Noname manuscript No.
(will be inserted by the editor)

Gibbs estimates for the convergence to Hartree dynamics in the Hardy class of potentials

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Received: date / Accepted: date

Abstract We present a quantitative estimate for the derivation of the Hartree dynamics for boson particles at low temperatures in arbitrary dimension. This achievement is obtained by the normal mode decomposition of the boson field operator evolved under the many body quantum dynamics, and estimates on Wick symbols through an $L^2(\mu)$ - norm with Gaussian thermal measures μ. This is directly linked to the estimate by the Gibbs measure. The rate of convergence is explicitly written in terms of the temperature and the number of particles. The interaction potential is supposed to be in the Hardy class, thus containing the Coulomb type, and it is not rescaled with respect to the number of particles. The dependence on time in the main estimates is shown to be globally linear.

Keywords Hartree dynamics · many body theory · Gibbs estimates

1 Introduction

The experimental observation of Bose-Einstein condesation (BEC), see [1] and [33], led to a great growth of activity in the physics of Bose gases. The approach was based on laser cooling techniques and magneto-optical traps, first introduced in the 80's, and represents a cornerstone in the field of interacting bosons, whose study has attracted increasing interest from experimental, numerical and theoretical communities (an excellent review is [40]).

This is still a very active field of research, improved by various fundamental results in the mathematical analysis of condensation for interacting bosons.
A first reference work is [43], where the author discusses a plenty of classical

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and quantum models for which kinetic equations can be derived rigorously. A second reference paper is [29], where the author shows that in the many body framework the classical limit of the expectation values of products of Weyl operators, translated in time by the quantum dynamics and taken on coherent states centered in (x, p)-space, are shown to become the exponentials of coordinate functions of the classical orbit in phase space. Such results have been extended in [27], and a recent review of this method is given in [18].
For a class of singular interaction potentials including the Coulomb potential, in [24] the authors show the convergence of the quantum dynamics to the Hartree dynamics when the number of particles becomes large.
Methods for deriving higher order corrections to the mean field asymptotics for the quantum systems are provided in the works [38], [13].
A reference role in the literature is played by those recent works dealing with the rigorous version of the Bogoliubov theory of superfluids; see e.g. the review [42]. The Gross-Pitaevskii equation is rigorously deduced for example in [9], whereas the fluctuations around it are studied in the works [17] and [14].
We also recall that the convergence to the limiting Hartree dynamics is studied in [41] and [5], and in [11] the Hartree-Fock-Bogoliubov is derived by the method of the quasi-free reduction. A general discussion of the role played by scaling of the physical parameters in BEC is given in [21].

In the recent paper [26], the authors prove that the grand canonical Gibbs state of an interacting quantum Bose gas converges to the Gibbs measure of a nonlinear Schrödinger equation in the mean-field limit, where the density of the gas becomes large and the interaction strength is assumed proportional to the inverse density. Moreover, in [30] the authors prove that the grand-canonical Gibbs state of a large bosonic quantum system converges to the Gibbs measure of a nonlinear Schrödinger-type classical field theory, in terms of partition functions and reduced density matrices. This gives a further derivation of nonlinear Gibbs measures in two and three space dimensions, starting from many-body quantum systems in thermal equilibrium.

With respect to these last results, we have a different target and thus we exhibit different results. Indeed, in our paper we deal with a family of flow invariant Gaussian measures and the Gibbs measure, not as the result of a mean field limit, but as a tool in order to study the rate of convergence of the quantum many body evolutive problem to the related Hartree effective dynamics in the low temperature asymptotics.
Concerning the two-body interaction potentials v, we consider the whole class of (positive) Hardy potentials v with Hardy constant $0 < C_v < +\infty$ ([6], [31]) and we do not assume any scaling of v in terms of the number of particles N, which here is supposed to be finite (see Sect. 2.2). This ensures also Coulomb type repulsive interactions.
More precisely, our approach is based on the normal mode decomposition together with an 'ultra violet' (UV) regularization of the quantum many body dynamics of the boson field Ψ, with respect to a fixed orthonormal basis (see Sect. 3.1). This means to study the reduced quantum dynamics on Bargmann-Fock space $\mathcal{F}_B(\mathbb{C}^\ell)$, $\ell \in \mathbb{N}$, and then to control such an UV - regularization

by estimating explicitly the remainder of the approximated dynamics with respect to the full Fock dynamics.
The advantage to consider the quantum dynamics of Ψ within this setting is that we can apply time global and rigorous results about the propagation of Wick operators on Bargmann-Fock space and related Wick symbols. This approach is inspired from the works [2], [3], [4] where the flow of the Hartree equation is recovered as mean field limit, thanks to infinite dimensional phase-space analysis through Wick operators on the Fock space, and by the use of infinite dimensional Wigner measures.
In addition, this framework allows to apply operator Gibbs estimates that control some Gaussian thermal $L^2(\mu)$ - norms on phase space where furthermore μ are invariant under the discrete Hartree flow (see Sect. 3.2). The use of these invariance properties, and such norms for the convergence, allows us to avoid the assumption of a scaling for the interaction potential in terms of N, to avoid the application of the Grönwall Lemma and various related arguments that frequently appears in the literature for derivation of Hartree dynamics and that give time exponential growth estimates.

Our main result (Theorem 1) proves the convergence of the (quantum evolved) Wick symbol of the boson field to the Wick symbol moved under the Hartree flow, with respect to $L^2(\mu)$ - gaussian norms. Moreover, an elliptic property on the Hamiltonian ensures also the L^2 - estimate by the Gibbs measure. The rate of convergence is explicitly written in terms of the temperature $T > 0$ which here is the 'small' asymptotic parameter, the fixed number of particles N and the Hardy constant $C_v > 0$ linked to the interaction potential v. Our choice of this $L^2(\mu)$ - convergence notion allows a globally linear estimate on the difference between the quantum many body dynamics and the effective one.

The contents of our paper are inspired by our previous results in [39]. The focus of [39] was on Bose-Hubbard models for the derivation of the discrete NLS flow in the mean field regime. In the present paper we consider the low temperature asymptotics for many body models on $\mathbb{R}^d$ far beyond the simple case of periodic external potentials in the tight binding approximation (here we assume confining ones with at most polynomial growth) and for the derivation of the continuous Hartree flow.
Furthermore, here we show a link between the one particle density operator associated to the field operators, the related Hilbert-Schmidt norm convergence, and the gaussian norm of fields we used in the paper (see Sect. 3.7). This allows us to discuss both the differences and the similarities between our main result and the ones in the existing literature that prove the Bose-Einstein condensation through one particle density operator and mean field asymptotics.
Within this discussion, we also suggest an index for the growth of particle correlations (see Remark 5) by exhibiting a lower bound for the density operator associated to the deviation field between Hartree flow and the quantum dynamics. Such an achievement is a novelty in the study of correlations, since related lower bounds are usually difficult to recover.

2 Preliminaries about the model

2.1 Many body operator and normal mode decomposition

We consider the Hamiltonian operator of identical spinless trapped interacting bosons of mass $m = 1$ in $\mathbb{R}^d$ with $1 \le d \le 3$ defined on the bosonic Fock space $\mathcal{F} := \bigoplus_{n\ge 0} L^2_s(\mathbb{R}^{dn})$, written in terms of the annihilation operator distribution $\Psi(x)$ (see sect. 10.2.2 in [18]),

$$\begin{aligned} \mathsf{H} &:= \mathsf{H}_{ext} + \mathsf{H}_{int} \\ &= \int_{\mathbb{R}^d} \Psi^\dagger(x)\,\mathsf{h}\,\Psi(x)dx + \frac{1}{2}\int_{\mathbb{R}^{2d}} \Psi^\dagger(x)\Psi^\dagger(y)v(x-y)\Psi(y)\Psi(x)dxdy \end{aligned} \tag{1}$$

where $\mathsf{h} := -\frac{1}{2}\Delta_x + u(x)$ is the single-particle operator, u and v are the trapping external potential and the positive interaction potential in the Hardy class (see Sect. 2.2). All over the paper we assume $\hbar = 1$. Under these assumptions, H is selfadjoint on $\mathcal{F}$ and the unitary map $U(t) := \exp\{-i\mathsf{H}t\} : \mathcal{F} \to \mathcal{F}$ is wellposed (see for example Prop. 122 in [18]).
Let us consider the time dependent operator distribution $\Psi(t) \equiv \Psi(t,x)$ that fulfills the Heisenberg equation

$$i\dot{\Psi}(t) = [\Psi(t), \mathsf{H}] \tag{2}$$

with the initial data $\Psi(0,x) := \Psi(x)$. This is directly solved by $\Psi(t,x) := U(t)^\dagger\Psi(x)U(t)$. At any fixed time, the canonical commutation relations hold true $[\Psi(t,x), \Psi^\dagger(t,y)] = \delta(x-y)\mathbb{1}$, $[\Psi(t,x), \Psi(t,y)] = 0$. In particular, the commutation $[\mathsf{H}, \mathsf{N}] = 0$ gives the operator valued conservation law

$$\mathsf{N} := \int_{\mathbb{R}^d} \Psi^\dagger(x)\Psi(x)dx = \int_{\mathbb{R}^d} \Psi^\dagger(t,x)\Psi(t,x)dx \qquad \forall t \ge 0. \tag{3}$$

We may now expand the field operator in terms of an orthonormal basis of the single particle Hilbert space $L^2(\mathbb{R}^d)$.

Remark 1 In what follows we denote $\{\varphi_k\}_{k\in\mathbb{N}^d} \subset L^2(\mathbb{R}^d)$ the orthonormal basis of eigenfunctions of the single particle harmonic oscillator $-\frac{1}{2}\Delta_x + \frac{1}{2}|x|^2$. The choice of the eigenfunctions φ_k is motivated from the knowledge of explicit bounds for $\|\nabla\varphi_k\|_{L^2(\mathbb{R}^d)}$ in term of k that will be useful for our estimates.

The Hamiltonian H given in (1) can be also expressed in the form (see [18])

$$\mathsf{H} = \sum_{km} u_{km}\, \mathsf{a}_k^\dagger \mathsf{a}_m + \frac{1}{2}\sum_{klmn} v_{klmn}\, \mathsf{a}_k^\dagger \mathsf{a}_l^\dagger \mathsf{a}_m \mathsf{a}_n\,, \tag{4}$$

where $k, l, m, n \in \mathbb{Z}^d$ and $\mathsf{a}_m := \int_{\mathbb{R}^d} \bar{\varphi}_m(x)\,\Psi(x)\,dx$. The coefficients of the quadratic part are given by

$$u_{km} := \langle \varphi_k, \mathsf{h}\,\varphi_m\rangle_{L^2(\mathbb{R}^d)} \tag{5}$$

and the quartic terms are the entries of the matrix of the selfadjoint two body interaction operator defined on the two particles symmetric space $\widehat{v} : L^2_s(\mathbb{R}^{2d}) \to L^2_s(\mathbb{R}^{2d})$

$$v_{klmn} := \langle \varphi_k \vee \varphi_l, \widehat{v}\, \varphi_m \vee \varphi_n \rangle_{L^2_s(\mathbb{R}^{2d})} \tag{6}$$

where $\varphi_m \vee \varphi_n := \frac{1}{2}(\varphi_m \otimes \varphi_n + \varphi_n \otimes \varphi_m) \in L^2_s(\mathbb{R}^{2d})$ is the symmetric tensor product. Such coefficients satisfy the following relation $\bar{v}_{klmn} = v_{mnkl}$.
The number operator N given in (3) can be equivalently written as

$$\mathsf{N} = \sum_k \mathsf{a}_k^\dagger \mathsf{a}_k \,. \tag{7}$$

Now consider the normal mode expansion with index $k \in \mathbb{N}^d$ through the time dependent $\mathsf{a}_k(t) := \int_{\mathbb{R}^d} \bar{\varphi}_k(x) \Psi(t,x) dx$,

$$\Psi(t,x) = \sum_k \mathsf{a}_k(t) \varphi_k(x) \tag{8}$$

We can now say that these time dependent operators $\mathsf{a}_k(t) : \mathcal{F} \to \mathcal{F}$ satisfy (formally) the following infinite family of coupled operator equations

$$i\dot{\mathsf{a}}_k(t) = [\mathsf{a}_k(t), \mathsf{H}] = \sum_m u_{km}\, \mathsf{a}_m(t) + \sum_{lmn} v_{klmn}\, \mathsf{a}_l^\dagger(t) \mathsf{a}_m(t) \mathsf{a}_n(t). \tag{9}$$

Notice that $\mathsf{a}_k(t) \mapsto [\mathsf{a}_k(t), \mathsf{H}]$ is a well posed operator map on Fock. In the next section, we are going to show a regularized (and rigorous) version for the righthand side of this equation.

2.2 Assumptions on the physical potentials

The external potential $u \in \mathcal{C}^2(\mathbb{R}^d; \mathbb{R}_+)$ is such that

$$c\|x\|^q \le u(x) \le \Omega \|x\|^{2p}, \tag{10}$$

for some $c, \Omega, q > 0$ and $p \in \mathbb{N}$. The simplest example is the isotropic harmonic trap $u(x) := \frac{1}{2}\varpi^2 \|x\|^2$.
The interaction potential $v : \mathbb{R}^d \to \mathbb{R}_+$ is assumed to be a measurable positive function such that $v(x) = v(-x)$ and belonging to the Hardy class, i.e. satisfying

$$\|v\psi\|_{L^2(\mathbb{R}^d)} \le C_v \|\psi\|_{H^1(\mathbb{R}^d)}, \quad \forall \psi \in H^1(\mathbb{R}^d), \tag{11}$$

for some $C_v > 0$ that we will call the Hardy constant of v. We recall that the H^1 - Sobolev norm reads $\|\psi\|^2_{H^1(\mathbb{R}^d)} := \|\psi\|^2_{L^2(\mathbb{R}^d)} + \|\nabla\psi\|^2_{L^2(\mathbb{R}^d)}$. In the bounded case $v \in L^\infty(\mathbb{R}^d)$ we can set $C_v = \|v\|_{L^\infty(\mathbb{R}^d)}$. For the Coulomb case $v(x) := 1/\|x\|$ with $d = 3$ the optimal Hardy constant reads $C_v = 2$.

In particular, the well known Hardy inequality (see [6], [31]) for the Coulomb potential and $d \geq 3$ reads

$$\int_{\mathbb{R}^d} \frac{\|\psi(x)\|^2}{\|x\|^2} dx \leq \frac{4}{(d-2)^2} \int_{\mathbb{R}^d} \|\nabla_x \psi(x)\|^2 dx.$$

In Appendix 4.1 we prove that, if v fulfills (11) then the same inequality (in the sense of semipositive operators) is realized for the bosonic field $\Psi(x)$, namely

$$\int_{\mathbb{R}^d} v(x)^2 \Psi^\dagger(x)\Psi(x)dx \;\leq\; C_v^2 \Big(\int_{\mathbb{R}^d} \Psi^\dagger(x)\Psi(x)dx + \int_{\mathbb{R}^d} \nabla_x\Psi^\dagger(x)\nabla_x\Psi(x)dx \Big). \\ =: C_v^2 \Big(\mathsf{N} + \mathsf{H}_{free}\Big). \tag{12}$$

Moreover, a consequence (see Proposition 8) is that H_{int} given in (1) fulfills

$$0 < \mathsf{H}_{int} \leq \frac{1}{2} C_v \mathsf{N}(\mathsf{N} + \mathsf{H}_{free}). \tag{13}$$

Such an operator inequality, which implies $\mathsf{H} \leq \mathsf{H}_{ext} + \frac{1}{2} C_v \mathsf{N}(\mathsf{N} + \mathsf{H}_{free}) \leq \Omega'_p \mathsf{N}^p_\Lambda$ (see Lemma 5) provides, for $\beta > 0$ and $\Omega'_p := 2((1+\Omega)3^p + C_v)$, the lower bound

$$e^{-\beta \Omega'_p \mathsf{N}^p} \leq e^{-\beta \mathsf{H}} \tag{14}$$

that will be useful in the determination of a quantitative estimate for the critical temperature shown in Section 3.6.

3 Main results

3.1 UV - regularization

The normal modes $\mathsf{a}_k(t)$, $k \in \mathbb{N}^d$, of the field operator $\Psi(t,x)$ solves equation (9) which is an infinite system of countably many strongly coupled operator equations. We regularize this system by inserting an ultra-violet (UV) cut-off $\Lambda \in \mathbb{N}$, requiring the sum to run only on multi-indices of norm lesser than Λ. We thus define the following norm $|k| := \max_{i \in \{1,\dots,d\}} k_i$ so that the set of multi-indices having norm lesser than a positive integer Λ is an hypercube of side Λ, with volume easy to compute by

$$\sum_{k \in \mathbb{N}^d}^{\Lambda} 1 \equiv \sum_{k \,:\, |k| < \Lambda} 1 = \Lambda^d =: \ell. \tag{15}$$

Notice that the norm $\|k\| := k_1 + \cdots + k_d$ is equivalent to the above one, since $|k| \leq \|k\| \leq d|k|$ and $\|k\|/d \leq |k| \leq \|k\|$. The Hamiltonian and number operators in this setting are given by

$$\mathsf{N}_\Lambda := \sum_k^\Lambda \mathsf{a}_k^\dagger \mathsf{a}_k\,, \tag{16}$$

$$\mathsf{H}_\Lambda := \sum_{km}^\Lambda u_{km}\,\mathsf{a}_k^\dagger \mathsf{a}_m + \frac{1}{2}\sum_{klmn}^\Lambda v_{klmn}\mathsf{a}_k^\dagger \mathsf{a}_l^\dagger \mathsf{a}_m \mathsf{a}_n = \mathsf{H}_{ext,\Lambda} + \mathsf{H}_{int,\Lambda}\,. \tag{17}$$

We stress that, for any $\Lambda \geq 1$,

$$[\mathsf{N}_\Lambda, \mathsf{H}_\Lambda] = 0. \tag{18}$$

Then, employing the short-hand notation $\sum_k^\Lambda \equiv \sum_{k\,:\,|k|<\Lambda}$ we consider the Heisenberg equations $\forall\ |k| < \Lambda$ for initial data $\mathsf{a}_{k,\Lambda}(0) := \mathsf{a}_k$

$$i\dot{\mathsf{a}}_{k,\Lambda}(t) = [\mathsf{a}_{k,\Lambda}(t), \mathsf{H}_\Lambda] = \sum_m^\Lambda u_{km}\,\mathsf{a}_{m,\Lambda}(t) + \sum_{lmn}^\Lambda v_{klmn}\,\mathsf{a}_{l,\Lambda}^\dagger(t)\mathsf{a}_{m,\Lambda}(t)\mathsf{a}_{n,\Lambda}(t). \tag{19}$$

Here the operators N_Λ, H_Λ and $\mathsf{a}_{k,\Lambda}(t)$ can be represented as Wick operators defined on the Bargmann-Fock space $\mathcal{F}_B(\mathbb{C}^\ell)$, see the brief review in [39] or the textbooks [12], section 10.2.2 in [18], section 1.6 in [23]. We recall that $\mathcal{F}_B(\mathbb{C}^\ell)$ is isomorphic to a subset of the bosonic Fock space $\mathcal{F} := \bigoplus_{n\geq 0} L^2_s(\mathbb{R}^{dn}) = \bigoplus_{n\geq 0}(L^2(\mathbb{R}^d))^{\otimes_s n}$. The Bargmann-Fock space is isomorphic (see pp. 48-49 in [23]) to the direct sum of the symmetric tensor products

$$\mathcal{F}_B(\mathbb{C}^\ell) \simeq \bigoplus_{n\geq 0} \mathfrak{h}_\Lambda^{\otimes_s n} \subset \mathcal{F} \tag{20}$$

where the (finite dimensional) subspace $\mathfrak{h}_\Lambda$ of the single particle space reads

$$\mathfrak{h}_\Lambda := \mathrm{Span}\Big\{\varphi_1, \varphi_2, ..., \varphi_k \in L^2(\mathbb{R}^d)\ |\ k \in \mathbb{N}^d,\ |k| < \Lambda\Big\}$$
$$\ell := \dim(\mathfrak{h}_\Lambda) = \Lambda^d.$$

Let us now define the related orthogonal projector

$$\pi_\Lambda : \mathcal{F} \to \mathcal{F}_B(\mathbb{C}^\ell) \tag{21}$$

(acting on any single sector) and notice that

$$\mathsf{H}_\Lambda = \pi_\Lambda\,\mathsf{H}\,\pi_\Lambda$$

and in particular $\mathsf{H}_{ext,\Lambda} = \pi_\Lambda \mathsf{H}_{ext} = \mathsf{H}_{ext}\pi_\Lambda$, $\mathsf{N}_\Lambda = \pi_\Lambda \mathsf{N} = \mathsf{N}\pi_\Lambda$. Whereas in general $\pi_\Lambda \mathsf{H}_{int} \neq \mathsf{H}_{int,\Lambda}$ thus we have only the general link $\mathsf{H}_{int,\Lambda} = \pi_\Lambda \mathsf{H}_{int}\,\pi_\Lambda$. We stress moreover that

$$\pi_\Lambda(\mathsf{H} - \mathsf{H}_\Lambda)^2\pi_\Lambda = \pi_\Lambda \mathsf{H}(\mathbb{1} - \pi_\Lambda)\mathsf{H}\pi_\Lambda = \pi_\Lambda \mathsf{H}_{int}(\mathbb{1} - \pi_\Lambda)\mathsf{H}_{int}\pi_\Lambda \neq 0. \tag{22}$$

Such an inequality tells us that, for general interaction terms H_{int}, the many body quantum dynamics does not preserve $\mathcal{F}_B(\mathbb{C}^\ell)$ nor the operators defined on it and moved under unitary conjugation in the Heisenberg picture. In particular, we have the inequality between semipositive operators

$$\pi_\Lambda(\mathsf{H}-\mathsf{H}_\Lambda)^4\pi_\Lambda \geq \Big(\pi_\Lambda(\mathsf{H}-\mathsf{H}_\Lambda)^2\pi_\Lambda\Big)^2 > 0. \tag{23}$$

The lefthand side of (23) will appear in the trace (52) to estimate the UV remainder of the quantum evolved annihilation operator, namely the difference $\Psi(t)-\Psi_\Lambda(t)$ where the UV - regularized field operator is defined as

$$\Psi_\Lambda(t) \equiv \mathsf{\Psi}_\Lambda(t,x) := \sum_k^\Lambda \mathsf{a}_{k,\Lambda}(t)\varphi_k(x). \tag{24}$$

We now denote $\omega > 0$ as the lowest eigenvalue of the semipositive operator $\mathsf{h} := -\frac{1}{2}\Delta_x + u(x)$ defined in Section 2.2, then it is smaller than the lowest eigenvalue of the matrix $u_{km} := \langle\varphi_k, \mathsf{h}\,\varphi_m\rangle_{L^2(\mathbb{R}^d)}$ where we assume $|k|,|m| < \Lambda$. Taking into account the interaction part in (17) which is the restriction of the semipositive interaction part H_{int} contained in (4), we have

$$\mathsf{H}_\Lambda \geq \omega\,\mathsf{N}_\Lambda \tag{25}$$

namely the operator $\mathsf{H}_\Lambda - \omega\,\mathsf{N}_\Lambda$ is semipositive definite. In the harmonic trap case $u(x) = \frac{1}{2}\varpi^2\|x\|^2$, we have $\omega = \frac{1}{2}d\varpi$.

Remark 2 Since the notation becomes heavy, the subscript Λ on the operators $\mathsf{a}_{k,\Lambda}$, marking these latter operators as satisfying the finite system of equations (19), will not be carried in the following Sections. Still, to remind us of the presence of the cut-off, all other relevant quantities will maintain the subscript.

In order to turn the coupled equations of Wick operators in (19) into coupled equations of Wick symbols we need to use coherent states $\phi_\alpha \in \mathcal{F}_B(\mathbb{C}^\ell)$, see Section 4.4, which are given by the eigenvectors of the annihilation operators

$$\mathsf{a}_k\phi_\alpha = \alpha_k\phi_\alpha, \quad \alpha_k \in \mathbb{C}.$$

Now, by taking the quantum expectation of a_k solving (19) over coherent states (namely, computing the Wick symbol)

$$a_k(t,\alpha,\bar\alpha) := \langle\phi_\alpha, \mathsf{a}_k(t)\phi_\alpha\rangle \tag{26}$$

we obtain the following problem with initial data $a_k(0,\alpha,\bar\alpha) := \alpha_k$,

$$i\dot a_k(t,\alpha,\bar\alpha) = \sum_m^\Lambda u_{km}\,a_m(t,\alpha,\bar\alpha) + \sum_{lmn}^\Lambda v_{klmn}\,\langle\phi_\alpha, \mathsf{a}_l^\dagger(t)\mathsf{a}_m(t)\mathsf{a}_n(t)\phi_\alpha\rangle \tag{27}$$

This would correspond to the scalar and discrete coupled Hartree equations (45), if not for the generic failure of the Wick symbol of operator product to map onto the pointwise one, namely

$$\langle\phi_\alpha, \mathsf{a}_l^\dagger(t)\mathsf{a}_m(t)\mathsf{a}_n(t)\phi_\alpha\rangle \neq \langle\phi_\alpha, \mathsf{a}_l^\dagger(t)\phi_\alpha\rangle\cdot\langle\phi_\alpha, \mathsf{a}_m(t)\phi_\alpha\rangle\cdot\langle\phi_\alpha, \mathsf{a}_n(t)\phi_\alpha\rangle.$$

To obtain a closed system of equations for the family of functions $a_k(t,\alpha,\bar\alpha)$ we will make use of Wick deformation quantization, see Proposition 2.

3.2 Gaussian thermal measures

Motivated from the inequality (25), a candidate to define a weighted trace is the following Gibbsian operator

$$\varrho_\Lambda := \frac{e^{-\beta\omega \mathsf{N}_\Lambda}}{\mathrm{Tr}(e^{-\beta\omega\mathsf{N}_\Lambda})}, \quad \beta := (k_B T)^{-1}, \tag{28}$$

for which we recall the next useful result (see Prop. 1 in [39]).

Lemma 1 *Consider the number operator* $\mathsf{N}_\Lambda := \sum_k^\Lambda \mathsf{a}_k^\dagger \mathsf{a}_k$*, then for any* λ *in* $\mathbb{R}$*, and any Wick operator* $\mathsf{F} : \mathcal{F}_B(\mathbb{C}^\ell) \to \mathcal{F}_B(\mathbb{C}^\ell)$ *we have that*

$$\begin{aligned}\frac{\mathrm{Tr}(\mathsf{F}e^{-\lambda\mathsf{N}_\Lambda})}{\mathrm{Tr}(e^{-\lambda\mathsf{N}_\Lambda})} &= \int_{\mathbb{C}^\ell} \langle \phi_\alpha, \mathsf{F}\phi_\alpha\rangle \, \frac{\sigma_{AW}(e^{-\lambda\mathsf{N}_\Lambda})}{\mathrm{Tr}(e^{-\lambda\mathsf{N}_\Lambda})} \, d\alpha \wedge d\bar\alpha \\ &= \int_{\mathbb{C}^\ell} \langle \phi_\alpha, \mathsf{F}\phi_\alpha\rangle \, (e^\lambda - 1)^\ell e^{-(e^\lambda-1)|\alpha|^2} \, d\alpha\wedge d\bar\alpha\end{aligned}$$

where $\alpha := q + ip$ *and* $d\alpha \wedge d\bar\alpha := \pi^{-\ell} dqdp$*.*

The above result follows from the trace formula involving Wick and anti-Wick operators (see section 7.6.1 in [18]) and thanks to the direct computation $\sigma_{AW}(e^{-\lambda\mathsf{N}_\Lambda}) = e^{\lambda\ell}e^{-(e^\lambda-1)|\alpha|^2}$ and $\mathrm{Tr}(e^{-\lambda\mathsf{N}_\Lambda}) = \int \sigma_{AW}(e^{-\lambda\mathsf{N}_\Lambda})\pi^{-\ell} d\alpha d\bar\alpha = (e^\lambda/(e^\lambda - 1))^\ell$.

A direct consequence of Lemma 1 is that tracing an operator F against ϱ_Λ, for $\lambda := \beta\omega$, is equivalent to averaging its Wick symbol $f(\alpha,\bar\alpha) := \langle\phi_\alpha, \mathsf{F}\phi_\alpha\rangle$ over $\mathbb{C}^\ell \simeq \mathbb{R}^{2\ell}$ with respect to a normalized gaussian measure μ, that is

$$\mathrm{Tr}(\mathsf{F}\varrho_\Lambda) = \int_{\mathbb{C}^\ell} f(\alpha,\bar\alpha)\, d\mu(\alpha,\bar\alpha), \tag{29}$$

with the *Gaussian thermal measure* linked to $B := e^{\beta\omega} - 1$

$$d\mu(\alpha,\bar\alpha) := B^\ell \, e^{-B|\alpha|^2} \, d\alpha\wedge d\bar\alpha\,. \tag{30}$$

When $B \to +\infty$ (i.e. $T \to 0^+$) the measure concentrates at $\alpha = 0$, or equivalently the trace becomes the projector onto the ground state ϕ_0 of the number operator, namely $\mathrm{Tr}(\mathsf{F}\rho_\Lambda) \to \langle\phi_0, \mathsf{F}\phi_0\rangle$.

Definition 1 Let $\omega > 0$ be the lowest eigenvalue of the single particle operator h. We introduce a (quantum invariant) weighted norm of fields Θ by

$$\|\Theta\|_\star^2 := \mathrm{Tr}\Big(\varrho_\Lambda \int_{\mathbb{R}^d} \Theta^\dagger(x)\Theta(x)dx\Big). \tag{31}$$

Remark 3 We stress the inequality between Wick symbols

$$\langle\phi_\alpha, \mathsf{F}^\dagger\mathsf{F}\phi_\alpha\rangle = |f(\alpha,\bar\alpha)|^2 + \sum_{n=1}^\infty \frac{\partial^n \bar f}{\partial\alpha^n}\frac{\partial^n f}{\partial\bar\alpha^n}(\alpha,\bar\alpha) \geq |f(\alpha,\bar\alpha)|^2. \tag{32}$$

Thus, the weighted operator norm $\|\Theta\|_\star$ in Def. 1 has a lower bound given by sum of the $L^2(\mu)$-norms of the Wick symbols $\vartheta_k(\alpha,\bar\alpha) := \langle \phi_\alpha, \mathsf{e}_k \phi_\alpha\rangle$, $\mathsf{e}_k := \int_{\mathbb{R}^d} \bar\varphi_k(x)\Theta(x)dx$,

$$\|\Theta\|_\star^2 \geq \sum_k^\Lambda \int_{\mathbb{C}^\ell} |\,\langle \phi_\alpha \mathsf{e}_k \phi_\alpha\rangle\,|^2\, d\mu(\alpha,\bar\alpha) \equiv \sum_k^\Lambda \|\vartheta_k\|^2_{L^2(\mu)}. \tag{33}$$

Notice that also in the case of arbitrary operators on Fock, $\mathsf{e} : \mathcal{F} \to \mathcal{F}$, the bracket with respect to the coherent states $\phi_\alpha \in \mathcal{F}_B(\mathbb{C}^\ell) \subset \mathcal{F}$ fulfills $\langle \phi_\alpha, \mathsf{e}^\dagger \mathsf{e} \phi_\alpha\rangle = \langle \phi_\alpha, \mathsf{e}^\dagger(\pi_\Lambda + \mathbb{1} - \pi_\Lambda)\mathsf{e}, \phi_\alpha\rangle = \langle \phi_\alpha, \pi_\Lambda \mathsf{e}^\dagger \pi_\Lambda \mathsf{e} \pi_\Lambda \phi_\alpha\rangle + \langle \phi_\alpha, \pi_\Lambda \mathsf{e}^\dagger(\mathbb{1} - \pi_\Lambda)\mathsf{e}\pi_\Lambda \phi_\alpha\rangle \geq \langle \phi_\alpha, \pi_\Lambda \mathsf{e}^\dagger \pi_\Lambda \mathsf{e} \pi_\Lambda \phi_\alpha\rangle$. Whence (33) still holds true.

In view of the above observations, we have the next

Proposition 1 *The following identities hold*

$$\mathrm{Tr}(\mathsf{a}_k^\dagger \mathsf{a}_k \varrho_\Lambda) = \int_{\mathbb{C}^\ell} |\alpha_k|^2 d\mu(\alpha,\bar\alpha) = \frac{1}{B}, \tag{34}$$

$$\mathrm{Tr}((\mathsf{a}_k^\dagger)^2 \mathsf{a}_k^2 \varrho_\Lambda) = \int_{\mathbb{C}^\ell} |\alpha_k|^4 d\mu(\alpha,\bar\alpha) = \frac{2}{B^2}. \tag{35}$$

More in general, for any $\sigma \geq 0$

$$\int_{\mathbb{C}^\ell} |\alpha_k|^\sigma d\mu(\alpha,\bar\alpha) = \frac{1}{B^{\sigma/2}}\, \Gamma\left(\frac{\sigma}{2}+1\right), \tag{36}$$

where $\Gamma : \mathbb{R}^+ \to \mathbb{R}^+$ *is the Euler Gamma function.*

Definition 2 ($\mathbf{L^2}$ - Gaussian thermal norm) Let $\Theta(x) := \sum_k \varphi_k(x)\,\mathsf{e}_k$ and related Wick symbols $\vartheta_k(\bar\alpha,\alpha) := \langle \alpha, \mathsf{e}_k \alpha\rangle$. Let μ be the Gaussian thermal measure (30). In view of Def. 1 and Remark 3, we now introduce

$$\|\Theta\|_\mu^2 := \frac{1}{\Lambda^d} \sum_k^\Lambda \|\vartheta_k\|^2_{L^2(\mu)} \leq \|\Theta\|_\star^2. \tag{37}$$

We remind the one to one correspondence between Wick operators and related Wick symbols. Thus, $\|\Theta\|_\mu$ is another well posed norm definition for field operators, which is furthermore sharper than the norm $\|\Theta\|_\star$ as shown in (33). Here we choose to normalize the sum since $\sum_k^\Lambda 1 = \Lambda^d$. Notice also that this norm is invariant under discrete Hartree flow (45) thanks to the invariance of the measure $d\mu$ (see Prop 4).

3.3 The Gibbs measure

We show that the L^2 - norm computed through the Gibbs measure can be controlled by the Gaussian measure introduced above. Let us consider the Hamiltonian function defined on the complex phase space $\mathbb{C}^\ell \simeq \mathbb{R}^{2\ell}$ given by $\mathcal{H}(\alpha,\bar\alpha) := \langle \phi_\alpha, \mathsf{H}_\Lambda \phi_\alpha\rangle$, where H_Λ is the many body operator (17), which reads

$$\mathcal{H}(\alpha,\bar\alpha) = \sum_{km}^{\Lambda} u_{km}\, \bar\alpha_k \alpha_m + \frac{1}{2}\sum_{klmn}^{\Lambda} v_{klmn}\, \bar\alpha_k\bar\alpha_l\alpha_m\alpha_n \,. \tag{38}$$

The Gibbs measure is defined as

$$d\mathcal{G}(\alpha,\bar\alpha) := \frac{1}{\int e^{-\lambda\mathcal{H}(\alpha,\bar\alpha)} d\alpha\wedge d\bar\alpha}\; e^{-\lambda\mathcal{H}(\alpha,\bar\alpha)}\; d\alpha\wedge d\bar\alpha \tag{39}$$

where $\alpha := q+ip$ and $d\alpha\wedge d\bar\alpha := \pi^{-\ell} dq dp$. Notice that the ellipticity property $\mathcal{H}(\alpha,\bar\alpha) \geq \tau_{min}|\alpha|^2$ implies the upper bound

$$0 < e^{-\lambda\mathcal{H}(\alpha,\bar\alpha)} \leq e^{-\lambda\tau_{min}|\alpha|^2} \tag{40}$$

where the value $\tau_{min} > 0$ is the smallest eigenvalue of the positive definite matrix $(u_{kj})_{1\leq k,j\leq \ell}$ linked to the quadratic part of $\mathcal{H}$ and given the anti-Wick symbol (see [23]) of $e^{-\lambda_0\widehat{N}}$ with $e^{\lambda_0} := \lambda\,\tau_{min}+1$ we have

$$\sigma_{AW}(e^{-\lambda_0\widehat{N}})(\alpha,\bar\alpha) = (\lambda\,\tau_{min}+1)^\ell\; e^{-\lambda\tau_{min}|\alpha|^2}\,. \tag{41}$$

Now define

$$dm(\alpha,\bar\alpha) := \frac{\sigma_{AW}(e^{-\lambda_0\widehat{N}})}{\mathrm{Tr}(e^{-\lambda_0\widehat{N}})}\; d\alpha\wedge d\bar\alpha = (\lambda\,\tau_{min})^\ell\; e^{-\lambda\tau_{min}|\alpha|^2} d\alpha\wedge d\bar\alpha. \tag{42}$$

and notice that there is $c_{\mathcal{H}} > 0$ such that (see Lemma 8)

$$(\lambda\,\tau_{min})^{-\ell} \leq c_{\mathcal{H}} \int e^{-\lambda\mathcal{H}(\alpha,\bar\alpha)}\, d\alpha\wedge d\bar\alpha. \tag{43}$$

A consequence of (40) - (43) is that $\forall\ f \in L^2(m)$ with the gaussian measure dm we have the following inequality between norms

$$\|f\|_{L^2(\mathcal{G})} \leq \sqrt{c_{\mathcal{H}}}\ \|f\|_{L^2(m)} \equiv \sqrt{c_{\mathcal{H}}}\ \|f\|_\mu \tag{44}$$

where the equivalence with the Gaussian thermal measure μ defined in (30) is obtained by setting $\lambda\tau_{min} \equiv B$. A direct application of (44) is that the upper bounds in Theorem 1 can be seen as L^2 - Gibbs estimates on the Wick symbols of the operators.

3.4 Convergence to the effective field

Let us consider the Hamiltonian flow on the phase space linked to (38), namely the solution of the (finite) system of coupled discrete Hartree equations

$$i\dot{c}_k = \frac{\partial \mathcal{H}}{\partial \bar{\alpha}_k}(c,\bar{c}) = \sum_m^{\Lambda} u_{km}\, c_m + \sum_{lmn}^{\Lambda} v_{klmn}\, \bar{c}_l c_m c_n \tag{45}$$

with initial data $c_k(0,\alpha,\bar{\alpha}) := \alpha_k$. The Hartree functional on $\psi \in L^2(\mathbb{R}^d)$

$$\mathcal{E}(\psi,\bar{\psi}) := \langle \psi, \mathsf{h}\psi\rangle + \frac{1}{2}\langle \psi, v * |\psi|^2\psi\rangle \tag{46}$$

when restricted on the span $\psi_\Lambda(x) = \sum_k^\Lambda \lambda_k\, \varphi_k(x)$ gives the reduced scalar Hartree equation on $L^2(\mathbb{R}^d)$

$$i\dot{\psi}_\Lambda = \frac{\partial \mathcal{E}}{\partial \bar{\psi}_\Lambda}(\psi_\Lambda, \bar{\psi}_\Lambda) \tag{47}$$

with initial data $\psi_\Lambda(0,x) := \sum_k^\Lambda \alpha_k\, \varphi_k(x)$. This corresponds to the whole family of equations (45). Notice that (unless $v = 0$) this is not the projected Hartree dynamics, namely $\psi_\Lambda(t) \neq \pi_\Lambda \psi(t)$ but it is still a nice approximating version of the full Hartree flow solving

$$i\dot{\psi} = \left(\mathsf{h} + v * |\psi|^2\right)\psi. \tag{48}$$

The regularized *effective field* is defined by

$$\Psi_\Lambda^{(0)}(t) \equiv \mathsf{\Psi}_\Lambda^{(0)}(t,x) := \sum_k^\Lambda \mathsf{c}_k(t)\varphi_k(x) \tag{49}$$

where the Wick operators $\mathsf{c}_k(t) : \mathcal{F}_B(\mathbb{C}^\ell) \to \mathcal{F}_B(\mathbb{C}^\ell)$ are the ones such that the symbols read $\langle \phi_\alpha, \mathsf{c}_k(t)\phi_\alpha\rangle = c_k(t,\alpha,\bar{\alpha})$. In other words, $\mathsf{c}_k(t)$ are the operators that can be associated, by the link between Wick operators and Wick symbols (see Section 4.4), to the components c_k of the above Hamiltonian flow. The well posedness of these operators is studied in details in the Section 3 of [45]. In view of this setting, the bracket with respect to the coherent state $\phi_\alpha \in \mathcal{F}_B(\mathbb{C}^\ell)$ associated to a fixed point $\alpha \in \mathbb{C}^\ell$ gives

$$\langle \phi_\alpha, \mathsf{\Psi}_\Lambda^{(0)}(t,x)\phi_\alpha\rangle = \sum_k^\Lambda c_k(t,\alpha,\bar{\alpha})\, \varphi_k(x) = \psi_\Lambda(t,x) \tag{50}$$

namely the Wick symbol of $\mathsf{\Psi}_\Lambda^{(0)}$, evaluated at α, is the solution ψ_Λ of the reduced Hartree equation (47).

Remark 4 The initial data $\alpha \in \mathbb{C}^\ell$ will be then distributed over the measure μ: this is our approach to introduce a Gaussian thermal estimate for the deviation of the quantum field $\Psi(t)$ from the effective field $\Psi_\Lambda^{(0)}(t)$.

We can now state the main result of the paper, to estimate the deviation of quantum field dynamics from the effective one, through the norm $\|\cdot\|_\mu$ given in (37), by the next

Theorem 1 (Main Result) *Let $\Psi(t)$ be the solution of (2), let $\Psi_\Lambda(t)$ be as in (24), $\Psi_\Lambda^{(0)}(t)$ given in (49) and v the interaction potential in the Hardy class with constant C_v. Then, for $B \geq 1$, $\Lambda \geq 1$ and $t \geq 0$ we have*

$$\|\Psi_\Lambda^{(0)}(t)\|_\mu = \frac{1}{\sqrt{B}}\,, \tag{51}$$

where $B := e^{\beta\omega} - 1$ and $\beta := (k_B T)^{-1}$. The UV - remainder satisfies

$$\begin{aligned} \|\Psi(t) - \Psi_\Lambda(t)\|_\mu &\leq \|\Psi(t) - \Psi_\Lambda(t)\|_\star \\ &\leq \frac{2}{\sqrt{B}}\,\mathrm{Tr}(\varrho_\Lambda |(\mathbb{1} - \pi_\Lambda)\mathsf{H}_{int}|^4)^{\frac{1}{4}}\ t \qquad (52) \\ &\leq \frac{4}{\sqrt{B}}\,C_v\left(\frac{\Lambda^d}{B}\right)^2 t \qquad (53) \end{aligned}$$

with $\varrho_\Lambda := e^{-\beta\omega\mathsf{N}_\Lambda}/\mathrm{Tr}(e^{-\beta\omega\mathsf{N}_\Lambda})$ and $|\mathsf{A}| := \sqrt{\mathsf{A}^\dagger\mathsf{A}}$. The quantum fluctuation around the effective field $\Psi_\Lambda^{(0)}(t)$ fulfills

$$\|\Psi_\Lambda(t) - \Psi_\Lambda^{(0)}(t)\|_\mu \ \leq \frac{2^9}{B}\,C_v\,(1+2\Lambda)^{4d+\frac{d}{2}}\ t \tag{54}$$

With respect to the above results, we have the following observations:

- The inequality $\|f\|_{L^2(\mathcal{G})} \leq \sqrt{c_{\mathcal{H}}}\ \|f\|_\mu$ between Gibbs and gaussian norms (see Section 3.3) ensures that the main Theorem can be rewritten in terms of Gibbs estimates.
- By the simple application of the triangular inequality taking into account (53) - (54) one gets an explicit bound for $\|\Psi(t) - \Psi_\Lambda^{(0)}(t)\|_\mu$.
- The norm can be rescaled as $\sqrt{B}\,\|\cdot\|_\mu$, so that $\Psi_\Lambda^{(0)}(t)$ has norm one and both (53) - (54) still work as vanishing estimates as $B \to +\infty$.
- By (54) we have thus proved there is a plenty of effective (and not equivalent) dynamics for any fixed value ℓ, coming from the low temperature asymptotics of the UV - regularized quantum many body Hamiltonian operators on Bargmann-Fock space.
- The bound (52) is obtained thanks to $\|\Psi(t) - \Psi_\Lambda(t)\|_\mu \leq \|\Psi(t) - \Psi_\Lambda(t)\|_\star$, where the bigger one is the weighted operator norm given in Def. 1. This norm is then estimated from above by the operator version of the L^4 - norm, i.e. the Schatten norm with index 4 and weight given by the semipositive trace one ϱ_Λ. The projector from Fock to Bargmann $\pi_\Lambda : \mathcal{F} \to \mathcal{F}_B(\mathbb{C}^\ell)$, see (21), weakly converges to identity as $\Lambda \to +\infty$. The term $(\mathbb{1} - \pi_\Lambda)\mathsf{H}_{int}$ measures the way H_{int} does not preserve the UV - cutoff, since for $\mathsf{A} := (\mathbb{1} - \pi_\Lambda)\mathsf{H}_{int}$ we have the nonzero term $|\mathsf{A}|^4 = \pi_\Lambda\mathsf{H}_{int}(\mathbb{1}-\pi_\Lambda)\mathsf{H}_{int}^2(\mathbb{1}-\pi_\Lambda)\mathsf{H}_{int}\pi_\Lambda$ in the trace (52).

- Our estimates by $L^2(\mu)$ - norm is linear in time, whereas the trace norm used to prove the convergence of the one particle density operator usually exhibits bounds with exponential growth in time. See also Sect. 3.7 on the link between these different notions of convergence.
- The interaction potential v is not rescaled with respect to the number of particles N, nor depending on the temperature T.
- The upper bound (53) is not sharp since we do expect, by a more refined analysis, a vanishing behavior $\mathrm{Tr}(\varrho_\Lambda |(\mathbb{1} - \pi_\Lambda)\mathsf{H}_{int}|^4) \to 0^+$ as $\Lambda \to +\infty$ for any fixed B. For (54) we can expect a better upper bound uniform with respect to Λ.

Bogoliubov, in [15], (see also [19]) assumed an expansion of the time dependent field operator by the operator valued distribution

$$\Psi(t,x) = \Psi^{(0,b)}(t,x) + \Theta(t,x)$$

where $\Psi^{(0,b)}(t,x) := \psi(t,x)\,\mathbb{1}$, for $\psi(t,x)$ solving the scalar Gross-Pitaevskii equation, $\mathbb{1}$ is the identity on the Fock space, and Θ the so-called normal fluid excitation field. Notice also the strong similarity between the Bogoliubov superfluid order parameter $\Psi^{(0,b)}(t,x)$ decomposed by

$$\Psi^{(0,b)}(t,x) = \sum_{k\in\mathbb{N}^d} \Big(\langle \varphi_k, \psi(t) \rangle_{L^2(\mathbb{R}^d)} \mathbb{1} \Big) \varphi_k(x)$$

and the effective field $\Psi_\Lambda^{(0)}(t,x) = \sum_k^\Lambda \mathsf{c}_k(t)\varphi_k(x)$ we introduced in (49). In particular, the operator $\langle \varphi_k, \psi(t)\rangle_{L^2(\mathbb{R}^d)}\mathbb{1}$ is a Wick operator whose symbol $\langle \phi_\alpha, \langle \varphi_k, \psi(t)\rangle_{L^2(\mathbb{R}^d)}\mathbb{1}\,\phi_\alpha\rangle = \langle \varphi_k, \psi(t)\rangle_{L^2(\mathbb{R}^d)}$ is the k-th Fourier component of $\psi(t)$ solving Hartree

$$i\dot{\psi} = \big(\mathsf{h} + v * |\psi|^2\big)\,\psi \tag{55}$$

for a fixed initial data $\psi_0 \in L^2(\mathbb{R}^d)$. Such components solves the family of discrete Hartree equations (45) without a cut-off, i.e. $\Lambda = +\infty$, and initial data $\langle \varphi_k, \psi_0\rangle_{L^2(\mathbb{R}^d)}$. Whereas we recall that the scalar terms $c_k(t,\alpha,\bar{\alpha}) = \langle \phi_\alpha, \mathsf{c}_k(t)\phi_\alpha\rangle$ determined by solving (45) are the k-th components of the Λ - regularized coupled discrete Hartree equations but with arbitrary initial data $\alpha_k \in \mathbb{C}$. This means that $\Psi_\Lambda^{(0)}(t,x)$ contains the information of the regularized Hartree flow for any fixed initial data.

3.5 The first order correction

The first order correction $\Psi^{(1)}(t,x)$ to the effective field $\Psi^{(0)}(t,x)$, that easily comes from the iteration of the integral equation for the quantum evolved Wick symbols

$$a_k(t,\omega,\bar{\omega}) = e^{-it\mathcal{L}_0} a_k(0,\omega,\bar{\omega}) + \int_0^t e^{-i(t-s)\mathcal{L}_0}\mathcal{L}_1 a_k(s,\omega,\bar{\omega})ds$$

reads, in the operator valued distribution form, as

$$\Psi^{(1)}(t,x) := \sum_{k}^{\Lambda} \varphi_k(x)\ \mathrm{Op}_{\mathrm{wick}}(\sigma_k^{(1)}(t,\cdot))$$

$$\sigma_k^{(1)}(t,\omega,\bar\omega) := e^{-it\mathcal{L}_0} a_k(0,\omega,\bar\omega) + \int_0^t e^{-i(t-s)\mathcal{L}_0}\mathcal{L}_1 e^{-is\mathcal{L}_0} a_k(0,\omega,\bar\omega) ds.$$

Clearly, $\Psi^{(1)}(t,x)$ is the first order correction of the effective quantum field $\Psi^{(0)}(t,x) := \sum_k^\Lambda \varphi_k(x)\ \mathrm{Op}_{\mathrm{wick}}(e^{-it\mathcal{L}_0}a_k(0,\cdot))$ that we have considered in Theorem 1. The straightforward generalization of Theorem 1 is the estimate for the quantum fluctuation of $\Psi_\Lambda^{(1)}(t,x)$ with respect to the quantum dynamics $\Psi_\Lambda(t,x) = \sum_k^\Lambda \varphi_k(x)\ \mathrm{Op}_{\mathrm{wick}}(a_k(t,\cdot))$ given by

$$\|\Psi_\Lambda - \Psi_\Lambda^{(1)}\|_\mu \le 2\sqrt{C_v}\frac{\Lambda^d}{B^{1+\frac14}}\Big(\frac{2^9}{B}\, C_v\, (1+2\Lambda)^{4d+\frac{d}{2}}\Big)\,\frac{t^2}{2}$$
$$\simeq 2^9 C_v^{1+\frac12}\,\frac{\Lambda^{5d+\frac{d}{2}}}{B^{2+\frac14}}\, t^2$$

and where $\Big(\frac{2^9}{B}\, C_v\, (1+2\Lambda)^{4d+\frac{d}{2}}\Big)$ is the constant appearing in the estimate for the zero-th order fluctuation $\|\Psi_\Lambda - \Psi_\Lambda^{(0)}\|_\mu$ showed in Theorem 1.
This proves that the first order correction $\|\Psi_\Lambda - \Psi_\Lambda^{(1)}\|_\mu$ is thus better with respect to the zero-th order effective fluctuation: both from the dependence on B as well as with respect to dependence on the Hardy constant $0 < C_v \le 1$ (now supposed smaller than one) of the interaction potential.

3.6 A bound on the temperature

We now set a fortiori bound between the total number $N \ge 2$ of bosons and the average of N_Λ with respect to the canonical Gibbs operator

$$\mathsf{G}_\Lambda := \frac{e^{-\beta \mathsf{H}_\Lambda}}{\mathrm{Tr}(e^{-\beta \mathsf{H}_\Lambda})} \tag{56}$$

with energy operator H_Λ as in (17). We recall that the number operator N_Λ is defined on the Bargmann-Fock space $\mathcal{F}_B(\mathbb{C}^\ell)$, $\ell := \Lambda^d$, which is isomorphic to a subset of the bosonic Fock space $\mathcal{F}$, see (20). For $\Lambda \to +\infty$ we have that $\mathcal{F}_B(\mathbb{C}^\ell)$ asymptotically recovers $\mathcal{F}$ as well as H_Λ, N_Λ weakly converge to H, N (see Sect. 3.1). We require $\forall \Lambda \ge 1$ that the number of expected particles with respect to the operator G_Λ cannot be bigger than N

$$\mathrm{Tr}(\mathsf{G}_\Lambda \mathsf{N}_\Lambda) \le N. \tag{57}$$

By defining $\beta := 1/(k_B T)$, $\hbar := 1$ and in analogy with the case of the free gas of bosons in the box (see page 16) we now exhibit an interval $0 < T \le \widetilde{T}$ of temperatures such that (57) is fulfilled $\forall \Lambda \ge 1$. In our work we are interested in the setting given in Sect. 2.2 where the interaction potential belongs to the Hardy class on $\mathbb{R}^d$. In Lemma 5 we prove the functional inequality between semipositive operators $\omega \mathsf{N}_\Lambda \leq \mathsf{H}_\Lambda \leq \Omega'_p \mathsf{N}_\Lambda^p$, for $\Omega'_p := 2(1+\Omega)3^p + 2C_v$, where the external potential fulfills $0 \leq u(x) \leq \Omega \|x\|^p$, $\Omega > 0$, $p \geq 2$. This allows, by the setting $B := e^{\beta\omega} - 1$, and thanks to the inequality (see Lemma 7)

$$\mathrm{Tr}(\mathsf{G}_\Lambda \mathsf{N}_\Lambda) \leq \frac{\mathrm{Tr}(e^{-\beta\omega \mathsf{N}_\Lambda} \mathsf{N}_\Lambda)}{\mathrm{Tr}(e^{-\beta \Omega'_p \mathsf{N}_\Lambda^p})}$$

to consider the new and more restrictive condition

$$\frac{\mathrm{Tr}(e^{-\beta\omega \mathsf{N}_\Lambda} \mathsf{N}_\Lambda)}{\mathrm{Tr}(e^{-\beta \Omega'_p \mathsf{N}_\Lambda^p})} \leq N. \tag{58}$$

In particular, we also prove (see Lemma 7) that the following interval of temperatures

$$0 < T \leq \frac{\omega}{k_B \ln\left(1 + \frac{1}{A_p N^2}\right)} =: \widetilde{T}\ , \qquad A_p := \left(\frac{\Gamma(\frac{1}{p})}{p}\right)^2 \left(\frac{\omega}{\Omega'_p}\right)^{\frac{2}{p}} \tag{59}$$

imply, $\forall \Lambda \geq 1$, the inequality (58) and thus also (57). Easily notice that $\sup_{p\geq 2} A_p < +\infty$, and that the function $\|x\|^p$ approximate, as $p \to +\infty$, the box potential for the volume $\|x\| \leq 1$. We stress moreover that, for large values of N,

$$\widetilde{T} \approx \frac{\omega}{k_B} A_p\, N^2. \tag{60}$$

The inequality in (59) can be rewritten as

$$B \geq \frac{1}{A_p\, N^2}, \qquad . \tag{61}$$

We have thus shown an interval of temperatures uniform with respect to the UV cutoff parameter $\Lambda \geq 1$. The setting $B \geq 1$ used in Theorem 1 can be replaced with (61) and the related estimates (53) - (54) can be modified in the proof of the main Theorem and become dependant on both B than N. Alternatively, one can fix the additional condition $A_p\, N^2 \geq 1$ and take into account the same estimates.
On the other hand, the value (59) is obtained by a (Λ - independent) lower bound for $\mathrm{Tr}(\mathsf{G}_\Lambda \mathsf{N}_\Lambda)$ and gives a larger interval of temperatures than the sharp one for the non regularized condition

$$\mathrm{Tr}(\mathsf{GN}) \leq N. \tag{62}$$

Indeed, in the free case of the box potential, the optimal value of this bound for the temperatures grows as $N^{2/3}$.

The free case in the box. We remind that the temperature under which there is experimental evidence of BEC for N non-interacting (i.e. $v = 0$) bosons confined by a 3d box potential in a volume $V = L^3$ is given by the interval $0 < T \leq T_c$ where

$$T_c := \Big(\frac{1}{\zeta(3/2)}\Big)^{\frac{2}{3}} \Big(\frac{N}{V}\Big)^{\frac{2}{3}} \frac{2\pi\hbar^2}{mk_B} \tag{63}$$

where ζ is the Riemann zeta function, m the mass of the particles and k_B the Boltzmann constant. Among all various approaches, (63) can be derived from the inequality

$$\mathrm{Tr}\Big(\frac{e^{-\beta \mathsf{H}}}{\mathrm{Tr}(e^{-\beta \mathsf{H}})}\mathsf{N}\Big) = \sum_{(n_x,n_y,n_z)\in\mathbb{N}^3\setminus\{0\}} \frac{1}{e^{\beta\varepsilon_1(n_x^2+n_y^2+n_z^2)}-1} \leq N \tag{64}$$

where $\mathsf{H} = \mathsf{H}_{free}$ on $\mathcal{F}$ with single particle state space $L^2(\mathbb{T}^3)$, and thus the first excited energy value reads $\varepsilon_1 := \frac{\hbar^2\pi^2}{2mL^2}$ linked to all the other energies by $\varepsilon_n := \varepsilon_1(n_x^2+n_y^2+n_z^2)$. The equality in the lefthand side of (64) can be proved by application the so-called quantum Wick Theorem, as it can be seen in the formula (1.28) and (1.31) in [25].

3.7 One particle density operator, thermal estimates and correlations

The one particle density operator $\Gamma : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ associated to $\sqrt{N}$-rescaled coherent states on Fock $\phi_{\sqrt{N}\alpha_0} \in \mathcal{F}$ centered on a fixed $\alpha_0 \in L^2(\mathbb{R}^d)$ with $\|\alpha_0\|_{L^2} = 1$, evaluated with respect to the field operator $\Psi(t,x)$ solving (2) has integral kernel

$$\begin{aligned}\Gamma(t,x,y) &= \frac{1}{\langle \phi_{\sqrt{N}\alpha_0}, \mathsf{N}\phi_{\sqrt{N}\alpha_0}\rangle}\langle \phi_{\sqrt{N}\alpha_0}, \Psi^\dagger(t,y)\Psi(t,x)\phi_{\sqrt{N}\alpha_0}\rangle \\ &= \frac{1}{N}\mathrm{Tr}(\pi_0\Psi^\dagger(t,y)\Psi(t,x))\end{aligned}$$

where π_0 is defined as the rank one projector into $\phi_{\sqrt{N}\alpha_0}$. In particular, it is easily seen that $\Gamma \geq 0$ and $\mathrm{Tr}(\Gamma) = 1$.
The term $\Psi^{(0,b)}(t,x) := \psi(t,x)\,\mathbb{1}$ has one particle density operator with integral kernel

$$\Gamma_0(t,x,y) = \frac{1}{N}\overline{\psi}(t,y)\psi(t,x)$$

namely is the rank one projector $\Pi_{\psi(t)}$ onto the single particle state $\psi(t)$ solving the scalar Hartree equation (55) with a fixed initial data $\psi(0) \in L^2(\mathbb{R}^d)$ and $\|\psi(t)\|_{L^2}^2 = N$.
A standard result in the literature (see for example [9] and references therein) can be given by the trace norm $\mathrm{Tr}|A| := \mathrm{Tr}\sqrt{A^\dagger A}$ convergence

$$\mathrm{Tr}|\Gamma(t) - \Pi_{\psi(t)}| \lesssim \frac{\exp(\exp(C\,t))}{\sqrt{N}} \tag{65}$$

for some $C > 0$. This can be proved under various assumptions on initial data and about the scaling of the interaction potential v in terms of N.

Notice that the operator $\delta\Gamma$ associated to ϱ_Λ (in place of π_0) and evaluated for the deviation field Θ_Λ with integral kernel

$$\delta\Gamma(t,x,y) := \mathrm{Tr}(\varrho_\Lambda \Theta^\dagger_\Lambda(t,y)\Theta_\Lambda(t,x)), \quad \Theta_\Lambda := \Psi_\Lambda - \Psi^{(0)}_\Lambda,$$

fulfills the inequalities (see Proposition 10)

$$\mathrm{Tr}|\delta\Gamma| \geq \|\delta\Gamma\|_{\mathrm{HS}} \geq \frac{1}{\Lambda^d}\mathrm{Tr}\Big(\varrho_\Lambda \int_{\mathbb{R}^d} \Theta^\dagger_\Lambda(t,x)\Theta_\Lambda(t,x)\,dx\Big). \tag{66}$$

The righthand side of (66) is the rescaled norm $\|\Theta\|^2_\star$ introduced in Def. 1,

$$\|\Theta_\Lambda\|^2_\star := \mathrm{Tr}\Big(\varrho_\Lambda \int_{\mathbb{R}^d} \Theta^\dagger_\Lambda(t,x)\Theta_\Lambda(t,x)\,dx\Big).$$

The norm $\|\Theta\|_\mu$ is sharp with respect to $\|\Theta\|_\star$ (see Remark 3). It follows that

$$\|\delta\Gamma\|_{\mathrm{HS}} \geq \frac{1}{\Lambda^d}\|\Theta_\Lambda\|^2_\star \geq \|\Theta_\Lambda\|^2_\mu\,. \tag{67}$$

Remark 5 The lower bound for $\|\delta\Gamma\|_{\mathrm{HS}}$ can be interpreted as an indicator for the growth of particles correlation in the framework of thermal measures. Since (54) is an upper bound, a sharp value for $\|\Theta_\Lambda\|_\mu$ grows at most linearly in time. Such an observation is a novelty in the study of lower bounds for quantum correlations, which are difficult to recover by quantitative bounds.

Notice also that the UV-regularized one particle density operator

$$\Gamma_\Lambda(t,x,y) := \mathrm{Tr}(\varrho_\Lambda \Psi^\dagger_\Lambda(t,y)\Psi_\Lambda(t,x)) \tag{68}$$

and the regularized effective one

$$\Gamma^{(0)}_\Lambda(t,x,y) := \mathrm{Tr}(\varrho_\Lambda \Psi^{(0),\dagger}_\Lambda(t,y)\Psi^{(0)}_\Lambda(t,x)) \tag{69}$$

satisfy the Hilbert-Schmidt operator estimate

$$\begin{aligned}
&\|\Gamma_\Lambda - \Gamma^{(0)}_\Lambda\|^2_{\mathrm{HS}} = \int_{\mathbb{R}^d} |\Gamma_\Lambda(t,x,y) - \Gamma^{(0)}_\Lambda(t,x,y)|^2\,dxdy \\
&\leq \|\Theta_\Lambda\|^4_\star + 2\|\Theta_\Lambda\|^3_\star \mathrm{Tr}(\varrho_\Lambda \mathsf{N}_\Lambda) + 4\|\Theta_\Lambda\|^{2+\frac{3}{2}}_\star (\mathrm{Tr}(\varrho_\Lambda \mathsf{N}_\Lambda))^{\frac{1}{2}} + 2\|\Theta_\Lambda\|_\star \mathrm{Tr}(\varrho_\Lambda \mathsf{N}_\Lambda) \\
&\leq \|\Theta_\Lambda\|^4_\star + 2\|\Theta_\Lambda\|^3_\star \frac{\Lambda^d}{B} + 4\|\Theta_\Lambda\|^{\frac{7}{2}}_\star \Big(\frac{\Lambda^d}{B}\Big)^{\frac{1}{2}} + 2\|\Theta_\Lambda\|_\star \frac{\Lambda^d}{B}.
\end{aligned} \tag{70}$$

It is remarkable that $\|\Theta_\Lambda\|_\star$ can be used to control from above the quantity $\|\Gamma_\Lambda - \Gamma^{(0)}_\Lambda\|^2_{\mathrm{HS}}$, as it can be used to control from the bottom the norm $\|\delta\Gamma\|_{\mathrm{HS}}$. Moreover, we do expect that this estimate can be improved with respect to the dependence on Λ.

3.8 Assessment of the convergence

In order to prove Theorem 1, we first need to write explicitly the equations of motion for the Wick symbols a_k given in (26).

Remark 6 In this subsection, since we have long computations within various proofs, we use the compact notation $\langle\alpha|\mathsf{F}|\alpha\rangle \equiv \langle\phi_\alpha, \mathsf{F}\phi_\alpha\rangle$ where $\phi_\alpha \in \mathcal{F}_B(\mathbb{C}^\ell)$ are the coherent states in the Bargmann-Fock space and $\langle\ ,\ \rangle$ is the scalar product on $\mathcal{F}_B(\mathbb{C}^\ell)$ as given in (107).

Proposition 2 *a_k satisfies the following Cauchy problem*

$$i\dot{a}_k = (\mathcal{L}_0 + \mathcal{L}_1)a_k \tag{71}$$

where $a_k(0) = \alpha_k$, $\mathcal{L}_0 := \{\,\cdot\,, \mathcal{H}\}$ is the Lie derivative along the Hamiltonian flow associated to $\mathcal{H}$ given in (38) and

$$\mathcal{L}_1 := \frac{1}{2}\sum_{ij}^{\Lambda} \frac{\partial^2\mathcal{H}}{\partial\bar{\alpha}_i\partial\bar{\alpha}_j}\frac{\partial^2}{\partial\alpha_i\partial\alpha_j} - \frac{\partial^2\mathcal{H}}{\partial\alpha_i\partial\alpha_j}\frac{\partial^2}{\partial\bar{\alpha}_i\partial\bar{\alpha}_j}.$$

Proof. The operator Heisenberg equation $i\dot{\mathsf{a}}_k(t) = [\mathsf{a}_k(t), \mathsf{H}_\Lambda]$ with initial data $\mathsf{a}_k(0) := \mathsf{a}_k$ turns, in terms of Wick symbols, into

$$i\dot{a}_k(t) = \{a_k(t), \mathcal{H}\}_w, \qquad a_k(0) := \alpha_k.$$

Recalling the explicit form of the Wick brackets, we obtain

$$i\dot{a}_k(t) = \{a_k(t), \mathcal{H}\}_w = \{a_k(t), \mathcal{H}\} + \frac{1}{2}\sum_{ij}^{\Lambda} \frac{\partial^2\mathcal{H}}{\partial\bar{\alpha}_i\partial\bar{\alpha}_j}\frac{\partial^2 a_k(t)}{\partial\alpha_i\partial\alpha_j} - \frac{\partial^2\mathcal{H}}{\partial\alpha_i\partial\alpha_j}\frac{\partial^2 a_k(t)}{\partial\bar{\alpha}_i\partial\bar{\alpha}_j}$$

since $\{\,\cdot\,, \mathcal{H}\}_w$ does not contain $\mathcal{O}(\partial^3)$ terms because $\mathcal{H}$ is a polynomial of degree 2 in α and in $\bar{\alpha}$. The identification of $\mathcal{L}_0$ and $\mathcal{L}_1$ is now straight-forward. □

Proposition 3 *The deviation term $\delta_k(t) := a_k(t) - c_k(t)$ satisfies*

$$i\dot{\delta}_k = \mathcal{L}_0\delta_k + \mathcal{L}_1 a_k$$

for $\delta_k(0) := a_k(0) - c_k(0) = 0$. Furthermore,

$$\delta_k(t) = \int_0^t e^{-i(t-s)\mathcal{L}_0}\mathcal{L}_1 a_k\,(s)ds\,,$$

where $e^{-is\mathcal{L}_0}$ denotes the pull-back by the Hamiltonian flow $\Phi^s_{\mathcal{H}}$.

Proof. Since $i\dot{c}_k = \{c_k, \mathcal{H}\} = \mathcal{L}_0 c_k$ and the Lie derivative $\mathcal{L}_0$ is linear, it follows

$$i\dot{\delta}_k = i(\dot{a}_k - \dot{c}_k) = \mathcal{L}_0(a_k - c_k) + \mathcal{L}_1 a_k = \mathcal{L}_0 \delta_k + \mathcal{L}_1 a_k.$$

Thus, a standard argument for perturbed semigroups gives

$$\delta_k(t) = e^{-it\mathcal{L}_0}\delta_k(0) + \int_0^t e^{-i(t-s)\mathcal{L}_0}\mathcal{L}_1 a_k\,(s)ds = \int_0^t e^{-i(t-s)\mathcal{L}_0}\mathcal{L}_1 a_k\,(s)ds.$$

□

We can now compute an estimate for the $L^2(\mathbb{C}^\ell, d\mu_\Lambda)$-norm of δ_k, but first we need to bear in mind the following Proposition.

Proposition 4 *The gaussian measure μ is invariant under the discrete Hartree flow, that is*

$$d\mu(\Phi^t_{\mathcal{H}}(\alpha)) = d\mu(\alpha) \quad \forall t \geq 0.$$

Furthermore, averages with respect to μ are invariant under the full quantum evolution, namely for all $\mathsf{F} \in \mathsf{P}_\Lambda(\mathsf{a}, \mathsf{a}^\dagger)$

$$\int_{\mathbb{C}^\ell} \langle\alpha|\,\mathsf{F}(\mathsf{a}(t), \mathsf{a}^\dagger(t))\,|\alpha\rangle\, d\mu(\alpha) = \int_{\mathbb{C}^\ell} \langle\alpha|\,\mathsf{F}(\mathsf{a}, \mathsf{a}^\dagger)\,|\alpha\rangle\, d\mu(\alpha), \quad \ell := \Lambda^d.$$

Proof. Since $|\alpha|^2$ is the Wick symbol of N_Λ, the measure, interpreted as a volume form on coherent phase space, can be written as

$$d\mu(\alpha) = \frac{1}{Z}\, e^{-B|\alpha|^2} \prod_k^\Lambda d\alpha_k \wedge d\bar{\alpha}_k.$$

Since $\{N_\Lambda, \mathcal{H}\} \equiv 0$, we obtain (in short form)

$$\begin{aligned} d\mu(\Phi^t_{\mathcal{H}}) &= \frac{e^{-B|\alpha|^2}}{Z} \prod_k^\Lambda d(\Phi^t_{\mathcal{H}})_k \wedge d(\bar{\Phi}^t_{\mathcal{H}})_k = \det(d(\Phi^t_{\mathcal{H}}))d\mu \\ &= d\mu \end{aligned}$$

where $\det(\mathrm{d}(\Phi^t_{\mathcal{H}})) = 1$ since $\Phi^t_{\mathcal{H}}$ is a one-parametre group of symplectomorphisms.
The second result follows by recalling the definition of μ with respect to the Wick map.

$$\begin{aligned} \int_{\mathbb{C}^\ell} \langle\alpha|\,\mathsf{F}(\mathsf{a}(t), \mathsf{a}^\dagger(t))\,|\alpha\rangle\, d\mu(\alpha) &= \mathrm{Tr}(\mathsf{F}(\mathsf{a}(t), \mathsf{a}^\dagger(t))\varrho_\Lambda) \\ &= \mathrm{Tr}(e^{it\mathsf{H}_\Lambda}\mathsf{F}(\mathsf{a}, \mathsf{a}^\dagger)e^{-it\mathsf{H}_\Lambda}\varrho_\Lambda) = \mathrm{Tr}(\mathsf{F}(\mathsf{a}, \mathsf{a}^\dagger)\varrho_\Lambda) \end{aligned}$$

where in the last passage cyclicity of the trace and $[\mathsf{H}_\Lambda, \varrho_\Lambda] = 0$ were employed.
□

Proposition 5 *The norm of the deviation δ_k satisfies the following upper bound.*

$$\|\delta_k(t)\|_\mu^2 \le \left(\int_0^t \left(\int_{\mathbb{C}^\ell} |\mathcal{L}_1 a_k(s)|^2 d\mu\right)^{\frac{1}{2}} ds\right)^2. \tag{72}$$

Proof. By Proposition 3,

$$\|\delta_k(t)\|_\mu^2 = \int_{\mathbb{C}^\ell}\int_{[0,t]^2} \overline{e^{-i(t-s)\mathcal{L}_0}\mathcal{L}_1 a_k\,(s)} e^{-i(t-u)\mathcal{L}_0}\mathcal{L}_1 a_k\,(u)\, dsdud\mu$$

For fixed times s, u, employing the Cauchy-Schwarz inequality in $L^2(\mathbb{C}^\ell, d\mu)$ the previous expression becomes

$$\|\delta_k(t)\|_\mu^2 \le \int_{[0,t]^2} \left\|e^{-i(t-s)\mathcal{L}_0}\mathcal{L}_1 a_k\,(s)\right\|_\mu \left\|e^{-i(t-u)\mathcal{L}_0}\mathcal{L}_1 a_k\,(u)\right\|_\mu dsdu.$$

However, since by Proposition 4 the measure μ is invariant under the scalar flow, we have that

$$\left\|e^{-i(t-s)\mathcal{L}_0}\mathcal{L}_1 a_k\,(s)\right\|_\mu = \|\mathcal{L}_1 a_k\,(s)\|_\mu$$

hence

$$\|\delta_k(t)\|_\mu^2 \le \int_0^t\int_0^t \|\mathcal{L}_1 a_k\,(s)\|_\mu \|\mathcal{L}_1 a_k\,(u)\|_\mu\, dsdu = \left(\int_0^t \|\mathcal{L}_1 a_k\,(s)\|_\mu\, ds\right)^2.$$

□

3.9 Computation of the remainder $\|\mathcal{L}_1 a_q\,(s)\|_\mu$

In Proposition 2 we have the term

$$\mathcal{L}_1 a_q(s) := \frac{1}{2}\sum_{ij}^{\Lambda} \frac{\partial^2\mathcal{H}}{\partial\bar\alpha_i\partial\bar\alpha_j}\frac{\partial^2 a_q(s)}{\partial\alpha_i\partial\alpha_j} - \frac{\partial^2\mathcal{H}}{\partial\alpha_i\partial\alpha_j}\frac{\partial^2 a_q(s)}{\partial\bar\alpha_i\partial\bar\alpha_j};$$

therefore taking into account the explicit expression of $\mathcal{H}$ and employing equations (118) - (119) to transform the second derivatives of $a_q(s)$ into coherent expectations of commutators, through simple algebraic manipulations we may write

$$\mathcal{L}_1 a_q(s) = \frac{1}{2}\sum_{klmn}^{\Lambda} v_{klmn}\Big(\alpha_m\alpha_n \langle\alpha|\,\mathsf{a}_q(s)\mathsf{a}_k^\dagger\mathsf{a}_l^\dagger\,|\alpha\rangle - 2\bar\alpha_k\alpha_m\alpha_n\langle\alpha|\,\mathsf{a}_q(s)\mathsf{a}_l^\dagger\,|\alpha\rangle$$
$$-\bar\alpha_k\bar\alpha_l\langle\alpha|\,\mathsf{a}_m\mathsf{a}_n\mathsf{a}_q(s)\,|\alpha\rangle + 2\bar\alpha_k\bar\alpha_l\alpha_m\langle\alpha|\,\mathsf{a}_n\mathsf{a}_q(s)\,|\alpha\rangle\Big). \tag{73}$$

Lemma 2 *The following inequalities holds true.*

$$\langle\alpha|\, \mathsf{a}_i\mathsf{a}_j\mathsf{a}_i^\dagger\mathsf{a}_j^\dagger\, |\alpha\rangle^{\frac{1}{2}} \leq 2+2|\alpha_i|+2|\alpha_j|+|\alpha_i\alpha_j|,$$
$$\langle\alpha|\, \mathsf{a}_i\mathsf{a}_i^\dagger\, |\alpha\rangle^{\frac{1}{2}} \leq 1+|\alpha_i|.$$

Proof. The inequalities are obtained by using the CCRs to bring the as to the right and by sublinearity of the square root function,

$$\begin{aligned}\langle\alpha|\, \mathsf{a}_i\mathsf{a}_j\mathsf{a}_i^\dagger\mathsf{a}_j^\dagger\, |\alpha\rangle^{\frac{1}{2}} &= \langle\alpha|\, 1+\delta_{ij}+\mathsf{a}_i^\dagger\mathsf{a}_i+\mathsf{a}_j^\dagger\mathsf{a}_j+2\delta_{ij}\mathsf{a}_i^\dagger\mathsf{a}_i+\mathsf{a}_i^\dagger\mathsf{a}_j^\dagger\mathsf{a}_i\mathsf{a}_j\, |\alpha\rangle^{\frac{1}{2}}\\ &\leq 2+2|\alpha_i|+2|\alpha_j|+|\alpha_i\alpha_j|.\\ \langle\alpha|\, \mathsf{a}_i\mathsf{a}_i^\dagger\, |\alpha\rangle^{\frac{1}{2}} &= \sqrt{\langle\alpha|\, \mathsf{a}_i^\dagger\mathsf{a}_i+1\, |\alpha\rangle} \leq 1+|\alpha_i|.\end{aligned}$$

□

Proposition 6 *The squared-module of $\mathcal{L}_1 a_q(s)$ may be estimated from above by the product of a time-dependent Wick symbol and a time-independent polynomial*

$$|\mathcal{L}_1 a_q(s)|^2 \leq \langle\alpha|\, \mathsf{a}_q(s)\mathsf{a}_q^\dagger(s)\, |\alpha\rangle\, (p(\alpha,\bar\alpha))^2,$$

where $p:\mathbb{C}^\ell\to\mathbb{R}^+$ is defined as

$$p(\alpha,\bar\alpha) := 3\sum_{klmn}^{\Lambda} |v_{klmn}|(|\alpha_k\alpha_l\alpha_m\alpha_n|+|\alpha_k\alpha_l\alpha_m|+|\alpha_k\alpha_m\alpha_n|+|\alpha_k\alpha_l|+|\alpha_m\alpha_n|).$$

Proof. Employing the following form of the triangular inequality, $|z+w| \leq (|z|+|w|)\ \forall z,w\in\mathbb{C}$ and taking into account equation (73), we have that $|\mathcal{L}_1 a_q(s)|$ has the upper bound

$$\begin{aligned}&\leq \sum_{klmn}^{\Lambda} \frac{|v_{klmn}|}{2}\Big[|\alpha_m\alpha_n|\Big|\, \langle\alpha|\, \mathsf{a}_q(s)\mathsf{a}_k^\dagger\mathsf{a}_l^\dagger\, |\alpha\rangle\, \Big| + 2|\alpha_k\alpha_m\alpha_n|\Big|\, \langle\alpha|\, \mathsf{a}_q(s)\mathsf{a}_l^\dagger\, |\alpha\rangle\, \Big|\\ &+|\alpha_k\alpha_l|\Big|\, \langle\alpha|\, \mathsf{a}_m\mathsf{a}_n\mathsf{a}_q(s)\, |\alpha\rangle\, \Big| + 2|\alpha_k\alpha_l\alpha_m|\Big|\, \langle\alpha|\, \mathsf{a}_n\mathsf{a}_q(s)\, |\alpha\rangle\, \Big|\Big].\end{aligned}$$

Using the Cauchy-Schwarz inequality and the previous Lemma for the first two summands,

$$\begin{aligned}\Big|\langle\alpha|\, \mathsf{a}_q(s)\mathsf{a}_k^\dagger\mathsf{a}_l^\dagger\, |\alpha\rangle\Big| &\leq \langle\alpha|\, \mathsf{a}_q(s)\mathsf{a}_q^\dagger(s)\, |\alpha\rangle^{\frac{1}{2}}\, \langle\alpha|\, \mathsf{a}_k\mathsf{a}_l\mathsf{a}_k^\dagger\mathsf{a}_l^\dagger\, |\alpha\rangle^{\frac{1}{2}}\\ &\leq \langle\alpha|\, \mathsf{a}_q(s)\mathsf{a}_q^\dagger(s)\, |\alpha\rangle^{\frac{1}{2}}\, (2+2|\alpha_k|+2|\alpha_l|+|\alpha_k\alpha_l|).\end{aligned}$$

and moreover

$$\Big|\langle\alpha|\, \mathsf{a}_q(s)\mathsf{a}_l^\dagger\, |\alpha\rangle\Big| \leq \langle\alpha|\, \mathsf{a}_q(s)\mathsf{a}_q^\dagger(s)\, |\alpha\rangle^{\frac{1}{2}}\, \langle\alpha|\, \mathsf{a}_l\mathsf{a}_l^\dagger\, |\alpha\rangle^{\frac{1}{2}} \leq \langle\alpha|\, \mathsf{a}_q(s)\mathsf{a}_q^\dagger(s)\, |\alpha\rangle^{\frac{1}{2}}\, (1+|\alpha_l|)$$

obtaining analogous expressions for the third and fourth one, and taking into account index symmetrisation, eventually we are lead to

$$|\mathcal{L}_1 a_q(s)|^2 \le \langle\alpha|\, \mathsf{a}_q(s)\mathsf{a}_q^\dagger(s)\,|\alpha\rangle \Big\{ \sum_{klmn}^{\Lambda} |v_{klmn}| \Big[|\alpha_k\alpha_l\alpha_m\alpha_n| + 3|\alpha_k\alpha_l\alpha_m| + +3\,|\alpha_k\alpha_m\alpha_n| + 2|\alpha_k\alpha_l| + 2|\alpha_m\alpha_n|\Big]\Big\}^2 .$$

Overestimating each of the constant factors multiplying the polynomial summands with 3, the statement is proven. □

Proposition 7 *The norm $\|\mathcal{L}_1 a_q(s)\|_\mu$ has an upper bound not depending on q and s indeces. In particular, assuming $B \ge 1$ we have*

$$\|\mathcal{L}_1 a_q(s)\|_\mu \le \frac{2^8}{B} \sum_{klmn}^{\Lambda} |v_{klmn}|.$$

Proof. Due to the previous Proposition, it is clear that

$$\begin{aligned}
\|\mathcal{L}_1 a_q(s)\|_\mu^2 &\le \int_{\mathbb{C}^\ell} \langle\alpha|\, \mathsf{a}_q(s)\mathsf{a}_q^\dagger(s)\,|\alpha\rangle\, p^2(\alpha)\ \mathrm{d}\mu(\alpha) \\
&\le \Big(\int_{\mathbb{C}^\ell} \big|\langle\alpha|\, \mathsf{a}_q(s)\mathsf{a}_q^\dagger(s)\,|\alpha\rangle\big|^2\, \mathrm{d}\mu(\alpha)\Big)^{\frac12} \left\|p^2\right\|_\mu \\
&\le \Big(\int_{\mathbb{C}^\ell} \langle\alpha|\, \mathsf{a}_q(s)\mathsf{a}_q^\dagger(s)\mathsf{a}_q(s)\mathsf{a}_q^\dagger(s)\,|\alpha\rangle\, \mathrm{d}\mu(\alpha)\Big)^{\frac12} \left\|p^2\right\|_\mu \\
&= \Big(\int_{\mathbb{C}^\ell} \langle\alpha|\, \mathsf{a}_q\mathsf{a}_q^\dagger\mathsf{a}_q\mathsf{a}_q^\dagger\,|\alpha\rangle\, \mathrm{d}\mu(\alpha)\Big)^{\frac12} \left\|p^2\right\|_\mu
\end{aligned}$$

where in the second inequality we used Cauchy-Schwarz inequality with respect to μ, and in the last one the invariance of the measure under the full quantum evolution. Let us compute the two terms.

$$\begin{aligned}
&\int_{\mathbb{C}^\ell} \langle\alpha|\, \mathsf{a}_q\mathsf{a}_q^\dagger\mathsf{a}_q\mathsf{a}_q^\dagger\,|\alpha\rangle\, d\mu(\alpha) = \int_{\mathbb{C}^\ell} \langle\alpha, 1 + 3\mathsf{a}_q^\dagger\mathsf{a}_q + (\mathsf{a}_q^\dagger)^2\mathsf{a}_q^2, \alpha\rangle d\mu(\alpha) \\
&= \int_{\mathbb{C}^\ell} 1 + 3|\alpha_q|^2 + |\alpha_q|^4 d\mu(\alpha) = 1 + \frac{3}{B} + \frac{2}{B^2}
\end{aligned}$$

where in the last equality we applied Proposition 1. Meanwhile,

$$\left\|p^2\right\|_\mu^2 = \int_{\mathbb{C}^\ell} \Big\{ \sum_{klmn}^{\Lambda} 3|v_{klmn}| \Big[|\alpha_k\alpha_l\alpha_m\alpha_n| + |\alpha_k\alpha_l\alpha_m| + |\alpha_k\alpha_m\alpha_n| + |\alpha_k\alpha_l| + |\alpha_m\alpha_n|\Big]\Big\}^4 d\mu(\alpha)\,.$$

We now look for an upper bound for the integral

$$I := \int_{\mathbb{C}^\ell} \Big[|\alpha_k\alpha_l\alpha_m\alpha_n| + |\alpha_k\alpha_l\alpha_m| + |\alpha_k\alpha_m\alpha_n| + |\alpha_k\alpha_l| + |\alpha_m\alpha_n|\Big]^4 d\mu(\alpha)$$

which can be written as the sum of several terms. Moreover, the intergral over phase space does not depend on the index. Thus, we can exhibit the bound

$$\begin{aligned} I \ &\leq 5^4\Big(\frac{4!}{2!}\Big)^2 \int_{\mathbb{C}^\ell} |\alpha_m\alpha_n|^4 \ d\mu(\alpha) \\ &= 5^4\Big(\frac{4!}{2!}\Big)^2 \Big(\int_{\mathbb{C}^\ell} |\alpha_n|^4 \ d\mu(\alpha)\Big)^2 \leq \frac{(4!)^2 \cdot 5^4}{B^4}. \end{aligned} \tag{74}$$

The above norm has thus the bound

$$\|p^2\|_\mu^2 \leq \left(\sum_{klmn}^{\Lambda} 3|v_{klmn}|\right)^4 \frac{(4!)^2\cdot 5^4}{B^4}. \tag{75}$$

Then, in view of all the above calculations, and assuming $B \geq 1$,

$$\begin{aligned} \|\mathcal{L}_1 a_q(s)\|_\mu \ &\leq \|p^2\|_\mu^{\frac{1}{2}} \Big(1+\frac{3}{B}+\frac{2}{B^2}\Big)^{\frac{1}{2}} \\ &\leq \Big(\sum_{klmn}^{\Lambda} 3|v_{klmn}|\Big)\frac{\sqrt{4!}\cdot 5}{B}\Big(1+\frac{3}{B}+\frac{2}{B^2}\Big)^{\frac{1}{2}} \\ &< \Big(\sum_{klmn}^{\Lambda} 3|v_{klmn}|\Big)\frac{12\cdot 5}{B} < \frac{2^8}{B}\sum_{klmn}^{\Lambda}|v_{klmn}|. \end{aligned} \tag{76}$$

□

In view of the previous statements, we can now show the proof of the main result of the paper.
Proof of Theorem 1. Recall that

$$\|\Psi_\Lambda^{(0)}(t)\|_\mu^2 := \frac{1}{\Lambda^d}\sum_{k}^{\Lambda}\|c_k(t)\|_{L^2(\mu)}^2 \tag{77}$$

for $c_k(t)$ solving the family of coupled discrete Hartree equations (45) and where measure μ is invariant under the flow. Thus, by defining $n_k(\bar{\alpha},\alpha) := |\alpha_k|^2$ we get

$$\|\Psi_\Lambda^{(0)}(t)\|_\mu^2 = \|\Psi_\Lambda^{(0)}(0)\|_\mu^2 = \frac{1}{\Lambda^d}\sum_{k}^{\Lambda}\|n_k\|_{L^1(\mu)}^2 = \frac{1}{\Lambda^d}\,\Lambda^d\,\frac{1}{B} = \frac{1}{B} \tag{78}$$

namely the statement $\|\Psi_\Lambda^{(0)}\|_\mu = \frac{1}{\sqrt{B}}$.
As for the second statement, notice that

$$\begin{aligned} \|\Psi(t)-\Psi_\Lambda(t)\|_\mu^2 \ &\leq \frac{1}{\Lambda^d}\ \|\Pi_\Lambda\Psi(t)-\Psi_\Lambda(t)\|_\star^2 \leq \frac{1}{\Lambda^d}\ \|\Psi(t)-\Psi_\Lambda(t)\|_\star^2 \\ &= \frac{1}{\Lambda^d}\ \mathrm{Tr}\Big(\frac{e^{-\beta\omega\mathsf{N}_\Lambda}}{\mathrm{Tr}(e^{-\beta\omega\mathsf{N}_\Lambda})}\int_{\mathbb{R}^d}\mathsf{B}_\Lambda^\dagger(t,x)\mathsf{B}_\Lambda(t,x)dx\Big) \end{aligned} \tag{79}$$

for $\mathsf{B}_\Lambda(t,x) := \Psi(t,x) - \Psi_\Lambda(t,x)$. A standard semigroup argument gives

$$\mathsf{B}_\Lambda(t) = e^{i\mathsf{H}_\Lambda t}\mathsf{B}_\Lambda(0)e^{-i\mathsf{H}_\Lambda t} + \int_0^t e^{i\mathsf{H}_\Lambda(t-s)}\,[\mathsf{H}-\mathsf{H}_\Lambda, \Psi(s)]\,e^{-i\mathsf{H}_\Lambda(t-s)}ds$$

whence

$$\|\mathsf{B}_\Lambda(t)\|_\star \le \|e^{i\mathsf{H}_\Lambda t}\mathsf{B}_\Lambda(0)e^{-i\mathsf{H}_\Lambda t}\|_\star + \int_0^t \|e^{i\mathsf{H}_\Lambda(t-s)}[\mathsf{H}-\mathsf{H}_\Lambda,\Psi(s)]e^{-i\mathsf{H}_\Lambda(t-s)}\|_\star\,ds.$$

Thus, since $[\mathsf{N}_\Lambda,\mathsf{H}_\Lambda]=0$ and the trace is invariant under any unitary conjugation of operators,

$$\begin{aligned} \|\mathsf{B}_\Lambda(t)\|_\star &\le \|\mathsf{B}_\Lambda(0)\|_\star + \int_0^t \|[\mathsf{H}-\mathsf{H}_\Lambda,\Psi(s)]\|_\star\,ds \\ &\le \|\mathsf{B}_\Lambda(0)\|_\star + \int_0^t \|(\mathsf{H}-\mathsf{H}_\Lambda)\Psi(s)\|_\star + \|\Psi(s)(\mathsf{H}-\mathsf{H}_\Lambda)\|_\star\,ds. \end{aligned} \tag{80}$$

In particular, $\|\Psi(s)(\mathsf{H}-\mathsf{H}_\Lambda)\|_\star = \|(\mathsf{H}-\mathsf{H}_\Lambda)\Psi(s)\|_\star$ and

$$\|\Psi(s)(\mathsf{H}-\mathsf{H}_\Lambda)\|_\star^2 = \mathrm{Tr}\Big(\frac{e^{-\beta\omega\mathsf{N}_\Lambda}}{\mathrm{Tr}(e^{-\beta\omega\mathsf{N}_\Lambda})}(\mathsf{H}-\mathsf{H}_\Lambda)\int_{\mathbb{R}^d}\Psi(s,x)^\dagger\Psi(s,x)dx\,(\mathsf{H}-\mathsf{H}_\Lambda)\Big)$$

whence

$$\|\Psi(s)(\mathsf{H}-\mathsf{H}_\Lambda)\|_\star^2 = \mathrm{Tr}\Big(\frac{e^{-\beta\omega\mathsf{N}_\Lambda}}{\mathrm{Tr}(e^{-\beta\omega\mathsf{N}_\Lambda})}(\mathsf{H}-\mathsf{H}_\Lambda)\mathsf{N}(\mathsf{H}-\mathsf{H}_\Lambda)\Big).$$

Taking into account that $\mathsf{H}_\Lambda = \pi_\Lambda\mathsf{H}\pi_\Lambda$ we have $\mathsf{H}_\Lambda\mathsf{N} = \mathsf{H}_\Lambda\pi_\Lambda\mathsf{N} = \mathsf{H}_\Lambda\mathsf{N}_\Lambda = \mathsf{N}_\Lambda\mathsf{H}_\Lambda$. Moreover, $[\mathsf{H},\mathsf{N}]=0$ whence

$$\begin{aligned} \|\Psi(s)(\mathsf{H}-\mathsf{H}_\Lambda)\|_\star &= \mathrm{Tr}\Big(\frac{e^{-\beta\omega\mathsf{N}_\Lambda}}{\mathrm{Tr}(e^{-\beta\omega\mathsf{N}_\Lambda})}\mathsf{N}_\Lambda(\mathsf{H}-\mathsf{H}_\Lambda)^2\Big)^{\frac12} \\ &= \mathrm{Tr}\Big(\frac{e^{-\beta\omega\mathsf{N}_\Lambda}}{\mathrm{Tr}(e^{-\beta\omega\mathsf{N}_\Lambda})}(\mathsf{H}-\mathsf{H}_\Lambda)^2\mathsf{N}_\Lambda\Big)^{\frac12} \end{aligned}$$

Since $\mathsf{H} = \mathsf{H}_{ext} + \mathsf{H}_{int}$ we can write

$$\|\Psi(s)(\mathsf{H}-\mathsf{H}_\Lambda)\|_\star \le \|\Psi(s)(\mathsf{H}_{ext}-\mathsf{H}_{ext,\Lambda})\|_\star + \|\Psi(s)(\mathsf{H}_{int}-\mathsf{H}_{int,\Lambda})\|_\star$$

but $\pi_\Lambda\mathsf{H}_{ext} = \mathsf{H}_{ext,\Lambda} = \pi_\Lambda\mathsf{H}_{ext,\Lambda}$ so that

$$\pi_\Lambda(\mathsf{H}_{ext}-\mathsf{H}_{ext,\Lambda})^2\pi_\Lambda = \pi_\Lambda(\mathsf{H}_{ext}-\mathsf{H}_{ext,\Lambda})(\mathsf{H}_{ext}-\mathsf{H}_{ext,\Lambda})\pi_\Lambda = 0.$$

Thus, by the setting $\varrho_\Lambda := e^{-\beta\omega\mathsf{N}_\Lambda}/\mathrm{Tr}(e^{-\beta\omega\mathsf{N}_\Lambda})$ we can write down

$$\|\Psi(s)(\mathsf{H}-\mathsf{H}_\Lambda)\|_\star \le \mathrm{Tr}\Big(\varrho_\Lambda(\mathsf{H}_{int}-\mathsf{H}_{int,\Lambda})^2\mathsf{N}_\Lambda\Big)^{\frac12}$$

and the operator version of Hölder inequality gives

$$\le \Big[\mathrm{Tr}\Big(\varrho_\Lambda\mathsf{N}_\Lambda^2\Big)\Big]^{\frac14}\Big[\mathrm{Tr}\Big(\varrho_\Lambda(\mathsf{H}_{int}-\mathsf{H}_{int,\Lambda})^4\Big)\Big]^{\frac14}.$$

A direct computation (thanks to Proposition 1) shows that

$$\leq \frac{\Lambda^{\frac{d}{2}}}{B^{\frac{1}{2}}} \left[\mathrm{Tr}\Big(\varrho_\Lambda (\mathsf{H}_{int} - \mathsf{H}_{int,\Lambda})^4\Big)\right]^{\frac{1}{4}}.$$

The zero-time term in (80)

$$\|\mathsf{B}_\Lambda(0)\|_\star^2 := \mathrm{Tr}\Big(\frac{e^{-\beta\omega \mathsf{N}_\Lambda}}{\mathrm{Tr}(e^{-\beta\omega\mathsf{N}_\Lambda})} \int_{\mathbb{R}^d} \mathsf{B}_\Lambda^\dagger(0,x)\mathsf{B}_\Lambda(0,x)dx\Big)$$

fulfills $\|\mathsf{B}_\Lambda(0)\|_\star^2 = 0$. Indeed,

$$\begin{aligned}\mathsf{B}_\Lambda(0) &= \Psi - \Psi_\Lambda = \Psi - \pi_\Lambda \Psi \pi_\Lambda \\ &= (\mathbb{I} - \pi_\Lambda)\Psi\pi_\Lambda + \pi_\Lambda\Psi(\mathbb{I} - \pi_\Lambda) + (\mathbb{I} - \pi_\Lambda)\Psi(\mathbb{I} - \pi_\Lambda)\end{aligned}$$

whence the trace reduces to

$$\begin{aligned}&\|\mathsf{B}_\Lambda(0)\|_\star^2 = \mathrm{Tr}\Big(\frac{e^{-\beta\omega \mathsf{N}_\Lambda}}{\mathrm{Tr}(e^{-\beta\omega\mathsf{N}_\Lambda})} \int_{\mathbb{R}^d} \pi_\Lambda \Psi^\dagger(x)(\mathbb{I} - \pi_\Lambda)\Psi(x)\pi_\Lambda\, dx\Big) \\ &= \mathrm{Tr}\Big(\frac{e^{-\beta\omega \mathsf{N}_\Lambda}}{\mathrm{Tr}(e^{-\beta\omega\mathsf{N}_\Lambda})} (\pi_\Lambda \mathsf{N} \pi_\Lambda - \int_{\mathbb{R}^d} \pi_\Lambda \Psi^\dagger(x)\pi_\Lambda\Psi(x)\pi_\Lambda\, dx)\Big)\end{aligned}$$

From $\Psi(x) = \sum_k \mathsf{a}_k \varphi_k(x)$ we have $(\pi_\Lambda \Psi_\Lambda \pi_\Lambda)(x) = \sum_k^\Lambda \mathsf{a}_{k,\Lambda}\varphi_k(x)$ as well as the adjoint equality $(\pi_\Lambda \Psi_\Lambda^\dagger \pi_\Lambda)(x) = \sum_k^\Lambda \mathsf{a}_{k,\Lambda}^\dagger \varphi_k(x)$. Since $\mathsf{N}_\Lambda := \sum_k^\Lambda \mathsf{a}_{k,\Lambda}^\dagger \mathsf{a}_{k,\Lambda} = \pi_\Lambda \mathsf{N} \pi_\Lambda$ we have that $\|\mathsf{B}_\Lambda(0)\|_\star^2 = 0$.
In view of the above computations,

$$\begin{aligned}\|\Psi(t) - \Psi_\Lambda(t)\|_\mu &\leq \frac{1}{\Lambda^{\frac{d}{2}}} 2t \frac{\Lambda^{\frac{d}{2}}}{B^{\frac{1}{2}}} \left[\mathrm{Tr}\Big(\varrho_\Lambda (\mathsf{H}_{int} - \mathsf{H}_{int,\Lambda})^4\Big)\right]^{\frac{1}{4}} \\ &\leq \frac{2t}{B^{\frac{1}{2}}} \left[\mathrm{Tr}\Big(\varrho_\Lambda (\mathsf{H}_{int} - \mathsf{H}_{int,\Lambda})^4\Big)\right]^{\frac{1}{4}}.\end{aligned}$$

In view of Lemma 4 we can write

$$\|\Psi(t) - \Psi_\Lambda(t)\|_\mu \leq \frac{2t}{B^{\frac{1}{2}}}\, \mathrm{Tr}(\varrho_\Lambda |(\mathbb{1} - \pi_\Lambda)\mathsf{H}_{int}|^4)^{\frac{1}{4}},$$

and thanks to Lemma 6, the next bound reads

$$\|\Psi(t) - \Psi_\Lambda(t)\|_\mu \leq \frac{2t}{B^{\frac{1}{2}}}\; 2C_v \Big(\frac{\Lambda^d}{B}\Big)^2 = \frac{4C_v t}{B^{\frac{1}{2}}}\; \Big(\frac{\Lambda^d}{B}\Big)^2.$$

To conclude, by combining the results of Propositions 5 and 7, we find that the difference $\|a_q(t) - c_q(t)\|_{L^2(\mu)}$ fulfills the relation:

$$\|a_q(t) - c_q(t)\|_{L^2(\mu)} =: \|\delta_q(t)\|_{L^2(\mu)} \leq \|\mathcal{L}_1 a_q\|_{L^2(\mu)}\; t \tag{81}$$

where

$$\|\mathcal{L}_1 a_q\|_{L^2(\mu)} \le \frac{2^8}{B} \sum_{klmn}^{\Lambda} |v_{klmn}| \le 2^8 \Lambda^{4d} \sqrt{2} C_v (1+2\Lambda)^{\frac{d}{2}}$$
$$< \frac{2^9}{B} C_v (1+2\Lambda)^{4d+\frac{d}{2}} =: b_{\Lambda,B}. \tag{82}$$

Consequently, the deviation of the effective field $\Psi_\Lambda^{(0)}(t)$ from the quantum field $\Psi_\Lambda(t)$ is controlled by

$$\|\Psi_\Lambda(t) - \Psi_\Lambda^{(0)}(t)\|_\mu := \Big(\frac{1}{\Lambda^d} \sum_q^{\Lambda} \|a_q(t) - c_q(t)\|^2_{L^2(\mu)} \Big)^{\frac{1}{2}} \le b_{\Lambda,B}\ t\,. \tag{83}$$

□

4 Appendix

4.1 Hardy potentials and interaction operator inequalities

Lemma 3 *For the Hardy potentials $v \ge 0$ in (11) with Hardy constant C_v, the annihilation operator distribution $\Psi(x)$ fulfills*

$$\int_{\mathbb{R}^d} v(x)^2 \Psi^\dagger(x)\Psi(x)dx \le C_v^2 \Big(\int_{\mathbb{R}^d} \Psi^\dagger(x)\Psi(x)dx + \int_{\mathbb{R}^d} \nabla_x \Psi^\dagger(x) \nabla_x \Psi(x) dx \Big)$$

where $\le$ is the inequality between semipositive operators.

Proof. We first notice that, for any fixed $x \in \mathbb{R}^d$, the operators $\Psi^\dagger(x)\Psi(x)$ and $\nabla_x \Psi^\dagger(x) \nabla_x \Psi(x)$ preserve all the sectors $L^2_s(\mathbb{R}^{dn})$ of the Fock space. Now consider an arbitrary $\varphi \in L^2_s(\mathbb{R}^{dn})$, so that

$$\langle \varphi, \Psi^\dagger(x)\Psi(x)\varphi\rangle_{L^2_s(\mathbb{R}^{dn})} = \langle \Psi(x)\varphi, \Psi(x)\varphi\rangle_{L^2_s(\mathbb{R}^{d(n-1)})}$$
$$\langle \varphi, \nabla_x\Psi^\dagger(x)\nabla_x\Psi(x)\varphi\rangle_{L^2_s(\mathbb{R}^{dn})} = \langle \nabla_x\Psi(x)\varphi, \nabla_x\Psi(x)\varphi\rangle_{L^2_s(\mathbb{R}^{d(n-1)})}\,.$$

Moreover, $\nabla_x \Psi(x)\varphi = \nabla_x(\Psi(x)\varphi)$. Now define $\psi(x) := \Psi(x)\varphi$, and denote $\psi_j(x) := \langle \psi(x), e_j\rangle_{L^2_s(\mathbb{R}^{d(n-1)})}$ where e_j with $j \in \mathbb{N}$ is a complete orthonormal set in $L^2_s(\mathbb{R}^{d(n-1)})$. Thus, the above equalities turns into

$$\langle \varphi, \Psi^\dagger(x)\Psi(x)\varphi\rangle_{L^2_s(\mathbb{R}^{dn})} = \sum_{j=0}^{\infty} |\psi_j(x)|^2$$
$$\langle \varphi, \nabla_x\Psi^\dagger(x)\nabla_x\Psi(x)\varphi\rangle_{L^2_s(\mathbb{R}^{dn})} = \sum_{j=0}^{\infty} |\nabla_x \psi_j(x)|^2$$

Now apply Hardy inequality for all the functions ψ_j so that

$$\sum_{j=0}^{\infty} C_v^2 \Big(\int_{\mathbb{R}^d} |\psi_j(x)|^2 dx + \int_{\mathbb{R}^d} |\nabla_x \psi_j(x)|^2 dx \Big) \ge \sum_{j=0}^{\infty} \int_{\mathbb{R}^d} v(x)^2 |\psi_j(x)|^2 dx$$

where the lower bound can be rewritten as

$$\int_{\mathbb{R}^d} v(x)^2 \sum_{j=0}^{\infty} |\psi_j(x)|^2 dx = \int_{\mathbb{R}^d} v(x)^2 \langle \varphi, \Psi^\dagger(x)\Psi(x)\varphi\rangle_{L^2_s(\mathbb{R}^{dn})}\, dx$$
$$= \langle \varphi, \int_{\mathbb{R}^d} v(x)^2 \Psi^\dagger(x)\Psi(x)\, dx\, \varphi\,\rangle_{L^2_s(\mathbb{R}^{dn})}$$

from which the statement follows. □

Proposition 8 *For the Hardy potentials (11), the interaction operator* H_{int} *in (1) fulfills*

$$0 < \mathsf{H}_{int} \leq \frac{C_v}{2}\mathsf{N}(\mathsf{N} + \mathsf{H}_{free}) \tag{84}$$

where C_v *is the Hardy constant.*

Proof. We begin by noticing that

$$\mathsf{H}_{int} := \frac{1}{2}\int_{\mathbb{R}^{2d}} \Psi^\dagger(y)\Psi^\dagger(x)v(x-y)\Psi(x)\Psi(y)dxdy$$

can be rewritten as

$$\mathsf{H}_{int} = \frac{1}{2}\int_{\mathbb{R}^d} \Psi^\dagger(y)\Big(\int_{\mathbb{R}^d} \Psi^\dagger(x)v(x-y)\Psi(x)dx\Big)\Psi(y)dy.$$

The operator version of Hölder inequality gives a bound for

$$\int_{\mathbb{R}^d} \Psi^\dagger(x)v(x-y)\Psi(x)dx = \int_{\mathbb{R}^d} (\Psi^\dagger(x)\Psi(x))^{\frac{1}{2}} v(x-y)(\Psi^\dagger(x)\Psi(x))^{\frac{1}{2}} dx$$
$$\leq \Big(\int_{\mathbb{R}^d} \Psi^\dagger(x)\Psi(x)dx\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^d} v(x-y)^2\Psi^\dagger(x)\Psi(x)dx\Big)^{\frac{1}{2}}$$
$$= \mathsf{N}^{\frac{1}{2}}\Big(\int_{\mathbb{R}^d} v(x-y)^2\Psi^\dagger(x)\Psi(x)dx\Big)^{\frac{1}{2}}.$$

Furthermore, recalling Lemma 3 and by applying a simple argument of translation invariance we get

$$\int_{\mathbb{R}^d} v(x-y)^2 \Psi^\dagger(x)\Psi(x)dx \leq C_v^2(\mathsf{N} + \mathsf{H}_{free}), \quad \forall\, y \in \mathbb{R}^d. \tag{85}$$

Whence, we get the upper bound by the semipositive operator

$$\leq \mathsf{N}^{\frac{1}{2}}\Big(C_v^2(\mathsf{N} + \mathsf{H}_{free})\Big)^{\frac{1}{2}} \leq C_v\, \mathsf{N}^{\frac{1}{2}}(\mathsf{N} + \mathsf{H}_{free})^{\frac{1}{2}}$$
$$\leq C_v\, (\mathsf{N} + \mathsf{H}_{free}).$$

where for the last inequality we used $\mathsf{N}, \mathsf{H}_{free} \geq 0$ and $[\mathsf{N}, \mathsf{H}_{free}] = 0$. Thus,

$$\mathsf{H}_{int} \leq \frac{1}{2}\int_{\mathbb{R}^d} \Psi^\dagger(y)\Big(C_v\, (\mathsf{N} + \mathsf{H}_{free})\Big)\Psi(y)dy. \tag{86}$$

Now recall that $\Psi(y) = \sum_k \varphi_k(y)\mathsf{a}_k$, $\mathsf{H}_{free} = \sum_k \|\nabla\varphi_k\|^2_{L^2}\mathsf{a}^\dagger_k\mathsf{a}_k$ and $\mathsf{N} = \sum_k \mathsf{a}^\dagger_k\mathsf{a}_k$. Rewrite the estimate (86) as

$$\begin{aligned}\mathsf{H}_{int} &\le \frac{C_v}{2}\sum_k \mathsf{a}^\dagger_k(\mathsf{N}+\mathsf{H}_{free})\mathsf{a}_k \\ &= \frac{C_v}{2}\sum_k \mathsf{a}^\dagger_k\Big[(\mathsf{N}+\mathsf{H}_{free}),\mathsf{a}_k\Big] + \frac{C_v}{2}\sum_k \mathsf{a}^\dagger_k\mathsf{a}_k(\mathsf{N}+\mathsf{H}_{free})\end{aligned}$$

Thanks to commutation rules $[\mathsf{N},\mathsf{a}_k] = -\mathsf{a}_k$ and $[\mathsf{H}_{free},\mathsf{a}_k] = -\|\nabla\varphi_k\|^2_{L^2}\mathsf{a}_k$ we conclude

$$\mathsf{H}_{int} \le -\frac{C_v}{2}(\mathsf{N}+\mathsf{H}_{free}) + \frac{C_v}{2}\mathsf{N}(\mathsf{N}+\mathsf{H}_{free}) \le \frac{C_v}{2}\mathsf{N}(\mathsf{N}+\mathsf{H}_{free}).$$

□

Remark 7 Let K be the harmonic oscillator Hamiltonian on Fock space, then $\mathsf{H}_{free} \le \mathsf{K} = (\frac{3}{2}\mathbb{1}+\mathsf{N}) \le 3\mathsf{N}$ and thus

$$0 \le \mathsf{H}_{int} \le \frac{C_v}{2}\mathsf{N}(\mathsf{N}+3\mathsf{N}) = 2C_v\mathsf{N}^2. \tag{87}$$

Notice moreover that $[\mathsf{H}_{int},\mathsf{N}] = [\mathsf{H}-\mathsf{H}_{ext},\mathsf{N}] = 0$ and whence $[\mathsf{H}_{int},\mathsf{N}^2] = 0$ so that

$$0 \le \mathsf{H}^2_{int} \le 4C_v^2\mathsf{N}^4. \tag{88}$$

We stress anyway that $[\mathsf{H}_{int},\mathsf{N}_\Lambda] \ne 0$ since $[\mathsf{H}_{int},\pi_\Lambda] \ne 0$.

Lemma 4 *Let H_{int} and $\mathsf{H}_{int,\Lambda}$ be as in (1) - (17). Then,*

$$\mathrm{Tr}(\varrho_\Lambda(\mathsf{H}_{int}-\mathsf{H}_{int,\Lambda})^4)^{\frac14} = \mathrm{Tr}(\varrho_\Lambda|\mathsf{A}|^4)^{\frac14}$$

where $\mathsf{A} := (\mathbb{1}-\pi_\Lambda)\mathsf{H}_{int}$.

Proof. We begin by the identity $\mathsf{H}_{int,\Lambda} = \pi_\Lambda\mathsf{H}_{int}\pi_\Lambda$ so that

$$\begin{aligned}\pi_\Lambda(\mathsf{H}_{int}-\mathsf{H}_{int,\Lambda})^2 &= \pi_\Lambda\mathsf{H}_{int}(\mathbb{1}-\pi_\Lambda)\mathsf{H}_{int}\,, \\ (\mathsf{H}_{int}-\mathsf{H}_{int,\Lambda})^2\pi_\Lambda &= \mathsf{H}_{int}(\mathbb{1}-\pi_\Lambda)\mathsf{H}_{int}\pi_\Lambda\,.\end{aligned}$$

Thus,

$$\pi_\Lambda(\mathsf{H}_{int}-\mathsf{H}_{int,\Lambda})^4\pi_\Lambda = \pi_\Lambda\mathsf{H}_{int}(\mathbb{1}-\pi_\Lambda)\mathsf{H}^2_{int}(\mathbb{1}-\pi_\Lambda)\mathsf{H}_{int}\pi_\Lambda.$$

For $\mathsf{D} := \mathsf{H}_{int}(\mathbb{1}-\pi_\Lambda)\mathsf{H}_{int} = \mathsf{A}^\dagger\mathsf{A} = |\mathsf{A}|^2$ and $\mathsf{A} := (\mathbb{1}-\pi_\Lambda)\mathsf{H}_{int}$ we have

$$\pi_\Lambda(\mathsf{H}_{int}-\mathsf{H}_{int,\Lambda})^4\pi_\Lambda = \pi_\Lambda\mathsf{D}^2\pi_\Lambda = \pi_\Lambda|\mathsf{A}|^4\pi_\Lambda.$$

Whence,

$$\mathrm{Tr}(\varrho_\Lambda(\mathsf{H}_{int}-\mathsf{H}_{int,\Lambda})^4)^{\frac14} = \mathrm{Tr}(\varrho_\Lambda|\mathsf{A}|^4)^{\frac14}.$$

□

Lemma 5 *Let* H *be as in (1) under the assumption that* $u(x) \leq \Omega\|x\|^{2p}$, $\Omega > 0$, $p \geq 2$, *and* $v(x)$ *in the Hardy class with constant* C_v. *Then,*

$$\mathsf{H} \leq \Big(2(1+\Omega)3^p + 2C_v\Big)\mathsf{N}^p =: \Omega_p'\mathsf{N}^p.$$

Proof. Recall that $\mathsf{H}_{int} \leq 2C_v\mathsf{N}^2$, that H_{ext} preserves all the n-sectors of the Fock space, and that the related restriction reads

$$\begin{aligned}\mathsf{H}^{(n)}_{ext} &= \mathsf{H}^{(n)}_{free} + \sum_{i=1}^{n} u(x_i) \leq \mathsf{H}^{(n)}_{free} + \Omega\sum_{i=1}^{n}\|x_i\|^{2p} \\ &\leq \mathsf{H}^{(n)}_{free} + \Omega\Big(\sum_{i=1}^{n}\|x_i\|^2\Big)^p = \mathsf{H}^{(n)}_{free} + \Omega\Big(\mathsf{K} - \mathsf{H}^{(n)}_{free}\Big)^p.\end{aligned}$$

Moreover, $\mathsf{K} - \mathsf{H}^{(n)}_{free} \leq \mathsf{K} \leq 3\mathsf{N}$ and $[\mathsf{K} - \mathsf{H}^{(n)}_{free}, 3\mathsf{N}] = 0$. This allows the upper bound

$$\mathsf{H}^{(n)}_{ext} \leq \mathsf{H}^{(n)}_{free} + \Omega(3\mathsf{N}^{(n)})^p \leq 3\mathsf{N}^{(n)} + \Omega(3\mathsf{N}^{(n)})^p \leq 2(1+\Omega)3^p\,(\mathsf{N}^{(n)})^p$$

where $\mathsf{N}^{(n)}$ denotes the restriction of the number operator to the n-sector. Recalling (87), $\mathsf{H}_{int} \leq 2C_v\mathsf{N}^2$, from which we get the statement. □

Remark 8 Notice that $\mathrm{Tr}(e^{-\beta\omega\mathsf{N}_\Lambda}) = \int_{\mathbb{C}^\ell}\sigma_{AW}(e^{-\beta\omega\mathsf{N}_\Lambda})(\alpha,\bar{\alpha})\,\pi^{-\ell}d\alpha d\bar{\alpha}$ with $\ell = \Lambda^d$ and $\sigma_{AW}(e^{-\beta\omega\mathsf{N}_\Lambda}) := (B+1)^\ell\,e^{-B|\alpha|^2}$. Then

$$\mathrm{Tr}(e^{-\beta\omega\mathsf{N}_\Lambda}) = \Big(\frac{B+1}{B}\Big)^{\Lambda^d} = \Big(\frac{B+1}{B}\Big)^{B\frac{\Lambda^d}{B}} \leq e^{\frac{\Lambda^d}{B}}.$$

Since $\omega\mathsf{N}_\Lambda \leq \mathsf{H}_\Lambda \leq \Omega_p'\mathsf{N}_\Lambda^p$ for $\Omega_p' := (2(1+\Omega)3^p + 2C_v)$ and recalling the commutation rule $[\mathsf{H}_\Lambda, \mathsf{N}_\Lambda] = 0$ then $e^{-\beta\Omega_p'\mathsf{N}_\Lambda^p} \leq e^{-\beta\mathsf{H}_\Lambda} \leq e^{-\beta\omega\mathsf{N}_\Lambda}$ which directly gives the inequalities $1 \leq \mathrm{Tr}(e^{-\beta\mathsf{H}_\Lambda}) \leq \mathrm{Tr}(e^{-\beta\omega\mathsf{N}_\Lambda})$ as well as $\mathrm{Tr}(e^{-\beta\Omega_p'\mathsf{N}_\Lambda^p}) \leq \mathrm{Tr}(e^{-\beta\omega\mathsf{N}_\Lambda})$.

Lemma 6 *For* $\mathsf{A} := (\mathbb{1} - \pi_\Lambda)\mathsf{H}_{int}$ *we have*

$$\mathrm{Tr}(\varrho_\Lambda|\mathsf{A}|^4)^{\frac{1}{4}} \leq 2C_v\left(\frac{\Lambda^d}{B}\right)^2.$$

Proof. In view of the previous Lemma,

$$\mathrm{Tr}(\varrho_\Lambda|\mathsf{A}|^4) = \mathrm{Tr}(\varrho_\Lambda\mathsf{H}_{int}(\mathbb{1} - \pi_\Lambda)\mathsf{H}^2_{int}(\mathbb{1} - \pi_\Lambda)\mathsf{H}_{int}).$$

Recalling (88), we have

$$\mathrm{Tr}(\varrho_\Lambda|\mathsf{A}|^4) \leq 4C_v^2\mathrm{Tr}(\varrho_\Lambda\mathsf{H}_{int}(\mathbb{1} - \pi_\Lambda)\mathsf{N}^4(\mathbb{1} - \pi_\Lambda)\mathsf{H}_{int}).$$

Since $[\mathsf{N}, \pi_\Lambda] = 0$ and $[\mathsf{H}_{int}, \mathsf{N}^4] = 0$ then

$$\mathrm{Tr}(\varrho_\Lambda|\mathsf{A}|^4) \leq 4C_v^2\mathrm{Tr}(\varrho_\Lambda\mathsf{N}^4\mathsf{H}_{int}(\mathbb{1} - \pi_\Lambda)\mathsf{H}_{int})$$

In particular, $\mathrm{Tr}(\varrho_\Lambda \mathsf{N}_\Lambda^4 \mathsf{H}_{int}\pi_\Lambda \mathsf{H}_{int}) = \mathrm{Tr}(\sqrt{\varrho_\Lambda}\mathsf{N}_\Lambda^2 \mathsf{H}_{int}\pi_\Lambda \mathsf{H}_{int}\sqrt{\varrho_\Lambda}\mathsf{N}_\Lambda^2) \geq 0$. Thus,

$$\mathrm{Tr}(\varrho_\Lambda |\mathsf{A}|^4) \leq 4C_v^2 \,\mathrm{Tr}(\varrho_\Lambda \mathsf{N}_\Lambda^4 \mathsf{H}_{int}^2) = 4C_v^2 \,\mathrm{Tr}(\sqrt{\varrho_\Lambda}\mathsf{N}_\Lambda^2 \mathsf{H}_{int}^2 \sqrt{\varrho_\Lambda}\mathsf{N}_\Lambda^2)$$

Now apply $\mathsf{H}_{int} \leq 2C_v \mathsf{N}^2$, so that

$$\mathrm{Tr}(\varrho_\Lambda |\mathsf{A}|^4) \leq 4^2 C_v^4 \,\mathrm{Tr}(\varrho_\Lambda \mathsf{N}_\Lambda^8) \leq 4^2 C_v^4 \,\mathrm{Tr}(\varrho_\Lambda \mathsf{N}_\Lambda)^8 = 2^4 C_v^4 \left(\frac{\Lambda^d}{B}\right)^8.$$

□

Lemma 7 *Let* $\beta := 1/(k_B T)$, H_Λ *as in (17) and* $\mathsf{G}_\Lambda := \frac{e^{-\beta \mathsf{H}_\Lambda}}{\mathrm{Tr}(e^{-\beta \mathsf{H}_\Lambda})}$. *Then,*

$$\mathrm{Tr}(\mathsf{G}_\Lambda \mathsf{N}_\Lambda) \leq \frac{\mathrm{Tr}(e^{-\beta\omega \mathsf{N}_\Lambda}\mathsf{N}_\Lambda)}{\mathrm{Tr}(e^{-\beta \Omega'_p \mathsf{N}_\Lambda^p})}.$$

Moreover, the following (Λ - independent) interval of temperatures

$$0 < T \leq \frac{\omega}{k_B \ln(1 + \frac{1}{A_p N^2})} =: \widetilde{T} \ , \quad A_p := \left(\frac{\Gamma(\frac{1}{p})}{p}\right)^2 \left(\frac{\omega}{\Omega'_p}\right)^{\frac{2}{p}} \tag{89}$$

implies the next inequality $\forall \Lambda \geq 1$

$$\frac{\mathrm{Tr}(e^{-\beta\omega \mathsf{N}_\Lambda}\mathsf{N}_\Lambda)}{\mathrm{Tr}(e^{-\beta \Omega'_p \mathsf{N}_\Lambda^p})} \leq N, \tag{90}$$

and thus also $\mathrm{Tr}(\mathsf{G}_\Lambda \mathsf{N}_\Lambda) \leq N$.

Proof. In view of Remark 8, and recalling that $\ell := \Lambda^d$,

$$\mathrm{Tr}(\mathsf{G}_\Lambda \mathsf{N}_\Lambda) = \frac{\mathrm{Tr}(e^{-\beta\omega \mathsf{N}_\Lambda}\mathsf{N}_\Lambda)}{\mathrm{Tr}(e^{-\beta\omega \mathsf{N}_\Lambda})} \leq \frac{\mathrm{Tr}(e^{-\beta\omega \mathsf{N}_\Lambda}\mathsf{N}_\Lambda)}{\mathrm{Tr}(e^{-\beta \Omega'_p \mathsf{N}_\Lambda^p})} = \frac{\frac{\ell}{B}}{\mathrm{Tr}(e^{-\beta \Omega'_p \mathsf{N}_\Lambda^p})} \tag{91}$$

In particular, $\sharp\{k_1 + k_2 + ... + k_\ell = n \mid 0 \leq \alpha_i \leq n\} \geq \ell$ and

$$\mathrm{Tr}(e^{-\beta \Omega'_p \mathsf{N}_\Lambda^p}) = \sum_{n=0}^{\infty} e^{-\beta \Omega'_p n^p} \ \sharp\{k_1 + k_2 + ... + k_\ell = n\} \tag{92}$$

$$\geq \ell \sum_{n=0}^{\infty} e^{-\beta \Omega'_p n^p} \geq \ell \int_0^\infty e^{-\beta \Omega'_p x^p} \ dx.$$

More in details,

$$\int_0^\infty e^{-\beta \Omega'_p x^p} \ dx = \frac{\Gamma(\frac{1}{p})}{p} \frac{1}{(\beta \Omega'_p)^{\frac{1}{p}}} \,.$$

We now set the (uniform with respect to ℓ) bound

$$\mathrm{Tr}(\mathsf{G}_\Lambda \mathsf{N}_\Lambda) \leq \frac{\frac{1}{B}}{\frac{\Gamma(\frac{1}{p})}{p} \frac{1}{(\beta \Omega'_p)^{\frac{1}{p}}}} \leq N \tag{93}$$

so that $\mathrm{Tr}(\mathsf{G}_\Lambda \mathsf{N}_\Lambda) \leq N$ is consequently fulfilled. The righthand side is equivalent to

$$\frac{1}{NB} \leq \frac{\Gamma(\frac{1}{p})}{p} \frac{1}{(\beta \Omega'_p)^{\frac{1}{p}}} \tag{94}$$

hence, recalling $B := e^{\beta\omega} - 1$,

$$\Big(\frac{1}{NB}\Big)^p \leq \Big(\frac{\Gamma(\frac{1}{p})}{p}\Big)^p \frac{1}{\beta \Omega'_p} = \Big(\frac{\Gamma(\frac{1}{p})}{p}\Big)^p \frac{\omega}{\ln(B+1)\Omega'_p}, \tag{95}$$

$$\frac{\ln(B+1)}{B^p} \leq \Big(\frac{\Gamma(\frac{1}{p})}{p}\Big)^p \frac{\omega N^p}{\Omega'_p}, \tag{96}$$

Notice that, since $p \geq 2$, $g(B) := \ln(B+1)/B^p$ is a strictly decreasing function for $B \geq 1$, and thus can be inverted. Anyway, in order to simplify the study of the interval of temperatures, we fix the stronger condition (that still ensures $\mathrm{Tr}(\mathsf{G}_\Lambda \mathsf{N}_\Lambda) \leq N$)

$$\frac{\ln(B+1)}{B^p} \leq \frac{1}{B^{p/2}} \leq \Big(\frac{\Gamma(\frac{1}{p})}{p}\Big)^p \frac{\omega N^p}{\Omega'_p} \tag{97}$$

The righthand side reads

$$\frac{1}{B} \leq \Big(\frac{\Gamma(\frac{1}{p})}{p}\Big)^2 \Big(\frac{\omega}{\Omega'_p}\Big)^{\frac{2}{p}} N^2 \tag{98}$$

which turns into

$$0 < T \leq \frac{\omega}{k_B \ln(1 + \frac{1}{A_p N^2})}; \quad A_p := \Big(\frac{\Gamma(\frac{1}{p})}{p}\Big)^2 \Big(\frac{\omega}{\Omega'_p}\Big)^{\frac{2}{p}}. \tag{99}$$

□

Lemma 8 *Let us recall the multi-index notation* $\|k\| := k_1 + k_2 + ... + k_\ell$ *and* $k! := k_1!\, k_2! \ldots k_\ell!$ *for* $k \in \mathbb{N}^\ell$. *Then, for* $\mathcal{H}$ *as in (38) and constant*

$$c_{\mathcal{H}} := \frac{e^{\Omega'_p \cdot n(p,\ell)}}{2(\tau_{min})^\ell (\ell!)} \tag{100}$$

with $n(p,\ell) := \sum_{k\in\mathbb{N}^\ell} \|k\|^p/k!$ *and where* $\Omega'_p := 2((1+\Omega)3^p + C_v)$ *we have the inequality*

$$(\lambda\, \tau_{min})^{-\ell} \leq c_{\mathcal{H}} \int e^{-\lambda \mathcal{H}(\alpha,\bar{\alpha})}\, d\alpha \wedge d\bar{\alpha}. \tag{101}$$

Proof. Recall that $\mathcal{H}(\alpha,\bar\alpha) := \langle \alpha, \mathsf{H}_\Lambda \alpha\rangle$ and thanks to Lemma 5, $\mathsf{H}_\Lambda \le \Omega'_p \mathsf{N}_\Lambda^p$ so that $\mathcal{H}(\alpha,\bar\alpha) \le \Omega'_p \langle \alpha, \mathsf{N}_\Lambda^p \alpha\rangle$. Thus,

$$\frac{(\lambda\,\tau_{min})^{-\ell}}{\int e^{-\lambda \mathcal{H}(\alpha,\bar\alpha)}\, d\alpha\wedge d\bar\alpha} \le \frac{(\lambda\,\tau_{min})^{-\ell}}{\int e^{-\lambda \Omega'_p\langle\alpha,\mathsf{N}_\Lambda^p\alpha\rangle}\, d\alpha\wedge d\bar\alpha} = \frac{(\tau_{min})^{-\ell}}{\int e^{-\Omega'_p\langle\beta,\mathsf{N}_\Lambda^p\beta\rangle}\, d\beta\wedge d\bar\beta} \tag{102}$$

where in the last equality we used the change of variables $\sqrt{\lambda}\alpha = \beta$. A further upper bound is given integrating just over the ball $B \subset \mathbb{C}^\ell$ centered at zero with radius one. Thus,

$$\frac{(\lambda\,\tau_{min})^{-\ell}}{\int e^{-\lambda \mathcal{H}(\alpha,\bar\alpha)}\, d\alpha\wedge d\bar\alpha} \le \frac{(\tau_{min})^{-\ell}}{\int_B e^{-\Omega'_p\langle\beta,\mathsf{N}_\Lambda^p\beta\rangle}\, d\beta\wedge d\bar\beta}\,. \tag{103}$$

Now apply the decomposition of a coherent state into eigenfunctions of the number operator, namely $|\beta\rangle = e^{-|\beta|^2/2}\sum_{k\in\mathbb{N}^\ell}\frac{\beta^k}{\sqrt{k!}}|k\rangle$, $|\beta|^2 := \bar\beta\beta$ so that $\langle\beta,\mathsf{N}_\Lambda^p\beta\rangle = e^{-|\beta|^2}\sum_{k\in\mathbb{N}^\ell}\frac{|\beta|^{2k}}{k!}\|k\|^p$. This gives, for any $|\beta|\le 1$ the simplified upper bound $\langle\beta,\mathsf{N}_\Lambda^p\beta\rangle \le \sum_{k\in\mathbb{N}^\ell}\frac{1}{k!}\|k\|^p =: n(p,\ell)$. We conclude with the new bound

$$\le \frac{(\tau_{min})^{-\ell} e^{+\Omega'_p\cdot n(p,\ell)}}{\int_B \pi^{-\ell} d\beta d\bar\beta} = \frac{(\tau_{min})^{-\ell} e^{+\Omega'_p\cdot n(p,\ell)}}{2(\ell!)}. \tag{104}$$

□

4.2 An upper bound for the interaction coefficients

We show that the Hardy constant of the interaction potential allows an upper bound for the interaction coefficients.

Proposition 9 *Let $|k|, |l|, |m|, |n| < \Lambda$ and let v_{klmn} be as in (6). Then,*

$$|v_{klmn}| \le \sqrt{2}\, C_v\, (1+2\Lambda)^{\frac{d}{2}}.$$

Proof. Recall that

$$\begin{aligned} v_{klmn} &:= \langle \varphi_k\vee\varphi_l, \widehat{v}\,\varphi_m\vee\varphi_n\rangle_{L^2_s(\mathbb{R}^{2d})} \\ &= \int_{\mathbb{R}^{2d}} \bar\varphi_k(y)\bar\varphi_l(y) v(x-y)\varphi_m(x)\varphi_n(x)\, dxdy \end{aligned}$$

where $v(x-y) = v(y-x)$ and $\varphi_m\vee\varphi_n := (\varphi_m\otimes\varphi_n + \varphi_n\otimes\varphi_m)/2 \in L^2_s(\mathbb{R}^{2d})$ is the symmetric tensor product. As a consequence, it is easily seen that

$$\begin{aligned} |v_{klmn}| &\le \int_{\mathbb{R}^d} |\varphi_m(x)|\; |\varphi_n(x)| \Big(\int_{\mathbb{R}^d} |v(x-y)|\,|\varphi_k(y)|\,|\varphi_l(y)|\, dy\Big) dx \\ &\le \int_{\mathbb{R}^d} |\varphi_m(x)|\; |\varphi_n(x)|\, \|\varphi_l\|_{L^2(\mathbb{R}^d)} \Big(\int_{\mathbb{R}^d} |v(x-y)|^2\,|\varphi_k(y)|^2\, dy\Big)^{\frac12} dx \\ &\le \int_{\mathbb{R}^d} |\varphi_m(x)|\; |\varphi_n(x)|\, C_v (\|\varphi_k\|^2_{L^2(\mathbb{R}^d)} + \|\nabla\varphi_k\|^2_{L^2(\mathbb{R}^d)})^{\frac12}\, dx \\ &\le \|\varphi_m\|_{L^2(\mathbb{R}^d)}\|\varphi_n\|_{L^2(\mathbb{R}^d)}\, C_v\, (1+\|\nabla\varphi_k\|^2_{L^2(\mathbb{R}^d)})^{\frac12}. \end{aligned}$$

In particular, for $d=1$, the eigenfunctions of the harmonic oscillator fulfill

$$-\frac{1}{2}\frac{d^2 f_k}{dx^2}+\frac{1}{2}|x|^2 f_k=\Big(\frac{1}{2}+k\Big)f_k$$

so that normalization $\|f_k\|_{L^2(\mathbb{R})}=1$ and integration by parts imply

$$\int_{\mathbb{R}}\frac{1}{2}\Big|\frac{df_k}{dx}\Big|^2 dx\le\Big(\frac{1}{2}+k\Big)\Rightarrow\int_{\mathbb{R}}\Big|\frac{df_k}{dx}\Big|^2 dx\le(1+2k).$$

It follows, for $d=1,2,3$

$$\|\nabla\varphi_k\|^2_{L^2(\mathbb{R}^d)}\le(1+2\Lambda)^d.$$

We thus conclude with the following bound

$$|v_{klmn}|\le\sqrt{2}\,C_v\,(1+2\Lambda)^{\frac{d}{2}}.$$

□

4.3 A lower bound for the density operator

Proposition 10 *Let $\Pi:\mathcal{F}_B(\mathbb{C}^\ell)\to\mathcal{F}_B(\mathbb{C}^\ell)$ be a trace one semipositive operator, $\ell:=\Lambda^d$. Let $\Theta_\Lambda(t,x):=\sum_k^\Lambda \mathsf{e}_k\,\varphi_k(x)$ and $\mathsf{e}_k:\mathcal{F}_B(\mathbb{C}^\ell)\to\mathcal{F}_B(\mathbb{C}^\ell)$. Define $\delta\Gamma^{(1)}:L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ with integral kernel*

$$\delta\Gamma^{(1)}(t,x,y):=\mathrm{Tr}(\Pi\Theta^\dagger_\Lambda(t,y)\Theta_\Lambda(t,x)).$$

Then,

$$\|\delta\Gamma^{(1)}\|_{\mathrm{HS}}\ge\frac{1}{\Lambda^d}\,\mathrm{Tr}\Big(\Pi\int_{\mathbb{R}^d}\Theta^\dagger_\Lambda(t,x)\Theta_\Lambda(t,x)\,dx\Big).\tag{105}$$

Proof. Let $\mathcal{K}(t,x,y)$ be the integral kernel of $\Gamma^{(1),\dagger}\circ\Gamma^{(1)}$,

$$\|\delta\Gamma^{(1)}\|^2_{\mathrm{HS}}=\mathrm{Tr}(\delta\Gamma^{(1),\dagger}\circ\delta\Gamma^{(1)})=\int_{\mathbb{R}^d}\mathcal{K}(t,x,x)\,dx\,.$$

Thus, since $\delta\Gamma^{(1),\dagger}=\delta\Gamma^{(1)}$,

$$\begin{aligned}\mathcal{K}(t,x,y)&=\int_{\mathbb{R}^d}\delta\Gamma^{(1)}(t,x,z)\,\delta\Gamma^{(1)}(t,z,y)\,dz\\&=\int_{\mathbb{R}^d}\mathrm{Tr}(\Pi\Theta^\dagger_\Lambda(t,z)\Theta_\Lambda(t,x))\,\mathrm{Tr}(\Pi\Theta^\dagger_\Lambda(t,y)\Theta_\Lambda(t,z))\,dz.\end{aligned}$$

As a consequence, the kernel on the diagonal $x=y$ reads

$$\begin{aligned}\mathcal{K}(t,x,x)&=\int_{\mathbb{R}^d}\mathrm{Tr}(\Pi\Theta^\dagger_\Lambda(t,z)\Theta_\Lambda(t,x))\,\mathrm{Tr}(\Pi\Theta^\dagger_\Lambda(t,x)\Theta_\Lambda(t,z))\,dz\\&=\int_{\mathbb{R}^d}\mathrm{Tr}(\Theta^\dagger_\Lambda(t,z)\Theta_\Lambda(t,x)\Pi)\,\mathrm{Tr}(\Pi\Theta^\dagger_\Lambda(t,x)\Theta_\Lambda(t,z))\,dz\end{aligned}$$

The finite normal mode decomposition allows for

$$\begin{aligned}\mathcal{K}(t,x,x) &= \sum_{lk}^{\Lambda}\sum_{\mu m}^{\Lambda}\int_{\mathbb{R}^d}\mathrm{Tr}(\mathsf{e}_l^{\dagger}\mathsf{e}_k\Pi)\,\mathrm{Tr}(\Pi\mathsf{e}_{\mu}^{\dagger}\mathsf{e}_m)\,\bar{\varphi}_l(z)\varphi_k(x)\bar{\varphi}_{\mu}(x)\varphi_m(z)\,dz\\ &= \sum_{lk\mu}^{\Lambda}\mathrm{Tr}(\mathsf{e}_l^{\dagger}\mathsf{e}_k\Pi)\,\mathrm{Tr}(\Pi\mathsf{e}_{\mu}^{\dagger}\mathsf{e}_l)\,\varphi_k(x)\bar{\varphi}_{\mu}(x)\end{aligned}$$

and the related integral satisfies

$$\begin{aligned}&\int_{\mathbb{R}^d}\mathcal{K}(t,x,x)\,dx = \sum_{lk}^{\Lambda}\mathrm{Tr}(\mathsf{e}_l^{\dagger}\mathsf{e}_k\Pi)\,\mathrm{Tr}(\Pi\mathsf{e}_k^{\dagger}\mathsf{e}_l) = \sum_{lk}^{\Lambda}|\mathrm{Tr}(\Pi\mathsf{e}_k^{\dagger}\mathsf{e}_l)|^2\\ &\geq \sum_{k}^{\Lambda}|\mathrm{Tr}(\Pi\mathsf{e}_k^{\dagger}\mathsf{e}_k)|^2 = \sum_{k}^{\Lambda}(\mathrm{Tr}(\Pi\mathsf{e}_k^{\dagger}\mathsf{e}_k))^2\,.\end{aligned}$$

The next inequality then follows

$$\Big(\int_{\mathbb{R}^d}\mathcal{K}(t,x,x)\,dx\Big)^{\frac{1}{2}} \geq \Big(\sum_{k}^{\Lambda}(\mathrm{Tr}(\Pi\mathsf{e}_k^{\dagger}\mathsf{e}_k))^2\Big)^{\frac{1}{2}} \geq \frac{1}{\Lambda^d}\sum_{k}^{\Lambda}\mathrm{Tr}(\Pi\mathsf{e}_k^{\dagger}\mathsf{e}_k)$$

and thanks to the equivalence

$$\mathrm{Tr}\Big(\Pi\int_{\mathbb{R}^d}\Theta_{\Lambda}^{\dagger}(t,x)\Theta_{\Lambda}(t,x)\,dx\Big) = \sum_{k}^{\Lambda}\mathrm{Tr}(\Pi\mathsf{e}_k^{\dagger}\mathsf{e}_k)$$

we get the statement above. □

4.4 Bargmann-Fock space, Wick operators and coherent phase space

Let $\bar{\mathcal{A}}(\mathbb{C}^{\ell})$ be the set of the anti-analytic functions $\psi:\mathbb{C}^{\ell}\to\mathbb{C}$. The Bargmann-Fock space is defined as

$$\mathcal{F}_B(\mathbb{C}^{\ell}) := \Big\{\psi\in\bar{\mathcal{A}}(\mathbb{C}^{\ell})\ \big|\ \int|\psi(\bar{z})|^2\,e^{-|z|^2}dz\wedge d\bar{z} < +\infty\Big\} \tag{106}$$

with the scalar product (here $z := x + i\xi$ and $dz\wedge d\bar{z} := \pi^{-\ell}dxd\xi$)

$$\begin{aligned}\langle\psi,\varphi\rangle &:= \int\psi^{\star}(\bar{z})\varphi(\bar{z})\,e^{-|z|^2}dz\wedge d\bar{z}\\ &= \frac{1}{\pi^{\ell}}\int_{\mathbb{R}^{2\ell}}\psi^{\star}(x-i\xi)\varphi(x-i\xi)\,e^{-(|x|^2+|y|^2)}dxd\xi\end{aligned} \tag{107}$$

Coherent states in $\mathcal{F}_B(\mathbb{C}^{\ell})$ are, with normalization factor $e^{-\frac{1}{2}|\alpha|^2}$, given by

$$|\alpha\rangle \equiv \phi_{\alpha}(\bar{z}) := e^{\alpha\cdot\bar{z}-\frac{1}{2}\alpha\cdot\bar{\alpha}}. \tag{108}$$

The creation and annihilation operators on $\mathcal{F}_B(\mathbb{C}^\ell)$ are defined as

$$(\mathsf{a}_k\psi)(\bar{z}) := \frac{\partial\psi(\bar{z})}{\partial\bar{z}_k}, \qquad (\mathsf{a}_k^\dagger\psi)(\bar{z}) := \bar{z}_k\psi(\bar{z}). \tag{109}$$

The vector space $\mathbb{C}^\ell$ with $\ell := \Lambda^d$ and $d = 1, 2, 3$ equipped with linear coordinates $\alpha = (\alpha_k)_{|k|<\Lambda}$ can be called *coherent phase space*, since its points are the coherent state eigenvalues,

$$\mathsf{a}_k\phi_\alpha = \alpha_k\phi_\alpha \tag{110}$$

We denote the space of (finite) power series in $(\alpha, \bar{\alpha})$ as

$$\mathcal{P}_\Lambda(\alpha, \bar{\alpha}) := \Big\{c(\alpha, \bar{\alpha}) = \sum_{ij}^{\Lambda}\sum_{nm} c_{ij,nm}\bar{\alpha}_i^n\alpha_j^m\Big\}.$$

The space of power series in terms of $\mathsf{a}, \mathsf{a}^\dagger$ will be indicated with

$$\mathsf{P}_\Lambda(\mathsf{a}, \mathsf{a}^\dagger) := \Big\{\mathsf{c}(\mathsf{a}, \mathsf{a}^\dagger) = \sum_{ij}^{\Lambda}\sum_{nm} c_{ij,nm}(\mathsf{a}_i^\dagger)^n\mathsf{a}_j^m\Big\}.$$

These definitions should be read in terms of multi-indices; for instance $\alpha_i^n = \prod_{p=1}^d \alpha_{i_p}^{n_p}$.

Definition 3 The Wick quantization map $\mathsf{W} : \mathcal{P}_\Lambda(\alpha, \bar{\alpha}) \to \mathsf{P}_\Lambda(\mathsf{a}, \mathsf{a}^\dagger)$ is given by the following properties:

1. $\mathsf{W}[1] = 1_{\mathcal{H}}$;
2. $\mathsf{W}[af + bg] = a\mathsf{W}[f] + b\mathsf{W}[g]$ for all $f, g \in \mathcal{P}_\Lambda(\alpha, \bar{\alpha})$ and $\forall a, b \in \mathbb{C}$;
3. normal form compatibility, i.e. $\mathsf{W}[\bar{\alpha}_i^n\alpha_j^m] = \mathsf{W}[\alpha_j^m\bar{\alpha}_i^n] = (\mathsf{a}_i^\dagger)^n\mathsf{a}_j^m$;

The inverse map $f := \mathsf{W}^{-1}[\mathsf{F}]$ is the Wick symbol of the operator F, given by the expectation over coherent states,

$$\mathsf{W}^{-1} : \mathsf{P}_\Lambda(\mathsf{a}, \mathsf{a}^\dagger) \to \mathcal{P}_\Lambda(\alpha, \bar{\alpha})\,.$$

In particular, $\langle\phi_\alpha, (\mathsf{a}_i^\dagger)^n\mathsf{a}_j^m\phi_\alpha\rangle = \bar{\alpha}_i^n\alpha_j^m$. This map sends polynomials into polynomial operators expressed in normal form via the prescription $\alpha \to \mathsf{a}$, $\bar{\alpha} \to \mathsf{a}^\dagger$. For general Wick operators, namely $\mathsf{F} := \mathrm{Op}_{\mathrm{wick}}(f)$ when f is not a polynomial, one can set

$$(\mathrm{Op}_{\mathrm{wick}}(f)\psi)(\bar{z}) := \int f(\bar{z}, \alpha)\,\psi(\bar{\alpha})\,e^{-|\alpha|^2+\alpha\cdot\bar{z}}\;d\alpha \wedge d\bar{\alpha}, \tag{111}$$

$$f(\bar{z}, \alpha) \;=\; \frac{\langle\phi_z, \mathsf{F}\phi_\alpha\rangle}{\langle\phi_z, \phi_\alpha\rangle}, \quad \psi \in \mathcal{F}_B(\mathbb{C}^\ell), \tag{112}$$

for which we address the reader to [7], [8], [12], [16], [23].

The Wick star product is defined as:

$$f \star g := \mathsf{W}^{-1}[\,\mathsf{W}[f]\,\mathsf{W}[g]\,], \tag{113}$$

which is linear and associative.

Proposition 11 *By defining the Wick parenthesis as*

$$\{f, g\}_w := f \star g - g \star f \tag{114}$$

the following properties can be stated: $\forall a, b \in \mathbb{C}$,

1. *linearity:* $\{af + bg, h\}_w = a\{f, h\}_w + b\{g, h\}_w$;
2. *skew-symmetry:* $\{f, g\}_w = -\{g, f\}_w$;
3. $\star$*-Leibniz property:* $\{f, g \star h\}_w = \{f, g \star h\}_w + g \star \{f, h\}_w$;
4. *Jacobi identity:* $\{f, \{g, h\}_w\}_w + \{h, \{f, g\}_w\}_w + \{g, \{h, f\}_w\}_w = 0$.

The Wick product admits the asymptotic expansion (see [8])

$$f \star g \simeq \sum_{n=0}^{\infty} \sum_{k_1,\dots,k_n}^{\Lambda} \frac{1}{n!} \frac{\partial^n f}{\partial\alpha_{k_1}\cdots\partial\alpha_{k_n}} \frac{\partial^n g}{\partial\bar{\alpha}_{k_1}\cdots\partial\bar{\alpha}_{k_n}} \tag{115}$$

which shows that the $\star$ - product may be seen as an algebraic deformation of the point-wise product between coherent phase space functions. Denoting by $\mathcal{O}(\partial^2)$ terms containing derivatives of at least order 2, notice that Wick parenthesis can be seen as an algebraic deformation of the Poisson parenthesis,

$$\begin{aligned} \{f, g\}_w &= \{f, g\} + \mathcal{O}(\partial^2), \\ \{f, g\} &:= \sum_k^{\Lambda} \frac{\partial f}{\partial\alpha_k}\frac{\partial g}{\partial\bar{\alpha}_k} - \frac{\partial f}{\partial\bar{\alpha}_k}\frac{\partial g}{\partial\alpha_k}. \end{aligned}$$

A straightforward application of the above properties shows some useful relations between phase space derivation and operator multiplication:

$$\langle\phi_\alpha, \mathsf{a}_k\mathsf{F}(\mathsf{a}, \mathsf{a}^\dagger)\phi_\alpha\rangle = \left(\alpha_k + \frac{\partial}{\partial\bar{\alpha}_k}\right) f(\alpha, \bar{\alpha}) \tag{116}$$

$$\langle\phi_\alpha, F(\mathsf{a}, \mathsf{a}^\dagger)\mathsf{a}_k^\dagger\phi_\alpha\rangle = \left(\bar{\alpha}_k + \frac{\partial}{\partial\alpha_k}\right) f(\alpha, \bar{\alpha}) \tag{117}$$

$$\frac{\partial f}{\partial\bar{\alpha}_k}(\alpha, \bar{\alpha}) = \langle\phi_\alpha, [\mathsf{a}_k, \mathsf{F}(\mathsf{a}, \mathsf{a}^\dagger)]\phi_\alpha\rangle \tag{118}$$

$$\frac{\partial f}{\partial\alpha_k}(\alpha, \bar{\alpha}) = \langle\phi_\alpha, [\mathsf{F}(\mathsf{a}, \mathsf{a}^\dagger), \mathsf{a}_k^\dagger]\phi_\alpha\rangle. \tag{119}$$

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