Hamilton-Jacobi homogenization
and the isospectral problem

Lorenzo Zanelli
Dept. of Mathematics “Tullio Levi-Civita”
University of Padova

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Abstract
We study the homogenization theory of Hamilton-Jacobi equations on
the one dimensional flat torus in connection to isospectrality problem
of Schrödinger operators. In particular, we link the equivalence of
effective Hamiltonians provided by the weak KAM theory with the
class of the corresponding operators with the same spectrum.

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1 Introduction
Let us consider the one dimensional flat torus $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, two functions
$V, A \in C^\infty(\mathbb{T}; \mathbb{R})$ and define $H(x, p) := \frac{1}{2}|p + A(x)|^2 + V(x)$. Consider the
Hamilton-Jacobi equation

$$\frac{1}{2}P + \nabla_x S(P, x) + A(x)^2 + V(x) = \overline{H}(P), \quad P \in \mathbb{R},$$

(1)

where the convex map $P \mapsto \overline{H}(P)$ is the effective Hamiltonian (see for example [9], [10], [11], [12], [13], [15], [19], [30]) whereas $S$ is a viscosity solution, in this case unique (see [4], [6], [8], [13] and references therein).

We recall the following inf-sup formula:

$$\overline{H}(P) = \inf_{v \in C^{1,1}(\mathbb{T})} \sup_{x \in \mathbb{T}} \frac{1}{2}P + \nabla_x v(x) + A(x)^2 + V(x).$$

(2)

The target of this paper is to study the functions $H$ with the same effective Hamiltonian

$$H_1 = \overline{H}_2$$

(3)

*lzamelli@math.unipd.it
in connection to the class of Schrödinger operators
\[
\hat{H} := \frac{1}{2}(-i\hbar \nabla_x + A(x))^2 + V(x)
\] (4)
and related isospectral problem
\[
\text{Spec}(\hat{H}_1) = \text{Spec}(\hat{H}_2) \quad \forall \, 0 < \hbar \leq 1,
\] (5)
In order to do so, we will make use of the inf-sup formula (2) together with
the well known Bohr-Sommerfeld rules on the (discrete) spectrum of (4),
that here we recall in Section 2.2 to prove the main result of the paper.
The target is to show that the (semiclassical) isospectrality condition implies
a constraint on the related two effective Hamiltonians.

The content of our main result is the following

**Theorem 1.1.** Let \( H_\alpha(x,p) := \frac{1}{2}|p + A_\alpha(x)|^2 + V_\alpha(x) \) with \( \alpha = 1, 2 \) such that \( \max V_1 = \max V_2 \) and \( \int_T A_1(x)dx = \int_T A_2(x)dx \). If
\[
\text{Spec}(\hat{H}_1) = \text{Spec}(\hat{H}_2) \quad \forall \, 0 < \hbar \leq 1,
\] (6)
then
\[
\overline{H}_1(P) = \overline{H}_2(P) \quad \forall P \in \mathbb{R}.
\] (7)
Conversely, if (7) holds true then \( \text{Spec}(\hat{H}_1) = \text{Spec}(\hat{H}_2) \) mod \( \mathcal{O}(\hbar^2) \) for \( E > \min \overline{H} \).

The assumptions on \( V \) and \( A \) are not restrictive, since these are in fact
necessary conditions to have the equality (7), as shown in Remark 3.2.
We recall that in section 4 of [19] it is shown a result on the link between homogenization theory of Hamilton-Jacobi equation and the spectrum of the Hill operator \(-\frac{1}{2}d^2dx + V(x)\), namely when \( \hbar = 1 \). In this setting, the authors proved that the isospectrality implies the same ‘viscous’ effective Hamiltonian \( \overline{H}_{visc} \), namely the function such that
\[
-\Delta_x w(P, x) + \frac{1}{2}|P + \nabla_x w(P, x)|^2 + V(x) = \overline{H}_{visc}(P), \quad P \in \mathbb{R},
\] (8)
for a unique \( C^2 \) function \( w : \mathbb{R} \times \mathbb{T} \to \mathbb{R} \). Moreover, in the paper [19] and also in [20] various results on the inverse problem in the theory of periodic homogenization of Hamilton-Jacobi equations are shown.

Our Theorem 1.1 provides a further link between the homogenization theory and spectral problem, for a larger class of Hamiltonians and by using other arguments with respect to [19]. Moreover, the second statement in Theorem 1.1 shows also that the effective Hamiltonian can be associated to the equivalent class of operators with the same spectrum mod \( \mathcal{O}(\hbar^2) \) above the energy value \( \min \overline{H} \).
We now recall Thm. 5.2 in [23], that works in one dimension and under some nondegeneracy conditions on the Hamiltonian dynamics, which shows that for such isospectral operators there exists (locally in a neighborhood of energies) a symplectic map $\psi : \mathbb{R} \times T \to \mathbb{R} \times T$ such that
\[ H_2 = H_1 \circ \psi. \] (9)

With respect to this observation, we also underline a symplectic invariance property of $\bar{H}$ in arbitrary dimension $n$. As shown in [3] for all the time one Hamiltonian flows $\varphi \equiv \varphi^1 : T^n \times \mathbb{R}^n \to T^n \times \mathbb{R}^n$ with $C^1$ - regularity, we have the invariance
\[ \overline{H \circ \varphi} = \overline{H}. \] (10)

Unluckily, the equality (9) cannot be applied to recover the equivalence of effective Hamiltonians (10) since it is well known that a symplectic map could not be (in general) an Hamiltonian flow.

Our paper is the first temptative (in the simple one-dim case) to provide a direct connection between the Hamilton-Jacobi homogenization and the inverse spectral problem for operators of kind (4) without passing through the equation (8) but taking into account only (1). We hope that the full generalization of Theorem 1.1 towards the $n$-dim case and arbitrary smooth potentials can be given by following the same ideas of the current paper, by using more general tools of spectral theory in place of Bohr-Sommerfeld rules that works only in a integrable setting like our one-dim case.

The fact that the isospectrality in Thm. 1.1 is for all the values $0 < \hbar \leq 1$ is not a restrictive assumption. Indeed, this can be found for example also in the above mentioned [23], and also in [26] where such a semiclassical isospectrality is considered for quantum integrable $\Psi$do and related to the convex hull of the image of the classical momentum map.

We remind that the original idea to apply techniques of Weak KAM theory into the framework of quantum mechanics goes back to L.C. Evans in the papers [9], [10] in order to study certain semiclassical approximation problems of Schrödinger eigenfunctions as $\hbar \to 0$.

The work [16] deals with some inverse spectral results for operators of kind $\Delta_x + Q(x)$, namely the Laplacian for an Hermitian line bundle $L$ plus a potential function $Q$ on a manifold $M$. In particular, it is shown that the spectra $\text{Spec}(Q; \nabla, L)$ when $\nabla$ ranges over all the translation invariant connections, uniquely determines the potential $Q$. Notice that in our paper we consider operators like $-\hbar^2 \Delta_x/2 - i\hbar A(x) \cdot \nabla_x + Q(x)$ for $0 < \hbar \leq 1$ hence we deal with the family of connections given by $\hbar \nabla_x$ on $T$.

On the link between the Schrödinger spectral problem and KAM tori into the phase space, we recall that in [2] for Schrödinger operators on $\mathbb{T}^n$ and under the assumptions of the KAM Theorem, the authors provide semiclassical expansions for the eigenfunctions and eigenvalues.
To conclude, we underline that a complete study of the link between the effective Hamiltonian, viscosity solutions of Hamilton-Jacobi equation and Schrödinger eigenvalue problem should involve also the phase space Analysis of eigenfunctions (and energy quasimodes). In this direction, some preliminary results for the n-dim case have been obtained in [5], [24], [31]. For the time evolution of WKB-type wave functions we address the reader to [17], [21], [25] and references therein.

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2 Preliminaries and Settings

2.1 Hamilton-Jacobi equation

In this section we recall standard results about Hamilton-Jacobi equation and effective Hamiltonian on $T^n \times \mathbb{R}^n$.

Let $H$ be a Tonelli Hamiltonian, namely $H \in C^2(T^n \times \mathbb{R}^n; \mathbb{R})$ is such that the map $p \mapsto H(x, p)$ is convex with positive definite Hessian and in addition $H(x, p)/\|p\| \to +\infty$ as $\|p\| \to +\infty$.

For any $P \in \mathbb{R}^n$, it is known that there exists a unique real number $c = \bar{H}(P)$ such that the problem on $T^n$:

$$H(x, P + \nabla_x S) = c, \quad (11)$$

has a solution $S = S(P, x)$ in the viscosity sense (see for example [13] and the references therein). Moreover, as shown in [13], any viscosity solution is also a weak KAM solution of negative type and belongs to $C^{0,1}(T^n; \mathbb{R})$.

Furthermore, as shown in [27], any viscosity solution $S$ exhibits the $C^{1,1}_{\text{loc}}$ - regularity outside the closure of its singular set $\Sigma(S)$ and $T^n \setminus \Sigma(S)$ is an open and dense subset of $T^n$.

The function $\overline{H}$ is called the effective Hamiltonian, it is a convex function and can be represented or approximated in various ways (see for example [9], [10], [11], [12], [13], [15], [19]).

In particular (see [3], [7] and references therein) we have an useful inf-sup formula. Let $v \in C^{1,1}(T^n)$ and $\Gamma := \{(x, \nabla_x v(x)) \in T^n \times \mathbb{R}^n \mid x \in T^n\}$, denote by $\mathcal{G}$ the set of all $\Gamma$. The effective Hamiltonian can be represented by the formula

$$\overline{H}(P) = \inf_{1 \in \mathcal{G}} \sup_{(x, p) \in \Gamma} H(x, p + P). \quad (12)$$

Moreover, such a value equals the Mather’s $\alpha(H)$ function (see [13], [30]).

As shown in Proposition 1 of [3], for any fixed time one Hamiltonian flow
\( \varphi \equiv \varphi^1 : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n \times \mathbb{R}^n \) with \( C^1 \) regularity we have the following invariance property

\[
\overline{H} \circ \varphi = \overline{H}.
\] (13)

In the mechanical case \( \mathcal{H}(x,p) := \frac{1}{2}|p|^2 + V(x) \) we have

\[
\overline{\mathcal{H}}(P) = \inf_{v \in C^{1,1}(\mathbb{T}^n)} \sup_{x \in \mathbb{T}^n} \frac{1}{2} |P + \nabla_x v(x)|^2 + V(x)
\] (14)

and it is easily seen that \( \overline{\mathcal{H}}(0) = \max V \). The so-called viscous version of the Hamilton-Jacobi equation

\[
-\Delta_x w(P,x) + \frac{1}{2} |P + \nabla_x w(P,x)|^2 + V(x) = \overline{\mathcal{H}}_{\text{visc}}(P), \quad P \in \mathbb{R}^n,
\] (15)

has a unique \( C^2 \)-solution \( w : \mathbb{R}^n \times \mathbb{T}^n \to \mathbb{R} \), see Thm. 5 in [14].

As shown in [15], in one dimensional setting the effective Hamiltonian can be given by the inversion of the map

\[
E \mapsto J(E) := \frac{1}{2\pi} \int_0^{2\pi} \sqrt{2(E - V(x))} \, dx, \quad E \geq \max V,
\] (16)

namely

\[
\overline{\mathcal{H}}(P) = \begin{cases} 
\max V & \text{if } |P| \leq J(\max V), \\
J^{-1}(P) & \text{otherwise}. 
\end{cases}
\] (17)

In Lemma 3.1 we show the effective Hamiltonian for \( H(x,p) := \frac{1}{2}|p + A(x)|^2 + V(x) = \mathcal{H}(x,p + A(x)) \) which turns out to be directly related to formula (17).

### 2.2 Bohr-Sommerfeld rules

The leading term of the Weyl Law (see [18]) for the number of the eigenvalues smaller than \( E \) (which is here supposed to be greater than \( \max V \)) is given by

\[
\mathcal{J}(E) := \text{Vol}\{(x,p) \in \mathbb{T} \times \mathbb{R} : \mathcal{H}(x,p) \leq E\}.
\] (18)

The so-called Bohr-Sommerfeld Rules (see [22], [28], [29]) in our one dimensional and periodic setting take the form

\[
S_h(E_{h,\ell}) = 2\pi h \ell \quad \text{for } \ell = 1, 2, \ldots
\] (19)

where \( E_{h,\ell} \) are the eigenvalues of \(-\frac{1}{2}h^2 \Delta_x + V(x)\), and

\[
S_h(E) \sim \sum_{j=0}^{\infty} h^j S_j(E) = 2\pi \mathcal{J}_0(E) + \frac{1}{2} h\pi \mu(E) + O(h^2),
\] (20)

where

\[
\mathcal{J}_0(E) = \int_{\mathbb{T}_E} p \, dx
\] (21)
is the Action integral for the classical curve $\gamma_E$ at energy $E$ and positive momentum, i.e. $p > 0$. When $E > \max V$ it is easily seen that

$$ J_0(E) = J(E) \quad (22) $$

as given above. In particular, the value $\mu(E)$ is the Maslov index of $\gamma_E$ seen as a Lagrangian submanifold of $T \times \mathbb{R}$. The condition $E > \max V$ also implies that $\mu(E) = 0$. Moreover, for any fixed constant $a \in \mathbb{R}$ and the translated Hamiltonian $\mathcal{H}(x, p + a)$ we have that the related Action is modified as

$$ J_0^{(a)}(E) = J_0(E) - a. \quad (23) $$

As shown in Prop. 5.2 of [22] the semiclassical series in (19) is locally uniform in $E$, which implies that the remainder vanishes as an $O(\hbar^2)$ term when $E$ is in fixed a bounded interval. Thus, the above equalities (19) - (20) imply that two systems with the same Bohr-Sommerfeld Rules have necessarily the same effective Hamiltonian. This fact will be one of the main ingredients to use in the proof of Theorem 1.1.

### 3 Results

In this section we provide two preliminary results and then we show the proof of Theorem 1.1.

**Lemma 3.1.** Let $V, A \in C^\infty(\mathbb{T}; \mathbb{R})$, $\bar{A} := \frac{1}{2\pi} \int_\mathbb{T} A(y) dy$. The effective Hamiltonian of $H(x, p) := \frac{1}{2}(p + A(x))^2 + V(x)$ is linked to effective Hamiltonian of $\mathcal{H}(x, p) := \frac{1}{2}p^2 + \bar{V}(x)$ in the following way

$$ \bar{H}(P) = H(P + \bar{A}) \quad \forall P \in \mathbb{R}. \quad (24) $$

Furthermore,

$$ \min_{P \in \mathbb{R}} \bar{H}(P) = \max_{x \in \mathbb{T}} V(x). \quad (25) $$

**Proof.** We begin to recall that

$$ \bar{H}(P) = \inf_{v \in C^{1,1}(\mathbb{T})} \sup_{x \in \mathbb{T}} \frac{1}{2} |P + \nabla_x v(x) + A(x)|^2 + V(x). \quad (26) $$

On the one-dim flat torus $\mathbb{T}$ we can write $A(x) = \bar{A} + \nabla_x \phi(x)$ and define $u(x) := v(x) + \phi(x)$ so that

$$ \bar{H}(P) = \inf_{u \in C^{1,1}(\mathbb{T})} \sup_{x \in \mathbb{T}} \frac{1}{2} |P + \bar{A} + \nabla_x u(x)|^2 + V(x) \quad (27) $$

namely we have

$$ \bar{H}(P) = \bar{H}(P + \bar{A}). \quad (28) $$
where $\mathcal{H}$ is explicitly shown in (17). To conclude,

$$\min_{P \in \mathbb{R}} \mathcal{H}(P) = \min_{P \in \mathbb{R}} \mathcal{H}(P + \bar{A}) = \min_{Q \in \mathbb{R}} \mathcal{H}(Q) = \max_{x \in T} V(x).$$  \hfill (29)

**Remark 3.2.** In view of the above Lemma, we have that $\max V_1 = \max V_2$ and $\int_T A_1(x)dx = \int_T A_2(x)dx$ are necessary conditions for the equality

$$\mathcal{H}_1(P) = \mathcal{H}_2(P), \quad \forall P \in \mathbb{R}.$$  \hfill (30)

Indeed, $\min_{P \in \mathbb{R}} \mathcal{H}(P) = \max_{x \in T} V(x)$ implies that the maximum of $V$ must be the same. Moreover, $\mathcal{H}(P) = \mathcal{H}(P + \bar{A})$ and thus, recalling (17), $\mathcal{H}$ is a constant function on the interval $I := [-\bar{A} - J(\max V), -\bar{A} + J(\max V)]$. As a consequence, (30) could be fulfilled if $I_1 = I_2$. On the other hand the two functions must be the same also outside $I$ and this gives the equality of functions $J_1 = J_2$. We conclude that it must be $\bar{A}_1 = \bar{A}_2$.

**Lemma 3.3.** Let us define $\hat{H} := \frac{1}{2}(-i\hbar \nabla_x + A(x))^2 + V(x)$ and $\hat{G} := \frac{1}{2}(-i\hbar \nabla_x + \bar{A})^2 + V(x)$ where $\bar{A} := \frac{1}{2\pi} \int_T A(y)dy$. Then,

$$\text{Spec}(\hat{H}) = \text{Spec}(\hat{G}), \quad \forall 0 < \hbar \leq 1,$$  \hfill (31)

**Proof.** Define the unitary operator on $L^2(T)$

$$(U\phi)(x) := e^{-\frac{i}{\hbar}\phi(x)}\phi(x)$$  \hfill (32)

where $\phi \in C^\infty(T; \mathbb{R})$ is such that $A(x) - \bar{A} = \nabla \phi(x)$. This provides the unitary conjugation

$$U^\dagger \circ \hat{H} \circ U = \hat{G}.$$  \hfill (33)

The proof of this fact can be done by taking the orthogonal set $e_k(x) := e^{ikx}$ with $k \in \mathbb{Z}$ and then computing the action of the operators on the right-hand side and also on the left-hand side of (33) on $e_k$, to check the equality. Now easily observe that, since the functions $H(x,p) := \frac{1}{2}|p + A(x)|^2 + V(x)$ and $G(x,p) := \frac{1}{2}|p + \bar{A}|^2 + V(x)$ have compact sub-level sets in $T \times \mathbb{R}$, then both the spectrums of $\hat{H}$ and $\hat{G}$ are discrete (see for example [18] with a more general class of $\Psi$do on manifolds). Whence, the unitary equivalence (33) directly implies the same spectrum.

**Remark 3.4.** The unitary conjugation (33) still holds true for operators of this kind on Sobolev space $W^{2,2}(\mathbb{R}; \mathbb{C})$ and suitable assumptions on smooth $V$ and $A$ (see for example sect. 1 of [7]). However, we stress that on $\mathbb{R}$ one can always write $A(x) = \nabla_x \phi(x)$ whereas on $T$ we must take into account a possible nonzero $\bar{A}$. This makes not trivial our study on the equivalence of effective Hamiltonians, as well as the use of these operators.
Proof of Theorem 1.1.
As a consequence of Lemma 3.1 and Lemma 3.3 in what follows we can consider only the case $A(x) = \bar{A}$.
In view of (19) and (20) we can write
\[2\pi J(E,\ell) - 2\pi \bar{A} + \frac{1}{2}h\pi \mu(E,\ell) + \mathcal{O}(h^2) = 2\pi h \ell.\] (34)
Recalling (16) and (17), we have
\[H(J(E)) = E \quad \forall E \geq \max V.\] (35)
and thus the equality
\[H(J(E) - \bar{A} + \bar{A}) = E.\] (36)
As a consequence, for any $E,\ell > \max V$ we have $\mu(E,\ell) = 0$ and
\[H(h \ell - r_{h,\ell}) = E, \quad r_{h,\ell} = \mathcal{O}(h^2).\] (37)
Notice in particular that the remainder $r_{h,\ell}$ depends from $\bar{A}$. Any vector $P \in \mathbb{R}$ in a bounded interval can be approximated by a sequence of kind $\ell_j h_j$ where $j \to +\infty, \ell_j \to +\infty$ and $h_j \to 0^+$.
We now remind the continuity of the map $P \mapsto H(P)$ for any $P \in \mathbb{R}$ and the continuity of $P \mapsto \nabla H(P)$ for those $P$ such that $H(P) \geq \min H + \varepsilon$ for some fixed $\varepsilon > 0$. In particular, notice that $\min H = \max V$.
In what follows we consider
\[\Omega := \{P \in \mathbb{R} \mid H(P) > \min H + \varepsilon; \quad |\nabla H(P)| < \lambda \}\] (39)
for some fixed $\lambda > 0$ large enough (recall that $H$ is a convex map).
Thus, we can use the values $E_{h,\ell}$ of the spectrum to recover the value $H(P)$.
Namely,
\[H(P) = \lim_{j \to +\infty} H(\ell_j h_j)\] (40)
\[= \lim_{j \to +\infty} H(\ell_j h_j - r_{h_j} + r_{h,j})\] (41)
\[= \lim_{j \to +\infty} E_{h_j,\ell_j} + R_{h_j,\ell_j}.\] (42)
The above remainder is defined as
\[R_{h,\ell} := H(\ell h) - H(\ell h - r_{h,\ell}).\] (43)
We recall that $r_{h,\ell} = \mathcal{O}(h^2)$ when $E_{h,\ell}$ lies in a bounded interval. Moreover, also the sequence $\ell_j h_j$ lies in a bounded interval since we assume that $P \in \mathbb{R}$.
belongs to a fixed bounded interval. The continuity of $\nabla \overline{H}$ with its uniformly boundedness on the prescribed set of $P$ and the previous observations ensure that

$$R_{h,\ell} \leq \sup_{0 \leq \alpha \leq 1} |\nabla \overline{H}(\alpha[\ell h] + (1 - \alpha)[\ell h - r_h])| |r_h|$$

$$\leq \|\nabla \overline{H}\|_{C^0(\Omega)} |r_h|. \quad (44)$$

The above convex combination belongs, for any $j$, to a suitable large bounded set $\Omega' \supseteq \Omega$ since $\forall P \in \Omega$ we have $\ell_j h_j \rightarrow P$ and $r_{h_j,\ell_j} \rightarrow 0$ as $j \rightarrow +\infty$. The limit (42) together with (44) allow to recover

$$\overline{H}(P) = \lim_{j \rightarrow +\infty} E_{h_j,\ell_j}. \quad (45)$$

Since we are assuming that for two potentials with the same maximum we have Schrödinger operators with the same spectrum, thus we necessarily have the same equality (45). For any fixed $P$ as prescribed above we have

$$\overline{H}_1(P) = \overline{H}_2(P). \quad (46)$$

We recall that $\overline{H}(P) = \overline{H}(P - \bar{A})$ and $\overline{H}(P) = \max V$ when $|P| \leq J(\max V)$ (see Section 2.1). Thus we have also the equivalence of the minimum points of the two effective Hamiltonians.

To conclude, in order to prove that

$$\text{Spec}(\hat{H}_1) = \text{Spec}(\hat{H}_2) \mod O(h^2) \quad (47)$$

for $E > \min \overline{H}$, simply use the equivalence (46) and recall (38). Apply again the estimate (44) to have the remainder $O(h^2)$. □

References


