_{N_1} - \bar{a}_{iN_2} x_{N_2} - \dots - \bar{a}_{iN_{(n-m)}} x_{N_{(n-m)}} \quad (i = 1 \dots m) \end{aligned}$$

\bar{z}_B scalar (objective function value for the corresponding basic solution)

\bar{b}_i scalar (value of basic variable i)

B_i index of the i -th basic variable ($i = 1 \dots m$)

N_j index of the j -th non-basic variable ($j = 1 \dots n - m$)

\bar{c}_{N_j} coefficient of the j -th non-basic variable in the objective function (**reduced cost of the variable with respect to basis B**)

$-\bar{a}_{iN_j}$ coefficient of the j -th non-basic variable in the constraints that makes explicit the i -th basic variable

Simplex method: optimality check

- **Reduced cost** of a variable: marginal unit increment of the objective function
- The reduced cost of a basis variable is $\bar{c}_{B_i} = 0$

Theorem: Sufficient optimality conditions

Given an LP and a feasible basis B , if all the reduced costs with respect to B are ≥ 0 , then B is an optimal basis

$$\bar{c}_j \geq 0, \forall j = 1 \dots n \quad \Rightarrow \quad B \text{ optimal}$$

- Notice: the inverse is not true! [there may be optimal basic solutions with negative reduced costs]

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Simplex method: basis change

- From feasible basis B , obtain a \tilde{B} **adjacent, feasible, improving**
- **One** column (\approx variable) enters and one variable leaves the basis

- **Entering** variable (improvement): any $x_h : \bar{c}_h < 0$

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- From feasible basis B , obtain a \tilde{B} **adjacent, feasible, improving**
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Simplex method: check for unlimited LP

- Let x_h : $\bar{c}_h < 0$.

$$z = \bar{z}_B + \bar{c}_h x_h$$

$$x_{B_i} = \bar{b}_i - \bar{a}_{ih} x_h \quad (i = 1 \dots m)$$

- If $\bar{a}_{ih} \leq 0, \forall i = 1 \dots m$, feasible solution with $x_h \rightarrow +\infty$

Condition of unlimited LP

There exists a basis such that

$$\exists x_h : (\bar{c}_h < 0) \wedge (\bar{a}_{ih} \leq 0, \forall i = 1 \dots m)$$

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Entering variable: any $x_h : \bar{c}_h < 0$

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Simplex tableau

- Represent the canonical form, can be used to operate Gauss-Jordan
- **Objective function as a constraint** (imposing the value of a new variable z):

$$z = c_1x_1 + c_2x_2 + \dots + c_nx_n \rightsquigarrow c_1x_1 + c_2x_2 + \dots + c_nx_n - z = 0$$

	x_{B_1}	\dots	x_{B_m}	x_{N_1}	\dots	$x_{N_{n-m}}$	z	\bar{b}
riga 0	c_B^T			c_N^T			-1	0
riga 1	B			N			0	b
\vdots							\vdots	
riga m							0	

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	x_{B_1}	...	x_{B_m}	x_{N_1}	...	$x_{N_{n-m}}$	z	\bar{b}
riga 0	0	...	0	\square	...	\square	-1	\square
riga 1	1		0	\square	...	\square	0	\square
\vdots		\ddots		\square	...	\square	\vdots	\square
riga m	0		1	\square	...	\square	0	\square

Tableau in canonical form

- Elementary row (z included) operations: up to reading x_B (and z) as functions of x_N

Tableau and canonical form

	x_{B_1}	...	x_{B_m}	x_{N_1}	...	$x_{N_{n-m}}$	z	\bar{b}
$-z$	0	...	0	\square	...	\square	-1	\square
x_{B_1}	1		0	\square	...	\square	0	\square
x_{B_i}		\ddots		\square	...	\square	\vdots	\square
x_{B_m}	0		1	\square	...	\square	0	\square

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Tableau and canonical form

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$-z$	0	...	0	\bar{c}_{N_1}	...	$\bar{c}_{N_{n-m}}$	-1	$-\bar{z}_B$
x_{B_1}	1		0	$\bar{a}_{1 N_1}$...	$\bar{a}_{1 N_{n-m}}$	0	\bar{b}_1
x_{B_i}		...		$\bar{a}_{i N_1}$...	$\bar{a}_{i N_{n-m}}$	\vdots	\bar{b}_i
x_{B_m}	0		1	$\bar{a}_{m N_1}$...	$\bar{a}_{m N_{n-m}}$	0	\bar{b}_m

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Retrieving an initial feasible basis: **two-phases method**

- **Phase I:** solve an *artificial problem*

$$\begin{aligned} w^* = \min w = & \quad 1^T y = y_1 + y_2 + \cdots + y_m \\ \text{s.t.} \quad & \quad Ax + Iy = b \\ & \quad x, y \geq 0 \end{aligned} \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \in \mathbb{R}_+^m$$

If $w^* > 0$, the original problem is unfeasible, stop!

If $w^* = 0$, then $y = 0$

- ▶ if some y in the (degenerate) basis, change basis to put all y out, thus obtaining an x_B feasible for the original problem!

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Simplex algorithm with matrix operations: revised simplex

$$\min z = c^T x$$

$$\text{s.t. } Ax = b$$

$$x \geq 0$$

standard form

$$\min z = c_B^T x_B + c_N^T x_N$$

$$\text{s.t. } Bx_B + Nx_N = b$$

$$x_B, x_N \geq 0$$

with (feasible) basis

$$-z + \bar{c}_N^T x_N = -z_B$$

$$I x_B + \bar{N} x_N = \bar{b}$$

canonical form

$$\bullet \bar{b} = B^{-1}b$$

$$\bullet z_B = c_B^T \bar{b}$$

$$\bullet \bar{N} = B^{-1}N$$

$$\bullet \bar{c}_N^T = c_N^T - c_B^T B^{-1}N$$

The (revised) simplex algorithm

- 1 Let $\beta[1], \dots, \beta[m]$ be the column indexes of the **initial basis**
- 2 Let $B = [A_{\beta[1]} | \dots | A_{\beta[m]}]$ and compute B^{-1} e $u^T = c_B^T B^{-1}$
- 3 compute **reduced costs**: $\bar{c}_h = c_h - u^T A_h$ for non-basic variables x_h
- 4 If $\bar{c}_h \geq 0$ for all non-basic variables x_h , **STOP**: B is **optimal**
- 5 Choose any x_h having $\bar{c}_h < 0$
- 6 Compute $\bar{b} = B^{-1}b = [\bar{b}_i]_{i=1}^m$ e $\bar{A}_h = B^{-1}A_h = [\bar{a}_{ih}]_{i=1}^m$
- 7 If $\bar{a}_{ih} \leq 0, \forall i = 1 \dots m$, **STOP: unlimited**
- 8 Determine $t = \arg \min_{i=1 \dots m} \{\bar{b}_i / \bar{a}_{ih}, \bar{a}_{ih} > 0\}$
- 9 Change basis: $\beta[t] \leftarrow h$.
- 10 Iterate from Step 2

Iteration 1: steps 2–5

$$x_B^T = [x_4 \quad x_5 \quad x_6] \quad c_B^T = [0 \quad 0 \quad 0]$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$u^T = c_B^T B^{-1} = [0 \quad 0 \quad 0] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [0 \quad 0 \quad 0]$$

$$\bar{c}_1 = c_1 - u^T A_1 = -3 - [0 \quad 0 \quad 0] \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = -3 - 0 = -3$$

$$\bar{c}_2 = c_2 - u^T A_2 = -1 - [0 \quad 0 \quad 0] \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = -1 - 0 = -1 \quad h = 2 \text{ (} x_2 \text{ enters)}$$

$$\bar{c}_3 = c_3 - u^T A_3 = -3 - [0 \quad 0 \quad 0] \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = -3 - 0 = -3$$

Iteration 1: steps 6–9

$$\bar{b} = B^{-1}b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} \quad \begin{array}{l} x_4 \\ x_5 \\ x_6 \end{array}$$

$$\bar{A}_h = B^{-1}A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$t = \arg \min \left\{ \frac{2}{1} \quad \frac{5}{2} \quad \frac{6}{2} \right\} = \arg \left(\frac{2}{1} \right) = 1 \quad \rightsquigarrow x_4 \text{ leaves}$$

$$\beta[1] = 2 \quad (\text{column 2 replaces } \beta[1] \text{ that was 4})$$

Iteration 2: steps 2–5

$$x_B^T = [x_2 \quad x_5 \quad x_6] \quad c_B^T = [-1 \quad 0 \quad 0]$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$u^T = c_B^T B^{-1} = [-1 \quad 0 \quad 0] \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = [-1 \quad 0 \quad 0]$$

$$\bar{c}_1 = c_1 - u^T A_1 = -3 - [-1 \quad 0 \quad 0] \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = -3 - (-2) = -1$$

$$\bar{c}_3 = c_3 - u^T A_3 = -3 - [-1 \quad 0 \quad 0] \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = -3 + 1 = -2 \quad h = 3$$

(\hat{x}_3 enters)

$$\bar{c}_4 = c_4 - u^T A_4 = 0 - [-1 \quad 0 \quad 0] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0 - (-1) = 1$$

Iteration 2: steps 6–9

$$\bar{b} = B^{-1}b = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \quad \begin{matrix} x_2 \\ x_5 \\ x_6 \end{matrix}$$

$$\bar{A}_h = B^{-1}A_3 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$t = \arg \min \left\{ \frac{2}{1} \quad \frac{1}{1} \quad X \right\} = \arg \left(\frac{1}{1} \right) = 2 \quad \rightsquigarrow x_5 \text{ leaves}$$

$$\beta[2] = 3 \quad (\text{column 3 replaces column } \beta[2] \text{ that was 5})$$

Iteration 3: steps 2–5

$$x_B^T = [x_2 \quad \hat{x}_3 \quad x_6] \quad c_B^T = [-1 \quad -3 \quad 0]$$

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 0 \\ 2 & 1 & 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 0 \\ -4 & 1 & 1 \end{bmatrix}$$

$$u^T = c_B^T B^{-1} = [-1 \quad -3 \quad 0] \begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 0 \\ -4 & 1 & 1 \end{bmatrix} = [3 \quad -2 \quad 0]$$

$$\bar{c}_1 = c_1 - u^T A_1 = -3 - [3 \quad -2 \quad 0] \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = -3 - (4) = -7 \quad h = 1$$

(x_1 enters)

It is not necessary to compute all reduced costs, stop as soon **one of them** is negative!

Iteration 3: steps 6–9

$$\bar{b} = B^{-1}b = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 0 \\ -4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \quad \begin{matrix} x_2 \\ \hat{x}_3 \\ x_6 \end{matrix}$$

$$\bar{A}_h = B^{-1}A_1 = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 0 \\ -4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ -5 \end{bmatrix}$$

$$t = \arg \min \left\{ \frac{1}{5} \quad X \quad X \right\} = \arg \left(\frac{1}{5} \right) = 1 \quad \rightsquigarrow x_2 \text{ leaves}$$

$$\beta[1] = 1 \quad (\text{column 1 replaces column } \beta[1] \text{ that was 2})$$

Iteration 4

$$x_B^T = [x_1 \quad \hat{x}_3 \quad x_6] \quad c_B^T = [-3 \quad -3 \quad 0]$$

$$B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 2 & 1 & 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 3/5 & -1/5 & 0 \\ -1/5 & 2/5 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$u^T = c_B^T B^{-1} = [-3 \quad -3 \quad 0] \begin{bmatrix} 3/5 & -1/5 & 0 \\ -1/5 & 2/5 & 0 \\ -1 & 0 & 1 \end{bmatrix} = [-6/5 \quad -3/5 \quad 0]$$

$$\bar{c}_2 = c_2 - u^T A_2 = -1 - [-6/5 \quad -3/5 \quad 0] \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = -1 - (12/5) = 7/5$$

$$\bar{c}_4 = c_4 - u^T A_4 = 0 - [-6/5 \quad -3/5 \quad 0] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0 - (6/5) = 6/5$$

$$\bar{c}_5 = c_5 - u^T A_5 = 0 - [-6/5 \quad -3/5 \quad 0] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0 - (3/5) = 3/5$$

Optimal solution

Standard form (the one we solved by simplex method):

$$\bullet x_B^* \begin{bmatrix} x_1 \\ \hat{x}_3 \\ x_6 \end{bmatrix} = B^{-1}b = \begin{bmatrix} 3/5 & -1/5 & 0 \\ -1/5 & 2/5 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 8/5 \\ 4 \end{bmatrix}$$

$$\bullet x_1^* = 1/5; x_2^* = 0; \hat{x}_3^* = 8/5; x_4^* = 0; x_5^* = 0; x_6^* = 4$$

$$\bullet z_{MIN}^* = c^T x^* = c_B^T x_B^T = \begin{bmatrix} -3 & -3 & 0 \end{bmatrix} \begin{bmatrix} 1/5 \\ 8/5 \\ 4 \end{bmatrix} = -27/5$$

Optimal solution for the initial problem:

$$\bullet x_1^* = 1/5$$

$$\bullet x_2^* = 0$$

$$\bullet x_3^* = -\hat{x}_3^* = -8/5$$

• first constraint satisfied with equality (since $x_4^* = 0$)

• second constraint satisfied with equality (since $x_5^* = 0$)

• third constraint satisfied with a slack of 4 (since $x_6^* = 4$)

$$\bullet z_{MAX}^* = -z_{MIN}^* = 27/5.$$

