Polynomial algorithm for CDBP: trace of the execution
G. Andreatta, C. De Francesco, L. De Giovanni, P. Serafini

We are going to apply the algorithm\(^1\) for CDBP to the tree \(T(V,E)\) in Figure 1, considered as rooted in node 1. Tree \(T\) contains \(n=5\) nodes, \(\bar{N} = \{0,1,\lfloor n/2 \rfloor = 2\}\) and \(1 \leq p \leq \lfloor n/2 \rfloor\). For any node that is not a leaf, its sons are arbitrarily ordered from left to right.

![Figure 1: the tree \(T(V,E)\)](image)

**Initialization**
Declare all nodes as not examined.
For \(v = 4,5\) leaves, consider \(T_v = \{v\}\) and \(f = f(v) = 3\):
\[
K^+(T_v) : \chi(v) = \chi(3) = b = \emptyset; \\
K^+(T_v) : \chi(v) = b, \chi(3) = w = \{1\}; \\
S^+(T_v, k = 1) : \chi(v) = b, \chi(3) = w = \{1\}; \\
S^+(T_v, k = 0) : \chi(v) = w, \chi(3) = b = \emptyset; \\
K^+(T_v) : \chi(v) = w, \chi(3) = w = \emptyset.
\]
Declare \(v = 4,5\) examined.

**Main iteration**

**Step 1, with \(v = 3\) and \(\ell = 1\)**

**Case 1.1:** \(\chi(3) = b\).
\[
K(T(3,1) : \chi(3) = b) = \{1\} \oplus K^+(T_4) : \chi(4) = w, \chi(3) = b) = \{1\} \oplus \{0\} = \{1\}. \\
S(T(3,1), k = 1) : \chi(3) = b) = \{3\} \cup \emptyset = \{3\}. \\
K^-(T(3, 1) : \chi(3) = b) = \{1\} \oplus K^+(T_4) : \chi(4) = b, \chi(3) = b) = \{1\} \oplus \emptyset = \emptyset.
\]

**Case 1.2:** \(\chi(3) = w\).
\[
K(T(3,1) : \chi(3) = w) = K^+(T_4) : \chi(4) = b, \chi(3) = w) = \{1\}. \\
S(T(3,1), k = 1) : \chi(3) = w) = S^+(T_4, k = 1) : \chi(4) = b, \chi(3) = w) = \{4\}. \\
K^-(T(3, 1) : \chi(3) = w) = K^+(T_4) : \chi(4) = w, \chi(3) = w) = \emptyset.
\]

**Step 2, with \(v = 3\) and \(\ell = 2\)**

**Case 2.1:** \(\chi(3) = b\).
\[
K(T(3,2) : \chi(3) = b) = \\
= K(T(3,1) : \chi(3) = b) \oplus K^+(T_5) : \chi(5) = w, \chi(3) = b) \cup \\
\cup K(T(3,1) : \chi(3) = b) \oplus K^+(T_5) : \chi(5) = b, \chi(3) = b) \cup \\
\cup K^-(T(3, 1) : \chi(3) = b) \oplus K^+(T_5) : \chi(5) = w, \chi(3) = b) = \\
\]

\(^1\)G. Andreatta, C. De Francesco, L. De Giovanni, P. Serafini, Constrained Domatic Bipartition of a Tree, 2015
Let $k$ for $k \in K(T(3,2) \mid \chi(3) = b) = \{1\}$ and $k'' \in K^+(T_5) \mid \chi(5) = w, \chi(3) = b = \{0\}$ as in case (2.1.a): 

$$S(T(3,2), k = 1 \mid \chi(3) = b) = S(T(3,1), k' = 1 \mid \chi(3) = b) \cup S^+(T_5), k'' = 0 \mid \chi(5) = w, \chi(3) = b = \{3\} \cup \emptyset = \{3\}.$$

Let $K^-(T(3,2) \mid \chi(3) = b) = K^-(T(3,1) \mid \chi(3) = b) \oplus K^+(T_5) \mid \chi(5) = b, \chi(3) = b = \emptyset \oplus \emptyset = \emptyset$.

**Case 2.2:** $\chi(3) = w$.

$$K(T(3,2) \mid \chi(3) = w) = K(T(3,1) \mid \chi(3) = w) \oplus K^+(T_5) \mid \chi(5) = b, \chi(3) = w) \cup \cup K(T(3,1) \mid \chi(3) = w) \oplus K^+(T_5) \mid \chi(5) = w, \chi(3) = w) \cup \cup K^-(T(3,1) \mid \chi(3) = w) \oplus K^+(T_5) \mid \chi(5) = b, \chi(3) = w) = \cup \{1\} \oplus \{1\} \cup \{1\} \oplus \emptyset \cup \emptyset \oplus \{1\} = \{1\} \oplus \{1\} = \{2\}.$$

For $k \in K(T(3,2) \mid \chi(3) = w = \{2\}$, it is $k = k' + k''$ with $k' \in K(T(3,1) \mid \chi(3) = w) = \{1\}$ and $k'' \in K^+(T_5) \mid \chi(5) = b, \chi(3) = w = \{1\}$ as in case (2.2.a):

$$S(T(3,2), k = 2 \mid \chi(3) = w) = S(T(3,1), k' = 1 \mid \chi(3) = w) \cup S^+(T_5), k'' = 1 \mid \chi(5) = b, \chi(3) = w) = \{4\} \cup \{5\} = \{4, 5\}.$$

Let $K^-(T(3,2) \mid \chi(3) = w) = K^-(T(3,1) \mid \chi(3) = w) \oplus K^+(T_5) \mid \chi(5) = w, \chi(3) = w) = \emptyset \oplus \emptyset = \emptyset$.

**Step 3, with $v = 3$**

**Case 3.1:** $\chi(3) = b$ and $\chi(2) = b$.

$$K^+(T_3 \mid \chi(3) = b, \chi(2) = b) = K(T(3,2) \mid \chi(3) = b) = \{1\} \text{ and } S^+(T_3, k = 1 \mid \chi(3) = b, \chi(2) = b) = S(T(3,2), k = 1 \mid \chi(3) = b) = \{3\}.$$

**Case 3.2:** $\chi(3) = b$ and $\chi(2) = w$.

$$K^+(T_3 \mid \chi(3) = b, \chi(2) = w) = K(T(3,2) \mid \chi(3) = b) \cup K^-(T(3,2) \mid \chi(3) = w) = \{1\} \cup \emptyset = \{1\} \text{ and } S^+(T_3, k = 1 \mid \chi(3) = b, \chi(2) = w) = S(T(3,2), k = 1 \mid \chi(3) = b) = \{3\}.$$

**Case 3.3:** $\chi(3) = b$ and $\chi(2) = w$.

$$K^+(T_3 \mid \chi(3) = w, \chi(2) = b) = K(T(3,2) \mid \chi(3) = w) \cup K^-(T(3,2) \mid \chi(3) = w) = \{2\} \cup \emptyset = \{2\} \text{ and } S^+(T_3, k = 2 \mid \chi(3) = w, \chi(2) = b) = S(T(3,2), k = 2 \mid \chi(3) = w) = \{4, 5\}.$$

**Case 3.4:** $\chi(3) = w$ and $\chi(2) = w$.

$$K^+(T_3 \mid \chi(3) = w, \chi(2) = w) = K(T(3,2) \mid \chi(3) = w) = \{2\} \text{ and } S^+(T_3, k = 2 \mid \chi(3) = w, \chi(2) = w) = S(T(3,2), k = 2 \mid \chi(3) = w) = \{4, 5\}.$$

Declare node $v = 3$ examined.

**Step 1, with $v = 2$ and $\ell = 1$**

**Case 1.1:** $\chi(2) = b$.

$$K(T(2,1) \mid \chi(2) = b) = \{1\} \oplus K^+(T_3 \mid \chi(3) = w, \chi(2) = b) = \{1\} \oplus \{2\} = \emptyset.$$

$$K^-(T(2,1) \mid \chi(2) = b) = \{1\} \oplus K^+(T_3 \mid \chi(3) = b, \chi(2) = b) = \{1\} \oplus \{1\} = \{2\}.$$

$$S^- (T(2,1), k = 2 \mid \chi(2) = b) = \{2\} \cup S^+(T_3, k = 1 \mid \chi(3) = b, \chi(2) = b) = \{2\} \cup \{3\} = \{2, 3\}.$$
Case 1.2: \( \chi(2) = w \).
\( K(T(2,1) \mid \chi(2) = w) = K^+(T_3 \mid \chi(3) = b, \chi(2) = w) = \{1\} \).
\( S(T(2,1), k = 1 \mid \chi(2) = w) = S^+(T_3, k = 1 \mid \chi(3) = b, \chi(2) = w) = \{3\} \).
\( K^-(T(2,1) \mid \chi(2) = w) = K^+(T_3 \mid \chi(3) = w, \chi(2) = w) = \{2\} \).
\( S^-(T(2,1), k = 2 \mid \chi(2) = w) = S^+(T_3, k = 2 \mid \chi(3) = w, \chi(2) = w) = \{4,5\} \).

**Step 3, with \( v = 2 \)**

Case 3.1: \( \chi(2) = b \) and \( \chi(1) = b \).
\( K^+(T_2 \mid \chi(2) = b, \chi(1) = b) = K(T(2,1) \mid \chi(2) = b) = \emptyset \).

Case 3.2: \( \chi(2) = b \) and \( \chi(1) = w \).
\( K^+(T_2 \mid \chi(2) = b, \chi(1) = w) = K(T(2,1) \mid \chi(2) = b) \cup K^-(T(2,1) \mid \chi(2) = b) = \emptyset \cup \{2\} = \{2\} \).
\( S^+(T_2, k = 2 \mid \chi(2) = b, \chi(1) = w) = S^-(T(2,1), k = 2 \mid \chi(2) = b) = \{2,3\} \).

Case 3.3: \( \chi(2) = w \) and \( \chi(1) = b \).
\( K^+(T_2 \mid \chi(2) = w, \chi(1) = b) = K(T(2,1) \mid \chi(2) = w) \cup K^-(T(2,1) \mid \chi(2) = w) = \{1\} \cup \{2\} = \{1,2\} \).
\( S^+(T_2, k = 1 \mid \chi(2) = w, \chi(1) = b) = S(T(2,1), k = 1 \mid \chi(2) = w) = \{3\} \).
\( S^+(T_2, k = 2 \mid \chi(2) = w, \chi(1) = b) = S^-(T(2,1), k = 2 \mid \chi(2) = w) = \{4,5\} \).

Case 3.4: \( \chi(2) = w \) and \( \chi(1) = w \).
\( K^+(T_2 \mid \chi(2) = w, \chi(1) = w) = K(T(2,1) \mid \chi(2) = w) = \{1\} \).
\( S^+(T_2, k = 1 \mid \chi(2) = w, \chi(1) = w) = S(T(2,1), k = 1 \mid \chi(2) = w) = \{3\} \).

Declare node \( v = 2 \) examined.

**Step 1, with \( v = 1 \) and \( \ell = 1 \)**

Case 1.1: \( \chi(1) = b \).
\( K(T(1,1) \mid \chi(1) = b) = \{1\} \oplus K^+(T_2 \mid \chi(2) = w, \chi(1) = b) = \{1\} \oplus \{1,2\} = \{2\} \).
\( S(T(1,1), k = 2 \mid \chi(1) = b) = \{1\} \cup S^+(T_2, k = 1 \mid \chi(2) = w, \chi(1) = b) = \{1\} \cup \{3\} = \{1,3\} \).
\( K^-(T(1,1) \mid \chi(1) = b) = \{1\} \oplus K^+(T_2 \mid \chi(2) = b, \chi(1) = b) = \{1\} \oplus \emptyset = \emptyset \).

Case 1.2: \( \chi(1) = w \).
\( K(T(1,1) \mid \chi(1) = w) = K^+(T_2 \mid \chi(2) = b, \chi(1) = w) = \{2\} \).
\( S(T(1,1), k = 2 \mid \chi(1) = w) = S^+(T_2, k = 2 \mid \chi(2) = b, \chi(1) = w) = \{2,3\} \).
\( K^-(T(1,1) \mid \chi(1) = w) = K^+(T_2 \mid \chi(2) = w, \chi(1) = w) = \{1\} \).
\( S^-(T(1,1), k = 1 \mid \chi(1) = w) = S^+(T_2, k = 1 \mid \chi(2) = w, \chi(1) = w) = \{3\} \).

**Step 4 (Final Step)**

Case 4.1: \( \chi(1) = b \).
Set \( K_b = K(T(1,1) \mid \chi(1) = b) = \{2\} \) and \( S_b(2) = S(T(1,1), k = 2 \mid \chi(1) = b) = \{1,3\} \).

Case 4.2: \( \chi(1) = w \).
Set \( K_w = K(T(1,1) \mid \chi(1) = w) = \{2\} \) and \( S_w(2) = S(T(1,1), k = 2 \mid \chi(1) = w) = \{2,3\} \).

Declare node \( v = 1 \) examined.

CDBP has a solution for \( p \in K_b \cup K_w = \{2\} \) and the tree \( T \) has (at least) two domatic bipartitions, one with 2 black nodes in \( S_b(2) \), the other with 2 black nodes in \( S_w(2) \). CDBP on tree \( T \) has also a solution for \( p = 3 \).