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Mean field asymptotics and invariant measures for the flow of dNLS

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Abstract. We derive the flow of discrete NLS equations by the mean field asymptotics of a many body quantum model for N interacting particles as N becomes large. This is obtained through L^2 - estimates on Wick symbols with respect to a class of flow invariant measures. Furthermore, we show weighted Hilbert-Schmidt norm estimates for Wick operators evolved in the Heisenberg picture. This leads to an Egorov type result for Wick symbols, global in time and with quantitative estimates.

Keywords: dNLS, Wick and anti-Wick operators, mean field asymptotics

1. Introduction

Let us consider the family of discrete nonlinear Schrödinger equations (dNLS) written in the form

$$i\frac{d}{dt}u_k(t) = \sum_{1\leqslant j\leqslant \ell} \tau_{kj}\, u_j(t) + U_k\, |u_k(t)|^2 u_k(t) \tag{1}$$

with $u_k(0) := \omega_k \in \mathbb{C}$, $1 \leqslant k \leqslant \ell$. The real matrix $(\tau_{kj})_{1\leqslant k,j\leqslant \ell}$ is supposed symmetric and positive definite, and $0 \leqslant U_k \leqslant U$. It is well known that $u := (u_1, u_2, ..., u_\ell)$ can be written in terms of the (complex) Hamiltonian flow linked to

$$\mathcal{H}(\bar\omega, \omega) := \sum_{1\leqslant j,k\leqslant \ell} \tau_{kj}\, \bar\omega_k\, \omega_j \; + \frac{1}{2} \sum_{1\leqslant k\leqslant \ell} U_k\, |\omega_k|^4, \tag{2}$$

namely $u(t,\omega) = \Phi_t(\bar\omega, \omega)$ where $\Psi_t = (\bar\Phi_t, \Phi_t) : \Delta = \{(\bar\omega, \omega) \mid \omega \in \mathbb{C}^\ell\} \subset \mathbb{C}^{2\ell} \to \mathbb{C}^{2\ell}$ is the flow of the equation $\dot\gamma = i(\partial_\omega \mathcal{H}(\gamma), -\partial_{\bar\omega}\mathcal{H}(\gamma))$.

In our work we follow an approach based on the use of Wick operators, many body theory and semi-classical estimates, where N is the number of particles and $h := 1/N$ is the asymptotic parameter.
Our main outcome is an Egorov type result (see [7], [13]) written for Wick operators on Bargmann space $\mathcal{F}_B(\mathbb{C}^\ell)$ in order to recover the 'classical' flow of the dNLS (1). In particular, we show a semiclassical estimate which is global in time and with respect to the L^2-norm linked to flow invariant measures. This approach has the advantage to overcome the well known problem of the Ehrenfest time, and avoid any exponential in time upper bounds. Furthermore, we show a link between L^2-norms of Wick symbols and weighted Hilbert-Schmidt norms of the corresponding Wick operators, which is a novel result in the framework of Wick and anti-Wick operators theory (see [2], [11], [12], [15], [17], [33]).

The target of our paper is to derive the flow of the dNLS equation (1) directly from the mean field asymptotics for the quantum dynamics of the following many body operator

$$\widehat{H} := \sum_{1\leqslant j,k\leqslant \ell} \tau_{kj}\, \hat{b}_k^\dagger \hat{b}_j \;+ \frac{1}{2N}\sum_{1\leqslant k\leqslant \ell} U_k\, \hat{b}_k^\dagger \hat{b}_k^\dagger \hat{b}_k \hat{b}_k \equiv \widehat{H}_2 + \widehat{H}_4 \tag{3}$$

written in terms of creation and annihilation operators $[\hat{b}_k, \hat{b}_\mu^\dagger] = \delta_{k\mu}\,\mathrm{Id}$ here defined on the Bargmann space $\mathcal{F}_B(\mathbb{C}^\ell)$, see Sect. 3.1. To introduce more in details our mean field approach, we first notice that the rescaled operators $\hat{a}_k := \frac{1}{\sqrt{N}}\hat{b}_k$, $\hat{a}_k^\dagger := \frac{1}{\sqrt{N}}\hat{b}_k^\dagger$ fulfill the commutation rules $[\hat{a}_k, \hat{a}_\mu^\dagger] = \frac{\delta_{k\mu}}{N}\,\mathrm{Id}$ and the Heisenberg equation reads

$$i\frac{d}{dt}\hat{a}_k(t) = \sum_{1\leqslant j\leqslant \ell} \tau_{kj}\, \hat{a}_j(t) + U_k\, \hat{a}_k^\dagger(t)\, \hat{a}_k(t)\, \hat{a}_k(t) \tag{4}$$

with $\hat{a}_k(0) := \hat{a}_k$ for $1 \leqslant k \leqslant \ell$. Notice that in order to show the link with flow of (1) we define the (rescaled) Wick symbol

$$\rho_k(t,\bar{\omega},\omega) := \langle \phi_{\sqrt{N}\omega}, \hat{a}_k(t)\phi_{\sqrt{N}\omega}\rangle \tag{5}$$

where $\hat{a}_k\phi_{\sqrt{N}\omega} = \omega_k\, \phi_{\sqrt{N}\omega}$, namely ϕ_ω are the normalized coherent states in $\mathcal{F}_B(\mathbb{C}^\ell)$, see Sect. 3.1. The number operator $\widehat{N} := \sum_{k=1}^\ell \hat{b}_k^\dagger \hat{b}_k = N\sum_{k=1}^\ell \hat{a}_k^\dagger \hat{a}_k$ satisfies $[\widehat{N}, \widehat{H}] = 0$ and provides the expected number of particles for the states $\phi_{\sqrt{N}\omega}$ by $\langle \phi_{\sqrt{N}\omega}, \widehat{N}\phi_{\sqrt{N}\omega}\rangle = N|\omega|^2$. By an easy computation we have that $N\mathcal{H}(\bar{\omega},\omega) = \langle \phi_{\sqrt{N}\omega}, \widehat{H}\phi_{\sqrt{N}\omega}\rangle$ and this implies that (4) can be rewritten through the Wick bracket (see Sect. 3.1)

$$\frac{i}{N}\frac{d}{dt}\rho_k = \{\rho_k, \mathcal{H}\}_{\mathrm{Wick}} = \sum_{\alpha=1}^{2} \frac{1}{\alpha!}\Big(\frac{1}{N}\Big)^\alpha \Big(\frac{\partial^\alpha \mathcal{H}}{\partial\bar{\omega}^\alpha}\frac{\partial^\alpha \rho_k}{\partial\omega^\alpha} - \frac{\partial^\alpha \mathcal{H}}{\partial\omega^\alpha}\frac{\partial^\alpha \rho_k}{\partial\bar{\omega}^\alpha}\Big) \tag{6}$$

with initial data $\rho_k(0,\bar{\omega},\omega) = \omega_k$, and where the sum is reduced to the second order thanks to the polynomial behavior of $\mathcal{H}$ in (2). In view of this equation, we now define the 'semiclassical' parameter $h := 1/N$, and notice that the k-th component of the dNLS flow in (1) solves

$$i\,\frac{du_k}{dt} = \frac{\partial\mathcal{H}}{\partial\bar{\omega}}\frac{\partial u_k}{\partial\omega} - \frac{\partial\mathcal{H}}{\partial\omega}\frac{\partial u_k}{\partial\bar{\omega}} \tag{7}$$

namely involves the Poisson bracket, which is the first order approximation of the equation (6).

The objective is thus to prove that $\rho_k - u_k \to 0$ as $N \to +\infty$ in an appropriate measure sense and, as a consequence, also by a weighted operator norm estimate.
To this aim, we consider the semipositive definite operators $\widehat{P}$ on $\mathcal{F}_B(\mathbb{C}^\ell)$ in the class of anti-Wick operators given by $\widehat{P} := f(\widehat{N})$ where $f \in C^\infty(\mathbb{R})$, $f(0) = 0$ and $f(x) \geqslant C_0\, x\;\forall x \geqslant 1$ for some $C_0 > 0$. Typical examples in this setting are $\widehat{P} = \widehat{N}^\alpha$ with $\alpha \geqslant 1$, see Def. 3.5. In this framework we will use the Trace formula involving Wick operators $\mathrm{Op_W}(\sigma)$, see Sect. 3.5,

$$\mathrm{Tr}\Big(\frac{e^{-\lambda\widehat{P}}}{c_\lambda}\mathrm{Op_W}(\sigma)\Big) = \int \sigma\, dm_\lambda, \quad \lambda > 0, \quad c_\lambda := \mathrm{Tr}(e^{-\lambda\widehat{P}}), \tag{8}$$

with the probability measure dm_λ defined by the (positive) anti-Wick symbol of $e^{-\lambda\widehat{P}}$

$$dm_\lambda(\bar\omega,\omega) := \frac{1}{c_\lambda}\,\sigma_{AW}(e^{-\lambda\widehat{P}})(\bar\omega,\omega)\;d\bar\omega\wedge d\omega \tag{9}$$

for $\omega := x + i\xi$ and $d\bar\omega\wedge d\omega := \pi^{-\ell}dxd\xi$. In particular, we will show in Prop. 4.3 that dm_λ are invariant measures under the flow Ψ_t of the dNLS, namely

$$\Psi_t^\star(dm_\lambda) = dm_\lambda \tag{10}$$

$\forall t \geqslant 0$ and arbitrary fixed $\lambda > 0$.

The first result of the paper reads

Theorem 1.1. *Let u be the flow of the DNLS equation (1), let ρ_k be the solution of (7) for $1 \leqslant k \leqslant \ell$ and dm_λ given (9) with $e^{C_0\lambda} = N+1$. Then, for $\ell/N \leqslant D < +\infty$ we have $u_k, \rho_k \in L^2(m_\lambda)$ and*

$$\|\rho_k - u_k\|_{L^2(m_\lambda)} \leqslant \sqrt{3}\;5^3\;e^{\frac{D}{2}}\;\frac{\ell}{N}\;\frac{U\,t}{\sqrt{N}}\,. \tag{11}$$

In this result we show an explicit quantitative estimate in terms of the parameters of the problem. We can in addition prescribe $\ell \simeq N^\beta$ for some $0 \leqslant \beta < 1$ or $\ell \simeq \ln(N)$ so that $\ell/N \to 0$ as $N \to +\infty$. In the case $\widehat{P} = C_0\widehat{N}$ and $e^{C_0\lambda} = N+1$, we have

$$dm_\lambda(\bar\omega,\omega) = N^\ell\;e^{-N|\omega|^2}d\bar\omega\wedge d\omega \tag{12}$$

namely a gaussian probability measure. Notice that any convex linear combination of the measures dm_λ as in (9) gives the same upper bound in (11). We thus have also a uniform estimate with respect to all the measures in the convex hull, that moreover are not necessarily written in the form (9), namely it is a larger convex set of invariant measures. It is rather interesting the open problem to show that all the invariant measures for our dNLS flow belong to this set. In Sect. 2 we show that Thm. 1.1 implies also a Gibbs estimate, to get a further derivation of the dNLS flow.

Let us now recall that the Hilbert-Schmidt norm reads $\|\widehat{B}\|_{\mathrm{HS}} := (\mathrm{Tr}(\widehat{B}^\dagger\widehat{B}))^{\frac12}$ and consider $\widehat{B} := (c_\lambda^{-1}e^{-\lambda\widehat{P}})^{\frac12}\,\widehat{\Pi}_{(\leqslant N)}\,\mathrm{Op}^h_{\mathrm{W}}(\sigma)$ with the semiclassical Wick operator $\mathrm{Op}^h_{\mathrm{W}}(\sigma)$, $h := 1/N$, given in (36). The operator $\widehat{\Pi}_{(\leqslant N)} : \mathcal{F}_B(\mathbb{C}^\ell) \to \Lambda_{(\leqslant N)} \subset \mathcal{F}_B(\mathbb{C}^\ell)$ is the projector into the span of the eigenspaces of $\widehat{N}$ up to eigenvalue N, see Def. 3.3. In particular, $\widehat{\Pi}_{(\leqslant N)} \to \mathrm{Id}$ as $N \to +\infty$. By assuming the following growth condition $\|\widehat{\Pi}_{(\leqslant N)}\,\mathrm{Op}^h_{\mathrm{W}}(\sigma)\|_{\mathrm{HS}} \leqslant C_1\,h^{-Q}$ for some $C_1 > 0$ and $Q \in \mathbb{N}$, then (see Prop. 4.6)

$$\Big\|\Big(\frac{e^{-\lambda\widehat{P}}}{c_\lambda}\Big)^{\frac12}\,\widehat{\Pi}_{(\leqslant N)}\,\mathrm{Op}^h_{\mathrm{W}}(\sigma)\Big\|^2_{\mathrm{HS}} \leqslant 2\,\|\sigma\|^2_{L^2(m_\lambda)} + e^{-\frac{1}{4h}}\,e^D\,C_1^2\,h^{-2Q} \tag{13}$$

provided $2^6\ell \leqslant N$. We have thus shown that an L^2-norm on Wick symbols, plus an explicit $\mathcal{O}(h^\infty)$ -term, controls a weighted Hilbert-Schmidt norm. This can be regarded as a result of phase space Analysis for Wick operators on Bargmann space.

To apply (13) for our targets, we show in Prop. 4.16 that the semiclassical Wick operator $\mathrm{Op}^h_{\mathrm{W}}(\rho_k - u_k)$ satisfies the above growth condition, uniformly in time, for suitable $C_1(D)$ and $Q(\ell) = 1 + \ell/2$. The more restrictive condition $2^6\ell \leqslant N/(8\ln(N))$, $N \geqslant 2$, guarantees that the remainder on the righthand side of (13) is $\mathcal{O}(h^\infty)$ uniformly with respect to ℓ and $t \geqslant 0$, see Remark 4.17.

Furthermore, if $\tau_{min} > 0$ is the minimum eigenvalue of the positive matrix $(\tau_{kj})_{1\leqslant j,k\leqslant \ell}$ in $\widehat{H}$ then, thanks to simple spectral arguments,

$$\Big\| \Big(\frac{e^{-\lambda \widehat{H}}}{b_\lambda}\Big)^{\frac{1}{2}} \,\widehat{\Pi}_{(\leqslant N)}\, \mathrm{Op}^h_{\mathrm{W}}(\sigma) \Big\|_{\mathrm{HS}} \leqslant e^{\frac{D}{2}} \,\Big\| \Big(\frac{e^{-\lambda \tau_{min} \widehat{N}}}{c_\lambda}\Big)^{\frac{1}{2}} \,\widehat{\Pi}_{(\leqslant N)}\, \mathrm{Op}^h_{\mathrm{W}}(\sigma) \Big\|_{\mathrm{HS}} \tag{14}$$

where $b_\lambda := \mathrm{Tr}(e^{-\lambda \widehat{H}})$ and $c_\lambda := \mathrm{Tr}(e^{-\lambda \tau_{min} \widehat{N}})$, see Prop. 4.7.

In view of Theorem 1.1 and thanks to the upper bounds in (13) - (14), we can now state the next

Theorem 1.2. *Let $\widehat{P}$ be as in Def. 3.5 and let $\widehat{H}$ be as in (3). Assume that $2^6\ell \leqslant N/(8\ln(N))$. Let ρ_k, u_k be as in Thm. 1.1. If $e^{C_0\lambda} = N+1$ then*

$$\Big\| \Big(\frac{e^{-\lambda \widehat{P}}}{c_\lambda}\Big)^{\frac{1}{2}} \,\widehat{\Pi}_{(\leqslant N)}\, \mathrm{Op}^h_{\mathrm{W}}(\rho_k - u_k) \Big\|_{\mathrm{HS}} \leqslant \sqrt{6}\, 5^3\, e^{\frac{D}{2}}\, \frac{\ell}{N}\, \frac{Ut}{\sqrt{N}} + 4(1+D)^{\frac{1}{2}} e^{\frac{1}{2}+D} e^{-\frac{1}{16h}}, \tag{15}$$

where $c_\lambda := \mathrm{Tr}(e^{-\lambda \widehat{P}})$. For $\lambda \equiv 1/T$ and $0 < T \leqslant \tau_{min}/\ln(N+1)$ we have

$$\Big\| \Big(\frac{e^{-\lambda \widehat{H}}}{b_\lambda}\Big)^{\frac{1}{2}} \,\widehat{\Pi}_{(\leqslant N)}\, \mathrm{Op}^h_{\mathrm{W}}(\rho_k - u_k) \Big\|_{\mathrm{HS}} \leqslant \sqrt{6}\, 5^3\, e^{D}\, \frac{\ell}{N}\, \frac{Ut}{\sqrt{N}} + 4(1+D)^{\frac{1}{2}} e^{\frac{1+3D}{2}} e^{-\frac{1}{16h}} \tag{16}$$

where $b_\lambda := \mathrm{Tr}(e^{-\lambda \widehat{H}})$.

The parameter T can be interpreted as the temperature of the N particles, and the estimate (16) works if the bound $0 < T \leqslant \tau_{min}/\ln(N+1)$ is fulfilled, namely the temperature must be small enough with respect to N.

We underline that $\mathrm{Op}^h_{\mathrm{W}}(\rho_k) = \hat{a}_k(t)$, the annihilation operator evolved in the Heisenberg picture, see Remark 3.1. Moreover, in Sect. 3.2 we show that the semiclassical quantization of the k-th component of the dNLS flow $\mathrm{Op}^h_{\mathrm{W}}(u_k)$ is a well posed Wick operator and solves an evolution equation, which is a deformation of the Heisenberg equation beyond the simple quadratic case.
We stress that both $\mathrm{Op}^h_{\mathrm{W}}(\rho_k)$ than $\mathrm{Op}^h_{\mathrm{W}}(u_k)$ are unbounded operators, and whence the use of weighted norms in Thm. 1.2 is a nice way to get a semiclassical estimate of their convergence. In Thm 1.4 of [13] the convergence is proved in operator norm (without weights) for bounded quantum observables but the problem of the Ehrenfest time for the validity of the semiclassical approximation is clearly shown. Here we overcome such a problem by this different kind of convergence.

In [30], the same kind of estimate of Thm. 1.1 is shown for the L^p-norm with $p \geqslant 1$ and gaussian measure. The present paper extends such result towards three directions: for $p = 2$ to a larger class of invariant measures thanks to the role of anti-Wick symbols, adding the operator estimates in (13)-(14) used in Thm. 1.2 and moreover providing the Gibbs estimate in Sect. 2.

The use of Wick operators in many body problems and related semiclassical estimates is well known in the literature. Here we apply these tools to study a class of dNLS (instead of NLS or Hartree equation) and to get linear in time estimates. We remind that in various works (see [3], [4], [27], [28], [29],

[32]) the derivation of dNLS type equations is obtained from the NLS equation, but exponential in time estimates appears frequently or local in time results are shown. This occurs also in the literature of mean field derivation of NLS and Hartree.
In the paper [6] the the flow of the Hartree equation is recovered as mean field limit, and infinite dimensional phase-space analysis are carried out thanks to Wick and anti-Wick operators on the Fock space. In section 7.2 of [19] the authors discuss, thanks to the Wick quantization for a class of symbols, how the many-body quantum mechanics of bosons can be viewed as a deformation quantization of the Hartree theory. A semiclassical asymptotic expansion is shown, in [1], for Wick symbols of density operators governed by time dependent Hartree–Fock equation.

To conclude the Introduction, we also remind some of the many papers on the mean field derivation of the NLS equation, the Hartree equation, or more in general the study of many body theory by different techniques (see for example [5], [8], [10], [16], [18], [21], [22], [23], [24], [25], [26], [31] and references therein).

The content of the paper is the following: in Sect. 2 we discuss the use of the classical Gibbs measure. In Sect. 3 we recall the basic notions of Wick and anti-Wick operators together with some technical results useful for our approach. Section 4 is devoted to the proofs of Theorems 1.1 and 1.2 and related additional results.

2. The Gibbs estimate

In this section we use the 'classical' Gibbs measure as a tool to prove another derivation (with respect to Thm. 1.2) of the dNLS flow in (1) in terms of the temperature $T \to 0$ as asymptotic parameter.
With respect to this topic, we remind that the Gibbs measures have been intensively studied in connection to many-body quantum problems. In what follows, we recall only some of the recent results, but we also address the reader to the references therein. In the paper [18] the Gibbs measures of nonlinear Schrödinger equations arise as high-temperature limits of thermal states in many-body quantum mechanics, in particular for defocusing interactions in dimensions $1 \leqslant d \leqslant 3$. The authors of [26] have shown that Gibbs measures based on 1D defocusing nonlinear Schrödinger functionals with sub-harmonic trapping can be obtained as the mean-field, or large temperature limit, of the corresponding grand-canonical ensemble for many bosons. In the work [14] the aim is to study concentration of the Gibbs measures for a class of periodic Zakharov, KdV, NLS and Gross-Pitaevskii equations in dimensions $1 \leqslant d \leqslant 2$.

Here we look for a semiclassical estimate through the use of the flow invariant **Gibbs measure**

$$\mathrm{d}\mathbb{P}_\lambda := \frac{1}{\int e^{-\lambda \mathcal{H}(\bar{\omega},\omega)} d\bar{\omega} \wedge d\omega} \, e^{-\lambda \mathcal{H}(\bar{\omega},\omega)} d\bar{\omega} \wedge d\omega \tag{17}$$

where $\omega := q + ip$ and $d\omega \wedge d\bar{\omega} := \pi^{-\ell} dq dp$. To answer this question, we notice that $e^{-\lambda \mathcal{H}}$ has an upper bound given by the anti-Wick symbol of $e^{-\lambda_0 \widehat{N}}$,

$$0 < e^{-\lambda \mathcal{H}(\bar{\omega},\omega)} \leqslant \sigma_{AW}(e^{-\lambda_0 \widehat{N}})(\bar{\omega}, \omega), \qquad e^{\lambda_0} = \lambda \, \tau_{min} + 1 \tag{18}$$

where the value $\tau_{min} > 0$ is the smallest eigenvalue of the positive definite matrix $(\tau_{kj})_{1\leqslant k,j\leqslant \ell}$ linked to the quadratic part of $\mathcal{H}$ in (2) and

$$\sigma_{AW}(e^{-\lambda_0 \widehat{N}})(\bar{\omega},\omega) = (\lambda\,\tau_{min}+1)^{\ell}\; e^{-\lambda\tau_{min}|\omega|^2}. \tag{19}$$

We are interested to use such a gaussian measure inside the statement of Theorem 1.1, hence we now require $\lambda\tau_{min} \geqslant N$. Notice moreover that if $\lambda \geqslant \ell/\tau_{min}$ then $\mathrm{Tr}(e^{-\lambda_0\widehat{N}}) \leqslant e$. By defining $\tau_{max} > 0$ as the biggest eigenvalue of $(\tau_{kj})_{1\leqslant k,j\leqslant \ell}$ and the constant $\alpha_{\mathcal{H}} := e(2\tau_{max})^{\ell}$ it follows also that

$$c_{\lambda_0} := \mathrm{Tr}(e^{-\lambda_0\widehat{N}}) = \int \sigma_{AW}(e^{-\lambda_0\widehat{N}})(\bar{\omega},\omega)\, d\bar{\omega}\wedge d\omega \leqslant \alpha_{\mathcal{H}} \int e^{-\lambda\mathcal{H}(\bar{\omega},\omega)}\, d\bar{\omega}\wedge d\omega \tag{20}$$

provided also that $\lambda \geqslant \lambda_1 \equiv \lambda_1(U,\ell)$ and where this value is given by the integral inequality $\int e^{-\tau_{max}|v|^2 - U/2\lambda_1 \sum_j |v_j|^4} d\bar{v}\wedge dv \geqslant (1/2\tau_{max})^{\ell}$. Thus, we require the more restrictive condition $\lambda \geqslant \max\{N/\tau_{min}; \ell/\tau_{min}; \lambda_1(U,\ell)\}$.
As a consequence, in view of (18) - (20), we have for any $g \in L^2(m_{\lambda_0})$ with the gaussian measure $dm_{\lambda_0} := c_{\lambda_0}^{-1}\, \sigma_{AW}(e^{-\lambda_0\widehat{N}})\, d\bar{\omega}\wedge d\omega$ the following inequality

$$\|g\|_{L^2(\mathbb{P}_\lambda)} \leqslant \sqrt{\alpha_{\mathcal{H}}}\; \|g\|_{L^2(m_{\lambda_0})}\,. \tag{21}$$

In view of (11) - (21), we have directly our classical Gibbs estimate by the next

Proposition 2.1. *For $g_k := \rho_k - u_k$ as in Thm. 1.1 and $\lambda \equiv 1/T$ it follows*

$$\|g_k\|_{L^2(\mathbb{P}_\lambda)} \leqslant \sqrt{2}\; 5^3\; e^D\; \frac{\ell}{N}\; U\; t\, \sqrt{\frac{\alpha_{\mathcal{H}}\, T}{\tau_{min}}} \tag{22}$$

where $0 < T \leqslant 1/\max\{N/\tau_{min}; \ell/\tau_{min}; \lambda_1(U,\ell)\}$ and $\ell/N \leqslant D$.

We underline the differences with respect to the Hilbert-Schmidt estimate (16) where $0 < T \leqslant \tau_{min}/\ln(N+1)$ and the upper bound is written only in terms of N. Here we have a different interval for T which is (asympotically) smaller since it decreases with the same order than $1/N$ or faster.

3. Settings and Preliminaries

3.1. Bargmann space and Wick Quantization

In this section we mainly follow the notations and some standard results of [11], but we also address the reader to [17] and [15]. In addition, we show some further properties on Wick and anti-Wick operators useful in the context of our paper.
Let $\bar{\mathcal{A}}(\mathbb{C}^{\ell})$ be the set of the anti-analytic functions $\psi : \mathbb{C}^{\ell} \to \mathbb{C}$. The Bargmann space is defined as

$$\mathcal{F}_B(\mathbb{C}^{\ell}) := \left\{ \psi \in \bar{\mathcal{A}}(\mathbb{C}^{\ell}) \mid \int |\psi(\bar{z})|^2\, e^{-|z|^2} dz\wedge d\bar{z} < +\infty \right\} \tag{23}$$

with the scalar product (here $z := x + i\xi$ and $dz \wedge d\bar{z} := \pi^{-\ell} dx d\xi$)

$$\langle \psi, \varphi \rangle := \int \psi^\star(\bar{z})\varphi(\bar{z})\, e^{-|z|^2} dz \wedge d\bar{z} = \frac{1}{\pi^\ell} \int_{\mathbb{R}^{2\ell}} \psi^\star(x - i\xi)\varphi(x - i\xi)\, e^{-(|x|^2+|y|^2)} dx d\xi$$

Coherent states in $\mathcal{F}_B(\mathbb{C}^\ell)$ are, with normalization factor $e^{-\frac{1}{2}|\omega|^2}$, given by

$$\phi_\omega(\bar{z}) := e^{\omega\cdot\bar{z} - \frac{1}{2}\omega\cdot\bar{\omega}}. \tag{24}$$

The creation and annihilation operators on $\mathcal{F}_B(\mathbb{C}^\ell)$ are defined as

$$(\hat{b}_k\psi)(\bar{z}) := \frac{\partial \psi(\bar{z})}{\partial \bar{z}_k}, \quad (\hat{b}_k^\dagger \psi)(\bar{z}) := \bar{z}_k \psi(\bar{z}). \tag{25}$$

For a given operator $\widehat{A} : \mathcal{F}_B(\mathbb{C}^\ell) \to \mathcal{F}_B(\mathbb{C}^\ell)$ its Wick symbol is defined by

$$\sigma_W(\widehat{A})(\bar{\omega}, \omega) := \langle \phi_\omega, \widehat{A}\phi_\omega \rangle \tag{26}$$

whereas outside the diagonal $(\bar{\omega}, \omega)$ the Wick symbol reads

$$\sigma_W(\widehat{A})(\bar{z}, \omega) := \frac{\langle \phi_z, \widehat{A}\phi_\omega \rangle}{\langle \phi_z, \phi_\omega \rangle}. \tag{27}$$

The **Wick quantization** of an entire function $\sigma : \mathbb{C}^\ell \times \mathbb{C}^\ell \to \mathbb{C}$ is given by

$$\mathrm{Op}_W(\sigma)(\psi)(\bar{z}) := \int \sigma(\bar{z}, \omega)\, \psi(\bar{\omega})\, e^{-|\omega|^2 + \omega\cdot\bar{z}}\, d\omega \wedge d\bar{\omega}. \tag{28}$$

In view of these settings, we have $\widehat{A} = \mathrm{Op}_{\mathrm{W}}(\sigma_W(\widehat{A}))$.
To be more precise about the set of these operators, we follow the arguments shown in [17] - pg. 139. Suppose that $\widehat{A}$ (possibly unbounded) is defined on $\mathcal{F}_B(\mathbb{C}^\ell)$ together with its adjoint $\widehat{A}^\dagger$, and assume that for all $\omega \in \mathbb{C}^\ell$, ϕ_ω belongs to the domains of $\widehat{A}$ and $\widehat{A}^\dagger$. Then, $\omega \mapsto \sigma_W(\widehat{A})(\bar{\omega}, \omega)$ is a smooth function on $\mathbb{C}^\ell$ and moreover $\sigma_W(\widehat{A})(\bar{\omega}, \omega)$ is the restriction on the diagonal of $\sigma_W(\widehat{A})(\bar{z}, \omega)$ as in (27), which is furthermore an entire function. As shown in Prop. 1.69 of [17], any entire function $K(\bar{z}, \omega)$ is uniquely determined by its restriction to $\{\bar{z} = \bar{\omega}\}$.
Thanks to these observations, $\widehat{A} = \mathrm{Op}_{\mathrm{W}}(\sigma) : \mathcal{F}_B(\mathbb{C}^\ell) \to \mathcal{F}_B(\mathbb{C}^\ell)$ is uniquely related to the symbol on the diagonal, and for this reason frequently in the literature one directly refers to $\widehat{A}$ as the Wick quantization of the function in (26).
A simple computation shows that

$$\sigma_W(\hat{b}_k) = \omega_k, \quad \sigma_W(\hat{b}_k^\dagger) = \bar{\omega}_k, \quad \sigma_W((\hat{b}_k^\dagger)^\alpha \circ (\hat{b}_\mu)^\beta) = \bar{\omega}_k^\alpha \omega_\mu^\beta. \tag{29}$$

These equalities allow to write the Wick symbol of $\widehat{H}$ as in (3)

$$\langle \phi_\omega, \widehat{H}\phi_\omega \rangle = \sum_{1 \leqslant j,k \leqslant \ell} \tau_{kj}\, \bar{\omega}_k\, \omega_j \; + \frac{1}{2N} \sum_{1 \leqslant j \leqslant \ell} U_j\, |\omega_j|^4. \tag{30}$$

The set of Wick operators is closed under composition, and the Wick-$\star$ product is defined as the symbol of the composition of two operators,

$$(\sigma^{(1)} \star_{\text{Wick}} \sigma^{(2)})(\bar{\omega}, \omega) := \langle \phi_\omega, \text{Op}_{\text{W}}(\sigma^{(1)}) \circ \text{Op}_{\text{W}}(\sigma^{(2)}) \phi_\omega \rangle. \tag{31}$$

It can be shown (see [11]) the following asymptotics (in multi-index notation)

$$\begin{aligned} \sigma^{(1)} \star_{\text{Wick}} \sigma^{(2)} &\simeq \sum_{r=0}^{\infty} \frac{1}{r!} \frac{\partial^r \sigma^{(1)}}{\partial \omega^r} \frac{\partial^r \sigma^{(2)}}{\partial \bar{\omega}^r}, \qquad (32) \\ &\simeq \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{i_1, i_2, \ldots i_r = 1}^{\ell} \frac{\partial^r \sigma^{(1)}}{\partial \omega_{i_1} \partial \omega_{i_2} \ldots \partial \omega_{i_r}} \frac{\partial^r \sigma^{(2)}}{\partial \bar{\omega}_{i_1} \partial \bar{\omega}_{i_2} \ldots \partial \bar{\omega}_{i_r}}, \end{aligned}$$

where $\partial \omega^r := \partial \omega_{i_1} \partial \omega_{i_2} \ldots \partial \omega_{i_r}$. About the well posedness of the righthand side and the semiclassical asymptotics in a set of Wick symbols we address the reader to [9]. The Wick bracket is defined as the symbol of the commutator

$$\{\sigma^{(1)}, \sigma^{(2)}\}_{\text{Wick}} := \sigma^{(1)} \star_{\text{Wick}} \sigma^{(2)} - \sigma^{(2)} \star_{\text{Wick}} \sigma^{(1)}. \tag{33}$$

In (32) the semiclassical parameter is $h = 1$ and the absence of the factor 2^r used in [9] is a consequence of the setting of the scalar product in (24). The semiclassical Wick-$\star$ product reads

$$\sigma^{(1)} \star_{\text{Wick}} \sigma^{(2)} \simeq \sum_{r=0}^{\infty} \frac{h^r}{r!} \frac{\partial^r \sigma^{(1)}}{\partial \omega^r} \frac{\partial^r \sigma^{(2)}}{\partial \bar{\omega}^r}. \tag{34}$$

The above defined Bargmann space $\mathcal{F}_B(\mathbb{C}^\ell)$ can be equipped with the scalar product $\langle \psi, \varphi \rangle_h := h^{-\ell} \int \psi^\star(\bar{z}) \varphi(\bar{z})\, e^{-\frac{1}{h}|z|^2} dz \wedge d\bar{z}$, the h-Wick quantization is given by

$$\text{Op}_{\text{W}}^h(\sigma)(\psi)(\bar{z}) := h^{-\ell} \int \sigma(\bar{z}, \omega)\, \psi(\bar{\omega})\, e^{-\frac{1}{h}(|\omega|^2 + \omega \cdot \bar{z})}\, d\omega \wedge d\bar{\omega} \tag{35}$$

for which the Wick-$\star$ product of symbols is (34). The semiclassical coherent states $\phi_\omega^h(\bar{z}) := e^{\frac{1}{h}(\omega \bar{z} - \frac{1}{2}\bar{\omega}\omega)}$ are normalized with respect to h - scalar product. Notice that for $\psi_h(\bar{v}) := \psi(\sqrt{h}\bar{v})$ then $\|\psi\|_h = \|\psi_h\|$ (the norm in Bargmann for $h = 1$) and $\forall z_0 \in \mathbb{C}^\ell$

$$\text{Op}_{\text{W}}^h(\sigma)(\psi)(\sqrt{h}\bar{z}_0) = \int \sigma(\sqrt{h}\bar{z}_0, \sqrt{h}v)\, \psi_h(\bar{v})\, e^{-(|v|^2 + v \cdot \bar{z}_0)}\, dv \wedge d\bar{v}$$

This equality gives a bijection between the h-Wick operators and the ones with $h = 1$.

$$\text{Op}_{\text{W}}^h(\sigma)(\psi)(\sqrt{h}\bar{z}_0) = \text{Op}_{\text{W}}(\sigma_h)(\psi_h)(\sqrt{h}\bar{z}_0), \quad \sigma_h(\bar{z}_0, v) := \sigma(\sqrt{h}\bar{z}_0, \sqrt{h}v). \tag{36}$$

Remark 3.1. *As a consequence of the above settings, by defining the rescaled symbol of any Wick operator* $\rho(\bar{\omega}, \omega) := \langle \phi_{\sqrt{N}\omega}, \widehat{A} \phi_{\sqrt{N}\omega} \rangle$ *(to simplify, we write it only on the diagonal) we have* $\text{Op}_{\text{W}}^h(\rho)\psi = \widehat{A}\psi_h$ *for* $h = 1/N$*. This is precisely the framework of the Theorem 1.2 for* ρ_k *defined in (5) where*

$$\text{Op}_{\text{W}}^h(\rho_k)\psi = \hat{a}_k(t)\psi_h\,. \tag{37}$$

Remark 3.2. *The family* $\psi_\alpha(\bar z) := \frac{1}{\sqrt{\alpha!}}\,\bar z^\alpha$ *where* $\alpha \in \mathbb{Z}^\ell_+$ *is an orthonormal set in* $\mathcal{F}_B(\mathbb{C}^\ell)$ *(see [11]). The number operator* $\widehat N = \sum_{j=1}^\ell \hat n_j := \sum_{j=1}^\ell \hat b_j^\dagger\, \hat b_j$ *fulfills* $\widehat N \psi_\alpha = |\alpha|\, \psi_\alpha$ *where* $|\alpha| := \alpha_1 + \alpha_2 + ... + \alpha_\ell$.

Definition 3.3. *We define the subsector up to N particles by the set* $\Lambda_{(\leqslant N)} := \mathrm{Span}\{\ \psi_\alpha \mid |\alpha| \leqslant N\}$. *Notice that if* $\psi \in \Lambda_{(\leqslant N)}$ *and* $\|\psi\| = 1$ *then* $\psi = \sum_{|\alpha|\leqslant N} c_\alpha\, \psi_\alpha$ *and* $\langle \psi, \widehat N \psi\rangle \leqslant N$. *One can also define in the same way the set* $\Lambda_{(N)}$ *given by the homogeneous polynomials of degree N in the variable* $\bar z$, *and denote it as the N-sector of the Bargmann space (see pg. 48 of [17] for the link with the sectors of the Fock space). We define the operator*

$$\widehat\Pi_{(\leqslant N)} : \mathcal{F}_B(\mathbb{C}^\ell) \to \Lambda_{(\leqslant N)} \subset \mathcal{F}_B(\mathbb{C}^\ell)$$

as the orthogonal projector into $\Lambda_{(\leqslant N)}$ *with respect to the scalar product on* $\mathcal{F}_B(\mathbb{C}^\ell)$. *Easily see that* $\widehat\Pi_{(\leqslant N)} \to \mathrm{Id}$ *weakly as* $N \to +\infty$.

3.2. The quantization of the dNLS flow

We show that the quantization of the dNLS flow is well posed; this means that for the solution u_k of the equation (7) the related operator $\mathrm{Op}_W(u_k)$ is a well defined Wick operator in the sense of (28). To this aim, we first prove that the equation

$$\begin{cases} i\frac{d}{dt}\hat u_k(t) = [\hat u_k(t), \widehat H_2] + \sum_{j=1}^\ell U_j\, \hat b_j^\dagger\, [\hat u_k(t), \hat n_j]\, \hat b_j\,, \quad \hat n_j := \hat b_j^\dagger \hat b_j\,, \\ \\ \hat u_k(0) := \hat b_k\,, \quad \text{with fixed} \quad 1 \leqslant k \leqslant \ell\,, \end{cases} \tag{38}$$

can be solved in a class of Wick operators.
Consider the Hilbert space of linear operators from $\mathcal{F}_B(\mathbb{C}^\ell)$ into itself, equipped with the scalar product $\langle \widehat A, \widehat B\rangle := \mathrm{Tr}(e^{-\widehat N}\widehat A^\dagger \widehat B)$. Notice that $\langle \hat b_k, \hat b_k\rangle < +\infty$.
By the same arguments shown after (196) with $e^{-\widehat N}$ in place of $\widehat\Pi_{(\leqslant N)}$ we get the conservation law

$$\frac{d}{dt}\langle \hat u_k, \hat u_k\rangle = 0. \tag{39}$$

Standard arguments of evolution equations on Hilbert spaces ensure the existence $\forall t \geqslant 0$ of the unitary flow e^{-iZt} where $Z(\hat u) := [\hat u, \widehat H_2] + \sum_{j=1}^\ell U_j\, \hat b_j^\dagger\, [\hat u, \hat n_j]\, \hat b_j$. In short notation, we write

$$\hat u_k(t) = e^{-iZt}\, \hat u_k(0)\,. \tag{40}$$

In particular, notice that if $U_j = 0$ then one gets the usual unitary conjugation $\hat u_k(t) = e^{i\widehat H_2 t}\, \hat u_k(0) e^{-i\widehat H_2 t}$. By taking the adjoint on both sides of (38) we get for $\hat u_k^\dagger(t)$ the same equation with Z, and thus

$$\hat u_k^\dagger(t) = e^{-iZt}\, \hat u_k^\dagger(0)\,. \tag{41}$$

It follows that $\hat u_k(t)$ and its adjoint are both defined with the domain $\mathcal{F}_B(\mathbb{C}^\ell)$, whence containing also coherent states. In view of subsection 3.1 we deduce that $\hat u_k(t)$ is a Wick operator.

We can now state that

$$\hat{u}_k(t) = \mathrm{Op}_W(u_k), \tag{42}$$

since taking the brackets on both sides of (38) with respect to coherent states ϕ_ω (i.e. looking at the Wick symbol) we get

$$i\,\frac{du_k}{dt} = \sum_{j=1}^{\ell}\left(\tau_{kj}\,\omega_j + U_j|\omega_j|^2\,\bar{\omega}_j\right)\frac{\partial u_k}{\partial \omega_j} - \sum_{j=1}^{\ell}\left(\tau_{kj}\,\bar{\omega}_j + U_j|\omega_j|^2\,\omega_j\right)\frac{\partial u_k}{\partial \bar{\omega}_j}\,. \tag{43}$$

which is precisely the equation (7), with the (complex) Poisson bracket. This is solved uniquely through the flow related to the Hamiltonian $\mathcal{H}$ given in (2) exhibiting bounded sublevel sets. This means that (38) is the operator counterpart of (43).

Remark 3.4. *With respect to this subsection we stress some facts.*

(i) *The semiclassical Wick quantization* $\mathrm{Op}^h_{\mathrm{W}}(u_k)$ *with* $0 < h \leqslant 1$ *can now be written by using the formula (36). The evolutive equation now reads*

$$\begin{cases} i\frac{d}{dt}\mathrm{Op}^h_{\mathrm{W}}(u_k) = [\mathrm{Op}^h_{\mathrm{W}}(u_k), \widehat{H}_2] + \sum_{j=1}^{\ell} U_j\,\hat{a}^\dagger_j\,[\mathrm{Op}^h_{\mathrm{W}}(u_k), \hat{n}_j]\,\hat{a}_j\,, \quad \hat{n}_j := \hat{b}^\dagger_j \hat{b}_j\,, \\ \mathrm{Op}^h_{\mathrm{W}}(u_k)(0) := \hat{a}_k\,, \quad \text{with fixed} \quad 1 \leqslant k \leqslant \ell\,. \end{cases} \tag{44}$$

To prove (44), compute the semiclassical Wick symbols on both sides by the use of semiclassical version of coherent states and of the Wick-$\star$ product. The result is again the equation (43).

(ii) *If* $U_j = 0\;\forall\, 1 \leqslant j \leqslant \ell$ *then the equation (38) becomes the Heisenberg equation with the quadratic operator* $\widehat{H}_2$. *This is not unexpected, indeed the classical flow solves the equation with the Poisson bracket that equals in this case the Wick bracket. On the other hand, it is remarkable that* $\mathrm{Op}^h_W(u_k)$ *solves an equation which is a linear deformation of Heisenberg.*

(iii) *The problem to quantize an Hamiltonian flow, namely to associate a well posed operator, is a nontrivial task and requires a lot of work in terms of estimates and analytical properties of the flow. See for example Lemma 3.1 and Remark 1.6 in [13] and Lemma 2.3 in [7]. In our paper, we avoid these difficulties since we deal with the polynomial Hamiltonian function* $\mathcal{H}$ *in (2), a suitable class of Wick operators and passing through other kind of arguments (we start our approach with an evolution equation whose solution is a 'candidate' to be the quantization of the flow).*

3.3. Our class of semipositive operators

We first introduce the general class of the **anti-Wick operators** on $\psi \in \mathcal{F}_B(\mathbb{C}^\ell)$ by

$$\widehat{P}(\psi)(\bar{z}) := \int \sigma_{AW}(\widehat{P})(\bar{\omega}, \omega)\,\psi(\bar{\omega})\,e^{-|\omega|^2 + \omega\cdot\bar{z}}\,d\omega \wedge d\bar{\omega}. \tag{45}$$

The anti-Wick symbols σ_{AW} can be taken (for example) in the class of bounded measurable functions or polynomial functions. For a wider discussion about the anti-Wick operators and the well posedness of the setting (45) we address the reader to Section 2.7 in [17]. In particular, we have

$$\sigma_{AW}((\hat{b}_k)^s(\hat{b}_k^\dagger)^s)(\bar{\omega},\omega) = |\omega_k|^{2s}. \tag{46}$$

Moreover, if $e^\lambda = N+1$ then (see Lemma 4.1)

$$\sigma_{AW}(e^{-\lambda\widehat{N}})(\bar{\omega},\omega) = (N+1)^\ell\, e^{-N|\omega|^2}. \tag{47}$$

Here need to focus our attention to the following class of operators

Definition 3.5. *Let us consider the semipositive operators $\widehat{P}$ on $\mathcal{F}_B(\mathbb{C}^\ell)$ in the class of anti-Wick operators given by $\widehat{P} := f(\widehat{N})$ where $f \in C^\infty(\mathbb{R})$, $f(0) = 0$ and $f(x) \geqslant C_0\, x\ \forall x \geqslant 1$ for some $C_0 > 0$.*

Thanks to Proposition 4.3 we prove that $e^{-\lambda\widehat{P}}$ is an anti-Wick operator, whose symbol is a continuous bounded function.

Remark 3.6. *In this family one has for example $\widehat{P} = \widehat{N}^\alpha$ with $\widehat{N} := \sum_{k=1}^\ell \widehat{b}_k^\dagger \widehat{b}_k$, $\alpha \geqslant 1$ and $C_0 = 1$. As a consequence of this setting, it follows $[\widehat{P},\widehat{N}] = 0$ and for the ground state of the harmonic oscillator $\psi_0(\bar{z}) \equiv 1$ we have $\widehat{P}\psi_0 = 0$.*

3.4. A spectral lowerbound

Any operator $\widehat{P}$ as in Definition 3.5 is selfadjoint on the Hilbert space $\mathcal{F}_B(\mathbb{C}^\ell)$, and $[\widehat{P},\widehat{N}] = 0$ implies the existence of a complete orthonormal set providing both eigenfunctions of $\widehat{P}$ than $\widehat{N}$. In particular, since $\widehat{P} = f(\widehat{N})$ we have the common eigenfunctions by $\psi_\alpha(\bar{z}) := \bar{z}^\alpha/\sqrt{\alpha!}$ in multi-index notation $\alpha \in \mathbb{N}^\ell$ and normalized with respect to scalar product (24); whereas the eigenvalues are given by $E_\alpha = f(|\alpha|)$ where $|\alpha| := \alpha_1 + ... + \alpha_\ell$.
Since $f(x) \geqslant C_0\, x\ \forall x \geqslant 1$, it follows the lowerbound

$$E_\alpha \geqslant C_0\,|\alpha|, \qquad \forall \alpha \in \mathbb{N}^\ell. \tag{48}$$

The inequality (48) will be useful to connect the Trace formula related to (negative) exponentials of $\widehat{N}$ and $\widehat{P}$. We now recall that $[\widehat{H},\widehat{N}] = 0$ and that the eigenspaces of $\widehat{N}$ are degenerate. Thus, to get the spectral lowerbound in this case, we denote by Φ_n with $n \in \mathbb{N}$ (notice we use now the 1-dim index) the common base of eigenfunctions for $\widehat{H}$ and $\widehat{N}$. Denote also by E_n^H the related eigenvalues of $\widehat{H}$. Now compute

$$E_n^H = \langle \Phi_n, \widehat{H}\Phi_n\rangle = \langle \Phi_n, \widehat{H}_2\Phi_n\rangle + \langle \Phi_n, \widehat{H}_4\Phi_n\rangle \tag{49}$$

where $\widehat{H}_2 := \sum_{1\leqslant j,k\leqslant\ell} \tau_{kj}\, \hat{b}_k^\dagger\, \hat{b}_j$ and $\widehat{H}_4 := \frac{1}{2N}\sum_{1\leqslant j\leqslant\ell} U_j\, \hat{b}_j^\dagger \hat{b}_j^\dagger \hat{b}_j \hat{b}_j$ with $U_j > 0$ and the matrix τ positive definite. It is easy to see that the quartic term is semipositive definite, which gives

$$E_n^H \geqslant \langle \Phi_n, \widehat{H}_2\Phi_n\rangle. \tag{50}$$

Also the quartic term $\widehat{H}_2$ is semipositive definite, since it can be rewritten as $\widehat{H}_2 := \sum_{1\leqslant i\leqslant\ell} T_{ii}\, \widehat{B}_i^\dagger \widehat{B}_i$ where the matrix T is diagonal and $T_{ii} > 0$ are the eigenvalues of the matrix τ; whereas $\widehat{B}_i :=$

$\sum_{1\leqslant k\leqslant \ell} R_{ki}\,\hat{b}_i$ are defined with the unitary matrix R such that $R^t\,\tau\,R\;=\;T$. Now define $\tau_{min}\;:=\;\min_{1\leqslant i\leqslant \ell} T_{ii}$ namely as the lowest eigenvalue of τ. Hence the matrix $\widetilde{\tau}\;:=\;\tau-\tau_{min}\,\mathrm{id}$ is semipositive definite and the operator $\widehat{H}_2-\tau_{min}\widehat{N}$ is a semipositive operator. This gives $\forall\; n\in\mathbb{N}$

$$E^H_n=\langle\Phi_n,(\widehat{H}_2-\tau_{min}\widehat{N})\Phi_n\rangle+\tau_{min}\langle\Phi_n,\widehat{N}\Phi_n\rangle\geqslant\tau_{min}\langle\Phi_n,\widehat{N}\Phi_n\rangle. \tag{51}$$

3.5. The Trace Formula

We recall the Trace formula involving Wick and anti-Wick operators defined on the Bargmann space $\mathcal{F}_B(\mathbb{C}^\ell)$ as shown in sect. 5.3 of [11] (see also sect. 7.6.1 in [15]),

$$\mathrm{Tr}\Big(\mathrm{Op_W}(g_1)\,\mathrm{Op_{AW}}(g_2)\Big)=\int g_1(\bar{\omega},\omega)\,g_2(\bar{\omega},\omega)\;d\bar{\omega}\wedge d\omega \tag{52}$$

where $\omega:=x+i\xi$, $d\bar{\omega}\wedge d\omega:=\pi^{-\ell}dxd\xi$. The general assumptions to make on g_1,g_2 are the ones that ensures the well posedness (see Sections 3.1 - 3.3) of the corresponding operators $\mathrm{Op_{AW}}(g_2)$, $\mathrm{Op_W}(g_1)$. In our paper we apply this formula for $\widehat{P}$ as in Definition 3.5,

$$\mathrm{Tr}\Big(\frac{e^{-\lambda\widehat{P}}}{c_\lambda}\mathrm{Op_W}(g)\Big)=\frac{1}{c_\lambda}\int g(\bar{\omega},\omega)\,\sigma_{AW}(e^{-\lambda\widehat{P}})(\bar{\omega},\omega)\;d\bar{\omega}\wedge d\omega. \tag{53}$$

The normalization constant $c_\lambda:=\mathrm{Tr}(e^{-\lambda\widehat{P}})=\sum_\alpha e^{-\lambda E_\alpha}>0$ is convergent since $E_\alpha\geqslant C_0\,\mathcal{E}_\alpha=C_0|\alpha|$ as seen in (48).
In Proposition 4.3 we prove the existence of $\sigma_{AW}(e^{-\lambda\widehat{P}})$ and an L^∞ - estimate. In view of (53) we define the normalized measure

$$dm_\lambda(\bar{\omega},\omega):=\frac{1}{c_\lambda}\,\sigma_{AW}(e^{-\lambda\widehat{P}})(\bar{\omega},\omega)\;d\bar{\omega}\wedge d\omega. \tag{54}$$

We underline that when $\widehat{P}=\widehat{N}:=\sum_{k=1}^{\ell}\hat{b}_k^\dagger\hat{b}_k$ and we set $e^\lambda=N+1$ then we have $\sigma_{AW}(e^{-\lambda\widehat{N}})(\bar{\omega},\omega)=(N+1)^\ell e^{-N|\omega|^2}$ and $c_\lambda=\mathrm{Tr}(e^{-\lambda\widehat{N}})=((N+1)/N)^\ell$. In our general setting, dm_λ is a normalized measure since

$$c_\lambda:=\mathrm{Tr}(e^{-\lambda\widehat{P}})=\int\sigma_{AW}(e^{-\lambda\widehat{P}})(\bar{\omega},\omega)\;d\bar{\omega}\wedge d\omega. \tag{55}$$

Notice also that the spectral lower bound in the previous section gives

$$1\leqslant\mathrm{Tr}(e^{-\lambda\widehat{P}})=\sum_{\alpha\in\mathbb{N}^\ell}e^{-\lambda E_\alpha}\leqslant\sum_{\alpha\in\mathbb{N}^\ell}e^{-\lambda C_0|\alpha|^2}=\mathrm{Tr}(e^{-\lambda C_0\widehat{N}}). \tag{56}$$

If ψ_0 is the ground state of the harmonic oscillator $\psi_0(\bar{z})=1$ (in the Bargmann representation), then our assumption $\widehat{P}\,\psi_0=0$ implies that $E_0=0$ in the above sum.
In particular, if $e^{\lambda C_0}\;=\;N+1$ then we have the same formula as above, namely the trace reads $\mathrm{Tr}(e^{-\lambda C_0\widehat{N}})=((N+1)/N)^\ell$. Thanks to the assumption $\ell/N\leqslant D$ it follows

$$\mathrm{Tr}(e^{-\lambda C_0\widehat{N}})\leqslant e^D. \tag{57}$$

Since the selfadjoint $\widehat{H}$ as in (3) fulfills $\widehat{H}\psi_0 = 0$, pick $(\Psi_n)_{n\in\mathbb{N}}$ as an orthonormal base of eigenfunctions of $\widehat{H}$ with $\Psi_0 = \psi_0$ and (necessarily real) eigenvalues E_n^H. Then, we have the lowerbound

$$\mathrm{Tr}(e^{-\lambda\widehat{H}}) = \sum_{n=0}^{\infty} e^{-\lambda E_n^H} = 1 + \sum_{n=1}^{\infty} e^{-\lambda E_n^H} \geqslant 1. \tag{58}$$

Remark 3.7. *The Hilbert-Schmidt norm* $\|\widehat{B}\|_{\mathrm{HS}} := (\mathrm{Tr}(\widehat{B}^{\dagger}\widehat{B}))^{\frac{1}{2}}$ *for any operator of kind* $\widehat{B} := (c_\lambda^{-1} e^{-\lambda\widehat{P}})^{\frac{1}{2}}\widehat{A}$*, where* $\widehat{A}$ *is a linear operator on* $\mathcal{F}_B(\mathbb{C}^\ell)$*,* $c_\lambda := \mathrm{Tr}(e^{-\lambda\widehat{P}})$*, fulfills*

$$\| (c_\lambda^{-1} e^{-\lambda\widehat{P}})^{\frac{1}{2}}\, \widehat{A}\, \|_{\mathrm{HS}}^2 = \frac{1}{c_\lambda}\mathrm{Tr}((e^{-\frac{\lambda}{2}\widehat{P}}\widehat{A})^{\dagger} e^{-\frac{\lambda}{2}\widehat{P}}\widehat{A}) = \frac{1}{c_\lambda}\mathrm{Tr}(\widehat{A}^{\dagger} e^{-\frac{\lambda}{2}\widehat{P}} e^{-\frac{\lambda}{2}\widehat{P}}\widehat{A}) = \mathrm{Tr}\Big(\frac{e^{-\lambda\widehat{P}}}{c_\lambda}\widehat{A}\widehat{A}^{\dagger}\Big).$$

4. Main Results

To begin, we show the gaussian form of the anti-Wick symbol of the operator $e^{-\lambda\widehat{N}}$, which turns out to be very useful in the proof of Theorem 1.1.

Lemma 4.1. *Let* $\widehat{N} := \sum_{k=1}^{\ell} \hat{b}_k^{\dagger}\hat{b}_k$ *and* $e^\lambda = N + 1$*. Then,*

$$\sigma_{AW}(e^{-\lambda\widehat{N}})(\bar{\omega}, \omega) = (N + 1)^\ell e^{-N|\omega|^2}. \tag{59}$$

Moreover, $c_\lambda := \mathrm{Tr}(e^{-\lambda\widehat{N}}) = ((N + 1)/N)^\ell$ *and thus*

$$dm_\lambda(\bar{\omega}, \omega) := \frac{1}{c_\lambda}\, \sigma_{AW}(e^{-\lambda\widehat{N}})(\bar{\omega}, \omega)\, d\bar{\omega} \wedge d\omega = N^\ell e^{-N|\omega|^2} d\bar{\omega} \wedge d\omega. \tag{60}$$

Proof. The Wick symbol of $e^{-\lambda\widehat{N}}$ reads

$$\sigma_W(e^{-\lambda\widehat{N}})(\bar{\omega}, \omega) = e^{-\mu|\omega|^2}, \quad \mu := 1 - e^{-\lambda}. \tag{61}$$

Thus,

$$\mathrm{Tr}(e^{-\lambda\widehat{N}}) = \int \sigma_W(e^{-\lambda\widehat{N}})(\bar{\omega}, \omega) d\omega \wedge d\bar{\omega} = \mu^{-\ell}. \tag{62}$$

Formula (2.38) in [11] shows a link between Wick and anti-Wick symbols

$$e^{-\mu|\omega|^2} = e^{\Delta_{\bar{\omega}\omega}} \sigma_{\mathrm{AW}}(e^{-\lambda\widehat{N}}) = \int e^{-(z-\omega)(\bar{z}-\bar{\omega})}\, \sigma_{\mathrm{AW}}(z, \bar{z})\, dz \wedge d\bar{z}, \tag{63}$$

where $\Delta_{\bar{\omega}\omega} := \sum_{k=1}^{\ell} \frac{\partial^2}{\partial\bar{\omega}_k \partial\omega_k}$. Remind that Wick and anti-Wick symbols of $e^{-\lambda\widehat{N}}$ are unique. Our target is to prove

$$e^{-\mu|\omega|^2} = e^{\Delta_{\bar{\omega}\omega}} \Big(a_{N,L}\, e^{-N|\omega|^2}\Big). \tag{64}$$

namely

$$e^{-\mu|\omega|^2} = a_{N,\ell} \int e^{-(z-\omega)(\bar z-\bar\omega)}\, e^{-N|z|^2}\, dz \wedge d\bar z \tag{65}$$

$$= a_{N,\ell}\, e^{-\frac{N}{N+1}|\omega|^2} \int e^{-(N+1)|z|^2}\, dz \wedge d\bar z = \frac{a_{N,\ell}}{(N+1)^\ell}\, e^{-\frac{N}{N+1}|\omega|^2}. \tag{66}$$

Since $\mu = (e^\lambda - 1)/e^\lambda = N/(N+1)$, we recover $a_{N,\ell} = (N+1)^\ell$. $\square$

Remark 4.2. *In view of the Lemma 4.1, we now denote by*

$$d\mu_N(\omega,\bar\omega) := N^\ell\, e^{-N|\omega|^2} d\omega \wedge d\bar\omega, \tag{67}$$

which is the measure dm_λ in (9) when $e^{C_0\lambda} = N+1$ and $\widehat{P} = C_0\widehat{N}$. In this way, we underline the dependence from N, and thus the L^2 - estimates are explicitly depending from the number of particles.

We now show a result on the existence and representation of the anti-Wick symbol for the operator $e^{-\lambda\widehat{P}}$ when $\widehat{P}$ belongs to our class of semipositive definite operators introduced in Section 3.3.

Proposition 4.3. *Let $\widehat{P} = f(\widehat{N}) : \mathcal{F}_B(\mathbb{C}^\ell) \to \mathcal{F}_B(\mathbb{C}^\ell)$ be as in Def. 3.5. Then, for $e^{C_0\lambda} = N+1$ the exponential $e^{-\lambda\widehat{P}}$ is an anti-Wick operator whose symbol is a bounded continuous function and fulfills*

$$0 \leqslant \sigma_{AW}(e^{-\lambda\widehat{P}})(\bar\omega,\omega) \leqslant e^{-\lambda f(-\ell + \frac{1}{C_0\lambda}N|\omega|^2)}. \tag{68}$$

The measure $\sigma_{AW}(e^{-\lambda\widehat{P}}) d\omega \wedge d\bar\omega$ is invariant under the dNLS flow.

Proof. Let us define

$$g_\lambda(\theta) := \begin{cases} 0, & \theta = 0, \\ e^{-\lambda f(-\frac{1}{C_0\lambda}\ln(\theta))}, & 0 < \theta \leqslant 1, \\ \theta, & \theta > 1. \end{cases} \tag{69}$$

Easily check that $g_\lambda \in C^0([0,T];\mathbb{R})$, for an arbitrary fixed $T \geqslant 1$, since we recall that $f \in C^\infty(\mathbb{R})$, $f(0) = 0$ and $f(x) \geqslant C_0\, x$, $\forall x \geqslant 1$, for some $C_0 > 0$. Furthermore, $g_\lambda(\theta) \leqslant \theta\, \forall\, 0 < \theta \leqslant 1$ and $\forall \lambda > 0$. Now observe that the family of values $e^{-\lambda C_0|\alpha|} \geqslant e^{-\lambda E_\alpha}$ with $\alpha \in \mathbb{N}^\ell$, $E_\alpha = f(|\alpha|)$ are the (strictly positive) eigenvalues of the operators $e^{-\lambda C_0\widehat{N}}$ and $e^{-\lambda\widehat{P}}$ related to the common base of eigenfunctions ψ_α with $\alpha \in \mathbb{N}^\ell$. Furthermore,

$$g_\lambda(e^{-\lambda C_0|\alpha|}) = e^{-\lambda f(|\alpha|)}. \tag{70}$$

Since the spectrum of these operators is discrete (and in particular all the eigenspaces are finite dimensional), it follows directly that

$$g_\lambda(e^{-\lambda C_0\widehat{N}}) = e^{-\lambda\widehat{P}}. \tag{71}$$

Now apply the Weierstrass Approximation Theorem, so that for any $\varepsilon(n) := 1/n$ with $n \in \mathbb{N}$ there exists a polynomial $p_\varepsilon(\theta) := \sum_{k=0}^K c_k\, \theta^k$ (depending from n and λ) such that

$$g_\lambda(\theta) = \sum_{0\leqslant k\leqslant K} c_k\, \theta^k + r_\lambda(\theta) \tag{72}$$

and $r_\lambda \in C^0([0,T];\mathbb{R}_+)$ with $\|r_\lambda\|_{C^0} < \varepsilon(n)$. Notice that we can choose this approximation in such a way that $r_\lambda \geqslant 0$. As a consequence,

$$g_\lambda(e^{-\lambda C_0 \widehat{N}}) = \lim_{n\to+\infty} \sum_{0\leqslant k\leqslant K} c_k\, (e^{-\lambda C_0 \widehat{N}})^k \tag{73}$$

in operator norm, since $\|r(e^{-\lambda C_0 \widehat{N}})\|_2 \leqslant \|r_\lambda\|_{C^0} \to 0$. Recall that $e^{C_0\lambda} = N+1$ and the anti-Wick symbol reads

$$\sigma_{AW}(e^{-\lambda C_0 \widehat{N}})(\bar{\omega},\omega) = (N+1)^\ell e^{-N|\omega|^2} =: \tau_N(\bar{\omega},\omega). \tag{74}$$

As a consequence, for $N_k := e^{k\lambda C_0} - 1$,

$$\sum_{0\leqslant k\leqslant K} c_k\, e^{-k\lambda C_0 \widehat{N}} = \sum_{0\leqslant k\leqslant K} c_k \operatorname{Op}_{AW}(\tau_{N_k}) = \operatorname{Op}_{AW}\Big(\sum_{0\leqslant k\leqslant K} c_k\, \tau_{N_k} \Big). \tag{75}$$

Since $e^{\lambda C_0} = N+1$ and this gives the link $N_k + 1 = (N+1)^k = \sum_{s=0}^k \binom{k}{s} N^s = 1 + kN + \sum_{s=2}^k \binom{k}{s} N^s$. Thus, $N_k = kN + \sum_{s=2}^k \binom{k}{s} N^s$. We are now in the position to study the finite sum of anti-Wick symbols

$$\sum_{0\leqslant k\leqslant K} c_k\, \tau_{N_k}(\bar{\omega},\omega) \leqslant \sum_{0\leqslant k\leqslant K} c_k\, ((N+1)^L\, e^{-N|\omega|^2})^k. \tag{76}$$

Now fix $T := (N+1)^\ell$ so that the approximation scheme (72) can be applied for $\theta = (N+1)^\ell\, e^{-N|\omega|^2}$, which gives in C^0 - norm

$$\lim_{n\to\infty} \sum_{0\leqslant k\leqslant K} c_k\, (\tau_N)^k = g_\lambda(\tau_N). \tag{77}$$

In particular, notice the inequalities $\sum_k c_k\, \tau_{N_k} \leqslant \sum_k c_k\, (\tau_N)^k \leqslant g_\lambda(\tau_N)$. Thanks to the setting of g_λ, we have

$$g_\lambda\, (\tau_N(\bar{\omega},\omega)) = e^{-\lambda f(-\ell + \frac{1}{C_0\lambda} N|\omega|^2)}. \tag{78}$$

Now define, in view of (77), the finite value $R := \|g_\lambda\,(\tau_N)\|_{C^0} < +\infty$. In view of (76), we have $\sum_{0\leqslant k\leqslant K} c_k\, \tau_{N_k} \in B_R(0) \subset C_b^0(\mathbb{C}^{2\ell};\mathbb{R}_+)$. Now apply the Banach-Alaoglu Theorem, so that for a suitable subsequence $\varepsilon(n(m)) \to 0$ as $m \to \infty$ we have the existence of the limit

$$\lim_{m\to+\infty} \sum_{0\leqslant k\leqslant K} c_k\, \tau_{N_k} =: \sigma_{AW}(e^{-\lambda \widehat{P}}) \geqslant 0 \tag{79}$$

with respect to the weak-$\star$ convergence in $C_b^0(\mathbb{C}^{2\ell};\mathbb{R}_+)$. Notice also that $\mu_m := \sum_k c_k\,\tau_{N_k} d\omega\wedge d\bar\omega$ is a sequence of flow invariant measures, $\mu_m = (\Phi_t)_\star\mu_m$. Thus, the weak-$\star$ convergence of measures $\mu_m \rightharpoonup \star\ \sigma_{AW}(e^{-\lambda\widehat{P}})d\omega\wedge d\bar\omega =: \mu$ implies the flow invariance for μ.
It remains to prove that we have in fact recovered the anti-Wick symbol of $e^{-\lambda\widehat{P}}$, namely that $\forall\ \psi\in\mathcal{F}_B(\mathbb{C}^\ell)$ the following representation holds true

$$e^{-\lambda\widehat{P}}(\psi)(\bar z) = \int \sigma_{AW}(e^{-\lambda\widehat{P}})(\bar\omega,\omega)\,\psi(\bar\omega)\,e^{-|\omega|^2+\omega\cdot\bar z}\,d\omega\wedge d\bar\omega. \tag{80}$$

In order to do this, remind the operator equality (73) which gives in $\mathcal{F}_B(\mathbb{C}^\ell)$

$$e^{-\lambda\widehat{P}}\psi = \lim_{n\to+\infty}\sum_{0\leqslant k\leqslant K} c_k\,(e^{-\lambda C_0\widehat{N}})^k\psi. \tag{81}$$

Now recall (75), so that (81) reads

$$e^{-\lambda\widehat{P}}\psi(\bar z) = \lim_{n\to+\infty}\int\Big(\sum_{0\leqslant k\leqslant K} c_k\,\tau_{N_k}\Big)(\bar\omega,\omega)\,\psi(\bar\omega)\,e^{-|\omega|^2+\omega\cdot\bar z}\,d\omega\wedge d\bar\omega. \tag{82}$$

Here easily observe that $\Phi(\bar\omega,\omega) := \psi(\bar\omega)\,e^{-|\omega|^2+\omega\cdot\bar z}$ is a continous function such that $|\int\Phi\,\varphi\,d\omega\wedge d\bar\omega\,| \leqslant R\|\Phi\|_{L^1}\ \forall\,\varphi\in B_R(0)\subset C_b^0(\mathbb{C}^{2\ell};\mathbb{R}_+)$. Thus, the weak-$\star$ convergence shown in (79) ensures the equality (80).
The anti-Wick symbol is necessarily unique since it is bounded and thanks to Lemma 2.95 in [17]. To conclude, the (77) - (78) imply that $\forall\phi\in C_b^0(\mathbb{C}^{2\ell};\mathbb{R}_+)$

$$\int\sigma_{AW}(e^{-\lambda\widehat{P}})(\bar\omega,\omega)\phi(\bar\omega,\omega)\,d\omega\wedge d\bar\omega \leqslant \int e^{-\lambda f(-\ell+\frac{1}{C_0\lambda}N|\omega|^2)}\phi(\bar\omega,\omega)\,d\omega\wedge d\bar\omega \tag{83}$$

If we assume that $\sigma_{AW}(e^{-\lambda\widehat{P}})(\bar\omega_0,\omega_0) > e^{-\lambda f(-\ell+\frac{1}{C_0\lambda}N|\omega_0|^2)}$ then (since they are continuous functions) there is an open neighborhood Ω_0 of ω_0 on which the same inequality holds true. Now pick $\phi\geqslant 0$ compactly supported inside Ω_0 and get a contradiction. $\square$

Remark 4.4. *In view of the Proposition 4.3, we recall that* $g_\lambda(\theta)\leqslant\theta$ *and thus*

$$\sigma_{AW}(e^{-\lambda\widehat{P}}) \leqslant g_\lambda(\sigma_{AW}(e^{-\lambda C_0\widehat{N}})) \leqslant \sigma_{AW}(e^{-\lambda C_0\widehat{N}}). \tag{84}$$

Thus, for $c_\lambda := \mathrm{Tr}(e^{-\lambda\widehat{P}})$ *and* $b_\lambda := \mathrm{Tr}(e^{-\lambda C_0\widehat{N}})$, *for any continuous* f,

$$\int|f|^2\,\frac{\sigma_{AW}(e^{-\lambda\widehat{P}})}{c_\lambda}d\omega\wedge d\bar\omega \leqslant \int|f|^2\,\frac{\sigma_{AW}(e^{-\lambda C_0\widehat{N}})}{c_\lambda}d\omega\wedge d\bar\omega \tag{85}$$

$$\leqslant\frac{b_\lambda}{c_\lambda}\int|f|^2\,\frac{\sigma_{AW}(e^{-\lambda C_0\widehat{N}})}{b_\lambda}d\omega\wedge d\bar\omega \leqslant e^D\int|f|^2\,\frac{\sigma_{AW}(e^{-\lambda C_0\widehat{N}})}{b_\lambda}d\omega\wedge d\bar\omega$$

In short notation, for $e^{C_0\lambda} = N+1$ we can write

$$\int |f|^2 \, dm_\lambda \leqslant e^D \int |f|^2 \, d\mu_N. \tag{86}$$

This upper bound allows to make computations with the gaussian measure $d\mu_N$ and then get immediately the estimate with respect to the whole class of measures dm_λ.

We now show the link between the weighted Hilbert-Schmidt operator norm and L^2 - estimates with respect to the measures dm_λ. Before to do this, we introduce the next

Definition 4.5. *Given the projector $\widehat{\Pi}_{(\leqslant N)}$ as in Def. 3.3, consider the class of semiclassical Wick operators (35) exhibiting the growth condition*

$$\|\widehat{\Pi}_{(\leqslant N)} \operatorname{Op}^h_{\mathrm{W}}(\sigma)\|_{\mathrm{HS}} \leqslant C_1 \, h^{-Q}, \quad h := 1/N, \tag{87}$$

for $C_1 > 0$ and $Q \in \mathbb{N}$. We will show that (87) is fulfilled by $\operatorname{Op}^h_{\mathrm{W}}(\rho_k - u_k)$ used in Thm. 1.1, which are unbounded operators. Any polynomial function of creation and annihilation operators belongs to this class, as well as all bounded (uniformly on h) operators.

Proposition 4.6. *Let dm_λ be as in (9) with $e^{C_0\lambda} = N+1$. Let $\operatorname{Op}^h_{\mathrm{W}}(\sigma)$ be a semiclassical Wick operator as in (87) such that $\sigma \in L^2(m_\lambda)$, and $\widehat{\Pi}_{(\leqslant N)}$ given in Def. 3.3. Then,*

$$\| \, (c_\lambda^{-1} e^{-\lambda \widehat{P}})^{\frac{1}{2}} \, \widehat{\Pi}_{(\leqslant N)} \operatorname{Op}^h_{\mathrm{W}}(\sigma) \, \|^2_{\mathrm{HS}} \leqslant 2 \int |\sigma|^2 \, dm_\lambda + 2e^D e^{-\frac{1}{4h}} \, C_1^2 \, h^{-2Q} \tag{88}$$

provided $2^6 \ell / N \leqslant 1$.

Proof. In view of Remark 3.7,

$$\begin{aligned}
&\|(c_\lambda^{-1} e^{-\lambda \widehat{P}})^{\frac{1}{2}} \, \widehat{\Pi}_{(\leqslant N)} \operatorname{Op}^h_{\mathrm{W}}(\sigma)\|^2_{\mathrm{HS}} = \qquad (89)\\
&= \frac{1}{c_\lambda} \operatorname{Tr} \Big(e^{-\lambda \widehat{P}} \Big(\widehat{\Pi}_{(\leqslant N)} \operatorname{Op}^h_{\mathrm{W}}(\sigma) \Big) \Big(\widehat{\Pi}_{(\leqslant N)} \operatorname{Op}^h_{\mathrm{W}}(\sigma) \Big)^\dagger \Big) \\
&= \frac{1}{c_\lambda} \operatorname{Tr} \Big(e^{-\lambda \widehat{P}} \, \widehat{\Pi}_{(\leqslant N)} \operatorname{Op}^h_{\mathrm{W}}(\sigma) \operatorname{Op}^h_{\mathrm{W}}(\sigma)^\dagger \, \widehat{\Pi}_{(\leqslant N)} \Big).
\end{aligned}$$

Let $g := \sigma_{AW}(e^{-\lambda \widehat{P}})/c_\lambda \geqslant 0$ and define $\mathcal{X}(\bar{\omega}, \omega) := \chi(|\omega|^2)$ with $\chi \in C^\infty[0, +2]$, $0 \leqslant \chi(\theta) \leqslant 1$, $\operatorname{supp}(\chi) \subset [0,1]$. We decompose the above term as the sum of two parts

$$= \operatorname{Tr} \Big(\operatorname{Op}_{AW}(g\mathcal{X}) \, \widehat{\Pi}_{(\leqslant N)} \operatorname{Op}^h_{\mathrm{W}}(\sigma) \operatorname{Op}^h_{\mathrm{W}}(\sigma)^\dagger \, \widehat{\Pi}_{(\leqslant N)} \Big) \tag{90}$$

$$+ \operatorname{Tr} \Big(\operatorname{Op}_{AW}(g(1-\mathcal{X})) \, \widehat{\Pi}_{(\leqslant N)} \operatorname{Op}^h_{\mathrm{W}}(\sigma) \operatorname{Op}^h_{\mathrm{W}}(\sigma)^\dagger \, \widehat{\Pi}_{(\leqslant N)} \Big). \tag{91}$$

Notice that both $\operatorname{Op}_{AW}(g\mathcal{X})$ and $\operatorname{Op}_{AW}(g(1-\mathcal{X}))$ are semipositive (the anti-Wick symbol is nonnegative), whence selfadjoint. Use the property $[\operatorname{Op}_{AW}(g\mathcal{X}), \widehat{\Pi}_{(\leqslant N)}] = 0$ (see Lemma 4.15) to rewrite (90) as

$$\mathrm{Tr}\Big(\;\widehat{\Pi}_{(\leqslant N)}\,\mathrm{Op}_{AW}(g\mathcal{X})\,\mathrm{Op}^h_{\mathrm{W}}(\sigma)\,\mathrm{Op}^h_{\mathrm{W}}(\sigma)^\dagger\,\widehat{\Pi}_{(\leqslant N)}\Big). \tag{92}$$

In this form, it is easy to see that this is an increasing function of positive terms as N increases. Hence (90) has the upper bound

$$\leqslant \mathrm{Tr}\left(\mathrm{Op}_{AW}(g\mathcal{X})\,\mathrm{Op}^h_{\mathrm{W}}(\sigma)\,\mathrm{Op}^h_{\mathrm{W}}(\sigma)^\dagger\,\right) = \int \sigma \star_{\mathrm{Wick}} \bar{\sigma}\; g\mathcal{X}\, d\bar{\omega}\wedge d\omega. \tag{93}$$

Notice that the integral is computed in the region $|\omega|^2 \leqslant 1$. The Wick symbol of the composition reads

$$\sigma \star_{\mathrm{Wick}} \bar{\sigma} \simeq |\sigma|^2 + \sum_{r=0}^{\infty} \frac{h^r}{r!} \sum_{i_1,i_2,\dots i_r=1}^{\ell} \frac{\partial^r \sigma}{\partial\omega_{i_1}\partial\omega_{i_2}\dots\partial\omega_{i_r}}\,\frac{\partial^r\bar{\sigma}}{\partial\bar{\omega}_{i_1}\partial\bar{\omega}_{i_2}\dots\partial\bar{\omega}_{i_r}} \tag{94}$$

and we look for the estimate

$$\Big|\sum_{i_1,i_2,\dots i_r=1}^{\ell} \frac{\partial^r \sigma}{\partial\omega_{i_1}\partial\omega_{i_2}\dots\partial\omega_{i_r}}\,\frac{\partial^r\bar{\sigma}}{\partial\bar{\omega}_{i_1}\partial\bar{\omega}_{i_2}\dots\partial\bar{\omega}_{i_r}}\Big| \leqslant \sum_{i_1,i_2,\dots i_r=1}^{\ell} \Big|\frac{\partial^r \sigma}{\partial\omega_{i_1}\partial\omega_{i_2}\dots\partial\omega_{i_r}}\Big|\,\Big|\frac{\partial^r\bar{\sigma}}{\partial\bar{\omega}_{i_1}\partial\bar{\omega}_{i_2}\dots\partial\bar{\omega}_{i_r}}\Big|.$$

Moreover, $(\frac{\partial^r\sigma}{\partial\omega^r})^\star = \frac{\partial^r\bar{\sigma}}{\partial\bar{\omega}^r}$ and hence $|\frac{\partial^r\sigma}{\partial\omega^r}| = |\frac{\partial^r\bar{\sigma}}{\partial\bar{\omega}^r}|$. Now apply the inequality (see Lemma 4.14)

$$\Big|\frac{\partial^r\bar{\sigma}}{\partial\bar{\omega}^r}\Big| \leqslant \frac{2}{\sqrt{\pi}}\, 4^r\,\sqrt{r!}\,\|\mathrm{Op}^h_{\mathrm{W}}(\bar{\sigma})\phi_\omega\|\,. \tag{95}$$

Thus,

$$\Big|\sum_{i_1,i_2,\dots i_r=1}^{\ell} \frac{\partial^r\sigma}{\partial\omega^r}\frac{\partial^r\bar{\sigma}}{\partial\bar{\omega}^r}\Big| \leqslant \sum_{i_1,i_2,\dots i_r=1}^{\ell}\Big|\frac{\partial^r\bar{\sigma}}{\partial\bar{\omega}^r}\Big|^2 \leqslant \ell^r\,\|\mathrm{Op}^h_{\mathrm{W}}(\bar{\sigma})\phi_\omega\|^2\,\frac{4^{2r+1}}{\pi}\,r!\,. \tag{96}$$

We get

$$\sigma \star_{\mathrm{Wick}} \bar{\sigma} \leqslant |\sigma|^2 + \|\mathrm{Op}^h_{\mathrm{W}}(\bar{\sigma})\phi_\omega\|^2 \sum_{r=1}^{\infty}\frac{h^r}{r!}\,\frac{4^{2r+1}}{\pi}\,r!\,\ell^r \tag{97}$$

$$= |\sigma|^2 + \|\mathrm{Op}^h_{\mathrm{W}}(\bar{\sigma})\phi_\omega\|^2\,\frac{4}{\pi}\,\sum_{r=1}^{\infty}\Big(h4^2\ell\Big)^r.$$

Set $q := h4^2\ell$ and require $q<1$, so that $\bar{\sigma}\star_{\mathrm{Wick}}\sigma \leqslant |\sigma|^2 + \frac{4}{\pi}\frac{q}{(1-q)}\|\mathrm{Op}^h_{\mathrm{W}}(\bar{\sigma})\phi_\omega\|^2$. In particular, when $q \leqslant 1/4$ it follows

$$\bar{\sigma}\star_{\mathrm{Wick}}\sigma \leqslant |\sigma|^2 + \frac{4}{\pi}\|\mathrm{Op}^h_{\mathrm{W}}(\bar{\sigma})\phi_\omega\|^2\frac{4}{3}q\,. \tag{98}$$

This condition can be rewritten as $2^6\ell/N \leqslant 1$. Notice also that $\|\mathrm{Op}^h_{\mathrm{W}}(\bar\sigma)\phi_\omega\|^2 = \langle\phi_\omega, \mathrm{Op}^h_{\mathrm{W}}(\sigma)\mathrm{Op}^h_{\mathrm{W}}(\sigma)^\dagger\phi_\omega\rangle$. We can now estimate the trace

$$\begin{aligned}&\mathrm{Tr}\left(\mathrm{Op}_{AW}(g\mathcal{X})\,\mathrm{Op}^h_{\mathrm{W}}(\sigma)\,\mathrm{Op}^h_{\mathrm{W}}(\sigma)^\dagger\right)\\ &\leqslant \int|\sigma|^2\,dm_\lambda + \frac{4}{\pi}\frac{4q}{3}\mathrm{Tr}\left(\mathrm{Op}_{AW}(g\mathcal{X})\,\mathrm{Op}^h_{\mathrm{W}}(\sigma)\,\mathrm{Op}^h_{\mathrm{W}}(\sigma)^\dagger\right)\end{aligned} \tag{99}$$

and since $4q \leqslant 1$, we have that

$$\Big(1-\frac{4}{3\pi}\Big)\mathrm{Tr}\left(\mathrm{Op}_{AW}(g\mathcal{X})\,\mathrm{Op}^h_{\mathrm{W}}(\sigma)\,\mathrm{Op}^h_{\mathrm{W}}(\sigma)^\dagger\right) \leqslant \int|\sigma|^2\,dm_\lambda\,, \tag{100}$$

namely

$$\mathrm{Tr}\left(\mathrm{Op}_{AW}(g\mathcal{X})\,\mathrm{Op}^h_{\mathrm{W}}(\sigma)\,\mathrm{Op}^h_{\mathrm{W}}(\sigma)^\dagger\right) \leqslant \frac{3\pi}{3\pi-4}\int|\sigma|^2\,dm_\lambda < 2\int|\sigma|^2\,dm_\lambda\,.$$

To conclude, we need to handle the remainder in (91). Simply observe that we have the composition of a semipositive operator with a positive one. Thus the trace can be estimated by the product of two traces

$$\begin{aligned}&\leqslant \mathrm{Tr}\left(\mathrm{Op}_{AW}(g(1-\mathcal{X}))\right)\mathrm{Tr}\left(\,\widehat{\Pi}_{(\leqslant N)}\,\mathrm{Op}^h_{\mathrm{W}}(\sigma)\,\mathrm{Op}^h_{\mathrm{W}}(\sigma)^\dagger\,\widehat{\Pi}_{(\leqslant N)}\right)\\ &= \mathrm{Tr}\left(\mathrm{Op}_{AW}(g(1-\mathcal{X}))\right)\|\,\widehat{\Pi}_{(\leqslant N)}\,\mathrm{Op}^h_{\mathrm{W}}(\sigma)\|^2_{\mathrm{HS}}\,.\end{aligned} \tag{101}$$

The first trace can be estimated by recalling that $g := \sigma_{AW}(e^{-\lambda\widehat{P}})/c_\lambda \geqslant 0$ and, in view of Remark 4.4,

$$\mathrm{Tr}\left(\mathrm{Op}_{AW}(g(1-\mathcal{X}))\right) = \int g(1-\mathcal{X})\,d\bar\omega\wedge d\omega \leqslant e^D\int_{|\omega|>1}\frac{\sigma_{AW}(e^{-\lambda C_0\widehat{N}})}{b_\lambda}d\omega\wedge d\bar\omega$$

which reads (recall that $\omega := x+i\xi$ and $d\omega\wedge d\bar\omega := \pi^{-\ell}dxd\xi$) in spherical coordinates

$$= e^D\int_{|\omega|>1}N^\ell e^{-N|\omega|^2}d\omega\wedge d\bar\omega = e^D\Big(\frac{N}{\pi}\Big)^\ell\int_1^\infty e^{-Nr^2}r^{2\ell-1}dr\,S^{2\ell-1}(1)$$

Recall that $S^{2\ell-1}(1) = 2\pi^\ell/\Gamma(\ell) = 2\pi^\ell/(\ell-1)!$, and observe (by $\sqrt{N}r = x$)

$$\begin{aligned}\int_1^\infty e^{-Nr^2}r^{2\ell-1}dr &= \frac{1}{N^\ell}\int_{\sqrt{N}}^\infty e^{-x^2}x^{2\ell-1}dx \leqslant \frac{1}{N^\ell}\,e^{-\frac12 N}\int_{\sqrt{N}}^\infty e^{-\frac12 x^2}x^{2\ell-1}dx\\ &\leqslant \frac{1}{N^\ell}\,e^{-\frac12 N}\int_0^\infty e^{-\frac12 x^2}x^{2\ell-1}dx = \frac{1}{N^\ell}\,e^{-\frac12 N}2^{\ell-1}(\ell-1)!\end{aligned} \tag{102}$$

so that

$$\mathrm{Tr}\left(\mathrm{Op}_{AW}(g(1-\mathcal{X}))\right) \leqslant 2e^De^{-\frac12 N}2^{\ell-1} = 2e^De^{-\frac14 N}e^{-\frac14 N}2^{\ell-1}\,.$$

Now we require $-N/4 + (\ell - 1)\ln(2) \leqslant 0$, namely $(\ell - 1)/N \leqslant 1/(4\ln(2))$. This condition is directly fulfilled thanks to the above setting $\ell/N \leqslant 2^{-6}$, since we have that $\ell - 1 < \ell$ and moreover $2^{-6} < 1/(4\ln(2))$. We are in the position to conclude that

$$\mathrm{Tr}\left(\mathrm{Op}_{AW}(g(1-\mathcal{X}))\right) \leqslant 2e^{D}e^{-\frac{1}{4}N}. \tag{103}$$

□

Next result shows and upper bound for the Hilbert-Schmidt norm involving the operator $e^{-\lambda\widehat{H}}$. This result, applied together with Prop. 4.6, makes the proof of Theorem 1.2 as a direct consequence of Theorem 1.1. We then apply the result for $\mathrm{Op}_{\mathrm{W}}(g) := \widehat{\Pi}_{(\leqslant N)}\mathrm{Op}^h_{\mathrm{W}}(\sigma)$ since $\widehat{\Pi}_{(\leqslant N)}$ is bounded and whence the composition with $\mathrm{Op}^h_{\mathrm{W}}(\sigma)$ is also a Wick operator.

Proposition 4.7. *Let $\widehat{H}$ be as in (3) and $\mathrm{Op}_{\mathrm{W}}(g)$ a Wick operator as in (28). Then, $\forall\ \lambda > 0$*

$$\left\|\mathrm{Op}_{\mathrm{W}}(g)\ (c_\lambda^{-1}e^{-\lambda\widehat{H}})^{\frac{1}{2}}\right\|_{\mathrm{HS}} \leqslant e^{D/2}\left\|\mathrm{Op}_{\mathrm{W}}(g)\ (b_\lambda^{-1}e^{-\lambda\tau_{\min}\widehat{N}})^{\frac{1}{2}}\right\|_{\mathrm{HS}} \tag{104}$$

with $c_\lambda := \mathrm{Tr}(e^{-\lambda\widehat{H}})$ and $b_\lambda := \mathrm{Tr}(e^{-\lambda\tau_{\min}\widehat{N}})$.

Proof. We begin by

$$\|\mathrm{Op}_{\mathrm{W}}(g)\ (c_\lambda^{-1}e^{-\lambda\widehat{H}})^{\frac{1}{2}}\|^2_{\mathrm{HS}} = \mathrm{Tr}\Big(\mathrm{Op}_{\mathrm{W}}(g)^\dagger\,\mathrm{Op}_{\mathrm{W}}(g)\ c_\lambda^{-1}e^{-\lambda\widehat{H}}\Big). \tag{105}$$

We recall the spectral bound (see Section 3.4)

$$E_n^H \geqslant \tau_{min}\langle\Phi_n, \widehat{N}\Phi_n\rangle, \quad n \geqslant 0, \tag{106}$$

for a (common) orthonormal set $(\Phi_n)_{n\in\mathbb{N}}$. The above trace reads

$$\sum_{n=0}^{\infty}\langle\Phi_n, \mathrm{Op}_{\mathrm{W}}(g)^\dagger\,\mathrm{Op}_{\mathrm{W}}(g)\ \frac{e^{-\lambda\widehat{H}}}{c_\lambda}\Phi_n\rangle = \sum_n\langle\Phi_n, \mathrm{Op}_{\mathrm{W}}(g)^\dagger\,\mathrm{Op}_{\mathrm{W}}(g)\ \frac{e^{-\lambda E_n^H}}{c_\lambda}\Phi_n\rangle,$$

for which we have the upper bound

$$\leqslant \sum_{n=0}^{\infty}\langle\Phi_n, \mathrm{Op}_{\mathrm{W}}(g)^\dagger\,\mathrm{Op}_{\mathrm{W}}(g)\ \frac{e^{-\lambda\tau_{min}\langle\Phi_n,\widehat{N}\Phi_n\rangle}}{c_\lambda}\Phi_n\rangle. \tag{107}$$

Since $\frac{b_\lambda}{c_\lambda} \leqslant e^D$ (see Section 3.4) it follows the upper bound

$$\leqslant e^D\sum_{n=0}^{\infty}\langle\Phi_n, \mathrm{Op}_{\mathrm{W}}(g)^\dagger\,\mathrm{Op}_{\mathrm{W}}(g)\ \frac{e^{-\lambda\tau_{min}\langle\Phi_n,\widehat{N}\Phi_n\rangle}}{b_\lambda}\Phi_n\rangle$$

$$= e^D\,\mathrm{Tr}\Big(\mathrm{Op}_{\mathrm{W}}(g)^\dagger\,\mathrm{Op}_{\mathrm{W}}(g)\ \frac{e^{-\lambda\tau_{min}\widehat{N}}}{b_\lambda}\Big) = e^D\left\|\mathrm{Op}_{\mathrm{W}}(g)\ (b_\lambda^{-1}e^{-\lambda\tau_{\min}\widehat{N}})^{\frac{1}{2}}\right\|^2_{\mathrm{HS}}.$$

□

Now we show a formula for the time evolved operator $\widehat{G}(s) := U^\dagger(s)\widehat{G}U(s)$ where $U(s) = e^{-i\widehat{H}s}$ and $\widehat{G} = \mathrm{Op_W}(g)$ are Wick operators (see Sect. 3). This result will be applied within the proof of Theorem 1.1 for operators of type $\widehat{G} = (\hat{a}_k^\dagger \hat{a}_k + \frac{1}{N})^2$ with rescaled creation and annihilation operators. This tool allows to overcome here the well known the problem of Ehrenfest time, as well as to avoid the application of Grönwall Lemma (and thus exponential in time upper bounds) used in many papers on mean field estimates for NLS.

Proposition 4.8. *Let $\widehat{G} = \mathrm{Op_W}(g)$ be a Wick operator on $\mathcal{F}_B(\mathbb{C}^\ell)$ such that $g \in L^1(\mu_N)$. Let $\widehat{G}(s) := U^\dagger(s)\widehat{G}U(s)$ where $U(s) = e^{-i\widehat{H}s}$ with $\widehat{H}$ as in (3). Define $g(s,\bar{\omega},\omega) := \langle \phi_\omega, \widehat{G}(s)\phi_\omega\rangle$. Then, $\forall s \geqslant 0$*

$$\int g(s,\bar{\omega},\omega)\, d\mu_N(\bar{\omega},\omega) = \int g(\bar{\omega},\omega)\, d\mu_N(\bar{\omega},\omega). \tag{108}$$

Proof. We apply Trace formula

$$\int g(\bar{\omega},\omega)\, d\mu_N(\bar{\omega},\omega) = \mathrm{Tr}\Big(\frac{e^{-\beta\widehat{N}}}{\gamma_\beta}\mathrm{Op_W}(g)\Big), \tag{109}$$

and recall that the trace is invariant by unitary conjugations of operators, so that

$$\mathrm{Tr}\Big(\frac{e^{-\beta\widehat{N}}}{\gamma_\beta}\mathrm{Op_W}(g)\Big) = \mathrm{Tr}\Big(U^\dagger(s)\frac{e^{-\beta\widehat{N}}}{\gamma_\beta}\mathrm{Op_W}(g)U(s)\Big). \tag{110}$$

Now we recall that $[\widehat{N},\widehat{H}] = 0$ and whence $[\widehat{N}, U^\star(s)] = 0$, which gives

$$\mathrm{Tr}\Big(U^\dagger(s)\frac{e^{-\beta\widehat{N}}}{\gamma_\beta}\mathrm{Op_W}(g)U(s)\Big) = \mathrm{Tr}\Big(\frac{e^{-\beta\widehat{N}}}{\gamma_\beta}U^\dagger(s)\mathrm{Op_W}(g)U(s)\Big) \tag{111}$$

and applying again Trace formula for $\widehat{G}(s) := U^\dagger(s)\mathrm{Op_W}(g)U(s)$ and, in view of Remark 4.9, we conclude

$$\mathrm{Tr}\Big(\frac{e^{-\beta\widehat{N}}}{\gamma_\beta}\widehat{G}(s)\Big) = \int g(s,\bar{\omega},\omega)\, d\mu_N(\bar{\omega},\omega). \tag{112}$$

□

Remark 4.9. *The operator $\widehat{H}$ is selfadjoint on the Hilbert space $\mathcal{F}_B(\mathbb{C}^\ell)$ and thus, by the Stone Theorem, the $U(s) := e^{-i\widehat{H}s}$ is a one parameter group of unitary operators. Hence, $U(s)$ is bounded on $\mathcal{F}_B(\mathbb{C}^\ell)$ and this implies it is a Wick operator itself (see Sect. 3). It follows that $\widehat{G}(s) := U^\dagger(s)\mathrm{Op_W}(g)U(s)$ equals a composition of Wick operators. Since the set of Wick operators is closed under composition, we deduce that $\widehat{G}(s)$ is still a Wick operator, and whence we denote its symbol by $g(s,\bar{\omega},\omega)$.*

Remark 4.10. *Prop. 4.8 works also with $g(s,\sqrt{N}\bar\omega,\sqrt{N}\omega)$ and $g(\sqrt{N}\bar\omega,\sqrt{N}\omega)$. Indeed,*

$$\int g(s,\sqrt{N}\bar\omega,\sqrt{N}\omega)\,d\mu_N(\bar\omega,\omega) = \int g(s,\bar v,v)\,d\mu_1(\bar v,v)$$
$$= \int g(\bar v,v)\,d\mu_1(\bar v,v) = \int g(\sqrt{N}\bar\omega,\sqrt{N}\omega)\,d\mu_N(\bar\omega,\omega). \tag{113}$$

We now get an estimate for $|\rho_k(t,\bar\omega,\omega)-u_k(t,\omega)|$ for any fixed $\omega\in\mathbb{C}^\ell$.

Proposition 4.11. *Let $\Delta := \{(\bar\omega,\omega)\mid \omega\in\mathbb{C}^\ell\}\subset\mathbb{C}^{2\ell}$, $\Psi_t := (\bar\Phi_t,\Phi_t) : \Delta\subset\mathbb{C}^{2\ell}\to\mathbb{C}^{2\ell}$ the flow of $\dot\gamma = i(\partial_\omega\mathcal{H}(\gamma), -\partial_{\bar\omega}\mathcal{H}(\gamma))$ with $\mathcal{H}$ as in (3). Let $u=(u_1,\dots u_\ell)(t,\omega)$ be the solution of (1), and*

$$\rho_k(t,\bar\omega,\omega) := \langle \phi_{\sqrt{N}\omega}, \hat a_k(t)\phi_{\sqrt{N}\omega}\rangle \tag{114}$$

$$n_k(t,\bar\omega,\omega) := \langle \phi_{\sqrt{N}\omega}, \hat a_k^\dagger(t)\hat a_k(t)\phi_{\sqrt{N}\omega}\rangle. \tag{115}$$

$$\mathcal{P}(\bar v,v) := \sum_{j=1}^{\ell}\Big[3N\,|v_j|^4 + 4\sqrt{N}|v_j|^3 + \sqrt{2}|v_j|^2\Big]. \tag{116}$$

Then, for $0\leqslant U_j\leqslant U$,

$$|\rho_k(t,\bar\omega,\omega)-u_k(t,\omega)| \leqslant U\int_0^t \mathcal{P}(\bar v,v)\Big(n_k(s,\bar v,v)+\frac{1}{N}\Big)^{\frac12}\Big|_{(\bar v,v)=\Psi_{t-s}(\bar\omega,\omega)}\,ds. \tag{117}$$

Proof. The semigroup identity

$$e^{-iN(\mathcal{L}_1+\mathcal{L}_2)t} = e^{-iN\mathcal{L}_1 t} + \int_0^t e^{-iN\mathcal{L}_1(t-s)}(-iN)\mathcal{L}_2\, e^{-iN(\mathcal{L}_1+\mathcal{L}_2)s}\,ds \tag{118}$$

applied to our case gives

$$\rho_k(t,\bar\omega,\omega)-u_k(t,\omega) = \int_0^t (-iN)\mathcal{L}_2\rho_k(s,\bar v,v)\Big|_{(\bar v,v)=\Psi_{t-s}(\bar\omega,\omega)}\,ds, \tag{119}$$

where the operators $\mathcal{L}_1,\mathcal{L}_2$ read

$$\mathcal{L}_1\rho = \frac{1}{N}\sum_{j=1}^{\ell}\Big(\frac{\partial\rho}{\partial v_j}\frac{\partial\mathcal{H}}{\partial\bar v_j} - \frac{\partial\mathcal{H}}{\partial v_j}\frac{\partial\rho}{\partial\bar v_j}\Big) \tag{120}$$

$$\mathcal{L}_2\rho = \frac{1}{2}\frac{1}{N^2}\sum_{j=1}^{\ell}\Big(\frac{\partial^2\rho}{\partial v_j^2}\frac{\partial^2\mathcal{H}}{\partial\bar v_j^2} - \frac{\partial^2\mathcal{H}}{\partial v_j^2}\frac{\partial^2\rho}{\partial\bar v_j^2}\Big) \tag{121}$$

and thus

$$(-iN)\mathcal{L}_2\rho = (-i)\frac{1}{2N}\sum_{j=1}^{\ell}U_j\Big(v_j^2\frac{\partial^2\rho}{\partial v_j^2} - \bar v_j^2\frac{\partial^2\rho}{\partial\bar v_j^2}\Big). \tag{122}$$

We now recall the setting

$$\rho_k(s,\bar{v},v) := \langle \phi_{\sqrt{N}v}, \hat{a}_k(s)\phi_{\sqrt{N}v}\rangle \tag{123}$$

where $\phi_{\sqrt{N}v}(\bar{z}) = e^{\sqrt{N}v\bar{z}-\frac{1}{2}N|v|^2}$ and notice that

$$\frac{\partial \phi_{\sqrt{N}v}}{\partial v_j} = \Big(\sqrt{N}\bar{z}_j - \frac{N}{2}\bar{v}_j\Big)\phi_{\sqrt{N}v}(\bar{z}), \tag{124}$$

$$\frac{\partial \phi^\star_{\sqrt{N}v}}{\partial v_j} = -\frac{N}{2}\bar{v}_j\phi^\star_{\sqrt{N}v}(\bar{z}). \tag{125}$$

Thanks to Lemma 4.13,

$$\frac{\partial \rho_k}{\partial v_j} = \langle\Big(\frac{\partial \phi^\star_{\sqrt{N}v}}{\partial v_j}\Big)^\star, \hat{a}_k(s)\phi_{\sqrt{N}v}\rangle + \langle \phi_{\sqrt{N}v}, \hat{a}_k(s)\Big(\frac{\partial \phi_{\sqrt{N}v}}{\partial v_j}\Big)\rangle \tag{126}$$

we have

$$\begin{aligned}\frac{\partial \rho_k}{\partial v_j} &= \langle -\frac{N}{2}v_j\phi_{\sqrt{N}v}, \hat{a}_k(s)\phi_{\sqrt{N}v}\rangle + \langle \phi_{\sqrt{N}v}, \hat{a}_k(s)\Big(\sqrt{N}\bar{z}_j - \frac{N}{2}\bar{v}_j\Big)\phi_{\sqrt{N}v}\rangle \\ &= -N\bar{v}_j\langle \phi_{\sqrt{N}v}, \hat{a}_k(s)\phi_{\sqrt{N}v}\rangle + \sqrt{N}\langle \phi_{\sqrt{N}v}, \hat{a}_k(s)\bar{z}_j\phi_{\sqrt{N}v}\rangle. \end{aligned}\tag{127}$$

Notice that $\bar{z}_j\phi_{\sqrt{N}v}(\bar{z}) = \sqrt{N}\hat{a}^\dagger_j(0)\phi_{\sqrt{N}v}(\bar{z})$ and thus

$$\frac{\partial \rho_k}{\partial v_j} = -N\bar{v}_j\langle \phi_{\sqrt{N}v}, \hat{a}_k(s)\phi_{\sqrt{N}v}\rangle + N\langle \phi_{\sqrt{N}v}, \hat{a}_k(s)\hat{a}^\dagger_j(0)\phi_{\sqrt{N}v}\rangle. \tag{128}$$

Applying twice this formula we get

$$\begin{aligned}\frac{\partial^2 \rho_k}{\partial v_j^2} &= N^2\bar{v}_j^2\langle \phi_{\sqrt{N}v}, \hat{a}_k(s)\phi_{\sqrt{N}v}\rangle - N^2\bar{v}_j\langle \phi_{\sqrt{N}v}, \hat{a}_k(s)\hat{a}^\dagger_j(0)\phi_{\sqrt{N}v}\rangle \\ &\quad - N^2\bar{v}_j\langle \phi_{\sqrt{N}v}, \hat{a}_k(s)\hat{a}^\dagger_j(0)\phi_{\sqrt{N}v}\rangle + N^2\langle \phi_{\sqrt{N}v}, \hat{a}_k(s)\hat{a}^\dagger_j(0)\hat{a}^\dagger_j(0)\phi_{\sqrt{N}v}\rangle \\ &= N^2\bar{v}_j^2\langle \phi_{\sqrt{N}v}, \hat{a}_k(s)\phi_{\sqrt{N}v}\rangle - 2N^2\bar{v}_j\langle \phi_{\sqrt{N}v}, \hat{a}_k(s)\hat{a}^\dagger_j(0)\phi_{\sqrt{N}v}\rangle \\ &\quad + N^2\langle \phi_{\sqrt{N}v}, \hat{a}_k(s)\hat{a}^\dagger_j(0)\hat{a}^\dagger_j(0)\phi_{\sqrt{N}v}\rangle. \end{aligned}\tag{129}$$

Applying the same computations for the derivatives on $\bar{v}_j$ we get

$$\begin{aligned}\frac{\partial^2 \rho_k}{\partial \bar{v}_j^2} &= N^2 v_j^2\langle \phi_{\sqrt{N}v}, \hat{a}_k(s)\phi_{\sqrt{N}v}\rangle - 2N^2 v_j\langle \hat{a}^\dagger_j(0)\phi_{\sqrt{N}v}, \hat{a}_k(s)\phi_{\sqrt{N}v}\rangle \\ &\quad + N^2\langle \hat{a}^\dagger_j(0)\hat{a}^\dagger_j(0)\phi_{\sqrt{N}v}, \hat{a}_k(s)\phi_{\sqrt{N}v}\rangle. \end{aligned}\tag{130}$$

The sum in (122) can now be rewritten as

$$
\begin{aligned}
&\sum_{j=1}^{\ell} U_j \Big(v_j^2 \frac{\partial^2 \rho}{\partial v_j^2} - \bar{v}_j^2 \frac{\partial^2 \rho}{\partial \bar{v}_j^2} \Big) \\
&= \sum_{j=1}^{\ell} U_j \Big(N^2 |v_j|^4 \langle \phi_{\sqrt{N}v}, \hat{a}_k(s) \phi_{\sqrt{N}v} \rangle - 2N^2 v_j |v_j|^2 \langle \phi_{\sqrt{N}v}, \hat{a}_k(s) \hat{a}_j^\dagger(0) \phi_{\sqrt{N}v} \rangle \\
&\quad + N^2 v_j^2 \langle \phi_{\sqrt{N}v}, \hat{a}_k(s) \hat{a}_j^\dagger(0) \hat{a}_j^\dagger(0) \phi_{\sqrt{N}v} \rangle \Big) \qquad (131) \\
&- \sum_{j=1}^{\ell} U_j \Big(N^2 |v_j|^4 \langle \phi_{\sqrt{N}v}, \hat{a}_k(s) \phi_{\sqrt{N}v} \rangle - 2N^2 \bar{v}_j |v_j|^2 \langle \hat{a}_j^\dagger(0) \phi_{\sqrt{N}v}, \hat{a}_k(s) \phi_{\sqrt{N}v} \rangle \\
&\quad + N^2 \bar{v}_j^2 \langle \hat{a}_j^\dagger(0) \hat{a}_j^\dagger(0) \phi_{\sqrt{N}v}, \hat{a}_k(s) \phi_{\sqrt{N}v} \rangle \Big), \qquad (132)
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
&= \sum_{j=1}^{\ell} U_j \Big(-2N^2 v_j |v_j|^2 \langle \phi_{\sqrt{N}v}, \hat{a}_k(s) \hat{a}_j^\dagger(0) \phi_{\sqrt{N}v} \rangle + N^2 v_j^2 \langle \phi_{\sqrt{N}v}, \hat{a}_k(s) \hat{a}_j^\dagger(0) \hat{a}_j^\dagger(0) \phi_{\sqrt{N}v} \rangle \Big) \\
&+ \sum_{j=1}^{\ell} U_j \Big(2N^2 \bar{v}_j |v_j|^2 \langle \hat{a}_j^\dagger(0) \phi_{\sqrt{N}v}, \hat{a}_k(s) \phi_{\sqrt{N}v} \rangle - N^2 \bar{v}_j^2 \langle \hat{a}_j^\dagger(0) \hat{a}_j^\dagger(0) \phi_{\sqrt{N}v}, \hat{a}_k(s) \phi_{\sqrt{N}v} \rangle \Big).
\end{aligned}
$$

The sum exhibits the following upper bound (recall that $0 \leqslant U_j \leqslant U$)

$$
\begin{aligned}
&\leqslant U \sum_{j=1}^{\ell} 2N^2 |v_j|^3 \|\hat{a}_k^\dagger(s) \phi_{\sqrt{N}v}\| \|\hat{a}_j^\dagger(0) \phi_{\sqrt{N}v}\| + N^2 |v_j|^2 \|\hat{a}_k^\dagger(s) \phi_{\sqrt{N}v}\| \|\hat{a}_j^\dagger(0) \hat{a}_j^\dagger(0) \phi_{\sqrt{N}v}\| \\
&+ U \sum_{j=1}^{\ell} 2N^2 |v_j|^3 \|\hat{a}_j^\dagger(0) \phi_{\sqrt{N}v}\| \|\hat{a}_k(s) \phi_{\sqrt{N}v}\| + N^2 |v_j|^2 \|\hat{a}_j^\dagger(0) \hat{a}_j^\dagger(0) \phi_{\sqrt{N}v}\| \|\hat{a}_k(s) \phi_{\sqrt{N}v}\|,
\end{aligned}
$$

namely

$$
\begin{aligned}
&\leqslant U \sum_{j=1}^{\ell} \Big(2N^2 |v_j|^3 \; \|\hat{a}_j^\dagger(0) \phi_{\sqrt{N}v}\| + N^2 |v_j|^2 \; \|\hat{a}_j^\dagger(0) \hat{a}_j^\dagger(0) \phi_{\sqrt{N}v}\| \Big) \|\hat{a}_k^\dagger(s) \phi_{\sqrt{N}v}\| \\
&+ U \sum_{j=1}^{\ell} \Big(2N^2 |v_j|^3 \|\hat{a}_j^\dagger(0) \phi_{\sqrt{N}v}\| + N^2 |v_j|^2 \|\hat{a}_j^\dagger(0) \hat{a}_j^\dagger(0) \phi_{\sqrt{N}v}\| \Big) \|\hat{a}_k(s) \phi_{\sqrt{N}v}\|.
\end{aligned}
$$

We need to get an estimate for $\|\hat{a}_j^\dagger(0)\phi_{\sqrt{N}v}\|$ and $\|\hat{a}_j^\dagger(0)\hat{a}_j^\dagger(0)\phi_{\sqrt{N}v}\|$.

$$\|\hat{a}_j^\dagger(0)\phi_{\sqrt{N}v}\|^2 = \langle\phi_{\sqrt{N}v}, \hat{a}_k(0)\hat{a}_j^\dagger(0)\phi_{\sqrt{N}v}\rangle = \langle\phi_{\sqrt{N}v}, \Big(\hat{a}_j^\dagger(0)\hat{a}_j(0) + \frac{1}{N}\Big)\phi_{\sqrt{N}v}\rangle$$
$$= \langle\phi_{\sqrt{N}v}, \hat{a}_j^\dagger(0)\hat{a}_j(0)\phi_{\sqrt{N}v}\rangle + \frac{1}{N}. \tag{133}$$

Since $\hat{a}_j(0)\phi_{\sqrt{N}v} = v_j\phi_{\sqrt{N}v}$ and recalling that $\phi_{\sqrt{N}v}$ are normalized, it follows

$$\|\hat{a}_j^\dagger(0)\phi_{\sqrt{N}v}\|^2 = \langle\phi_{\sqrt{N}v}, \hat{a}_j(0)\hat{a}_j^\dagger(0)\phi_{\sqrt{N}v}\rangle = \langle\phi_{\sqrt{N}v}, \Big(\hat{a}_j^\dagger(0)\hat{a}_j(0) + \frac{1}{N}\Big)\phi_{\sqrt{N}v}\rangle$$
$$= \langle\phi_{\sqrt{N}v}, \hat{a}_j^\dagger(0)\hat{a}_j(0)\phi_{\sqrt{N}v}\rangle + \frac{1}{N} = |v_j|^2 + \frac{1}{N}, \tag{134}$$

and thus

$$\|\hat{a}_j^\dagger(0)\phi_{\sqrt{N}v}\| = \Big(|v_j|^2 + \frac{1}{N}\Big)^{\frac{1}{2}} \leqslant |v_j| + \frac{1}{\sqrt{N}}. \tag{135}$$

We now look at

$$\|\hat{a}_j^\dagger(0)\hat{a}_j^\dagger(0)\phi_{\sqrt{N}v}\|^2 = \langle\phi_{\sqrt{N}v}, \hat{a}_j(0)\hat{a}_j(0)\hat{a}_j^\dagger(0)\hat{a}_j^\dagger(0)\phi_{\sqrt{N}v}\rangle \tag{136}$$
$$= \langle\phi_{\sqrt{N}v}, \hat{a}_j(0)\Big(\hat{a}_j^\dagger(0)\hat{a}_j(0) + \frac{1}{N}\Big)\hat{a}_j^\dagger(0)\phi_{\sqrt{N}v}\rangle$$
$$= \langle\phi_{\sqrt{N}v}, \hat{a}_j(0)\hat{a}_j^\dagger(0)\hat{a}_j(0)\hat{a}_j^\dagger(0)\phi_{\sqrt{N}v}\rangle + \frac{1}{N}\|\hat{a}_j^\dagger(0)\phi_{\sqrt{N}v}\|^2$$
$$= \langle\phi_{\sqrt{N}v}, \Big(\hat{a}_j^\dagger(0)\hat{a}_j(0) + \frac{1}{N}\Big)\Big(\hat{a}_j^\dagger(0)\hat{a}_j(0) + \frac{1}{N}\Big)\phi_{\sqrt{N}v}\rangle + \frac{1}{N}\|\hat{a}_j^\dagger(0)\phi_{\sqrt{N}v}\|^2$$
$$= \|\hat{a}_j^\dagger(0)\hat{a}_j(0)\phi_{\sqrt{N}v}\|^2 + \frac{2}{N}\langle\phi_{\sqrt{N}v}, \hat{a}_j^\dagger(0)\hat{a}_j(0)\phi_{\sqrt{N}v}\rangle + \frac{1}{N^2} + \frac{1}{N}\|\hat{a}_j^\dagger(0)\phi_{\sqrt{N}v}\|^2.$$

By using again $\hat{a}_j(0)\phi_{\sqrt{N}v} = v_j\phi_{\sqrt{N}v}$ and (134), we have

$$\|\hat{a}_j^\dagger(0)\hat{a}_j^\dagger(0)\phi_{\sqrt{N}v}\|^2 \tag{137}$$
$$= |v_j|^2\Big(|v_j|^2 + \frac{1}{N}\Big) + \frac{2}{N}|v_j|^2 + \frac{1}{N^2} + \frac{1}{N}\Big(|v_j|^2 + \frac{1}{N}\Big)$$
$$= |v_j|^4 + \frac{4}{N}|v_j|^2 + \frac{2}{N^2},$$

and hence

$$\|\hat{a}_j^\dagger(0)\hat{a}_j^\dagger(0)\phi_{\sqrt{N}v}\| \leqslant |v_j|^2 + \frac{2}{\sqrt{N}}|v_j| + \frac{\sqrt{2}}{N}. \tag{138}$$

Inserting (135) - (138) into (133) we get

$$
\begin{aligned}
&\Big| \sum_{j=1}^{\ell} U_j \Big(v_j^2 \frac{\partial^2 \rho}{\partial v_j^2} - \bar{v}_j^2 \frac{\partial^2 \rho}{\partial \bar{v}_j^2} \Big) \Big| \qquad (139) \\
&\leqslant U \sum_{j=1}^{\ell} \Big(2N^2 |v_j|^3 \Big(|v_j| + \frac{1}{\sqrt{N}} \Big) + N^2 |v_j|^2 \, \Big(|v_j|^2 + \frac{2}{\sqrt{N}} |v_j| + \frac{\sqrt{2}}{N} \, \Big) \Big) \| \hat{a}_k^\dagger(s) \phi_{\sqrt{N}v} \| \\
&+ U \sum_{j=1}^{\ell} \Big(2N^2 |v_j|^3 \Big(|v_j| + \frac{1}{\sqrt{N}} \Big) + N^2 |v_j|^2 \Big(|v_j|^2 + \frac{2}{\sqrt{N}} |v_j| + \frac{\sqrt{2}}{N} \, \Big) \Big) \| \hat{a}_k(s) \phi_{\sqrt{N}v} \|.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\Big| \sum_{j=1}^{\ell} U_j \Big(v_j^2 \frac{\partial^2 \rho}{\partial v_j^2} - \bar{v}_j^2 \frac{\partial^2 \rho}{\partial \bar{v}_j^2} \Big) \Big| \qquad (140) \\
&\leqslant U N^2 \sum_{j=1}^{\ell} \Big(3|v_j|^4 + \frac{4}{\sqrt{N}} |v_j|^3 + \frac{\sqrt{2}}{N} |v_j|^2 \Big) (\| \hat{a}_k^\dagger(s) \phi_{\sqrt{N}v} \| + \| \hat{a}_k(s) \phi_{\sqrt{N}v} \|).
\end{aligned}
$$

We observe that

$$
\begin{aligned}
\| \hat{a}_k(s) \phi_{\sqrt{N}v} \| &= (\langle \phi_{\sqrt{N}v}, \hat{a}_k^\dagger(s) \hat{a}_k(s) \phi_{\sqrt{N}v} \rangle)^{\frac{1}{2}} \qquad (141) \\
&\leqslant \Big(\langle \phi_{\sqrt{N}v}, \hat{a}_k^\dagger(s) \hat{a}_k(s) \phi_{\sqrt{N}v} \rangle + \frac{1}{N} \Big)^{\frac{1}{2}},
\end{aligned}
$$

and

$$
\begin{aligned}
\| \hat{a}_k^\dagger(s) \phi_{\sqrt{N}v} \| &= (\langle \phi_{\sqrt{N}v}, \hat{a}_k(s) \hat{a}_k^\dagger(s) \phi_{\sqrt{N}v} \rangle)^{\frac{1}{2}} \qquad (142) \\
&= \Big(\langle \phi_{\sqrt{N}v}, \hat{a}_k^\dagger(s) \hat{a}_k(s) \phi_{\sqrt{N}v} \rangle + \frac{1}{N} \Big)^{\frac{1}{2}}.
\end{aligned}
$$

As a consequence

$$
\begin{aligned}
&\Big| \sum_{j=1}^{\ell} U_j \Big(v_j^2 \frac{\partial^2 \rho}{\partial v_j^2} - \bar{v}_j^2 \frac{\partial^2 \rho}{\partial \bar{v}_j^2} \Big) \Big| \qquad (143) \\
&\leqslant 2N^2 U \sum_{j=1}^{\ell} \Big(3|v_j|^4 + \frac{4}{\sqrt{N}} |v_j|^3 + \frac{\sqrt{2}}{N} |v_j|^2 \Big) \Big(n_k(s, \bar{v}, v) + \frac{1}{N} \Big)^{\frac{1}{2}}.
\end{aligned}
$$

Now can define $\mathcal{P}(\bar{v}, v) := \sum_{1 \leqslant j \leqslant \ell} (3N\, |v_j|^4 + 4\sqrt{N} |v_j|^3 + \sqrt{2} |v_j|^2)$. To conclude, thanks to (119) - (122) we directly obtain the statement (117). $\square$

In view of previous propositions, we can now provide the proof of the first main result stated in the Introduction.

Proof of Theorem 1.1
First Step. Here we prove the Mean Field estimate with respect to the gaussian measure $d\mu_N$ defined in (67). Recalling (117), we define the positive function

$$\psi(s) := U\mathcal{P}(\bar{v}, v)\Big(n_k(s, \bar{v}, v) + \frac{1}{N}\Big)^{\frac{1}{2}}\Big|_{(\bar{v},v)=\Psi_{t-s}(\bar{\omega},\omega)}, \tag{144}$$

and for the sake of simplicity we avoid to write the dependence from $(\bar{\omega}, \omega)$.
Thus, $|\rho_k - u_k| \leqslant \int_0^t \psi(s)\, ds$ and

$$\|\rho_k - u_k\|_{L^2(\mu_N)} \leqslant \Big\| \int_0^t \psi(s)\, ds \Big\|_{L^2(\mu_N)}. \tag{145}$$

More in details,

$$\Big\| \int_0^t \psi(s)\, ds \Big\|^2_{L^2(\mu_N)} = \int \Big(\int_0^t \psi(s)\, ds\Big)^2 d\mu_N. \tag{146}$$

The Hölder inequality $\|fg\|_{L^1} \leqslant \|f\|_{L^p}\|g\|_{L^q}$ with $1/q + 1/p = 1$, gives

$$\Big(\int_0^t \psi(s)\, ds\Big)^2 \leqslant t \int_0^t \psi^2(s)\, ds \tag{147}$$

and thus

$$\int \Big(\int_0^t \psi(s)\, ds\Big)^2 d\mu_N \leqslant t \int_0^t \Big(\int \psi^2(s)\, d\mu_N\Big)\, ds. \tag{148}$$

We now focus our attention to

$$\int \psi^2(s)\, d\mu_N = \int \Big(U\mathcal{P}(\bar{v}, v)\Big(n_k(s, \bar{v}, v) + \frac{1}{N}\Big)^{\frac{1}{2}}\Big|_{\Psi_{t-s}(\bar{\omega},\omega)}\Big)^2 d\mu_N. \tag{149}$$

The invariance of μ_N under the flow Ψ_{t-s} implies

$$\int \psi^2(s)\, d\mu_N = \int \Big(U\mathcal{P}(\bar{\omega}, \omega)\Big)^2 \Big(n_k(s, \bar{\omega}, \omega) + \frac{1}{N}\Big)\, d\mu_N \tag{150}$$

The Cauchy-Schwartz inequality gives

$$\leqslant \Big(\int U^4 \mathcal{P}(\bar{\omega}, \omega)^4\, d\mu_N\Big)^{\frac{1}{2}} \Big(\int \langle \phi_{\sqrt{N}\omega}, \Big(\hat{a}_k^\dagger(s)\hat{a}_k(s) + \frac{1}{N}\Big)\phi_{\sqrt{N}\omega}\rangle^2\, d\mu_N\Big)^{\frac{1}{2}} \tag{151}$$

$$\leqslant \Big(\int U^4 \mathcal{P}(\bar{\omega}, \omega)^4\, d\mu_N\Big)^{\frac{1}{2}} \Big(\int \langle \phi_{\sqrt{N}\omega}, \Big(\hat{a}_k^\dagger(s)\hat{a}_k(s) + \frac{1}{N}\Big)^2 \phi_{\sqrt{N}\omega}\rangle\, d\mu_N\Big)^{\frac{1}{2}} \tag{152}$$

The last inequality is ensured by the fact that $\langle \phi_{\sqrt{N}\omega}, \widehat{A}\phi_{\sqrt{N}\omega}\rangle^2 \leqslant \langle \phi_{\sqrt{N}\omega}, \widehat{A}^2\phi_{\sqrt{N}\omega}\rangle$ for any selfadjoint $\widehat{A}$. Now apply Proposition 4.8 and Remark 4.10 and get

$$= \Big(\int U^4 \mathcal{P}(\bar{\omega}, \omega)^4 \, d\mu_N \Big)^{\frac{1}{2}} \Big(\int \langle \phi_{\sqrt{N}\omega}, \Big(\hat{a}_k^\dagger(0)\hat{a}_k(0) + \frac{1}{N} \Big)^2 \phi_{\sqrt{N}\omega} \rangle \, d\mu_N \Big)^{\frac{1}{2}}. \tag{153}$$

Integrating the first term (see Lemma 4.12) we have

$$\Big(\int \mathcal{P}(\bar{\omega}, \omega)^4 \, d\mu_N \Big)^{\frac{1}{2}} \leqslant (5^3)^2 \Big(\frac{\ell}{N} \Big)^2. \tag{154}$$

Whereas a direct computation shows that

$$\Big(\int \langle \phi_{\sqrt{N}\omega}, \Big(\hat{a}_k^\dagger(0)\hat{a}_k(0) + \frac{1}{N} \Big)^2 \phi_{\sqrt{N}\omega} \rangle \, d\mu_N \Big)^{\frac{1}{2}} = \Big(\frac{1}{N^3} + \frac{4}{N^2} \Big)^{\frac{1}{2}} < \frac{3}{N}. \tag{155}$$

Thus,

$$\int \psi^2(s) \, d\mu_N \leqslant U^2 \; (5^3)^2 \; \Big(\frac{\ell}{N} \Big)^2 \, \frac{3}{N}. \tag{156}$$

We are now in the position to conclude

$$\|\rho_k - u_k\|^2_{L^2(\mu_N)} \leqslant t \int_0^t U^2 \; (5^3)^2 \Big(\frac{\ell}{N} \Big)^2 \frac{2}{N} ds = t^2 \; U^2 \; (5^3)^2 \; \Big(\frac{\ell}{N} \Big)^2 \frac{3}{N} \tag{157}$$

namely

$$\|\rho_k - u_k\|_{L^2(\mu_N)} \leqslant t \; U \; \sqrt{2} \, 5^3 \; \frac{\ell}{N} \; \frac{1}{\sqrt{N}} = \sqrt{3} \; 5^3 \; \frac{\ell}{N} \; \frac{Ut}{\sqrt{N}}. \tag{158}$$

We now prove that $\rho_k, \; u_k \in L^2(\mu_N)$. Recall that $u_k(t, \omega) = \Phi_t^{(k)}(\bar{\omega}, \omega)$ and that μ_N is invariant under $\Psi_t = (\bar{\Phi}_t, \Phi_t)$. Hence,

$$\int |u_k(t, \omega)|^2 d\mu_N(\bar{\omega}, \omega) = \int |\Phi_t^{(k)}(\bar{\omega}, \omega)|^2 d\mu_N(\bar{\omega}, \omega) \tag{159}$$

$$= \int |\omega^k|^2 \, (\Psi_t)_\star d\mu_N(\bar{\omega}, \omega) = \int |\omega^k|^2 d\mu_N(\bar{\omega}, \omega) < +\infty \tag{160}$$

where the last inequality is guaranteed since μ_N is a gaussian type measure and $|\omega^k|^2$ is a polynomial term. Inequality $\|\rho_k - u_k\|_{L^2(\mu_N)} < +\infty$ implies also $\|\rho_k\|_{L^2(\mu_N)} < +\infty$.

Second Step. We now prove the mean field estimate with respect to the (more general) measure dm_λ. Recalling Remark 4.4, we have

$$\|\rho_k - u_k\|_{L^2(m_\lambda)} \leqslant e^{D/2} \; \|\rho_k - u_k\|_{L^2(\mu_N)} \tag{161}$$

The previous part of the proof directly gives

$$\|\rho_k - u_k\|_{L^2(m_\lambda)} \leqslant \sqrt{3}\, 5^3\, e^{D/2} \frac{\ell}{N}\, \frac{Ut}{\sqrt{N}}\,. \tag{162}$$

To conclude, we realize that both $\rho_k, u_k \in L^2(m_\lambda)$ thanks to (161) and previous step on the estimate with the gaussian measure μ_N. □

We give two technical Lemma that we have used above.

Lemma 4.12. *Let $\mathcal{P}(\bar\omega, \omega)$ be as in (116), then*

$$\Big(\int \mathcal{P}(\bar\omega,\omega)^4\, d\mu_N\Big)^{\frac{1}{2}} \leqslant (5^3)^2 \,\Big(\frac{\ell}{N}\Big)^2. \tag{163}$$

Proof. We first notice that for any fixed $\omega \in \mathbb{C}^\ell$, $\mathcal{P}(\bar\omega, \omega)$ is a sum of real non-negative numbers

$$\mathcal{P}(\bar\omega,\omega) = \sum_{j=1}^{\ell} f(\bar\omega_j, \omega_j) \tag{164}$$

where $f(\bar\omega_j, \omega_j) := 3N|\omega_j|^4 + 4\sqrt{N}|\omega_j|^3 + \sqrt{2}|\omega_j|^2$, so by using Hölder inequality

$$\mathcal{P}(\bar\omega,\omega)^4 \leqslant \ell^3 \sum_{j=1}^{\ell} f(\bar\omega_j, \omega_j)^4. \tag{165}$$

Since for any $v \in \mathbb{C}$, $f(\bar v/\sqrt{N}, v/\sqrt{N}) = N^{-1} g(\bar v, v)$ for $g(\bar v, v) = 3|v|^4 + 4|v|^3 + \sqrt{2}|v|^2$, integrating with respect to gaussian measure and performing the change of variables $\omega'_j = \sqrt{N}\omega_j$ we have

$$\begin{aligned}
&\int_{\mathbb{C}^\ell} \mathcal{P}(\bar\omega,\omega)^4 d\mu_N \leqslant \ell^3 \sum_j c_{N,\ell} \int_{\mathbb{C}^\ell} f(\bar\omega_j,\omega_j)^4 e^{-N|\omega|^2} d\bar\omega \wedge d\omega = \\
&= \frac{\ell^3}{N^4} \sum_{j=1}^{\ell} \frac{c_{N,\ell}}{N^\ell} \int_{\mathbb{C}^\ell} \Big(3|\omega'_j|^4 + 4|\omega'_j|^3 + \sqrt{2}|\omega'_j|^2\Big)^4 e^{-|\omega'|^2} d\bar\omega' \wedge d\omega'.
\end{aligned} \tag{166}$$

For each $j = 1, \ldots, \ell$ we factorize the integrals not containing ω_j, so introducing the variable $v \in \mathbb{C}$ and its corresponding measure $d\bar v \wedge dv$ we have

$$\begin{aligned}
&= \frac{\ell^3}{N^4} \sum_{j=1}^{\ell} \Big(\int_{\mathbb{C}} e^{-|v|^2} d\bar v \wedge dv\Big)^{\ell-1} \int_{\mathbb{C}} \Big(3|v|^4 + 4|v|^3 + \sqrt{2}|v|^2\Big)^4 e^{-|v|^2} d\bar v \wedge dv \\
&= \frac{\ell^4}{N^4} \int_{\mathbb{C}} \Big(3|v|^4 + 4|v|^3 + \sqrt{2}|v|^2\Big)^4 e^{-|v|^2} d\bar v \wedge dv < \frac{\ell^4}{N^4}(5^3)^4.
\end{aligned} \tag{167}$$

□

Lemma 4.13. *Let* $\mathrm{Op}_W(g)$ *be a Wick operator,* $\rho(\bar{v}, v) := \langle \phi_{\sqrt{N}v}, \mathrm{Op}_W(g)\phi_{\sqrt{N}v} \rangle$. *Then,*

$$\frac{\partial \rho}{\partial v_j} = \langle \Big(\frac{\partial \phi^{\star}_{\sqrt{N}v}}{\partial v_j} \Big)^{\star}, \mathrm{Op}_W(g)\phi_{\sqrt{N}v} \rangle + \langle \phi_{\sqrt{N}v}, \mathrm{Op}_W(g) \Big(\frac{\partial \phi_{\sqrt{N}v}}{\partial v_j} \Big) \rangle. \tag{168}$$

Proof. We begin by

$$\begin{aligned} \frac{\partial \rho}{\partial v_j} &= \frac{\partial}{\partial v_j} \int \phi^{\star}_{\sqrt{N}v}(\bar{z}) \mathrm{Op}_W(g)\phi_{\sqrt{N}v}(\bar{z})\, e^{-|z|^2} dz \wedge d\bar{z} \\ &= \int \frac{\partial}{\partial v_j} \phi^{\star}_{\sqrt{N}v}(\bar{z}) \cdot \mathrm{Op}_W(g)\phi_{\sqrt{N}v}(\bar{z})\, e^{-|z|^2} dz \wedge d\bar{z} \\ &\quad + \int \phi^{\star}_{\sqrt{N}v}(\bar{z}) \cdot \frac{\partial}{\partial v_j} \mathrm{Op}_W(g)\phi_{\sqrt{N}v}(\bar{z})\, e^{-|z|^2} dz \wedge d\bar{z}. \end{aligned} \tag{169}$$

In particular, the second term can be rewritten

$$\begin{aligned} &\int \phi^{\star}_{\sqrt{N}v}(\bar{z}) \cdot \frac{\partial}{\partial v_j} \mathrm{Op}_W(g)\phi_{\sqrt{N}v}(\bar{z})\, e^{-|z|^2} dz \wedge d\bar{z} \\ &= \Big(\int \phi^{\star}_{\sqrt{N}v}(\bar{z}) \cdot \frac{\partial}{\partial w_j} \mathrm{Op}_W(g)\phi_{\sqrt{N}w}(\bar{z})\, e^{-|z|^2} dz \wedge d\bar{z} \Big)\Big|_{w=v} \\ &= \Big(\frac{\partial}{\partial w_j} \int \phi^{\star}_{\sqrt{N}v}(\bar{z}) \cdot \mathrm{Op}_W(g)\phi_{\sqrt{N}w}(\bar{z})\, e^{-|z|^2} dz \wedge d\bar{z} \Big)\Big|_{w=v} \\ &= \Big(\frac{\partial}{\partial w_j} \int \Big(\mathrm{Op}_W(g)^{\dagger} \phi_{\sqrt{N}v} \Big)^{\star}(\bar{z}) \cdot \phi_{\sqrt{N}w}(\bar{z})\, e^{-|z|^2} dz \wedge d\bar{z} \Big)\Big|_{w=v} \\ &= \Big(\int \Big(\mathrm{Op}_W(g)^{\dagger} \phi_{\sqrt{N}v} \Big)^{\star}(\bar{z}) \cdot \frac{\partial}{\partial w_j} \phi_{\sqrt{N}w}(\bar{z})\, e^{-|z|^2} dz \wedge d\bar{z} \Big)\Big|_{w=v} \\ &= \int \Big(\mathrm{Op}_W(g)^{\dagger} \phi_{\sqrt{N}v} \Big)^{\star}(\bar{z}) \cdot \frac{\partial}{\partial v_j} \phi_{\sqrt{N}v}(\bar{z})\, e^{-|z|^2} dz \wedge d\bar{z}, \end{aligned} \tag{170}$$

and this last form equals $\langle \phi_{\sqrt{N}v}, \mathrm{Op}_W(g) \Big(\frac{\partial \phi_{\sqrt{N}v}}{\partial v_j} \Big) \rangle$. $\square$

Proof of Theorem 1.2

In view of Proposition 4.6 and Remark 4.17 we have

$$\| \, (c_{\lambda}^{-1} e^{-\lambda \widehat{P}})^{\frac{1}{2}} \, \widehat{\Pi}_N \, \mathrm{Op}^h_{\mathrm{W}}(\rho_k - u_k) \, \|^2_{\mathrm{HS}} \leqslant 2 \| \rho_k - u_k \|^2_{L^2(m_{\lambda})} + 4^2 (1+D) e^{(1+2D)} e^{-\frac{1}{8h}}. \tag{171}$$

Moreover, thanks to Theorem 1.1 and the setting $e^{\lambda C_0} = N + 1$ we obtain

$$\| \, (c_{\lambda}^{-1} e^{-\lambda \widehat{P}})^{\frac{1}{2}} \, \widehat{\Pi}_N \, \mathrm{Op}^h_{\mathrm{W}}(\rho_k - u_k) \, \|_{\mathrm{HS}} \leqslant \sqrt{6}\, 5^3 \, e^{\frac{D}{2}} \, \frac{\ell}{N} \, \frac{Ut}{\sqrt{N}} + 4 (1+D)^{\frac{1}{2}} e^{\frac{1}{2} + D} e^{-\frac{1}{16h}}. \tag{172}$$

To prove the second estimate of the theorem, recall again Proposition 4.7 that ensures that for $e^{\tau_{min}\lambda} = N+1$

$$\| (b_\lambda^{-1} e^{-\lambda \widehat{H}})^{\frac{1}{2}} \, \widehat{\Pi}_N \, \mathrm{Op}_{\mathrm{W}}^h(\rho_k - u_k) \|_{\mathrm{HS}} \leqslant e^{\frac{D}{2}} \, \| (c_\lambda^{-1} e^{-\lambda \widehat{P}})^{\frac{1}{2}} \, \widehat{\Pi}_N \, \mathrm{Op}_{\mathrm{W}}^h(\rho_k - u_k) \|_{\mathrm{HS}}. \tag{173}$$

Now apply (171) with $\widehat{P} = \tau_{min} \widehat{N}$ and get

$$\| (b_\lambda^{-1} e^{-\lambda \widehat{H}})^{\frac{1}{2}} \, \widehat{\Pi}_N \, \mathrm{Op}_{\mathrm{W}}^h(\rho_k - u_k) \|_{\mathrm{HS}} \leqslant \sqrt{6}\, 5^3 \, e^D \, \frac{\ell}{N} \, \frac{Ut}{\sqrt{N}} + 4(1+D)^{\frac{1}{2}} e^{\frac{1+3D}{2}} e^{-\frac{1}{16h}}. \tag{174}$$

If we denote $\lambda \equiv \frac{1}{T}$ then $N = e^{\tau_{min}/T} - 1$ or equivalently $T = \tau_{min}/\ln(N+1)$. The estimate becomes

$$\leqslant \sqrt{6}\, 5^3 \, e^D \, \frac{\ell}{N} \, \frac{Ut}{\sqrt{e^{\tau_{min}/T} - 1}} + \mathcal{O}(h^\infty). \tag{175}$$

For the interval $0 < T \leqslant \tau_{min}/\ln(N+1)$ the term $1/\sqrt{e^{\tau_{min}/T} - 1}$ is a decreasing function as $T \to 0^+$. Thus, for this interval of temperatures we can write

$$\leqslant \sqrt{6}\, 5^3 \, e^D \, \frac{\ell}{N} \, \frac{Ut}{\sqrt{N}} + \mathcal{O}(h^\infty). \tag{176}$$

□

Lemma 4.14. *Let us denote $\widehat{\mathcal{Y}} = \mathrm{Op}_{\mathrm{W}}(\eta)$. Then, for $|\omega| \leqslant 1$ and $\forall r \geqslant 0$*

$$\Big| \frac{\partial^r \eta}{\partial \bar{\omega}_{k_1} \partial \bar{\omega}_{k_2} \ldots \partial \bar{\omega}_{k_r}} (\omega, \bar{\omega}) \Big| \leqslant \frac{2}{\sqrt{\pi}} 4^r \sqrt{r!} \, \| \widehat{\mathcal{Y}} \phi_\omega \| \, . \tag{177}$$

Proof. We notice that $\eta = \langle \phi_\omega, \widehat{\mathcal{Y}} \phi_\omega \rangle$ and $\|\phi_\omega\| = 1$ which gives $|\eta| \leqslant \|\widehat{\mathcal{Y}} \phi_\omega\|$. Moreover,

$$\frac{\partial \eta}{\partial \bar{\omega}_{k_1}} = \langle \phi_\omega, [\widehat{b}_k, \widehat{\mathcal{Y}}] \phi_\omega \rangle, \quad \frac{\partial^2 \eta}{\partial \bar{\omega}_{k_1} \partial \bar{\omega}_{k_2}} = \langle \phi_\omega, [\widehat{b}_{k_1}, [\widehat{b}_{k_2}, \widehat{\mathcal{Y}}]] \phi_\omega \rangle \tag{178}$$

and iterating r-times the commutator $[\widehat{b}_k, \cdot\,]$

$$\frac{\partial^r \eta}{\partial \bar{\omega}_{k_1} \partial \bar{\omega}_{k_2} \ldots \partial \bar{\omega}_{k_r}} = \langle \phi_\omega, [\, \widehat{b}_{k_1}, [\, \widehat{b}_{k_2}, \ldots [\, \widehat{b}_{k_r}, \, \widehat{\mathcal{Y}} \,] \, \phi_\omega \rangle. \tag{179}$$

Recalling that $\widehat{b}_k \phi_\omega = \omega_k \phi_\omega$, a simple estimate based on binomial formula and thanks to the assumption $|\omega| \leqslant 1$ gives

$$\Big| \frac{\partial^r \eta}{\partial \bar{\omega}_k^r} \Big| \leqslant \| \widehat{\mathcal{Y}} \phi_\omega \| \sum_{s=0}^{r} \binom{r}{s} \| \widehat{b}_{k_1}^\dagger \widehat{b}_{k_2}^\dagger \ldots \widehat{b}_{k_s}^\dagger \phi_\omega \| \, . \tag{180}$$

In particular,

$$\| \widehat{b}_{k_1}^\dagger \widehat{b}_{k_2}^\dagger \ldots \widehat{b}_{k_s}^\dagger \phi_\omega \|^2 = \langle \phi_\omega, \widehat{b}_{k_s} \ldots \widehat{b}_{k_2} \widehat{b}_{k_1} \widehat{b}_{k_1}^\dagger \widehat{b}_{k_2}^\dagger \ldots \widehat{b}_{k_s}^\dagger \phi_\omega \rangle = \sigma_W(\widehat{b}_{k_s} \ldots \widehat{b}_{k_2} \widehat{b}_{k_1} \widehat{b}_{k_1}^\dagger \widehat{b}_{k_2}^\dagger \ldots \widehat{b}_{k_s}^\dagger)(\bar{\omega}, \omega)$$

and (see formula (2.38) in [11]) by setting $\widehat{B} := \widehat{b}_{k_s}...\widehat{b}_{k_2}\widehat{b}_{k_1}\widehat{b}^{\dagger}_{k_1}\widehat{b}^{\dagger}_{k_2}...\widehat{b}^{\dagger}_{k_s}$ we have

$$\sigma_W(\widehat{B})(\bar{\omega},\omega) = \int e^{-|\omega-z|^2}\sigma_{AW}(\widehat{B})(\bar{z},z)\, d\bar{z}\wedge dz\,. \tag{181}$$

Since $\sigma_{AW}(\widehat{B})(\bar{z},z) = |z_{k_1}|^2|z_{k_2}|^2...\,|z_{k_s}|^2$ we have

$$\begin{aligned}
\sigma_W(\widehat{B})(\bar{\omega},\omega) &= \int e^{-|\omega-z|^2}|z_{k_1}|^2|z_{k_2}|^2...\,|z_{k_s}|^2\; d\bar{z}\wedge dz \\
&= \int e^{-|u|^2}|\omega_{k_1}-u_{k_1}|^2...\,|\omega_{k_s}-u_{k_s}|^2\; d\bar{u}\wedge du \\
&\leqslant \int e^{-|u|^2}(|\omega_{k_1}|+|u_{k_1}|)^2...\,(|\omega_{k_s}|+|u_{k_s}|)^2\; d\bar{u}\wedge du \\
&\leqslant \int e^{-|u|^2}(1+|u_{k_1}|)^2...\,(1+|u_{k_s}|)^2\; d\bar{u}\wedge du\,.
\end{aligned} \tag{182}$$

We observe that each of the indexes $(k_1,k_2,...\,k_s)$ can take a value between 1 and ℓ. Thus, we can have same values, for example, like $k_1 = k_2$ and so on. In view of this observation, and thanks to the normalization $\int e^{-|x|^2}d\bar{x}\wedge dx = 1$, the integral on the righthand side can be written as the product

$$\prod_{q(j)}\int e^{-|\theta|^2}(1+|\theta|)^{2q(j)}\; d\bar{\theta}\wedge d\theta, \quad \theta\in\mathbb{C},$$

where $q(1)+q(2)+...\,q(J) = s$ for some $1\leqslant J\leqslant s$ depending on the indexes $(k_1,k_2,...\,k_s)$.

$$\begin{aligned}
&\int e^{-|\theta|^2}(1+|\theta|)^{2q(j)}\; d\bar{\theta}\wedge d\theta = \frac{1}{\pi}\int_0^\infty e^{-\rho^2}(1+\rho)^{2q(j)}\; d\rho \\
&\leqslant \frac{1}{\pi}2^{2q(j)} + \frac{1}{\pi}\int_1^\infty e^{-\rho^2}(1+\rho)^{2q(j)}\; d\rho \leqslant \frac{1}{\pi}2^{2q(j)} + \frac{1}{\pi}2^{2q(j)}\int_1^\infty e^{-\rho^2}\rho^{2q(j)}\; d\rho \\
&\leqslant \frac{1}{\pi}2^{2q(j)}\Big(1+\int_1^\infty e^{-\rho^2}\rho^{2s}\; d\rho\Big) \leqslant \frac{1}{\pi}2^{2q(j)}\Big(1+\int_0^\infty e^{-\rho^2}\rho^{2s}\; d\rho\Big)\,.
\end{aligned} \tag{183}$$

It follows

$$\int_0^\infty e^{-\rho^2}\rho^{2s}\; d\rho \leqslant \frac{1}{2}\,\Gamma\Big(\frac{2s+1}{2}\Big) < \frac{1}{2}\Gamma(s+1) = \frac{1}{2}s!\,. \tag{184}$$

In view of (182) - (183) we have (for $|\omega|\leqslant 1$)

$$\sigma_W(\widehat{B})(\bar{\omega},\omega) \leqslant \frac{1}{\pi}4^{q(1)+q(2)+...\,q(J)}\;\Big(1+\frac{1}{2}s!\Big) = \frac{1}{\pi}4^s\;\Big(1+\frac{1}{2}s!\Big). \tag{185}$$

To conclude,

$$\Big|\frac{\partial^r \eta}{\partial\bar\omega_{k_1}\partial\bar\omega_{k_2}\ldots\partial\bar\omega_{k_r}}\Big| \leqslant \|\widehat{\mathcal{Y}}\phi_\omega\| \sum_{s=0}^{r} \binom{r}{s} \sqrt{\frac{1}{\pi}4^s\left(1+\frac{1}{2}s!\right)} \leqslant \|\widehat{\mathcal{Y}}\phi_\omega\| 2^r \sqrt{\frac{1}{\pi}4^r\left(1+\frac{1}{2}r!\right)} \tag{186}$$

$$\leqslant \|\widehat{\mathcal{Y}}\phi_\omega\| \frac{4^r}{\sqrt{\pi}} \sqrt{\left(1+\frac{1}{2}r!\right)} < \|\widehat{\mathcal{Y}}\phi_\omega\| \frac{4^r}{\sqrt{\pi}} 2\sqrt{r!}\,. \tag{187}$$

□

Lemma 4.15. *Let* $g := \sigma_{AW}(e^{-\lambda\widehat{P}})/c_\lambda \geqslant 0$ *with* $\widehat{P}$ *as in Def. 3.5, and let* $\mathcal{X}(\bar\omega,\omega) = \chi(|\omega|^2)$ *with* $\chi \in C^\infty[0,+2]$, $0 \leqslant \chi(\theta) \leqslant 1$, $\mathrm{supp}(\chi) \subset [0,1]$. *Let* $\widehat{\Pi}_{(\leqslant N)}$ *be as in (3.3). Then,*

$$[\mathrm{Op}_{AW}(g\mathcal{X}), \widehat{\Pi}_{(\leqslant N)}] = 0\,. \tag{188}$$

Proof. For $\widehat{N} := \sum_{k=1}^{\ell} \hat b_k^\dagger \hat b_k$, we first prove that $[\mathrm{Op}_{AW}(g\mathcal{X}), \widehat{N}] = 0$. As a consequence of this property, we have the existence of a common basis φ_ν, $\nu \geqslant 1$, for $\mathrm{Op}_{AW}(g\mathcal{X})$ and $\widehat{N}$ (both selfadjoint). Since $\widehat{N}\psi_\alpha = |\alpha|\psi_\alpha$ then for any $\alpha \in \mathbb{N}^\ell$ with $|\alpha| \leqslant N$ we can write $\psi_\alpha = \sum_{\nu=1}^{d(N)} \langle \varphi_\nu, \psi_\alpha\rangle \varphi_\nu$ where $d(N) := \sharp\{|\alpha| \leqslant N\}$. Hence, $\widehat{\Pi}_{(\leqslant N)}$ can be written as a finite sum of projectors associated to φ_ν and this gives the commutation (188).
In order to prove the first statement, we rewrite this equality for the Wick symbols, $\{e^{\Delta}(g\mathcal{X}), \sigma_0\}_{\mathrm{Wick}} = 0$ where $\sigma_0(\bar\omega,\omega) = |\omega|^2$. In particular, thanks to the form of σ_0, the Wick bracket becomes the (complex) Poisson bracket. Thus, we need to prove that $\mathcal{L}_1(e^{\Delta}(g\mathcal{X})) := \{e^{\Delta}(g\mathcal{X}), \sigma_0\}_{\mathrm{P}} = 0$, where $\mathcal{L}_1 = \omega\partial_\omega - \bar\omega\partial_{\bar\omega}$. This is equivalent to prove the invariance under the linear flow $\Psi_t(\bar\omega,\omega) := (\bar\Phi_t, \Phi_t)(\bar\omega,\omega)$ of the Hamiltonian $\mathcal{H}_0 = |\omega|^2$. By the explicit form

$$(e^{\Delta}(g\mathcal{X}))(\bar\omega,\omega) = \int e^{-|\omega-v|^2} g(\bar v, v)\mathcal{X}(\bar v, v)\, d\bar v\wedge dv \tag{189}$$
$$= \int e^{-|y|^2} g(\bar\omega-\bar y, \omega-y)\mathcal{X}(\bar\omega-\bar y,\omega-y)\, d\bar y\wedge dy$$
$$= \int e^{-|y|^2} g(\Omega-Y)\mathcal{X}(\Omega-Y)\, d\bar y\wedge dy$$

where we have set $\Omega := (\bar\omega,\omega)$ and $Y := (\bar y, y)$. Thus, the invariance of g, $\mathcal{X}$, $|\omega|^2$ and linearity of Ψ_t give

$$(e^{\Delta}(g\mathcal{X}))\Psi_t(\bar\omega,\omega) = \int e^{-|y|^2} g(\Psi_t(\Omega)-Y)\mathcal{X}(\Psi_t(\Omega)-Y)\, d\bar y\wedge dy$$
$$= \int e^{-|y|^2} g(\Psi_t(\Omega-\Psi_t^{-1}(Y)))\mathcal{X}(\Psi_t(\Omega-\Psi_t^{-1}(Y)))\, d\bar y\wedge dy$$
$$= \int e^{-|\Psi_t(y)|^2} g(\Psi_t(\Omega-Y))\mathcal{X}(\Psi_t(\Omega-Y))\, d\bar y\wedge dy$$
$$= \int e^{-|y|^2} g(\Omega-Y)\mathcal{X}(\Omega-Y)\, d\bar y\wedge dy = (e^{\Delta}(g\mathcal{X}))(\bar\omega,\omega). \tag{190}$$

The invariance under Ψ_t now implies $[\mathrm{Op}_{AW}(g\mathcal{X}), \widehat{N}] = 0$. $\square$

Lemma 4.16. *The operators* $\mathrm{Op}_W^h(\rho_k)$ *and* $\mathrm{Op}_W^h(u_k)$ *satisfy the growth condition in Def. 4.5, with the constants* $C_1 := 2(1+D)^{\frac{1}{2}} e^{\frac{1}{2}(1+D)}$ *and* $Q := \frac{1}{2}(\ell+2)$.

Proof. We begin by the identity $\mathrm{Op}_W^h(\rho_k) = \hat{a}_k(t)$ shown in Remark 3.1, and

$$\|\widehat{\Pi}_{(\leqslant N)}\hat{a}_k(t)\|^2_{\mathrm{HS}} = \mathrm{Tr}(\widehat{\Pi}_{(\leqslant N)}\hat{a}_k(t)(\widehat{\Pi}_{(\leqslant N)}\hat{a}_k(t))^\dagger) = \mathrm{Tr}(\widehat{\Pi}_{(\leqslant N)}\hat{a}_k(t)\hat{a}_k(t)^\dagger).$$

Moreover, $\hat{a}_k(t) = e^{i\widehat{H}t}\hat{a}_k e^{-i\widehat{H}t}$ and $[e^{\pm i\widehat{H}t}, \widehat{\Pi}_{(\leqslant N)}] = 0$. Since the trace is invariant under unitary conjugations of operators, we get

$$\|\widehat{\Pi}_{(\leqslant N)}\hat{a}_k(t)\|^2_{\mathrm{HS}} = \mathrm{Tr}(\widehat{\Pi}_{(\leqslant N)}\hat{a}_k\hat{a}_k^\dagger) = \mathrm{Tr}(\widehat{\Pi}_{(\leqslant N)}(\frac{1}{N}\mathrm{Id} + \hat{a}_k^\dagger\hat{a}_k)).$$

In particular, in view of Definition 3.3,

$$\begin{aligned}\mathrm{Tr}(\widehat{\Pi}_{(\leqslant N)}) = \dim(\Lambda_{(\leqslant N)}) &= \sum_{n=0}^N \sum_{|\alpha|=n} 1 = \sum_{n=0}^N \binom{\ell-1+n}{n} = \sum_{n=0}^N \frac{(\ell-1+n)!}{n!(\ell-1)!}. \\ &\leqslant 1 + N\frac{(\ell-1+N)!}{N!(\ell-1)!} = 1 + N\frac{(\ell-1+N)!}{N!(\ell-1)!}\end{aligned}$$

Now apply the two estimates $e(m/e)^m \leqslant m! \leqslant e((m+1)/e)^{m+1}$, $m \in \mathbb{N}$, to get

$$\begin{aligned}\mathrm{Tr}(\widehat{\Pi}_{(\leqslant N)}) &\leqslant 1 + \frac{N}{(\ell-1)!}\frac{(\ell+N)^{\ell+N}}{N^N}\frac{e^N}{e^{\ell+N}} \\ &= 1 + \frac{N}{(\ell-1)!}\Big(1+\frac{\ell}{N}\Big)^N\Big(1+\frac{\ell}{N}\Big)^\ell\frac{N^\ell}{e^\ell} \leqslant 1 + \frac{N}{(\ell-1)!}e^\ell(1+D)^\ell\frac{N^\ell}{e^\ell}\end{aligned} \tag{191}$$

where we used the setting $\ell/N \leqslant D$ of Thm. 1.1.

$$\mathrm{Tr}(\widehat{\Pi}_{(\leqslant N)}) \leqslant 1 + \frac{(1+D)^\ell}{(\ell-1)!}N^{\ell+1} \leqslant \Big(1+\frac{(1+D)^\ell}{(\ell-1)!}\Big)N^{\ell+1} \tag{192}$$

Notice that

$$\sup_{\ell\geqslant 1}\Big(1+\frac{(1+D)^\ell}{(\ell-1)!}\Big) < 1 + (1+D)e^{(1+D)} < 2(1+D)e^{(1+D)}. \tag{193}$$

Moreover, $\widehat{\Pi}^2_{(\leqslant N)} = \widehat{\Pi}_{(\leqslant N)}$ and $\mathrm{Tr}(\widehat{\Pi}_{(\leqslant N)}\,\hat{a}_k^\dagger\hat{a}_k) = \mathrm{Tr}(\widehat{\Pi}_{(\leqslant N)}\,\hat{a}_k^\dagger\hat{a}_k\widehat{\Pi}_{(\leqslant N)})$. Hence,

$$\mathrm{Tr}(\widehat{\Pi}_{(\leqslant N)}\,\hat{a}_k^\dagger\hat{a}_k) = \sum_{n=0}^N\sum_{|\alpha|=n}\langle\psi_\alpha, \hat{a}_k^\dagger\hat{a}_k\psi_\alpha\rangle = \sum_{n=0}^N\sum_{|\alpha|=n}\frac{\alpha_k^2}{N} < N\mathrm{Tr}(\widehat{\Pi}_{(\leqslant N)}). \tag{194}$$

Thus, $\forall t \geqslant 0$

$$\|\widehat{\Pi}_{(\leqslant N)}\hat{a}_k(t)\|^2_{\mathrm{HS}} \leqslant \Big(\frac{1}{N}+N\Big)\mathrm{Tr}(\widehat{\Pi}_{(\leqslant N)}) \leqslant 2N\mathrm{Tr}(\widehat{\Pi}_{(\leqslant N)}) \leqslant 4(1+D)e^{(1+D)}N^{\ell+2}. \tag{195}$$

We now focus the attention on $\mathrm{Op}^h_W(u_k)$, and for the sake of simplicity we consider $\hat{u}_k(t) := \mathrm{Op}_W(u_k)$, i.e. with $h=1$. Then, we derive the general case as a consequence.
This operator is time dependent, but we show that the following quantity does not depend on time

$$\|\widehat{\Pi}_{(\leqslant N)}\hat{u}_k(t)\|^2_{\mathrm{HS}} = \mathrm{Tr}(\widehat{\Pi}_{(\leqslant N)}\hat{u}_k(t)\hat{u}_k^\dagger(t)) \tag{196}$$

Define the following two selfadjoint operators $\hat{v}_k(t) := (\hat{u}_k(t)+\hat{u}_k^\dagger(t))/2$ and $\hat{w}_k(t) := -i(\hat{u}_k(t)-\hat{u}_k^\dagger(t))/2$. Then, $\hat{u}_k(t) = (\hat{v}_k(t)+i\hat{w}_k(t))$ and $\hat{u}_k^\dagger(t) = (\hat{v}_k(t)-i\hat{w}_k(t))$. Moreover, $\hat{u}_k(t)\hat{u}_k^\dagger(t) = \hat{v}_k^2(t)+\hat{w}_k^2(t)+i[\hat{w}_k(t),\hat{v}_k(t)]$.
Recall that $\hat{u}_k(t)$ solves the evolution equation (38); as a consequence easily check that $\hat{v}_k(t)$ and $\hat{w}_k(t)$ solve the same equation with initial data $\hat{v}_k(0) := \hat{x}_k$ and $\hat{w}_k(0) := \hat{p}_k$. Indeed,

$$\begin{cases} i\frac{d}{dt}\hat{u}_k(t) = [\hat{u}_k(t),\widehat{H}_2] + \frac{1}{2}\sum_{j=1}^{\ell} U_j\,\hat{b}_j^\dagger\,[\hat{u}_k(t),\hat{n}_j]\,\hat{b}_j\,, \\ -i\frac{d}{dt}\hat{u}_k^\dagger(t) = [\widehat{H}_2,\hat{u}_k^\dagger(t)] + \frac{1}{2}\sum_{j=1}^{\ell} U_j\,\hat{b}_j^\dagger\,[\hat{n}_j,\hat{u}_k^\dagger(t)]\,\hat{b}_j\,, \end{cases}$$

implies

$$\begin{cases} i\frac{d}{dt}\hat{v}_k(t) = [\hat{v}_k(t),\widehat{H}_2] + \frac{1}{2}\sum_{j=1}^{\ell} U_j\,\hat{b}_j^\dagger\,[\hat{v}_k(t),\hat{n}_j]\,\hat{b}_j\,, \\ i\frac{d}{dt}\hat{w}_k(t) = [\hat{w}_k(t),\widehat{H}_2] + \frac{1}{2}\sum_{j=1}^{\ell} U_j\,\hat{b}_j^\dagger\,[\hat{w}_k(t),\hat{n}_j]\,\hat{b}_j\,. \end{cases}$$

Thus, we can treat the three contributions

$$\|\widehat{\Pi}_{(\leqslant N)}\hat{u}_k(t)\|^2_{\mathrm{HS}} = \mathrm{Tr}(\widehat{\Pi}_{(\leqslant N)}\hat{v}_k^2(t)) + \mathrm{Tr}(\widehat{\Pi}_{(\leqslant N)}\hat{w}_k^2(t)) + i\mathrm{Tr}(\widehat{\Pi}_{(\leqslant N)}[\hat{w}_k(t),\hat{v}_k(t)]) \tag{197}$$

where the first two terms work in the same way, whereas the third one is zero since given by the trace of a symmetric operator composed by an antisymmetric one.
In particular,

$$\frac{d}{dt}\mathrm{Tr}(\widehat{\Pi}_{(\leqslant N)}\hat{v}_k^2(t)) = \mathrm{Tr}(\widehat{\Pi}_{(\leqslant N)}\,\dot{\hat{v}}_k(t)\,\hat{v}_k(t) + \widehat{\Pi}_{(\leqslant N)}\,\hat{v}_k(t)\,\dot{\hat{v}}_k(t)) \tag{198}$$

that equals

$$= -i\mathrm{Tr}\Big(\widehat{\Pi}_{(\leqslant N)}\Big([\hat{v}_k(t),\widehat{H}_2] + \frac{1}{2}\sum_{j=1}^{\ell} U_j\,\hat{b}_j^\dagger\,[\hat{v}_k(t),\hat{n}_j]\,\hat{b}_j\Big)\hat{v}_k(t)\Big)$$
$$-i\mathrm{Tr}\Big(\widehat{\Pi}_{(\leqslant N)}\hat{v}_k(t)\Big([\hat{v}_k(t),\widehat{H}_2] + \frac{1}{2}\sum_{j=1}^{\ell} U_j\,\hat{b}_j^\dagger\,[\hat{v}_k(t),\hat{n}_j]\,\hat{b}_j\Big)\Big)\,. \tag{199}$$

This can be rewritten in terms of commutators and anti-commutators

$$= -i\mathrm{Tr}\Big([\hat{v}_k(t), \widehat{\Pi}_{(\leqslant N)}]_a \, [\hat{v}_k(t), \widehat{H}_2] \Big) - i\mathrm{Tr}\Big(\frac{1}{2}\sum_{j=1}^{\ell} U_j \, \hat{b}_j^\dagger \, [\hat{v}_k(t), \hat{n}_j] \, \hat{b}_j \, [\hat{v}_k(t), \widehat{\Pi}_{(\leqslant N)}]_a \Big) . \tag{200}$$

We deduce that the result is zero, since $[\hat{v}_k(t), \widehat{H}_2]$ and $\hat{b}_j^\dagger \, [\hat{v}_k(t), \hat{n}_j] \, \hat{b}_j$ are antisymmetric operators, whereas $[\hat{v}_k(t), \widehat{\Pi}_{(\leqslant N)}]_a$ is symmetric. Recalling that $\hat{u}_k(0) = \hat{b}_k$ then

$$\|\widehat{\Pi}_{(\leqslant N)} \hat{u}_k(t)\|_{\mathrm{HS}}^2 = \|\widehat{\Pi}_{(\leqslant N)} \hat{u}_k(0)\|_{\mathrm{HS}}^2 = \mathrm{Tr}(\widehat{\Pi}_{(\leqslant N)} \hat{b}_k \hat{b}_k^\dagger) = N\mathrm{Tr}(\widehat{\Pi}_{(\leqslant N)} \hat{a}_k \hat{a}_k^\dagger) .$$

About the semiclassical quantization of the flow, we remind that the time dependent $\mathrm{Op}_W^h(u_k)$ solves the equation (44) with initial data $\mathrm{Op}_W^h(u_k)(0) := \hat{a}_k$. Hence, the above arguments thus work also in this setting, and

$$\|\widehat{\Pi}_{(\leqslant N)} \mathrm{Op}_W^h(u_k)\|_{\mathrm{HS}}^2 = \|\widehat{\Pi}_{(\leqslant N)} \mathrm{Op}_W^h(u_k)(0)\|_{\mathrm{HS}}^2 = \mathrm{Tr}(\widehat{\Pi}_{(\leqslant N)} \hat{a}_k \hat{a}_k^\dagger). \tag{201}$$

To conclude, we notice that we have the same upper bound as in (195). $\square$

Remark 4.17. *Let $C_1 := 2(1+D)^{\frac{1}{2}} e^{\frac{1}{2}(1+D)}$ and $Q := \frac{1}{2}(\ell+2)$. Set the condition $\ell + 2 \leqslant N/(8\ln(N))$. Then, the remainder term in (13) for the operator $\mathrm{Op}_W^h(\rho_k - u_k)$ reads*

$$e^{-\frac{1}{4h}} \, e^D \, (2C_1)^2 \, h^{-2Q} \leqslant e^{-\frac{1}{8h}} e^{-\frac{1}{8h}} \, e^D \, 4^2(1+D) e^{(1+D)} \, h^{-(2+\ell)} \leqslant 4^2(1+D) e^{(1+2D)} e^{-\frac{1}{8h}} .$$

This estimate is thus $\mathcal{O}(h^\infty)$. Moreover, since we are interested to deal with the whole righthand side of (13), we require the more restrictive condition $2^6\ell \leqslant N/(8\ln(N))$, $N \geqslant 2$, that guarantees $\ell + 2 \leqslant N/(8\ln(N))$ as well as that $2^6\ell \leqslant N$.

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