Convergent and divergent series, solutions of the Prolate Spheroidal differential equation

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Abstract

The prolate spheroidal wave functions, \( \{ \varphi_{n,\sigma,\tau} \} \), constitute an orthonormal basis of the space of \( \sigma \)-bandlimited functions on the real line, i.e. functions whose Fourier transforms have support on the interval \([-\sigma, \sigma]\). They can be characterized as the eigenfunctions of a differential operator of order 2:

\[
(\tau^2 - t^2)\varphi''_{n,\sigma,\tau} - 2t\varphi'_n,\sigma,\tau - \sigma^2 t^2 \varphi_{n,\sigma,\tau} = \mu_{n,\sigma,\tau} \varphi_{n,\sigma,\tau}.
\]

In this talk we will present some new results obtained on the formal solutions of this equation. For this purpose, we specialize to particular values of the two parameters: \( \sigma = \tau = 1 \).

We use the MAPLE package DESIR to compute the formal solutions in the neighborhood of the singularities (the regular ones \( \pm 1 \), and the irregular one, infinity) and to do some numerical experiments: computation of Stokes matrices [1] and of monodromy. This leads to the conjecture that the following properties are equivalent:

- \( \mu \) is an eigenvalue of the differential operator \( L = (t^2 - 1) \frac{d^2}{dt^2} + 2t \frac{d}{dt} + t^2 \);
- the series solutions near \( \pm 1 \) of the equation \( L(y) = \mu y \) are entire functions (and so, eigenfunctions);
- the series appearing in the solutions near infinity of the equation \( L(y) = \mu y \) are convergent;
- the Stokes phenomenon of the operator \( L - \mu \) at infinity is trivial;
- the monodromy around \([-1, 1]\) of the operator \( L - \mu \) is trivial.

The second part of the talk will give the proof of the conjecture.

References