Abstract

We provide a result on the isospectrality for a class of one dimensional periodic Schrödinger operators in relation to the homogenization theory of Hamilton-Jacobi equation defined on the flat torus.

Keywords: Schrödinger operators, Spectral problem, Homogenization theory

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1. Introduction

Let us consider the one dimensional flat torus $T := \mathbb{R}/2\pi\mathbb{Z}$, $V, A \in C^\infty(T; \mathbb{R})$ in order to define the Hamiltonian function $H(x, p) := \frac{1}{2}|p - A(x)|^2 + V(x)$. We consider the class of Schrödinger operators

$$\tilde{H} := \frac{1}{2}(-ih\nabla_x - A(x))^2 + V(x)$$

with domain $W^{2,2}(T; \mathbb{C})$. In this setting, the spectrum of the operator is discrete, which we denote by $E_{\tilde{H}}$, $\ell \in \mathbb{N}$. Our target is the study of the isospectrality problem

$$\text{Spec}(\tilde{H}_1) = \text{Spec}(\tilde{H}_2) \quad \forall 0 < \hbar \leq 1,$$

in connection to the homogenization theory applied to the Hamilton-Jacobi equation

$$\frac{1}{2}|P + \nabla_x S(P, x) - A(x)|^2 + V(x) = \Pi(P), \quad P \in \mathbb{R},$$

where $\Pi(P)$ is called the effective Hamiltonian and is a convex map (see for example [10], [11], [15], [18]) and $S$ is any viscosity solution. The following inf-sup formula (which works for a rather general class of p-convex and superlinear Hamiltonians) provides a way to represent the effective Hamiltonian,

$$\Pi(P) = \inf_{v \in C^1(T)} \sup_{x \in T} \frac{1}{2}|P + \nabla_x v(x) - A(x)|^2 + V(x).$$

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The target of our paper is to show that the isospectrality condition (2) implies a constraint on the related two effective Hamiltonians.

The content of our main result is the following

**Theorem 1.1.** Let \( H_\alpha(x, p) := \frac{1}{2} |p - A_\alpha(x)|^2 + V_\alpha(x) \) with \( \alpha = 1, 2 \) and such that \( \max V_1 = \max V_2 \). If

\[
\text{Spec}(\tilde{H}_1) = \text{Spec}(\tilde{H}_2) \quad \forall 0 < h \leq 1,
\]

then

\[
\overline{H}_1(P) = \overline{H}_2(P) \quad \forall P \in \mathbb{R}.
\]

In the Section 4 of [18] it is shown a result on the link between the homogenization of Hamilton-Jacobi equation and the spectrum of the Hill operator \(-\frac{1}{2} \frac{d^2}{dx^2} + V(x)\), namely when \( h = 1 \). In this setting, the authors proved that the isospectrality implies the same ‘viscous’ effective Hamiltonian \( \overline{H}_{\text{visc}} \), namely the function such that

\[
-\Delta_x w(P, x) + |P + \nabla_x w(P, x)|^2 + V(x) = \overline{H}_{\text{visc}}(P), \quad P \in \mathbb{R},
\]

for a smooth function \( w : \mathbb{R} \times \mathbb{T} \to \mathbb{R} \). Our Theorem 1.1 provides another result on the link between spectral problem and homogenization theory, for a larger class of Hamiltonians and by using the Bohr-Sommerfeld rules (see Section 2). We also recall Thm. 5.2 in [21], working in one dimension and under some nondegeneracy conditions on the Hamiltonian dynamics, which shows that for such isospectral operators there exists a symplectic map \( \psi : \mathbb{R} \times \mathbb{T} \to \mathbb{R} \times \mathbb{T} \) such that

\[
H_2 = H_1 \circ \psi
\]

at least locally. With respect to this observation, we also underline a symplectic invariance property of \( \overline{H} \) in arbitrary dimension \( n \). As shown in [3], for all the time one Hamiltonian flows \( \varphi \equiv \varphi^1 : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n \times \mathbb{R}^n \) with \( C^1 \) - regularity, we have the invariance

\[
\overline{H} \circ \varphi = \overline{H}.
\]

Notice that (8) cannot be directly applied to (9) since a symplectic map could be not an Hamiltonian flows.

The work [16] deals with inverse spectral results for the Laplacian \( \Delta \) on a Hermitian line bundle \( L \) over a flat torus plus a potential function \( Q \) on \( M \). In particular, it is shown that the spectra \( \text{Spec}(Q; \nabla, L) \) when \( \nabla \) ranges over all the translation invariant connections, uniquely determines the potential \( Q \). Note that in our paper we consider operators like \(-h^2 \Delta_x + V \) for \( 0 < h \leq 1 \) hence we are considering the family of connections \( h \nabla_x \) on the flat torus \( \mathbb{T}^n := (\mathbb{R}/2\pi \mathbb{Z})^n \) which is not exhaustive for all the translation invariant ones.

On the link between the Schrödinger spectral problem and KAM tori into the phase space, we recall that in [1] for Schrödinger operators on \( \mathbb{T}^n \) and under the assumptions of the KAM Theorem, the authors provide semiclassical expansions for the eigenfunctions and eigenvalues, and showed the link with \( \overline{H}(P) \).
Our paper deals with the problem of isospectrality for a class of Schrödinger operators. The fact that the isospectrality is considered for any value of $\hbar$ is not a restrictive assumption, as shown for example in the reference [21] as mentioned above, or also in the paper [24] where isospectrality is connected with the convex hull of the image of the momentum map.

To conclude the Introduction, we stress that a study of the link between the above effective Hamiltonian $\mathcal{H}$, viscosity solutions of Hamilton-Jacobi equation and eigenfunctions related to the spectrum (or energy quasimodes) in a semiclassical framework should involve also their phase space localization. Some preliminary results, beyond the one dimensional case, have been obtained in [2], [5], [22], [28]. For the time evolution of WKB-type wave functions we address the reader for example to [19], [23] and references therein.

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### 2. Some remarks on Bohr-Sommerfeld rules

As shown in [15], in one dimensional setting the effective Hamiltonian can be given by the inversion of the map

$$ E \mapsto \mathcal{J}(E) := \frac{1}{2\pi} \int_{0}^{2\pi} \sqrt{2(E - V(x))} \, dx, \quad E \geq \max V, \quad (10) $$

namely

$$ \mathcal{H}(P) = \begin{cases} \max V & \text{if } |P| \leq \mathcal{J}(\max V), \\ \mathcal{J}^{-1}(P) & \text{otherwise.} \end{cases} \quad (11) $$

Moreover, we stress that $\mathcal{J}(E)$ equals the leading term of the Weyl Law for the number of the eigenvalues smaller than $E$, and thus can be written as

$$ \mathcal{J}(E) = \text{Vol}\{(x, p) \in T \times \mathbb{R} : \frac{1}{2} p^2 + V(x) \leq E\}. \quad (12) $$

The Bohr-Sommerfeld Rules (see [26], [20]) in our one dimensional and periodic setting take the form

$$ S_{\hbar}(E_{h, \ell}) = 2\pi \hbar \ell \quad \text{for } \ell = 1, 2, ... \quad (13) $$

where $E_{h, \ell}$ are the eigenvalues of $-\frac{1}{2} \hbar^2 \Delta_x + V(x)$, and

$$ S_{\hbar}(E) \sim \sum_{j=0}^{\infty} \hbar^j S_j(E) = 2\pi J_0(E) + \frac{1}{2} \hbar \pi \mu(E) + \mathcal{O}(\hbar^2), \quad (14) $$
where
\[ J_0(E) = \int_{\gamma_E} p \, dx \] (15)
is the Action integral for the classical curve \( \gamma_E \) at energy \( E \) and positive momentum, i.e. \( p > 0 \). For \( E > \max V \) it is easily seen that
\[ J_0(E) = J(E) \] (16)
as given above. The value \( \mu(E) \) is the Maslov index of \( \gamma_E \) seen as a Lagrangian submanifold of \( T \times \mathbb{R} \); for \( E > \max V \) we have \( \mu(E) = 1 \). As shown in Prop. 5.2 of [20] such a semiclassical series is locally uniform in \( E \), which implies that the remainder vanishes as \( O(h^2) \) when \( E \) is in bounded interval. Thus, the above equalities (13) - (14) imply that two systems with the same Bohr-Sommerfeld Rules have necessarily the same effective Hamiltonian.

3. Hamilton-Jacobi equation

Let \( H \) be a Tonelli Hamiltonian, namely \( H \in C^2(T^n \times \mathbb{R}^n; \mathbb{R}) \) is such that the map \( p \mapsto H(x, p) \) is convex with positive definite Hessian and \( H(x, p)/\|p\| \to +\infty \) as \( \|p\| \to +\infty \).

For any \( P \in \mathbb{R}^n \), there exists a unique real number \( c = \mathcal{H}(P) \) such that the following cell problem on \( T^n \):
\[ H(x, P + \nabla_x S) = c, \] (17)
has a solution \( S = S(P, x) \) in the viscosity sense (see [14] and the references therein). As shown in [14], any viscosity solution is also a weak KAM solution of negative type and belongs to \( C^{0,1}(\mathbb{T}^n) \). Moreover, as shown in [25], any viscosity solution \( S \) exhibits \( C^{1,1}_{\text{loc}} \) - regularity outside the closure of its singular set \( \Sigma(S) \). In particular, \( \mathbb{T}^n \backslash \Sigma(S) \) is an open and dense subset of \( \mathbb{T}^n \). The function \( \mathcal{H} \) is called the effective Hamiltonian, it is a convex function and can be represented or approximated in various ways (see for example [3], [11], [12], [13]). In particular (see [3] and references therein) we have the following inf-sup formula. Let \( v \in C^{1,1}(\mathbb{T}^n) \) and \( \Gamma := \{(x, \nabla_x v(x)) \in \mathbb{T}^n \times \mathbb{R}^n \mid x \in \mathbb{T}^n\} \). Let \( \mathcal{G} \) be the set of all such sets \( \Gamma \).
\[ \mathcal{H}(P) = \inf_{\Gamma} \sup_{(x, p) \in \Gamma} H(x, p + P). \] (18)
The effective Hamiltonian equals the Mather’s \( \alpha(H) \) function (see [14], [27]).

The inf-sup formula can be computed also over \( v \in C^1(\mathbb{T}^n) \) (see for example [11]) but the Lipschitz regularity of \( \nabla v \) has the advantage of being the highest one for which the infimum is a minimum (see [4]).

As shown in Proposition 1 of [3], for any time one Hamiltonian flows \( \varphi \equiv \varphi^1 : T^n \times \mathbb{R}^n \to T^n \times \mathbb{R}^n \) with \( C^1 \) - regularity we have the invariance property
\[ \mathcal{H} \circ \varphi = \mathcal{H}. \] (19)
In the mechanical case $H = \frac{1}{2}|p|^2 + V(x)$ the effective Hamiltonian can be written as

$$\mathcal{H}(P) = \inf_{v \in C^1(T^n)} \sup_{x \in T^n} \frac{1}{2} |P + \nabla_x v(x)|^2 + V(x) \quad (20)$$

It is easily seen that $\mathcal{H}(0) = \inf_{v \in C^1(T^n)} \sup_{x \in T^n} \frac{1}{2} |P + \nabla_x v(x)|^2 + V(x) = \max V$.

The so-called viscous version of the Hamilton-Jacobi equation can be written as

$$-\Delta_x w(P, x) + |P + \nabla_x w(P, x)|^2 + V(x) = \mathcal{H}_{\text{visc}}(P), \quad P \in \mathbb{R}, \quad (21)$$

which admits a smooth solution $w : \mathbb{R} \times T \rightarrow \mathbb{R}$. The literature about this topic is very large, so that we address the reader to the paper [18] and the references therein.

4. Main Result

Proof of Theorem 1.1.

In this one dimensional setting, the spectrum of $H := \frac{1}{2}(-i\hbar \nabla_x - A(x))^2 + V(x)$ does not depend on $A$, since the unitary operator $(U \phi)(x) := \exp(i\phi(x)/\hbar) \phi(x)$ with $\phi(x) := \int_0^x A(y)dy$ provides the conjugation

$$U^* \circ \hat{H} \circ U = -\frac{1}{2} \hbar^2 \Delta_x + V(x) \quad (22)$$

Since the spectrum of these operators is discrete, then the unitary equivalence implies the same spectrum.

Moreover, also the effective Hamiltonian is not affected by $A(x)$ as can be seen by the inf-sup formula

$$\mathcal{H}(P) = \inf_{v \in C^1(T^n)} \sup_{x \in T^n} \frac{1}{2} |P + \nabla_x v(x) - A(x)|^2 + V(x), \quad (23)$$

where we can write $A(x) = \nabla_x \phi(x)$ and define $u(x) := v(x) - \phi(x)$ so that

$$\mathcal{H}(P) = \inf_{u \in C^1(T^n)} \sup_{x \in T^n} \frac{1}{2} |P + \nabla_x u(x)|^2 + V(x) \quad (24)$$

namely equals the effective Hamiltonian of $\frac{1}{2}|p|^2 + V(x)$. As a consequence of the above observations, in what follows we can consider only the case $A(x) = 0$.

In view of (13) and (14) we can write

$$2\pi J(E_{h, \ell}) + \frac{1}{2} \hbar^2 \mu(E_{h, \ell}) + O(h^2) = 2\pi h \ell. \quad (25)$$

Recalling (10) and (11), we have

$$\mathcal{H}(J(E)) = E \quad \forall E \geq \max V. \quad (26)$$

Thus, for any $E_{h, \ell} > \max V$ we have $\mu(E_{h, \ell}) = 1$ and

$$\mathcal{H}(h \ell - \frac{1}{4} \hbar - r_{h, \ell}) = E_{h, \ell}, \quad r_{h, \ell} = O(h^2). \quad (27)$$
Any vector $P \in \mathbb{R}$ in a bounded interval can be approximated by a sequence of kind $(\ell_j - 1/4)\hbar_j$ where $j \to +\infty$, $\ell_j \to +\infty$ and $\hbar_j \to 0^+$. We now remind the continuity of the map $P \mapsto \overline{H}(P)$ for any $P \in \mathbb{R}$ and the continuity of $P \mapsto \nabla \overline{H}(P)$ for those $P$ such that $\overline{H}(P) \geq \min \overline{H} + \varepsilon$ for some fixed $\varepsilon > 0$. In particular, notice that $\min \overline{H} = \max V$. In what follows we consider $\Omega$ as the set of $P \in \mathbb{R}$ such that $\overline{H}(P) > \min \overline{H} + \varepsilon$ and $|\nabla \overline{H}(P)| < \lambda$ for some fixed $\lambda > 0$ large enough (recall that $\overline{H}$ is a convex map, see [10]).

Thus, we can use the values $E_{h,\ell}$ of the spectrum of the operator $O_{h\ell}(H) = -\frac{1}{2} \hbar^2 \frac{d^2}{dx^2} + V(x)$ to recover the value $\overline{H}(P)$. Namely,

$$\overline{H}(P) = \lim_{j \to +\infty} \overline{H}(\ell_j - \mu/4)\hbar_j)$$

$$= \lim_{j \to +\infty} \overline{H}(\ell_j - \mu/4)\hbar_j - r_{h_j} + r_{h_j})$$

$$= \lim_{j \to +\infty} E_{h,\ell} + R_{h,\ell}.$$ (30)

The above remainder is defined as

$$R_{h,\ell} := \overline{H}(\ell_j - \mu/4)\hbar_j) - \overline{H}(\ell_j - \mu/4)\hbar_j - r_{h_j}).$$ (31)

We recall that $r_{h,\ell} = O(h^2)$ when $E_{h,\ell}$ lies in a bounded interval. Moreover, also the sequence $(\ell_j - \mu/4)\hbar_j$ lies in a bounded interval since $P \in \mathbb{R}$ belongs to a fixed bounded interval. The continuity of $\nabla \overline{H}$ with its uniformly boundedness on the prescribed set of $P$ and the previous observations ensure that

$$R_{h,\ell} \leq \sup_{0 \leq \alpha \leq 1} |\nabla \overline{H}(\alpha(\ell_j - \mu/4)\hbar_j) + (1 - \alpha)((\ell_j - \mu/4)\hbar_j - r_{h_j})||r_{h_j}|$$

$$\leq \|\nabla \overline{H}\|_{C^0(\Omega')} |r_{h_j}|.$$ (32)

The above convex combination belongs, for any $j$, to a suitable large bounded set $\Omega' \supset \Omega$ since $\forall \, P \in \Omega$ we have $(\ell_j - 1/4)\hbar_j \to P$ and $r_{h,\ell} \to 0$ as $j \to +\infty$. The limit (30) together with (32) allow to recover

$$\overline{H}(P) = \lim_{j \to +\infty} E_{h,\ell}.$$ (33)

Since we are assuming that for two potentials with the same maximum we have Schrödinger operators with the same spectrum, thus we necessarily have the same equality (33). For any fixed $P$ as prescribed above we have thus

$$\overline{H}_1(P) = \overline{H}_2(P).$$ (34)

To conclude, we recall that $\overline{H}(P) = \max V$ when $|P| \leq J(\max V)$ which is in our 1-dim case a closed ball centered in the origin (see [15]). Thus for such vectors $\overline{H}_1(P) = \overline{H}_2(P)$, namely we have also the equivalence of the minimum points of the two effective Hamiltonians.

\[ \square \]
References


