A characterization of generalized existential completions

Maria Emilia Maietti[a], Davide Trotta[b]

[a]*Department of Mathematics, University of Padova, Italy*
[b]*Department of Computer Science, University of Pisa, Italy*

Abstract

This paper aims to provide an intrinsic characterization of the notion of generalized existential completion of a conjunctive doctrine P for a class Λ of morphisms of the base category of P. The cornerstone of this result consists of an algebraic description of the logical concept of existential free formulas closely connected to the validity of some choice principles.

The link between our characterization and choice principles is emphasized by the fact that an existential doctrine P is the generalized existential completion of itself for all the projections of its base if and only if P is equipped with Hilbert's epsilon operators.

Our characterization provides a useful tool to recognize a wide variety of examples of doctrines arising as generalized existential completions. These include the subobjects doctrine and the weak subobjects doctrine of a category with finite limits as well all realizability triposes and among localic triposes only the supercoherent ones.

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Email addresses: `maietti@math.unipd.it` (Maria Emilia Maietti), `trottadavide92@gmail.com` (Davide Trotta)

1. Introduction

The introduction of the notion of *hyperdoctrine* by Lawvere in [18, 19] marked the beginning of the field of categorical logic aiming to provide an algebraic presentation of both the notions of logical theory and model. Since then, several kinds of free constructions involving hyperdoctrines (here simply called doctrines) have been employed in various areas of mathematics and computer science.

We recall, for example, the well-known notion of *tripos-to-topos construction* of J.M.E. Hyland, P.T. Johnstone, A.M. Pitts in [13, 29] or that of *exact and regular completions* of lex or regular categories introduced in [3, 4] and viewed as instances of suitable completions of suitable doctrines in [23, 22, 24].

In this work, we consider the construction of the *generalized existential completion* of a conjunctive doctrine P (Definition 2.1) relative to a suitable class Λ of morphisms of the base category of P. Such a notion was originally introduced in [34]. The word *generalized* refers to the fact that, as customary in categorical logic, existential quantifiers are formulated for arbitrary terms rather than just for variables.

From a logical perspective, a conjunctive doctrine represents a model of the conjunctive logic over arbitrary sorts, and the generalized existential completion is the free construction of a model for conjunctive logic with existential quantifications formulated for arbitrary terms (in our case only those chosen Λ), namely a Λ-existential doctrine (Definition 3.4). Intuitively, the morphisms of the class Λ represent *generalized projections.*

Our main contribution is to provide a logical characterization of this construction (Theorem 4.16) through a categorical presentation of the notion of *existential-free formulas* (Definition 4.2).

This latter notion can be equivalently presented as the validity of certain choice principles (depending on the class Λ) called existence property in [39, Def. 5.4.1] (Proposition 4.4).

Employing this categorification of existential-free elements, we will show that a Λ-existential doctrine P is an instance of generalized existential completion (relative to Λ) if and only if it has *enough existential-free objects* (Definition 4.10), which are closed under binary conjunction and the top element, in which case P is (isomorphic to) the free completion of the conjunctive subdoctrine on such objects (Theorem 4.16), which is called *existential cover* (Definition 4.14).

Our characterization recalls Carboni's characterization of the construction of the exact completion of a lex category [3, Lem. 2.1] if we replace the notion of "lex category" with that of a conjunctive doctrine, the notion of "exact completion" with that of generalized existential completion relative to Λ, and the notion of "projective" with that of "existential-free formula".

As a byproduct of our analysis, we show that an existential doctrine is a generalized existential completion of itself relative to all the projections of its base if and only if it is equipped with *Hilbert's epsilon operators*, as defined in [20] (Theorem 5.1).

Our characterization allows us to recognize several examples of doctrines arising as generalized existential completions, including the following ones: every subobjects doctrine of a lex category, every $\mathcal{M}$-subobjects doctrine relative to a $\mathcal{M}$-category [5, 31] (Theorem 7.11), every weak subobjects doctrine of a lex category (Theorem 7.3), every realizability tripos [13] (Theorem 7.24) and, among localic triposes, exactly those associated with a supercoherent locale in the sense of [1] (Theorem 7.32).

In particular, both the subobjects doctrine $\mathrm{Sub}_{\mathcal{C}}$ and the weak subobjects doctrine $\Psi_{\mathcal{C}}$ of a category $\mathcal{C}$ with finite limits are generalized existential completions of the constant true doctrine, the first along the class of all the monomorphisms of $\mathcal{C}$ while the latter along all the morphisms of $\mathcal{C}$.

We conclude by underlying that a preliminary version of our characterization of generalized existential completion was presented in [35] in 2020, while a similar characterization, for the class of generalized existential completions relative to Λ as the whole class of base morphisms, was independently presented in [7, Thm 6.3] in terms of $\exists$-prime elements.

Furthermore, our characterization has already been fruitfully employed in recent works [36, 37, 38] to give a categorical description of the logical principles involved in Gödel's dialectica interpretation [8], by deepening the analysis in [10].

In future work, we intend to apply the results presented here to broaden the study of regular and exact completions of generalized existential completions initiated in [26].

2. Preliminary notions of doctrines

The notion of hyperdoctrine was introduced by F.W. Lawvere in a series of seminal papers [18, 19]. We recall from *loc. cit.* some definitions which will be useful in the following. The reader can find more details about the theory of elementary and existential doctrine also in [23, 22, 24, 20, 6].

In the following we adopt the notation fg to mean the composition of a morphism $f\colon Y \longrightarrow Z$ with another $g\colon X \longrightarrow Y$ within a category.

We indicate with Set the category of sets and functions which are formalizable within the classical axiomatic set theory ZFC.

We introduce the following basic notion of "conjunctive doctrine" as a generalization of that of "primary doctrine" in [23].

Definition 2.1. A **conjunctive doctrine** is a functor $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ from the opposite of the category $\mathcal{C}$ to the category of inf-semilattices.

Definition 2.2. A conjunctive doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is a **primary doctrine** if the category $\mathcal{C}$ has finite products.

We add the definition of **fibred subdoctrine** for conjunctive subdoctrines of conjunctive doctrines on the same base category:

Definition 2.3. A conjunctive doctrine $P'\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is said a **fibred subdoctrine** of a conjunctive doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ if each fibre of $P'(A)$ is a full sub-inf-semilattice of $P(A)$ for every object A.

Given a conjunctive doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, we will refer to an inf-semilattice $P(A)$ calling it a *fibre*.

Definition 2.4. A primary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is **elementary** if for every A in $\mathcal{C}$ there exists an object δ_A in $P(A \times A)$, called **fibred equality**, such that

1. the assignment
$$\exists_{\langle \mathrm{id}_A, \mathrm{id}_A \rangle}(\alpha) := P_{\mathrm{pr}_1}(\alpha) \wedge \delta_A$$
for an element α of $P(A)$ determines a left adjoint to $P_{\langle \mathrm{id}_A, \mathrm{id}_A \rangle}\colon P(A \times A) \longrightarrow PA$;

2. for every morphism e of the form $\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_2 \rangle\colon X \times A \longrightarrow X \times A \times A$ in $\mathcal{C}$, the assignment
$$\exists_e(\alpha) := P_{\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle}(\alpha) \wedge P_{\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle}(\delta_A)$$
for α in $P(X \times A)$ determines a left adjoint to $P_e\colon P(X \times A \times A) \longrightarrow P(X \times A)$.

Definition 2.5. A primary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is **pure existential** if, for every object A_1 and A_2 in $\mathcal{C}$, for any product projection $\mathrm{pr}_i\colon A_1 \times A_2 \longrightarrow A_i$, $i = 1, 2$, the functor
$$P_{\mathrm{pr}_i}\colon P(A_i) \longrightarrow P(A_1 \times A_2)$$
has a left adjoint $\exists_{\mathrm{pr}_i}$, and these satisfy:

(BCC) **Beck-Chevalley condition:** for any pullback diagram

$$\begin{array}{ccc} X' & \xrightarrow{\mathrm{pr}'} & A' \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ X & \xrightarrow[\mathrm{pr}]{} & A \end{array}$$

with pr and pr′ projections, for any β in $P(X)$ the canonical arrow
$$\exists_{\mathrm{pr}'} P_{f'}(\beta) \leq P_f \exists_{\mathrm{pr}}(\beta)$$
is an isomorphism;

(FR) **Frobenius reciprocity:** for any projection $\mathrm{pr}\colon X \longrightarrow A$, for any object α in $P(A)$ and β in $P(X)$, the canonical arrow
$$\exists_{\mathrm{pr}}(P_{\mathrm{pr}}(\alpha) \wedge \beta) \leq \alpha \wedge \exists_{\mathrm{pr}}(\beta)$$
in $P(A)$ is an isomorphism.

Remark 2.6. Pure existential doctrines are simply called **existential** in previous works of both authors including [34, 23, 24]. In this paper we call them “pure existential” to emphasize that they are particular instances of generalized existential doctrines.

Remark 2.7. For a logical formulation of the Beck-Chevalley and Frobenius reciprocity conditions for existential doctrines see respectively example 2.12 (a) of [23] and the section on quantification in [28]. For a more familiar characterization of elementary doctrines see [6, Prop. 2.5].

Moreover, notice that left adjoints required in Definitions 2.4 and 2.5 are not arrows in the category **InfSL** of inf-semilattices in general.

Example 2.8. The following examples are discussed in [18, 13].

1. Let $\mathcal{C}$ be a category with finite limits. The functor

$$\mathrm{Sub}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

assigns to an object A in $\mathcal{C}$ the poset $\mathrm{Sub}_{\mathcal{C}}(A)$ of subobjects of A in $\mathcal{C}$ and, for an arrow $B \xrightarrow{f} A$ the morphism $\mathrm{Sub}_{\mathcal{C}}(f) \colon \mathrm{Sub}_{\mathcal{C}}(A) \longrightarrow \mathrm{Sub}_{\mathcal{C}}(B)$ is given by pulling a subobject back along f. The fibred equalities are the diagonal arrows. This is an elementary doctrine and it is also pure existential if and only if the category $\mathcal{C}$ is regular. See [11].

2. Consider a category $\mathcal{D}$ with finite products and weak pullbacks: the doctrine is given by the functor of weak subobjects (or variations)

$$\Psi_{\mathcal{D}} \colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

where $\Psi_{\mathcal{D}}(A)$ is the poset reflection of the slice category $\mathcal{D}/A$, whose objects are indicated with $[f]$ for any arrow $B \xrightarrow{f} A$ in $\mathcal{D}$, and for an arrow $B \xrightarrow{f} A$, the homomorphism $\Psi_{\mathcal{D}}([f]) \colon \Psi_{\mathcal{D}}(A) \longrightarrow \Psi_{\mathcal{D}}(B)$ is given by the equivalence class of a weak pullback of an arrow $X \xrightarrow{g} A$ with f. This doctrine is pure existential and elementary, and the left adjoints are given by the post-composition. See [23].

3. Let $\mathbb{T}$ be a theory in a first order language $\mathcal{L}$. We define a primary doctrine

$$\mathrm{LT} \colon \mathcal{C}_{\mathcal{L}}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

where $\mathcal{C}_{\mathcal{L}}$ is the category of lists of variables and term substitutions:

- **objects** of $\mathcal{C}_{\mathcal{L}}$ are finite lists of variables $\vec{x} := (x_1, \ldots, x_n)$, and we include the empty list $()$;
- a **morphism** from $(x_1, \ldots, x_n)$ into $(y_1, \ldots, y_m)$ is a substitution $[t_1/y_1, \ldots, t_m/y_m]$ where the terms t_i are built in $\mathcal{L}$ on the variable $x_1, \ldots, x_n$;
- the **composition** of two morphisms $[\vec{t}/\vec{y}] \colon \vec{x} \longrightarrow \vec{y}$ and $[\vec{s}/\vec{z}] \colon \vec{y} \longrightarrow \vec{z}$ is given by the substitution

$$[s_1[\vec{t}/\vec{y}]/z_k, \ldots, s_k[\vec{t}/\vec{y}]/z_k] \colon \vec{x} \longrightarrow \vec{z}.$$

The functor $\mathrm{LT} \colon \mathcal{C}_{\mathcal{L}}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ sends a list $(x_1, \ldots, x_n)$ to the partial order $\mathrm{LT}(x_1, \ldots, x_n)$ of equivalence classes $[\phi]$ of well formed formulas ϕ in the context

$(x_1, \ldots, x_n)$ where $[\psi] \leq [\phi]$ for $\phi, \psi \in \mathrm{LT}(x_1, \ldots, x_n)$ if $\psi \vdash_{\mathbb{T}} \phi$ and two formulas are equivalent if they are equiprovable in the theory. Given a morphism of $\mathcal{C}_{\mathcal{L}}$

$$[t_1/y_1, \ldots, t_m/y_m] \colon (x_1, \ldots, x_n) \longrightarrow (y_1, \ldots, y_m)$$

the functor $\mathrm{LT}_{[\vec{t}/\vec{y}]}$ acts as the substitution $\mathrm{LT}_{[\vec{t}/\vec{y}]}(\psi(y_1, \ldots, y_m)) = \psi[\vec{t}/\vec{y}]$.

The doctrine $\mathrm{LT} \colon \mathcal{C}_{\mathcal{L}}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is elementary exactly when $\mathcal{L}$ has an equality predicates and it is pure existential exactly when $\mathcal{L}$ has existential quantifiers. For more details we refer to [23], and for the case of a many sorted first order theory we refer to [28].

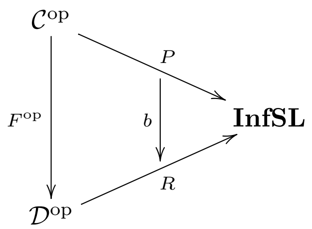
4. Let $\mathcal{A}$ be a locale, i.e. $\mathcal{A}$ is a poset with finite meets and arbitrary joins, satisfying the *infinite distributive law* $x \wedge (\bigvee_i y_i) = \bigvee_i (x \wedge y_i)$. The localic doctrine is given by the functor:

$$\mathcal{A}^{(-)} \colon \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

assigning $I \mapsto \mathcal{A}^I$. The partial order (and hence the propositional connectives) is provided by the pointwise partial order on functions $f \colon I \longrightarrow \mathcal{A}$. This doctrine is elementary and pure existential, where the existential quantifier along a given function $f \colon I \longrightarrow J$ maps a function $\phi \in \mathcal{A}^I$ to $\exists_f(\phi)$ given by $j \mapsto \bigvee_{\{i \in I | f(i) = j\}} \phi(i)$.

The category of primary doctrines **PD** is a 2-category, where:

- a **1-cell** is a pair (F, b)



  such that $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ is a functor preserving finite products and $b \colon P \longrightarrow R \circ F^{\mathrm{op}}$ is a natural transformation.

- a **2-cell** from (F, b) to (G, c) is a natural transformation $\theta \colon F \longrightarrow G$ such that for every A in $\mathcal{C}$ and every α in $P(A)$, we have

$$b_A(\alpha) \leq R_{\theta_A}(c_A(\alpha)).$$

We denote by **ExD** the 2-full subcategory of **PD** whose elements are pure existential doctrines, and whose 1-cells are those 1-cells of **PD** which preserve the existential structure.

We conclude this section by recalling some choice principles from [12, 24, 20, 25]. To this purpose, we recall the notion of *functional* and *entire* element of an elementary and existential doctrine.

Definition 2.9. Given an elementary and pure existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, an element $\alpha \in P(A \times B)$ is called **entire** from A to B if

$$\top_A \leq \exists_{\mathrm{pr}_A}(\alpha).$$

Moreover it is called **functional** if

$$P_{\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle}(\alpha) \wedge P_{\langle \mathrm{pr}_1, \mathrm{pr}_3\rangle}(\alpha) \leq P_{\langle \mathrm{pr}_2, \mathrm{pr}_3\rangle}(\delta_B)$$

in $P(A \times B \times B)$. Notice that for every relation $\alpha \in P(A \times B)$ and $\beta \in P(B \times C)$, the **relational composition of** α and β is given by the relation

$$\exists_{\langle \mathrm{pr}_1, \mathrm{pr}_3\rangle}(P_{\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle}(\alpha) \wedge P_{\langle \mathrm{pr}_2, \mathrm{pr}_3\rangle}(\beta)$$

in $P(A \times B)$, where pr_i are the projections from $A \times B \times C$.

Definition 2.10. Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an elementary and pure existential doctrine. We say that P satisfies the **Rule of Unique Choice** (RUC) if for every entire functional relation ϕ in $P(A \times B)$ there exists an arrow $f\colon A \longrightarrow B$ such that

$$\top_A \leq P_{\langle \mathrm{id}_A, f\rangle}(\phi)$$

Example 2.11. The subobjects doctrine $\mathrm{Sub}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ presented in Example 2.8 satisfies RUC as observed in [20].

Now we recall the notion of Extended Rule of Choice and its particular instance called Rule of Choice introduced and analyzed in [20, 25].

Definition 2.12. An elementary and pure existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ satisfies the **Extended Rule of Choice** (ERC) if for every $\phi \in P(B)$ and for every $g\colon B \longrightarrow A$ such that

$$\top_A \leq \exists_g(\phi)$$

there exists an arrow $f\colon A \longrightarrow B$ in $\mathcal{C}$ such that $gf = \mathrm{id}_A$ and

$$\top_A \leq P_f(\phi).$$

Definition 2.13. For a pure existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, we say that P satisfies the **Rule of Choice** (RC) if it satisfies the Extended Rule of Choice only for projections, namely for every $\phi \in P(A \times B)$ such that

$$\top_A \leq \exists_{\mathrm{pr}_1}(\phi)$$

there exists an arrow $f\colon A \longrightarrow B$ in $\mathcal{C}$ such that

$$\top_A \leq P_{\langle \mathrm{id}_A, f\rangle}(\phi).$$

Example 2.14. Recall from [20] that the doctrine of weak subobjects

$$\Psi_{\mathcal{D}}\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

presented in Example 2.8 (2) satisfies the Extended Rule of Choice.

We recall from [20] the notion of existential doctrine with Hilbert's ϵ-**operators**:

Definition 2.15. Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be a pure existential doctrine. An object B of $\mathcal{C}$ is equipped with Hilbert's ϵ-**operator** if, for any object A in $\mathcal{C}$ and any α in $P(A\times B)$ there exists an arrow $\epsilon_\alpha\colon A \longrightarrow B$ such that

$$\exists_{\mathrm{pr}_1}(\alpha) = P_{\langle \mathrm{id}_A, \epsilon_\alpha\rangle}(\alpha)$$

holds in $P(A)$, where $\mathrm{pr}_1\colon A\times B \longrightarrow A$ is the first projection.

Definition 2.16. We say that a pure existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is **equipped with** Hilbert's ϵ-**operators** if every object in $\mathcal{C}$ is equipped with ϵ-operator.

We recall from [20, Ex. 5.14] the following example of doctrine equipped with Hilbert's ϵ-operators.

Example 2.17. Let Set_* be the category of non-empty sets and let ξ be an ordinal with greatest element, and $\mathcal{H} := (\xi, \geq)$ be the frame given by the set ξ equipped with the reverse order. Following Example 2.8(4), we consider then the doctrine

$$\mathcal{H}^{(-)}\colon \mathrm{Set}_*^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

that is elementary and pure existential. In particular, we recall that for every $\alpha \in \mathcal{H}^{A\times B}$, the left adjoint $\exists_{\mathrm{pr}_A}$ is defined as

$$\exists_{\mathrm{pr}_A}(\alpha)(a) = \bigvee_{b\in B} \alpha(a,b)$$

and the equality predicate $\delta(i,j) \in \mathcal{H}^{A\times A}$ is defined as the top element if $i=j$, and the bottom otherwise. Moreover, the doctrine $\mathcal{H}^{(-)}\colon \mathrm{Set}_*^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is equipped with ϵ-operators. In particular for every element $\alpha \in \mathcal{H}^{A\times B}$, and for every $a\in A$ one can consider the (non empty) set

$$I_\alpha(a) = \{b \in B \mid \alpha(a,b) = \bigvee_{c\in B} \alpha(a,c)\}.$$

Then, by the axiom of choice, there exists a function $\epsilon_\alpha\colon A \longrightarrow B$ such that $\epsilon_\alpha(a) \in I_\alpha(a)$. Therefore we have that

$$\alpha(a, \epsilon_\alpha(a)) = \bigvee_{c\in B} \alpha(a,c) = \exists_{\mathrm{pr}_A}(\alpha)(a)$$

and this prove that $\mathcal{H}^{(-)}$ is equipped with Hilbert's ϵ-operators.

We conclude this section recalling from [18, 22, 23] the notion of doctrine with (full) comprehensions. This notion provides an abstract algebraic counterpart of the set-theoretic "comprehension axiom", we refer to [14, Sec. 4.6] for a complete introduction to this notion in the fibrational context.

Definition 2.18. Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be a conjunctive doctrine and α be an object of $P(A)$. A **comprehension** of α is an arrow $\{\!|\alpha|\!\}\colon X \longrightarrow A$ such that $P_{\{\!|\alpha|\!\}}(\alpha) = \top_X$ and, for every $f\colon Z \longrightarrow A$ such that $P_f(\alpha) = \top_Z$, there exists a unique map $g\colon Z \longrightarrow X$ such that $f = \{\!|\alpha|\!\} \circ g$.

Intuitively, the domain of the comprehension morphism of the predicate α represents the set $\{x \in A \mid \alpha(x)\}$ containing the elements of the object A satisfying α. Then, one says that P **has comprehensions** if every α has a comprehension, and that P **has full comprehensions** if, moreover, $\alpha \leq \beta$ in $P(A)$ whenever $\{\!|\alpha|\!\}$ factors through $\{\!|\beta|\!\}$.

Note that each morphism can be the full comprehension of a unique object and the fibre order of a conjunctive doctrine with full comprehensions is equivalent to the usual subobject order between comprehensions as one can easily check:

Remark 2.19. Notice that when $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is a conjunctive doctrine with full comprehensions, by applying in both directions the condition of comprehensions being full, we have that for all objects α_1, α_2 of $P(A)$:

- if $\{\!|\alpha_1|\!\} = \{\!|\alpha_2|\!\}$ then $\alpha_1 = \alpha_2$.
- $\alpha_1 \leq \alpha_2$ if and only if there exists a morphism t such that $\{\!|\alpha_1|\!\} = \{\!|\alpha_2|\!\} t$.

Remark 2.20. For every $f\colon B \longrightarrow A$ in $\mathcal{C}$ the mediating arrow between the comprehensions $\{\!|\alpha|\!\}\colon A_\alpha \longrightarrow A$ and $\{\!|P_f(\alpha)|\!\}\colon B_{P_f(\alpha)} \longrightarrow B$ produces a pullback

$$\begin{array}{ccc} B_{P_f(\alpha)} & \xrightarrow{f'} & A_\alpha \\ {\scriptstyle \{\!|P_f(\alpha)|\!\}}\downarrow & \lrcorner & \downarrow{\scriptstyle \{\!|\alpha|\!\}} \\ B & \xrightarrow[f]{} & A \end{array}$$

Thus comprehensions are stable under pullbacks. Moreover it is straightforward to verify that if $\{\!|\alpha|\!\}\colon X \longrightarrow A$ is a comprehension of α, then $\{\!|\alpha|\!\}$ is monic.

Notation: given an element $\alpha \in P(A)$, we will denote by A_α the domain of the comprehension $\{\!|\alpha|\!\}$.

We show here a general version of a useful lemma regarding comprehensions (see [21, Prop. 4.5]):

Lemma 2.21. *Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be a conjunctive doctrine with full comprehensions, and suppose that P has left adjoints along comprehensions, satisfying BCC, i.e. for every pullback*

$$\begin{array}{ccc} B_{P_f(\alpha)} & \xrightarrow{f'} & A_\alpha \\ {\scriptstyle \{\!|P_f(\alpha)|\!\}}\downarrow & \lrcorner & \downarrow{\scriptstyle \{\!|\alpha|\!\}} \\ B & \xrightarrow[f]{} & A \end{array}$$

we have $P_f \exists_{\{\!|\alpha|\!\}} = \exists_{\{\!|P_f(\alpha)|\!\}} P_{f'}$. *Then we have*

$$\alpha = \exists_{\{\!|\alpha|\!\}}(\top_{A_\alpha})$$

for every element α *in* $P(A)$.

Proof. Let α be an element of $P(A)$, and let us consider the comprehension $\{\!|\alpha|\!\} \colon A_\alpha \longrightarrow A$. First, it is direct to see that $\exists_{\{\!|\alpha|\!\}}(\top_{A_\alpha}) \leq \alpha$ since $\exists_{\{\!|\alpha|\!\}}$ is left adjoint to $P_{\{\!|\alpha|\!\}}$ and $P_{\{\!|\alpha|\!\}}(\top_A) = \top_{A_\alpha}$. Notice that since comprehensions are monomorphisms, and pullbacks along comprehensions always exist by Remark 2.20, we have that $P_{\{\!|\alpha|\!\}} \exists_{\{\!|\alpha|\!\}} = \mathrm{id}$ by BCC. In particular we have $P_{\{\!|\alpha|\!\}}(\exists_{\{\!|\alpha|\!\}}(\top_{A_\alpha})) = \top_{A_\alpha}$. So, by fullness, $\alpha \leq \exists_{\{\!|\alpha|\!\}}(\top_{A_\alpha})$, and then we can conclude that $\alpha = \exists_{\{\!|\alpha|\!\}}(\top_{A_\alpha})$. □

Corollary 2.22. *Let* $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be a conjunctive doctrine with full comprehensions, and suppose that* P *has left adjoints along comprehensions, satisfying BCC. Then* $\{\!|\alpha \wedge \beta|\!\} = \{\!|\alpha|\!\}\{\!|P_{\{\!|\alpha|\!\}}(\beta)|\!\}$.

Now we recall a class of doctrines whose fibred equality turns out to be equivalent to the morphism equality of their base morphisms (see [20, Prop. 2.2] and the original notion called "comprehensive equalizers" in [23]).

Definition 2.23. An elementary doctrine $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ has **comprehensive diagonals** if the arrow Δ_A is the comprehension of the element $\delta_A \in P(A \times A)$ for every objects A.

Recall from [20] that:

Definition 2.24. An elementary doctrine is called **m-variational** if it has full comprehensions and comprehensive diagonals.

We summarize some useful properties and results about pure existential m-variational doctrines.

Lemma 2.25. *Let* $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be a pure existential m-variational doctrine. Then*

1. *an arrow* $f \colon A \longrightarrow B$ *is monic if and only if* $P_{f \times f}(\delta_B) = \delta_A$;
2. *an element* $\phi \in P(A \times B)$ *is functional, i.e. it satisfies*

$$P_{\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle}(\phi) \wedge P_{\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle}(\phi) \leq P_{\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle}(\delta_B)$$

in $P(A \times B \times B)$ *if and only if* $\mathrm{pr}_A \{\!|\phi|\!\}$ *is monic, where* $\mathrm{pr}_A \colon A \times B \longrightarrow A$ *is the first projection since* P *has comprehensive diagonals.*

Proof. (*1*) If f is monic then $P_{f \times f}(\delta_B) = \delta_A$ follows by [23, Cor. 4.8], while the other direction follows from [20, Rem. 2.14].

(*2*) It is straightforward to check this point by employing the internal language of a doctrine and the previous point (1). For a complete algebraic proof we refer to [26, Lem. 2.22].

□

3. Generalized existential completion

We recall here from [34] how to complete a conjunctive doctrine to a generalized existential doctrine with respect to a class Λ of morphisms of $\mathcal{C}$ closed under composition, pullbacks and containing the identities.

Definition 3.1. A class of morphisms Λ of a category $\mathcal{C}$ is called a **left class of morphisms** if it satisfies the following conditions:

1. given an arrow fh of $\mathcal{C}$, if $f \in \Lambda$ and $h \in \Lambda$, then we have $fh \in \Lambda$;
2. pullbacks of arrows in Λ exist for every arrow of $\mathcal{C}$ and for every $f \in \Lambda$ and g of $\mathcal{C}$, for every pullback square

$$\begin{array}{ccc} D & \xrightarrow{f^*g} & A \\ {\scriptstyle g^*f}\downarrow & \lrcorner & \downarrow{\scriptstyle f\in\Lambda} \\ C & \xrightarrow[g]{} & B \end{array}$$

   we have that $g^*f \in \Lambda$;

3. every isomorphism is in Λ.

Example 3.2. For any category with finite products the class of product projections is an example of a left class of morphisms.

Actually, the class Λ represents *generalized projections* with respect to which we complete a conjunctive doctrine to a generalized existential one.

Definition 3.3. A **left class doctrine** is a pair (P, Λ), where $P : \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$ is a conjunctive doctrine and Λ is a left class of morphisms.

Now, we generalize the notion of "existential doctrine" to a doctrine closed under left adjoints to functors P_f for any morphism f in the left class Λ:

Definition 3.4. A left class doctrine (P, Λ) is called a **generalized existential doctrine** relative to a left class of morphisms Λ if, for any arrow $f : A \longrightarrow B$ of Λ, the functor

$$P_f : P(B) \longrightarrow P(A)$$

has a left adjoint $\exists_f$, and these satisfy:

(BCC) **Beck-Chevalley condition**: for any pullback

$$\begin{array}{ccc} X' & \xrightarrow{g'} & A' \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ X & \xrightarrow[g]{} & A \end{array}$$

with $g \in \Lambda$ (hence also $g' \in \Lambda$), for any $\beta \in P(X)$ the following equality holds

$$\exists_{g'} P_{f'}(\beta) = P_f \exists_g(\beta).$$

(FR) **Frobenius reciprocity**: for every morphism $f\colon X \longrightarrow A$ of Λ, for every element $\alpha \in P(A)$ and $\beta \in P(X)$, the following equality holds

$$\exists_f(P_f(\alpha) \wedge \beta) = \alpha \wedge \exists_f(\beta).$$

In order to simplify the notation, sometimes we will simply say that P is a **Λ-existential doctrine** to indicate a left class doctrine (P, Λ) that is a generalized existential doctrine.

The notion of *pure existential doctrine* presented in Definition 2.5 is a particular case of generalized existential doctrine in the sense of Definition 3.4 when we consider the class of all the projections of the base category.

Moreover, we adopt the following specific name for Λ-existential doctrines when Λ is the class of all the base morphisms:

Definition 3.5. Let $\mathcal{C}$ be a finite limit category. A **full existential doctrine** is a generalized existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ relative to the class of all the base morphisms.

Now, we define the 2-category $\mathbf{CD}_{\mathrm{lc}}$ as follows:

- **objects** are left class doctrines (P, Λ);
- **1-cells** are pairs $(F, b)\colon (P, \Lambda) \longrightarrow (R, \Lambda')$ where $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a functor such that for every $f \in \Lambda$, we have $F(f) \in \Lambda'$, F preserves pullbacks along morphisms of Λ and $b\colon P \longrightarrow R \circ F^{\mathrm{op}}$ is a natural transformation.
- a **2-cell** from (F, b) to (G, c) is a natural transformation $\theta\colon F \longrightarrow G$ such that for every A in $\mathcal{C}$ and every α in $P(A)$, we have
$$b_A(\alpha) \leq R_{\theta_A}(c_A(\alpha)).$$

Similarly, we denote by $\mathbf{ExCD}_{\mathrm{lc}}$ the 2-full subcategory of $\mathbf{CD}_{\mathrm{lc}}$ whose objects are Λ-existential doctrine (P, Λ), and a 1-cell of $\mathbf{ExCD}_{\mathrm{lc}}$ is a 1-cell (F, b) of $\mathbf{CD}_{\mathrm{lc}}$ such that the natural transformation b commutes with left adjoints of the left classes, i.e. given (P, Λ) and (R, Λ'), for every $f\colon A \longrightarrow B$ arrow of Λ, we have that the diagram

$$\begin{array}{ccc} P(A) & \xrightarrow{\exists_f} & P(B) \\ {\scriptstyle b_A}\downarrow & & \downarrow{\scriptstyle b_B} \\ RF(A) & \xrightarrow[\exists_{F(f)}]{} & RF(B) \end{array}$$

commutes, namely $b_B \exists_f = \exists_{F(f)} b_A$. As in the case of the ordinary existential doctrines, the 2-cell remains the same.

Given this setting, we recall from [34] the construction of the **generalized existential completion** of a left class doctrine (P, Λ) consisting of a Λ-existential doctrine $\mathsf{Ex}^{\Lambda}(P)\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ freely generated from P relative to Λ. Such a free construction provides a left 2-adjoint to the forgetful functor $U\colon \mathbf{ExCD}_{\mathrm{lc}} \longrightarrow \mathbf{CD}_{\mathrm{lc}}$.

Definition 3.6 (Generalized existential completion). For every object A of $\mathcal{C}$ consider the following preorder:

- the objects are pairs $(\, B \xrightarrow{g \in \Lambda} A\,,\ \alpha \in PB)$;
- $(\, B \xrightarrow{h \in \Lambda} A\,,\ \alpha \in PB) \leq (\, D \xrightarrow{f \in \Lambda} A\,,\ \gamma \in PD)$ if there exists $w \colon B \longrightarrow D$ such that

$$\begin{array}{ccc} B & \xrightarrow{\ w\ } & D \\ & \searrow^{h} & \downarrow f \\ & & A \end{array}$$

  commutes and $\alpha \leq P_w(\gamma)$.

It is easy to see that the previous data give a preorder. We denote by $\mathsf{Ex}^{\Lambda}(P)(A)$ the partial order obtained by identifying two objects when

$$(\, B \xrightarrow{h \in \Lambda} A\,,\ \alpha \in PB) \gtreqless (\, D \xrightarrow{f \in \Lambda} A\,,\ \gamma \in PD)$$

in the usual way. With abuse of notation we denote the equivalence class of an element in the same way.

Given a morphism $f \colon A \longrightarrow B$ in $\mathcal{C}$, let $\mathsf{Ex}^{\Lambda}(P)_f(\, C \xrightarrow{g \in \Lambda} B\,,\ \beta \in PC)$ be the object

$$(\, D \xrightarrow{f^* g} A\,,\ P_{g^* f}(\beta) \in PD)$$

where

$$\begin{array}{ccc} D & \xrightarrow{\ g^* f\ } & C \\ {\scriptstyle f^* g}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ A & \xrightarrow[\ f\ ]{} & B \end{array}$$

is a pullback because $g \in \Lambda$.

The assignment $\mathsf{Ex}^{\Lambda}(P) \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is called **the generalized existential completion of P with respect to Λ.**

Theorem 3.7. *For every left class doctrine (P, Λ) the doctrine $(\mathsf{Ex}^{\Lambda}(P), \Lambda)$ given by the assignment $\mathsf{Ex}^{\Lambda}(P) \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$is a Λ-existential doctrine.*

Proof. See [34, Thm 4.3].

□

Again, we fix the notation for the specific cases in which the left class of morphisms of the base category $\mathcal{C}$ is the class of all the projections of $\mathcal{C}$ or the class of all its morphisms:

Definition 3.8. Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be a conjunctive doctrine on a finite limit (lex) category $\mathcal{C}$. A **full existential completion** of P, denoted with the symbol $\mathsf{fEx}(P)$, is the generalized existential completion of P with respect to the class of all the base morphisms.

Definition 3.9. Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be a conjunctive doctrine on a category $\mathcal{C}$ with finite products. A **pure existential completion**, denoted with the symbol $\mathsf{pEx}(P)$, is the generalized existential completion of P with respect to the class of all the product projections in its base category.

Observe that we can define a canonical injection of left class doctrines

$$(\mathrm{id}_{\mathcal{C}}, \eta_P)\colon (P,\Lambda) \longrightarrow (\mathsf{Ex}^{\Lambda}(P),\Lambda) \tag{1}$$

where $(\eta_P)_A\colon P(A) \longrightarrow \mathsf{Ex}^{\Lambda}(P)(A)$ acts sending

$$\alpha \mapsto (\ A \xrightarrow{\mathrm{id}_A} A\, , \alpha \in P(A)).$$

Similarly, if P is Λ-existential, we can define the 1-cell of Λ-existential doctrines

$$(\mathrm{id}_{\mathcal{C}}, \varepsilon_P)\colon (\mathsf{Ex}^{\Lambda}(P),\Lambda) \longrightarrow (P,\Lambda)$$

which preserves the left-adjoints along morphisms of Λ, where $(\varepsilon_P)_A\colon P^{\mathrm{ex}}(A) \longrightarrow P(A)$ acts sending

$$(\ B \xrightarrow{f\in\Lambda} A\, , \beta \in P(B)) \mapsto \exists_f(\alpha).$$

It is direct to check that, if (P,Λ) is Λ-existential, then $\varepsilon_P\eta_P = \mathrm{id}_P$ and that $\mathrm{id}_{\mathsf{Ex}^{\Lambda}(P)} \leq \eta_P\varepsilon_P$.

Remark 3.10. Observe that every element $(\ B \xrightarrow{f} A\, , \beta)$ of $\mathsf{Ex}^{\Lambda}(P)(A)$ is equal to $\exists^{\Lambda}_f \eta_B(\beta)$ because $\exists^{\Lambda}_f$ acts as the post-composition. See [34, Prop. 4.2].

Notice that the universal properties of existential completion shown in [34], can be generalized for the arbitrary case of the generalized existential completion.

In particular, the proof of [34, Thm. 4.14] can be adapted to this more general setting, just observing that in the proof the fact that the class Λ is the class of projections is not concretely used. The proof depends only on the properties of closure under pullbacks, composition and identities of the class of projections, and hence it can be directly generalized as follows.

Theorem 3.11. *The forgetful 2-functor* $U\colon \mathbf{ExCD}_{\mathrm{lc}} \longrightarrow \mathbf{CD}_{\mathrm{lc}}$ *has a left 2-adjoint 2-functor* $E\colon \mathbf{CD}_{\mathrm{lc}} \longrightarrow \mathbf{ExCD}_{\mathrm{lc}}$*, acting on the objects as* $(P,\Lambda) \mapsto (\mathsf{Ex}^{\Lambda}(P),\Lambda)$*.*

We conclude this section recalling from [34, Ex. 5.9] the following example of pure existential completion.

Example 3.12 (The regular fragment of Intuitionistic Logic). Let $\mathcal{L}_{=,\exists}$ be the $(\top, \wedge, =, \exists)$-fragment of first-order Intuitionistic logic (also called regular in[16]), i.e. the fragment with top element, conjunction, equality and existential quantifiers. Then the elementary pure existential doctrine

$$\mathrm{LT}_{=,\exists}\colon \mathcal{C}^{\mathrm{op}}_{\mathcal{L}_{=,\exists}} \longrightarrow \mathbf{InfSL}$$

is the pure existential completion of the syntactic elementary doctrine

$$\mathrm{LT}_{=}\colon \mathcal{C}^{\mathrm{op}}_{\mathcal{L}_{=}} \longrightarrow \mathbf{InfSL}$$

associated with the Horn fragment $\mathcal{L}_{=}$, i.e. the $(\top, \wedge, =)$-fragment of first-order Intuitionistic logic. This result is a consequence of the fact that extending the language $\mathcal{L}_{=}$ with existential quantifications is a free operation, so by the known equivalence between doctrines and logic, the elementary pure existential doctrine $\mathsf{pEx}(\mathrm{LT}_{\mathcal{L}_{=}})$ must coincide with the syntactic doctrine $\mathrm{LT}_{=,\exists}$, since both completions are free.

4. A characterization of generalized existential completions

The main purpose of this section is to present a characterization of the generalized existential completion in logical terms, namely Theorem 4.16.

To achieve this goal, we introduce the notion of Λ-*existential-free* objects of a Λ-existential doctrine P to denote objects which are *free from the left adjoints* $\exists_f$ along Λ. This notion provides *an algebraic counterpart* of the logical concept of *existential-free formula*.

To this purpose we first define:

Definition 4.1. Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an Λ-existential doctrine. An object β of the fibre $P(B)$ is said to be a **Λ-existential splitting** if for every morphism in Λ

$$C \xrightarrow{g \in \Lambda} B$$

and for every element γ of the fibre $P(C)$, whenever $\beta = \exists_g(\gamma)$ holds then there exists an arrow $h\colon B \longrightarrow C$ such that

$$\beta = P_h(\gamma)$$

and $gh = \mathrm{id}$.

Notice that nothing guarantees that Λ-existential splitting objects as defined in Definition 4.1 are closed under reindexing along a morphism. Therefore, we introduce a stronger notion:

Definition 4.2. Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be a Λ-existential doctrine. An object α of the fibre $P(A)$ is said to be **Λ-existential-free** if for every morphism

$$B \xrightarrow{f} A$$

$P_f(\alpha)$ is a Λ-existential splitting.

Now, Λ-existential-free objects are closed under reindexing along any morphism:

Proposition 4.3. *Let* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be a* Λ*-existential doctrine. Let* α *be an element of the fibre* $P(A)$*. Then the following conditions are equivalent:*

1. α *is* Λ*-existential-free;*

2. *for every morphism* $g\colon B \longrightarrow A$ *of* Λ*,* $P_g(\alpha)$ *is* Λ*-existential-free.*

Proof. *(1)* $\Rightarrow$ *(2)* For any morphism $g\colon B \longrightarrow A$ of Λ, by definition of α as Λ-existential-free, we deduce that $P_g(\alpha)$ is Λ-existential splitting and that for any other morphism $h\colon C \longrightarrow B$ also $P_h(P_g(\alpha)) = P_{gh}(\alpha)$ is Λ-existential splitting, namely $P_g(\alpha)$ is Λ-existential-free.

(2) $\Rightarrow$ *(1)* Take for g the identity. ☐

The following proposition presents a useful equivalent characterization of Λ-existential splitting elements in terms of a form of *Existence Property* [39].

Proposition 4.4. *Let* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be a* Λ*-existential doctrine, let* β *be an element of the fibre* $P(B)$*. Then the following conditions are equivalent:*

1. β *is* Λ*-existential splitting;*

2. *for every morphism* $g\colon C \longrightarrow B$ *of* Λ *and for every element* γ *of the fibre* $P(C)$*, whenever* $\beta \leq \exists_g(\gamma)$ *holds then there exists an arrow* $h\colon B \longrightarrow C$ *such that* $\beta \leq P_h(\gamma)$ *and* $gh = \mathrm{id}$*.*

Proof. *(1)* $\Rightarrow$ *(2)* Suppose that $\beta \leq \exists_g(\gamma)$, with $g \in \Lambda$. In particular, we have that $\beta = \exists_g(\gamma) \wedge \beta$ and, applying FR, we have $\beta = \exists_g(\gamma \wedge P_g(\beta))$. Employing the assumption that β is Λ-existential splitting, and we have that there exists and arrow $h\colon B \longrightarrow C$ such that $\beta = P_h(\gamma \wedge P_g(\beta))$ and $gh = \mathrm{id}$. Therefore,

$$\beta = P_h(\gamma \wedge P_g(\beta)) = P_h(\gamma) \wedge P_{gh}(\beta) = P_h(\gamma) \wedge \beta$$

and then we can conclude that $\beta \leq P_h(\gamma)$.

(2) $\Rightarrow$ *(1)* If $\beta = \exists_g(\gamma)$ then, in particular, we have that $\beta \leq \exists_g(\gamma)$ and it follows by our assumption that there exists an arrow $h\colon B \longrightarrow C$ such that $\beta \leq P_h(\gamma)$ and $gh = \mathrm{id}$. Moreover, from $\exists_g(\gamma) \leq \beta$, then $\gamma \leq P_g(\beta)$ and finally $P_h(\gamma) \leq P_h(P_g(\beta)) = \beta$ because $fh = \mathrm{id}$. Hence, we conclude that $\beta = P_h(\gamma)$. ☐

Remark 4.5. The notion of Λ-**existential-free** object provides an algebraic version of the syntactic notion of *existential-free formula* because if a Λ-**existential-free** object β is equal to the existential quantification of an object γ along an arrow g in Λ, then β is equal to a reindexing of γ.

Here we fix some notation for the specific cases in which the left class of morphisms of the base category is the class of product projections or of all the morphisms. In particular:

- when Λ is the class of product projections we will speak of **pure-existential splitting** and **pure-existential-free**.
- when Λ is the class of all the morphisms of the base category we will speak of **full-existential splitting** and **full-existential-free**.

In order to provide a more precise logical intuition of the categorical notion of existential-free elements introduced in Definition 4.2, we anticipate this example which will follow from the main Theorem of this section (Theorem 4.16):

Example 4.6 (Regular fragment of Intuitionistic Logic). The pure-existential-free objects of the pure existential doctrine

$$\mathrm{LT}_{=,\exists}\colon \mathcal{C}^{\mathrm{op}}_{\mathcal{L}_{=,\exists}} \longrightarrow \mathbf{InfSL}$$

introduced in Example 3.12 are exactly the formulae which are free from the existential quantifier as a consequence of our main Theorem 4.16.

Now we are going to observe how the notion of Λ-existential-free object is related to well known choice principles. To this purpose we first introduce a generalization of the notion of Rule of Choice by relativizing the existence of a witness to the arrows of the class Λ.

Definition 4.7. For a Λ-existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, we say that P satisfies the **Generalized Rule of Choice** with respect to the left class Λ of morphisms, for short **Λ-RC** rule, if whenever

$$\top_A \leq \exists_g(\beta)$$

where $g\colon B \longrightarrow A$ is an arrow of Λ, then there exists an arrow $f\colon A \longrightarrow B$ such that

$$\top_A \leq P_f(\beta)$$

and $gf = \mathrm{id}$.

Remark 4.8. Observe that a Λ-existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ has Λ-RC if and only if for every object A of $\mathcal{C}$, the top element $\top_A \in P(A)$ is Λ-existential splitting. Indeed, if every top element is Λ-existential splitting then then every top element is Λ-existential-free because the reindexing along any morphism, which is an inf-semilattice morphism by definition, preserves the top element.

Definition 4.9. Given a Λ-existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, we also say that an element α of the fibre $P(A)$ is **Λ-covered** by an element $\beta \in P(B)$ if β is a Λ-existential-free object and there exists an arrow $f\colon B \longrightarrow A$ of Λ such that $\alpha = \exists_f(\beta)$.

Definition 4.10. We say that a Λ-existential doctrine P has **enough-Λ-existential-free objects** if for every object A of $\mathcal{C}$, any element $\alpha \in P(A)$ is **Λ-covered** by some element $\beta \in P(B)$ for some object B of $\mathcal{C}$, namely β is a Λ-existential-free element and

$$\alpha = \exists_g(\beta).$$

for an arrow $g\colon B \longrightarrow A$ in Λ.

Lemma 4.11. *If a Λ-existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow$ **InfSL** has enough-Λ-existential-free elements, then every Λ-existential splitting element is Λ-existential-free.*

Proof. Let us consider an element α of P that is Λ-existential splitting. Since P has enough-Λ-existential-free elements, we have that $\alpha = \exists_g(\beta)$ with $g \in \Lambda$ and β Λ-existential-free. Moreover, by definition of Λ-existential splitting, there exists an arrow h such that $\alpha = P_h(\beta)$. Since β is Λ-existential-free, we conclude that $P_h(\beta)$, which is α, is also Λ-existential-free by proposition 4.3. $\square$

Following the notation introduced before, we fix the notation for the specific cases in which the left class of morphisms of the base category is the class of product projections or of all the morphisms. In particular:

- when Λ is the class of product projections we will speak of **enough-pure-existential-free objects**;
- when Λ is the class of all the morphisms of the base category we will speak of **enough-full-existential-free objects**.

Now, we present some technical lemmas which will be useful to prove the main result of this section, namely Theorem 4.16 where we characterize doctrines arising as generalized existential completions.

Lemma 4.12. *Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow$ **InfSL** be a Λ-existential doctrine, and let us consider a Λ-existential-free element $\alpha \in P(B)$. If $\exists_f(\alpha) \leq \exists_g(\beta)$ where $f\colon B \longrightarrow A$ and $g\colon C \longrightarrow A$ are arrows of the base category $\mathcal{C}$ and $\beta \in P(C)$, then there exists an arrow $m\colon B \longrightarrow D$ where D is the vertex of the pullback of f along g such that*

- $(f^*g)m = \mathrm{id}$*;*
- $\alpha \leq P_{(g^*f)m}(\beta)$.

And hence also $f = g(g^*f)m$.

Proof. If $\exists_f(\alpha) \leq \exists_g(\beta)$, then $\alpha \leq P_f\exists_g(\beta)$, and applying BCC, we have $\alpha \leq \exists_{f^*g}P_{g^*f}(\beta)$, where

$$\begin{array}{ccc} D & \xrightarrow{\ f^*g\ } & B \\ {\scriptstyle g^*f}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ C & \xrightarrow[\ g\ ]{} & A \end{array}$$

is a pullback. Thus, since α is Λ-existential-free we can apply Proposition 4.4, and conclude that there exists a morphism $m\colon B \longrightarrow D$ such that $\alpha \leq P_{(g^*f)m}(\beta)$ and $(f^*g)m = \mathrm{id}$. In particular $f = g(g^*f)m$. $\square$

Recall that for every left class doctrine (P, Λ) we have a canonical morphism of $(\mathrm{id}, \eta)\colon P \longrightarrow \mathsf{Ex}^{\Lambda}(P)$ of left class doctrine relative to Λ as defined in (1).

Lemma 4.13. *Let (P, Λ) be a left class doctrine. If every element of the form $\eta_A(\alpha)$ for any object α of $P(A)$ with A object of $\mathcal{C}$ is Λ-existential splitting in $\mathsf{Ex}^{\Lambda}(P)$ then every element of the form $\eta_A(\alpha)$ is a Λ-existential-free object.*

Proof. It follows by the naturality of η_A. □

In the following definition we introduce in the context of Λ-existential doctrines a notion reminiscent of that of *projective cover* introduced in [4, Def. 2].

Definition 4.14. Let $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be a Λ-existential doctrine equipped with a fibred conjunctive subdoctrine $P' \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$. We say that P is an **Λ-existential cover** of P if for any object A, every element α' of $P'(A)$ is Λ-existential-free for P and every element α of $P(A)$ is Λ-covered by an element of P'.

Following the notation introduced in the previous definitions, when Λ is the class of product projections we will speak of **pure-existential cover** and when Λ is the class of all the morphisms of the base category we will speak of **full-existential cover**.

The existential cover doctrines of a doctrine P are unique as shown in the following:

Proposition 4.15. *Let $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be a Λ-existential doctrine equipped with a fibred subdoctrine $P' \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$. If P' is a Λ-existential cover of P then the elements of P' are exactly those elements of P which are Λ-existential-free. Hence if Q is another Λ-existential cover of P then $Q = P'$.*

Proof. Let $\alpha \in P(A)$ be a Λ-existential-free object. We have to prove that $\alpha \in P'(A)$. Since every element of P is covered by an element of P', we have that $\alpha = \exists_f(\beta)$ where β is Λ-existential-free, $\beta \in P'(B)$ and $f \colon B \longrightarrow A$ is in Λ. Therefore, we can conclude there exists an arrow $h \colon A \longrightarrow B$ such that $\alpha = P_h(\beta)$, and hence that α is an element of the fibre $P'(A)$ since β and its reindexings are in P'. □

Now we are ready to prove the main result of this section.

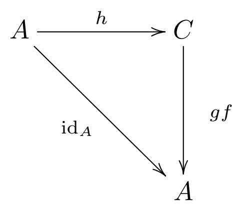
Theorem 4.16. *Let $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be a Λ-existential doctrine relative to a left-class Γ. Then the following are equivalent:*

1. *P is isomorphic to the generalized existential completion $\mathsf{Ex}^{\Lambda}(P')$ of a left class doctrine (P', Λ);*
2. *P has a (unique) Λ-existential cover in the sense of Definition 4.14.*
3. *P satisfies the following points:*
   (a) *P satisfies Λ-RC, i.e. each top element $\top_A$ of the fibre $P(A)$ is a Λ-existential-free object.*
   (b) *for every Λ-existential-free object α and β of $P(A)$, then $\alpha \wedge \beta$ is a Λ-existential-free object.*
   (c) *P has enough-Λ-existential-free objects.*

Proof. (*1*) $\Rightarrow$ (*2*) Suppose that P is a generalized existential completion $\mathsf{Ex}^{\Lambda}(P')$. We claim that the fibred subdoctrine P'' of P whole elements of the fibres are exactly the images of η (in particular $P' \cong P''$) is a Λ-existential cover of P. We first show that every element of the form $\eta_A(\alpha)$ is a Λ-existential splitting of P. Thus, let $\overline{\beta} := (C \xrightarrow{f} B, \beta)$ be an object of the fibre $\mathsf{Ex}^{\Lambda}(P')(B)$, and suppose that

$$\eta_A(\alpha) \leq \exists^{\Lambda}_{g}(\overline{\beta}) \tag{2}$$

where $g \colon B \longrightarrow A$. Recall that, by definition of the doctrine $\mathsf{Ex}^{\Lambda}(P')$, the inequality (2) means that there exists an arrow $h \colon A \longrightarrow C$ such that the following diagram commutes



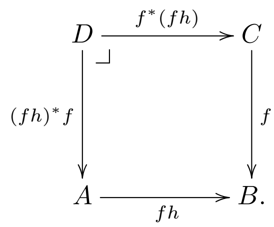
and

$$\alpha \leq P'_h(\beta) \tag{3}$$

We claim that

$$\eta_A(\alpha) \leq \mathsf{Ex}^{\Lambda}(P')_{fh}(\overline{\beta}). \tag{4}$$

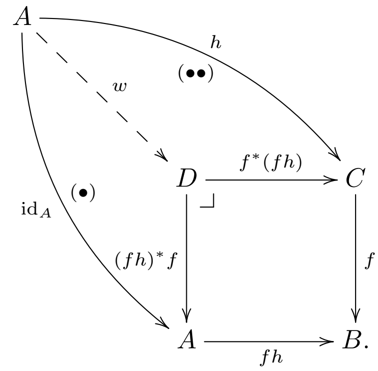
Thus, let us consider the pullback



Moreover,we have by definition of generalized existential completion

$$\mathsf{Ex}^{\Lambda}(P')_{fh}(\overline{\beta}) = (D \xrightarrow{(fh)^*f} A, P'_{f^*(fh)}(\beta)).$$

Thus, by the universal property of pullbacks, there exists an arrow $w \colon A \longrightarrow D$ such that the following diagram commutes



Hence, combining (3) with the triangle (••) we have that

$$\alpha \leq P'_h(\beta) = P'_w(P'_{f^*(fh)}(\beta)).$$

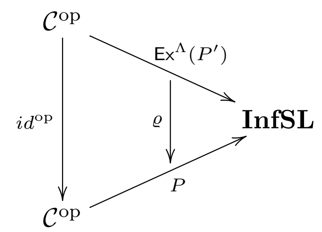
From this and the diagram (•) the claim (4) follows. By Proposition 4.4, this ends the proof that an element of the form $\eta_A(\alpha)$ is Λ-existential splitting. By Lemma 4.13 all such elements are also Λ-existential-free objects. Furthermore, observe that every element $(B \xrightarrow{f} A, \beta)$ of $\mathsf{Ex}^{\Lambda}(P')(A)$ is equal to $\exists^{\Lambda}_f \eta_B(\beta)$ because $\exists^{\Lambda}_f$ acts as the post-composition. Finally, by Remark 3.10, we have that every element of P is Λ-covered by an element of P''. This ends the proof that P'' is an existential cover of P and it is unique by Proposition 4.15.

(2) ⇒ *(3)* Let P' be a Λ-existential cover of P. Conditions $(a), (b)$ and (c) follow by definition of Λ-existential cover (Definition 4.14) and from the fact that, by Proposition 4.15, the elements of P' are exactly those elements of P which are Λ-existential-free for P.

(3) ⇒ *(1)* Let (P, Λ) be a Λ-existential doctrine satisfying $(a), (b)$ and (c). Then, we can define the conjunctive doctrine

$$P' \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

as the functor which sends an object A to the poset $P'(A)$ whose elements are the Λ-existential-free objects of $P(A)$ with the order induced from that of $P(A)$, and such that $P'_f = P_f$ for every arrow f of $\mathcal{C}$. Notice that this functor is a conjunctive doctrine because Λ-existential-free objects are stable under re-indexing by definition, and they are closed under the top element and binary conjunctions by the assumptions (a) and (b). Therefore, by the universal property of the generalized existential completion, there exists a 1-cell of $\mathbf{ExCD}_{\mathrm{lc}}$



where the map $\varrho_A\colon \mathsf{Ex}^{\Lambda}(P')(A) \longrightarrow P(A)$ sends

$$(B \xrightarrow{f} A , \alpha) \mapsto \exists_f(\alpha).$$

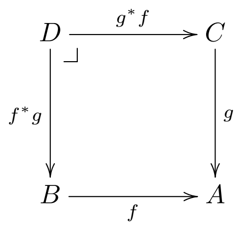
In particular, ϱ is a morphism of inf-semilattices and it is natural with respect to left adjoints along the class Λ. Moreover, notice that for every object A, we have that $\varrho_A\colon \mathsf{Ex}^{\Lambda}(P')(A) \longrightarrow P(A)$ is surjective on the objects since P has enough-Λ-existential-free objects, i.e. every object α of $P(A)$ is of the form $\exists_g(\beta)$ for some $g\colon B \longrightarrow A$ and $\beta \in P(B)$. Now we want to show that ϱ reflects the order, and hence that it is an isomorphism. Suppose that

$$\exists_f(\alpha) \leq \exists_g(\beta) \tag{5}$$

where $f\colon B \longrightarrow A$ and $g\colon C \longrightarrow A$ are arrows of $\mathcal{C}$, and $\alpha \in P(B)$ and $\beta \in P(C)$ are Λ-existential-free objets. Then we have to prove that

$$(B \xrightarrow{f} A , \alpha) \leq (C \xrightarrow{g} A , \beta).$$

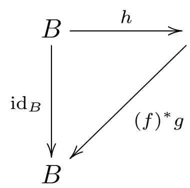
By (5) we have that $\alpha \leq P_f\exists_g(\beta)$. Now we can consider the pullback in $\mathcal{C}$



and, after applying BCC, we obtain

$$\alpha \leq \exists_{f^*g} P_{g^*f}(\beta).$$

Therefore, since α is a Λ-existential-free object, we can apply Proposition 4.4 and conclude that there exists an arrow $h\colon B \longrightarrow D$ such that



and

$$\alpha \leq P_h P_{g^* f}(\beta)$$

From this, it follows that the diagram

$$\begin{array}{ccc} B & \xrightarrow{(g^* f)h} & C \\ & \searrow^{f} & \downarrow{\scriptstyle g} \\ & & A \end{array}$$

commutes because $g(g^* f)h = f(f^* g)h = f$, and $\alpha \leq P_{(g^* f)h}(\beta)$. Thus, we have proved that $(\, B \xrightarrow{f} A, \alpha) \leq (\, C \xrightarrow{g} A, \beta)$. Therefore, we can conclude that (id, ϱ) is an invertible 1-cell of $\mathbf{ExCD}_{\mathrm{lc}}$, and then $\mathsf{Ex}^{\Lambda}(P') \cong P$.

□

A preliminary version of our characterization of generalized existential completion, namely Theorem 4.16, was presented in [35] in 2020, while a similar result, for the specific case in which the class Λ is the whole class of base morphisms, was independently presented in [7, Thm 6.3] in terms of $\exists$-prime elements.

Corollary 4.17. *If* Λ *is the class of all product projections of a primary doctrine* $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$*, then the pure existential completion* $\mathsf{pEx}(P)$ *satisfies the Rule of Choice of 2.13.*

Corollary 4.18. *If* Λ *is the class of all the base morphisms of a given left class doctrine* $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$*, then the full existential completion* $\mathsf{fEx}(P)$ *satisfies the Extended Rule of Choice in 2.12.*

Example 4.19 (Regular fragment of Intuitionistic Logic). From the fact that the elementary and pure existential doctrine

$$\mathrm{LT}_{=,\exists} \colon \mathcal{C}^{\mathrm{op}}_{\mathcal{L}_{=,\exists}} \longrightarrow \mathbf{InfSL}$$

introduced in Example 3.12 coincides with the pure existential completion of the syntactic elementary doctrine

$$\mathrm{LT}_{=} \colon \mathcal{C}^{\mathrm{op}}_{\mathcal{L}_{=}} \longrightarrow \mathbf{InfSL}$$

from Theorem 4.16 we can conclude that *the syntactic concept of existential-free formulas* coincides with our algebraic one in Definition 4.2 as anticipated in Example 4.6. Furthermore, the fact that the doctrine has enough-pure-existential-free objects gives an alternative proof that every formula of the $(\top, \wedge, =, \exists)$-fragment of first-order Intuitionistic logic can be presented in a prenex normal form. Moreover, the logic satisfies the choice principle RC by Corollary 4.17.

Finally, by Proposition 4.4, we have that for every syntactical existential-free formula $[a : A] \mid \alpha(a)$ if

$$[a : A] \mid \alpha(a) \vdash \exists b : B\ \beta(a, b)$$

then there exists a term $[a : A] \mid t(a) : B$ such that

$$[a : A] \mid \alpha(a) \vdash \beta(a, t(a)).$$

Observe that full existential completions come equipped with comprehensive diagonals defined in 2.4:

Corollary 4.20. *Let $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be a full existential completion. Then P has comprehensive diagonals.*

Proof. By [20, Prop. 2.21] showing that P has comprehensive diagonals is equivalent to showing that for any arrows $f, g \colon A \longrightarrow B$ we have that $f = g$ if and only if $\top_A \leq P_{\langle f,g\rangle}(\delta_B)$. Hence, suppose that $\top_A \leq P_{\langle f,g\rangle}(\delta_B)$, i.e. that $\exists_{\langle f,g\rangle}(\top_A) \leq \delta_B = \exists_{\Delta_B}(\top_B)$. By Theorem 4.16 we have that every top element is full-existential-free, so we can apply Lemma 4.12 and conclude that there exists an arrow m such that $\langle f, g\rangle = \Delta_B m$, and this means that $f = g$. The converse follows directly from the fact that $\exists_{\Delta_B} \dashv P_{\Delta_B}$ since if $g = f$ then $\top_A = P_f(\top_B) \leq P_f(P_{\Delta_B}\exists_{\Delta_B}(\top_B)) = P_{\langle f,f\rangle}(\delta_B) = P_{\langle f,g\rangle}(\delta_B)$. Therefore we can conclude that P has comprehensive diagonals. □

5. Doctrines with Hilbert's ϵ-operators as idempotent pure existential completion algebras

In [34] it is shown that the assignment $P \mapsto \mathsf{pEx}(P)$ of a primary doctrine to its pure existential completion extends to a lax-idempotent 2-monads

$$\mathrm{T}^{\mathrm{ex}} \colon \mathbf{PD} \longrightarrow \mathbf{PD}$$

on the 2-category of primary doctrines, and that the 2-category T^{ex}-**Alg** of algebras is isomorphic to the 2-category **ExD** of pure existential doctrines. This means that in the case Λ is the class of product projections, the pure existential doctrines are exactly the T^{ex}-algebras, also called pure existential completion algebras. In particular, the specialization of Theorem 4.16 provides a characterization of the *free algebras* of the monad T^{ex}.

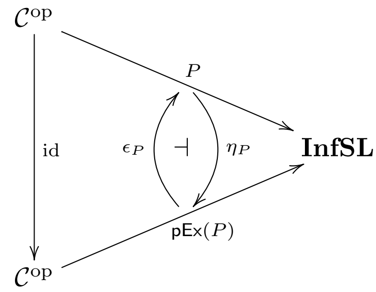
Here we focus our attention to the *idempotent* T^{ex}-*algebras*, namely those pure existential doctrines P such that $P \cong \mathsf{pEx}(P)$, and we show that they coincide with the pure existential doctrines equipped with Hilbert's epsilon operators (see [20] for relevant applications of such doctrines to tripos-to-topos constructions). This fact emphasizes the relationship between choice principles and our algebraic characterization of pure existential completions.

Theorem 5.1. *Let $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be a pure existential doctrine. Then P is isomorphic to $\mathsf{pEx}(P)$ if and only if P is equipped with Hilbert's ϵ-operators.*

Proof. We need to show that P is isomorphic to $\mathsf{pEx}(P)$ if and only if for every A and B of $\mathcal{C}$ and every $\alpha \in P(A \times B)$ there exists arrow $f \colon A \longrightarrow B$ such that

$$\exists_{\mathrm{pr}_1}(\alpha) \leq P_{\langle \mathrm{id}_A, f\rangle}(\alpha) \tag{6}$$

where $\mathrm{pr}_1 \colon A \times B \longrightarrow A$ is the first projection. Recall that by [34] we have the following adjunction



with

$$\mathrm{id}_P = \epsilon_P \eta_P \quad \text{and} \quad \mathrm{id}_{P^{\mathrm{ex}}} \leq \eta_P \epsilon_P.$$

In particular the doctrine P is isomorphic to $\mathsf{pEx}(P)$ if an only if $\eta_P \epsilon_P \leq \mathrm{id}_{P^{\mathrm{ex}}}$. By definition of η_P and ϵ_P, we have that for every object $(\, A \times B \xrightarrow{\mathrm{pr}_1} A, \alpha \in P(A \times B)) \in \mathsf{pEx}(P)(A)$, the element $\eta_P \epsilon_P(\, A \times B \xrightarrow{\mathrm{pr}_1} A, \alpha \in P(A \times B)) \in \mathsf{pEx}(P)(A)$ is given by

$$\eta_P \epsilon_P(\, A \times B \xrightarrow{\mathrm{pr}_1} A, \alpha \in P(A \times B)) = (\, A \xrightarrow{\mathrm{id}_A} A, \exists_{\mathrm{pr}_1}(\alpha) \in PA)$$

Then, by definition of the preorder in the doctrine $\mathsf{pEx}(P)$, the inequality $\eta_P \epsilon_P \leq \mathrm{id}_{P^{\mathrm{ex}}}$ holds if and only if for every object A of $\mathcal{C}$ and every $(\, A \times B \xrightarrow{\mathrm{pr}_1} A, \alpha \in P(A \times B)) \in \mathsf{pEx}(P)(A)$, there exists an arrow $f \colon A \longrightarrow B$ such that $\exists_{\mathrm{pr}_1}(\alpha) \leq P_{\langle \mathrm{id}_A, f \rangle}(\alpha)$. Therefore, we can conclude that P is isomorphic to $\mathsf{pEx}(P)$ if and only if P is equipped with Hilbert's ϵ-operators. □

Example 5.2. The well-known hyperdoctrine of subsets $\mathrm{P} \colon \mathrm{Set}^{\mathrm{op}} \longrightarrow$ **InfSL** over the usual category Set of sets formalizable in Zermelo-Fraenkel set theory with the Axiom of Choice is a notable example of doctrine equipped with Hilbert's ϵ-operators, and hence of a T^{ex}-idempotent algebra as shown in Theorem 5.1.

A second example of a T^{ex}-idempotent algebra is provided by the hyperdoctrine presented in Example 2.17.

6. Elementary pure existential completions must inherit their elementary structure

In [34] it was shown that the pure existential completion preserves the elementary structure. Here we show that elementary pure existential completions must inherit their elementary structure from their generating conjunctive doctrine, but this does not apply to generic generalized existential completions. Furthermore, any generalized existential completion relative to a class Λ containing product projections which inherits its elementary structure from its generating conjunctive doctrine must be a pure existential completion.

Theorem 6.1. *Let $P' \colon \mathcal{C}^{\mathrm{op}} \longrightarrow$ **InfSL** a primary doctrine and $P = \mathsf{pEx}(P')$ its pure existential completion. The following conditions are equivalent*

1. *P' is elementary*
2. *$\mathsf{pEx}(P')$ is elementary.*

Proof. For *(1)* $\Rightarrow$ *(2)* see [34, Prop. 6.1].

(2) $\Rightarrow$ *(1)* First of all recall from Definition 2.4 that left adjoints along the functors $P_{\Delta\times\mathrm{id}}$ are of the form

$$\exists_{\Delta\times\mathrm{id}}(\alpha) = P_{\langle\mathrm{pr}_1,\mathrm{pr}_2\rangle}(\alpha) \wedge P_{\langle\mathrm{pr}_2,\mathrm{pr}_2\rangle}(\delta_A)$$

Since P' is the fibred subdoctrine of pure-existential-free objects of P, in order to show that P' is elementary it is enough to show that the equality predicates δ_A are pure-existential-free objects so that $\exists_{\Delta\times\mathrm{id}}$ restricts to P'.

By Theorem 4.16, we have that P has enough-pure-existential-free elements hence, by Lemma 4.11, it is enough to show that δ_A is a pure-existential-splitting object to conclude that δ_A is pure-existential-free. Therefore, let us suppose that $\delta_A = \exists_{\Delta_A}(\top_A) \leq \exists_{\mathrm{pr}_{A\times A}}(\beta)$. Then by the equality adjunction

$$\top_A \leq P_{\Delta_A}\exists_{\mathrm{pr}_{A\times A}}(\beta). \tag{7}$$

Hence, by BCC, we have that

$$\top_A \leq \exists_{\mathrm{pr}_A} P_{\Delta_A\times\mathrm{id}_B}(\beta). \tag{8}$$

By Theorem 4.16 P satisfies the Rule of Choice, hence there exists an arrow $f \colon A \longrightarrow B$ such that $\top_A \leq P_{\langle\mathrm{id}_A,f\rangle}P_{\Delta_A\times\mathrm{id}_B}(\beta)$. Now since $(\Delta_A\times\mathrm{id}_B)\langle\mathrm{id}_A,f\rangle = (\Delta_A\times f)\Delta_A$ we obtain $\top_A \leq P_{\Delta_A}P_{\Delta_A\times f}(\beta)$ and by the equality adjunction we conclude

$$\delta_A = \exists_{\Delta_A}(\top) \leq P_{\Delta_A\times f}(\beta). \tag{9}$$

By Proposition 4.4, this ends the proof that δ_A is pure-existential-splitting object. Then, by Lemma 4.11, we can conclude that δ_A is a pure-existential-free object.

□

Generalized existential completions with an elementary structure inherited by their generating doctrine are essentially pure existential completions:

Proposition 6.2. *Let $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow$ **InfSL** be an elementary existential doctrine and Λ is a left class of morphisms in $\mathcal{C}$ containing the projections. If P' is a Λ-existential cover of P such that every fibred equality $\delta_A \in P(A\times A)$ is contained in $P'(A\times A)$, then P' is a pure-existential cover of P. Therefore $P \cong \mathsf{pEx}(P') \cong \mathsf{Ex}^{\Lambda}(P')$.*

Proof. Clearly every element of P' is pure-existential-free in P since Λ contains the product projections. Hence, to meet our purpose it is enough to show that every element $\alpha \in P(A)$ is of the form $\alpha = \exists_{\mathrm{pr}_A}(\gamma)$ with γ element of P'. Now, since P' is a Λ-existential cover of P, for every $\alpha \in P(A)$ there exists an arrow $g\colon B \longrightarrow A$ and an element $\beta \in P'(B)$ such that $\alpha = \exists_g(\beta)$. Moreover, since P is elementary and existential we have that

$$\alpha = \exists_g(\beta) = \exists_{\mathrm{pr}_A}(P_{\mathrm{pr}_B}(\beta) \wedge P_{\langle \mathrm{pr}_A, g\,\mathrm{pr}_B\rangle}(\delta_A))$$

which actually gives the claimed representation because, employing the assumption that every δ_B is an element of $P'(B \times B)$ and the fact that P' is a fibred subdoctrine of P, we can conclude that $P_{\mathrm{pr}_B}(\beta) \wedge P_{\langle \mathrm{pr}_1, g\,\mathrm{pr}_2\rangle}(\delta_B)$ is an element of P'.

$\square$

Remark 6.3. Theorem 6.1 does not hold for all generalized existential completions. This can be expected since the process of *freely* adding left adjoints destroys previously existing ones.

In fact, there exist elementary generalized existential completions which do not inherit their elementary structure from their generating elementary conjunctive doctrine, namely their existential cover. We will present concrete examples of this case in the following section, in Remark 7.5.

Remark 6.4 (Elementary and pure existential doctrines which are not full existential doctrines). Recall from [29, 20], that any elementary existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ has left adjoint of every re-indexing functor P_g given by the assignment

$$\exists_g(\alpha) = \exists_{\mathrm{pr}_1}(P_{\mathrm{pr}_2}(\alpha) \wedge P_{\langle \mathrm{pr}_1, g\,\mathrm{pr}_2\rangle}(\delta_A)).$$

There are some particular cases in which every $\exists_g$ satisfies BCC and FR, for example when P has full comprehensions and comprehensive diagonals (see [20, Lem. 5.8].).

However, as observed in [29, Rem. 4.6], these left adjoints do not necessarily satisfy BCC and FR in general. In particular, notice that we can employ the pure existential completion to construct examples of existential and elementary doctrines (with finite limit base) that do not satisfy BCC or FR along all the morphisms, i.e. that are not full existential. In fact, let us consider an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ with a finite limit base, and let us consider its pure existential completion $\mathsf{pEx}(P)$. Employing the universal properties of pure and full existential completions (and the fact that the pure existential completion preserves the elementary structure by Theorem 6.1), we have that $\mathsf{pEx}(P)$ satisfies BCC and FR along all the morphisms of the base, namely it is a full existential doctrine, if and only if $\mathsf{pEx}(P) \cong \mathsf{fEx}(P)$.

Now, observe that the embedding of $\mathsf{pEx}(P)$ into $\mathsf{fEx}(P)$, which holds by constructions, is an equivalence if and only if every element $(\, B \xrightarrow{f} A \,, \alpha) \in \mathsf{fEx}(P)(A)$ is equivalent to an element of the form $(\, A \times C \xrightarrow{\mathrm{pr}_A} A \,, \beta) \in \mathsf{pEx}(P)(A)$. By definition of the order of $\mathsf{fEx}(P)(A)$ this holds when there exist two arrows $g_1\colon B \longrightarrow A \times C$ and $g_2\colon A \times C \longrightarrow B$ such that $fg_2 = \mathrm{pr}_A$ and $\mathrm{pr}_A\, g_1 = f$ with $\alpha \leq P_{g_1}(\beta)$ and $\beta \leq P_{g_2}(\alpha)$. In particular, $fg_2 = \mathrm{pr}_A$ implies that f is epi. Therefore, each pure existential completion of an elementary doctrine on a finite limit category, where not all f

are epi, like for example Set, does not generally satisfy both BCC and FR along every morphism.

7. Examples of generalized existential completions

In this section we provide relevant examples of generalized existential completions.

7.1. Λ-Weak subobjects doctrines

Definition 7.1 (**Λ-weak subobjects doctrine).** Let $\mathcal{C}$ be a category and let Λ be a left class of morphisms of $\mathcal{C}$. We define the Λ-**weak subobjects** doctrine, or the doctrine of **Λ-weak subobjects**, as the functor

$$\Psi_\Lambda \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

assigning to every object A of $\mathcal{C}$ the poset

$$\Psi_\Lambda(A) := \{\ B \xrightarrow{[f]} A \mid f \in \Lambda\}$$

with the usual order given by the factorization, i.e. the partial order induced by the usual preorder on morphisms given by $f \leq g$ if there exists an arrow h such that $f = gh$. The re-indexing functors $\Psi_\Lambda(f)$ act by pulling back the elements of the fibres.

One can directly check that Ψ_Λ is a Λ-existential doctrine, since the left adjoints are given by the post-composition of arrows of Λ. Moreover, Ψ_Λ is clearly an example of a fibred subdoctrine of the doctrine $\Psi_{\mathcal{C}}$ of weak subobjects presented in Example 2.8(2).

Definition 7.2. Given a category $\mathcal{C}$, we call the **trivial doctrine** on $\mathcal{C}$ the functor

$$\Upsilon \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

assigning the poset with only one element to every object A of $\mathcal{C}$ denoted as $\Upsilon(A) = \{\top\}$.

By definition of generalized existential completion, we immediately obtain the following theorem.

Theorem 7.3. *The Λ-existential doctrine $\Psi_\Lambda \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is isomorphic to the generalized existential completion of the trivial doctrine $\Upsilon \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$.*

Proof. Observe that the top elements *cover* the weak subobjects doctrine, i.e. for every object A and for every element $[f] \colon B \longrightarrow A$ in the fibre $\Psi_{\mathcal{D}}(A)$ we have $[f] = \exists_f(\top_B)$, since the left adjoints $\exists_f$ are given by the post-composition and the top element $\top_B$ is the identity morphism $[\mathrm{id}_B]$. □

Example 7.4. Observe that the weak subobjects doctrine defined in Example 2.14 is a Λ-**weak subobjects** doctrine where Λ is the class of all the morphisms of a finite limit base category $\mathcal{C}$. Hence from Theorem 7.3 the weak subobjects doctrine is isomorphic to the *full existential completion* of the trivial doctrine $\Upsilon \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ and then, applying Corollary 4.18, we obtain a proof alternative to that in [20] that the doctrine $\Psi_{\mathcal{C}}$ of weak subobjects satisfies the Extended Rule of Choice in 2.12.

Remark 7.5. Notice that Λ-weak subobjects doctrines provide an example of generalized existential completions that do not preserve the elementary structure of the base. In fact, a doctrine $\Psi_\Lambda \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is elementary if Λ contains the diagonal arrow and, by Theorem 7.3, Ψ_Λ is generated from the trivial primary doctrine $\Upsilon \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ which is elementary, too. However, the elementary structure of Υ is not in general preserved by the generalized existential completion, like for example when $\mathcal{C}$ is Set and Λ is the class of all morphisms of Set.

7.2. Examples of generalized existential completions with full comprehensions

In this section we show that relevant examples of generalized existential doctrines are given by doctrines with full comprehensions closed under compositions.

Definition 7.6. Given a conjunctive doctrine

$$P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

with full comprehensions, we say that P has **composable comprehensions** if its comprehensions are closed under compositions, namely if for all objects α in $P(A)$ and β in $P(B)$ with $\{\!\{\alpha\}\!\} \colon B \longrightarrow A$ then we have $\{\!\{\alpha\}\!\}\{\!\{\beta\}\!\} = \{\!\{\gamma\}\!\} \colon C \longrightarrow A$ for γ in $P(C)$.

Note that γ is unique by Remark 2.19.

Our aim now is to show that a doctrine $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ with full composable comprehensions is an instance of the generalized existential completion construction with respect to the class Λ_{comp} of comprehensions.

To this purpose, we first show that such a P has left adjoints along comprehensions:

Proposition 7.7. *Let $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be a conjunctive doctrine with full comprehensions. Then the following conditions are equivalent:*

1. *P has composable comprehensions.*
2. *P has left adjoints along the comprehensions satisfying BCC.*

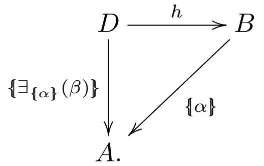
Proof. *(1)* $\Rightarrow$ *(2)* If comprehensions compose it is direct to check that then there exists a left adjoint along $\{\!\{\alpha\}\!\}$ defined as

$$\exists_{\{\!\{\alpha\}\!\}}(\beta) := \gamma$$

where γ is the *unique* element of $P(A)$ such that $\{\!\{\alpha\}\!\}\{\!\{\beta\}\!\} = \{\!\{\gamma\}\!\}$. This satisfies BCC after recalling from Remark 2.20 that the pullbacks of comprehensions along any map exist in $\mathcal{C}$ (see [26] for further details).

(2) $\Rightarrow$ *(1)* Suppose that the doctrine P has left adjoints along comprehensions, and consider two comprehensions $\{\!\{\beta\}\!\} \colon C \longrightarrow B$ and $\{\!\{\alpha\}\!\} \colon B \longrightarrow A$. We claim that the comprehension $\{\!\{\exists_{\{\!\{\alpha\}\!\}}(\beta)\}\!\} \colon D \longrightarrow A$ is the composition $\{\!\{\alpha\}\!\}\{\!\{\beta\}\!\}$.

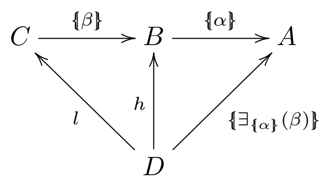
First of all observe that $\top_C \leq P_{\{\!\{\alpha\}\!\}\{\!\{\beta\}\!\}}\exists_{\{\!\{\alpha\}\!\}}(\beta)$ follows from the unit of the adjunction $\beta \leq P_{\{\!\{\alpha\}\!\}}\exists_{\{\!\{\alpha\}\!\}}(\beta)$ by full comprehension and hence and by Remark 2.19 there exists a unique $g \colon C \longrightarrow D$ such that $\{\!\{\exists_{\{\!\{\alpha\}\!\}}\}\!\}g = \{\!\{\alpha\}\!\}\{\!\{\beta\}\!\}$. Now we are going to prove that g is invertible. Observe that by Lemma 2.21, we have $\alpha = \exists_{\{\!\{\alpha\}\!\}}(\top_B)$ from which $\exists_{\{\!\{\alpha\}\!\}}(\beta) \leq \exists_{\{\!\{\alpha\}\!\}}(\top_B) \leq \alpha$ and hence by Remark 2.19 there exists a unique $h\colon D \longrightarrow B$ such that the following diagram commutes



Now, observe that $P_{\{\!\{\alpha\}\!\}}\exists_{\{\!\{\alpha\}\!\}}(\beta) = \beta$ holds by BCC and the fact that $\{\!\{\alpha\}\!\}$ is monic and hence we conclude

$$\top_D \leq P_{\{\!\{\exists_{\{\!\{\alpha\}\!\}}(\beta)\}\!\}}(\,\exists_{\{\!\{\alpha\}\!\}}(\beta)\,) = P_h(P_{\{\!\{\alpha\}\!\}}(\exists_{\{\!\{\alpha\}\!\}}(\beta)) \leq P_h\beta$$

which by comprehension yields the existence of a unique $l\colon D \longrightarrow C$ such that



(10)

commutes. It is immediate to observe that l is an inverse of g by the fact that comprehensions are monic.

□

Corollary 7.8. *Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow$ **InfSL** be a conjunctive doctrine with full composable comprehensions. Then every left adjoint $\exists_{\{\!\{\alpha\}\!\}}$ satisfies FR.*

Proof. We have to prove that $\exists_{\{\!\{\alpha\}\!\}}(P_{\{\!\{\alpha\}\!\}}(\beta) \wedge \gamma) = \beta \wedge \exists_{\{\!\{\alpha\}\!\}}(\gamma)$. Let use define $\sigma := \exists_{\{\!\{\alpha\}\!\}}(P_{\{\!\{\alpha\}\!\}}(\beta) \wedge \gamma)$ and $\sigma' := \beta \wedge \exists_{\{\!\{\alpha\}\!\}}(\gamma)$. To show the result it is enough to prove that $\{\!\{\sigma\}\!\} = \{\!\{\sigma'\}\!\}$ since comprehensions are full. By Proposition 7.7 we have that $\{\!\{\sigma\}\!\} := \{\!\{\alpha\}\!\}\{\!\{P_{\{\!\{\alpha\}\!\}}(\beta) \wedge \gamma\}\!\}$ and hence, applying Corollary 2.22, we have

$$\{\!\{\sigma\}\!\} = \{\!\{\alpha\}\!\}\{\!\{\gamma\}\!\}\{\!\{P_{\{\!\{\alpha\}\!\}\{\!\{\gamma\}\!\}}(\beta)\}\!\} = \{\!\{\exists_{\{\!\{\alpha\}\!\}}(\gamma)\}\!\}\{\!\{P_{\{\!\{\exists_{\{\!\{\alpha\}\!\}}(\gamma)\}\!\}}(\beta)\}\!\} = \{\!\{\sigma'\}\!\}$$

□

Example 7.9. Recall that a $\mathcal{M}$-*category* is a pair $(\mathcal{C}, \mathcal{M})$ where $\mathcal{C}$ is a category and $\mathcal{M}$ is a stable system of monics, i.e. $\mathcal{M}$ is a collection of monics which includes all isomorphisms and is closed under composition and pullbacks. Observe that a stable system of monics is essentially what was called a *dominion* in [32], an *admissible system of subobjects* in [30], a *notion of partial maps* in [31] and a *domain structure* in [27]. Given an $\mathcal{M}$-category $(\mathcal{C}, \mathcal{M})$ one can define the doctrine of $\mathcal{M}$-*subobjects*

$$\mathrm{Sub}_{\mathcal{M}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \textbf{InfSL}$$

where $\mathrm{Sub}_{\mathcal{M}}(X)$ if the inf-semilattice of $\mathcal{M}$-subobjects of $\mathcal{M}$, and the action of $\mathrm{Sub}_{\mathcal{M}}$ on a morphism $f\colon X \longrightarrow Y$ of $\mathcal{C}$ is given by pulling back the $\mathcal{M}$-subobjects of Y along f. It is direct to prove that given an $\mathcal{M}$-category $(\mathcal{C}, \mathcal{M})$, the doctrine of $\mathcal{M}$-subobjects $\mathrm{Sub}_{\mathcal{M}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ has full composable comprehensions and hence by prop. 7.7 it has left adjoints along comprehensions.

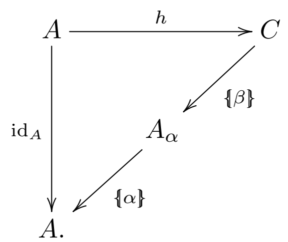
Proposition 7.10. *Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be a conjunctive doctrine with full composable comprehensions, and let Λ_{comp} be the class of the comprehensions. Then:*

1. *Λ_{comp} is a left class of morphisms of $\mathcal{C}$;*
2. *for every $\alpha \in P(A)$, $\alpha = \exists_{\{\!|\alpha|\!\}}(\top_{A_\alpha})$;*
3. *an element $\alpha \in P(A)$ is a Λ_{comp}-existential-free object if and only if it is the top element. In particular the doctrine P satisfies the Λ_{comp}-RC rule.*

Proof. (*1*) The class of comprehensions contains identities, and comprehensions are stable under pullbacks by Remark 2.20 while they compose by assumption.

(*2*) The second point follows from Proposition 7.7 and Lemma 2.21.

(*3*) First we show that every top element is Λ_{comp}-existential-free. If $\top_A \leq \exists_{\{\!|\alpha|\!\}}(\beta)$, then we have that $\{\!|\top_A|\!\} = \mathrm{id}_A$ factors on $\{\!|\exists_{\{\!|\alpha|\!\}}(\beta)|\!\}$, which is equal to the arrow $\{\!|\alpha|\!\}\{\!|\beta|\!\}$ by Proposition 7.7. Then there exists an arrow h such that the following diagram commutes



Now we define $f := \{\!|\beta|\!\}h$. Thus, we have that $\{\!|\alpha|\!\}f = \mathrm{id}_A$ and

$$P_f(\beta) = P_h P_{\{\!|\beta|\!\}}(\beta) = \top_A.$$

Now we prove the converse. By point 2. we have that $\alpha = \exists_{\{\!|\alpha|\!\}}(\top_{A_\alpha})$, and then if α is a Λ_{comp}-existential-free object, $\alpha \leq \exists_{\{\!|\alpha|\!\}}(\top_{A_\alpha})$ implies that there exists an arrow $f\colon A \longrightarrow A_{\{\!|\alpha|\!\}}$ such that $\alpha \leq P_f(\top_{A_\alpha}) = \top_A$ and $\{\!|\alpha|\!\}f = \mathrm{id}_A$. Since $\mathrm{id}_A = \{\!|\top_A|\!\}$ by fullness of comprehensions we conclude $\top_A \leq \alpha$ and hence $\top_A = \alpha$. $\square$

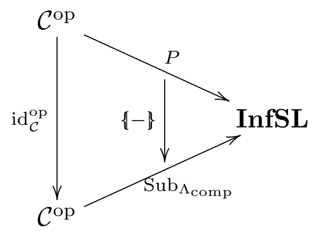
By employing Proposition 7.10 we derive the following theorem:

Theorem 7.11. *Every conjunctive doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ with full and composable comprehensions is an instance of the generalized existential completion of the trivial conjunctive doctrine*

$$\Upsilon\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

where for every object A the poset $\Upsilon(A)$ contains only the top element, with respect to the class Λ_{comp} of comprehensions. Therefore, it is isomorphic to the doctrine $\mathrm{Sub}_{\Lambda_{\text{comp}}}\colon \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$ *where* $\mathrm{Sub}_{\Lambda_{\text{comp}}}(A)$ *is the class of comprehensions of A.*

Proof. It follows by point (2) of Theorem 4.16 and Proposition 7.10. In particular, we have an isomorphism given by the 1-cell



where $\{\!\!\{-\}\!\!\}_A\colon P(A) \longrightarrow \mathrm{Sub}_{\Lambda_{\text{comp}}}(A)$ sends α to $\{\!\!\{\alpha\}\!\!\}$. ☐

Moreover, one can directly check that the previous result, together with Example 7.9, extends to an isomorphism of 2-categories.

Theorem 7.12. *We have an isomorphism of 2-categories*

$$\mathcal{M}\text{-}\mathbf{Cat} \cong \mathbf{CE}_{\text{c}}$$

*where $\mathcal{M}$-**Cat** is the 2-category of $\mathcal{M}$-categories and $\mathbf{CE}_{\text{c}}$ is the 2-category of doctrines with full composable comprehensions.*

Remark 7.13. Recall from [20, Prop. 2.19] that pure existential m-variational doctrines have left adjoints along all the morphisms of the base and these satisfy BCC. Hence, by Proposition 7.7, every existential m-variational doctrine has composable comprehensions. Moreover, by Corollary 7.8, we have that left adjoints along comprehensions satisfy FR.

Therefore, observe that existential m-variational doctrines are also a generalized existential completions thanks to Proposition 7.10 and Theorem 7.11:

Corollary 7.14. *Every existential m-variational doctrine P is an instance of the generalized existential completion construction of the conjunctive doctrine*

$$\Upsilon\colon \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$$

with respect to the class of comprehensions of P.

Remark 7.15. Recall that in the case of pure existential m-variational doctrines, the equivalence of Theorem 7.12 restricts to an equivalence between the 2-category of existential m-variational doctrines and the 2-category of proper stable factorization systems [17]. We refer to [25, 24] and [11] for a proof of this equivalence.

In the case of the subobjects doctrine, we have that the class of comprehensions is exactly that of all the monomorphims, and hence we conclude:

Corollary 7.16. *The subobjects doctrine* $\mathrm{Sub}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *of a category* $\mathcal{C}$ *with finite limits is isomorphic to the generalized existential completion of the trivial doctrine* $\Upsilon\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$*, with respect to the class of all the monomorphisms.*

By employing some results of this section, in particular Proposition 7.10, we provide the following characterization of existential m-variational doctrines satisfying the Rule of Unique Choice 2.10:

Proposition 7.17. *Let* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be a primary doctrine. Then the following are equivalent:*

1. *P is existential, m-variational, and it satisfies the Rule of Unique Choice RUC;*
2. *$\mathcal{C}$ is regular and $P = \mathrm{Sub}_{\mathcal{C}}$;*
3. *P is the $\mathcal{M}$-existential completion of the trivial doctrine $\Upsilon\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, where $\mathcal{M}$ is a class of morphisms of $\mathcal{C}$ such that*
   - *(a) there exists a class $\mathcal{E}$ of morphisms of $\mathcal{C}$ such that $(\mathcal{E},\mathcal{M})$ is a proper, stable, factorization system on $\mathcal{C}$;*
   - *(b) for every projection $\mathrm{pr}_A\colon A \times B \longrightarrow A$ of $\mathcal{C}$, if $\mathrm{pr}_A f$ is a monomorphism and $f \in \mathcal{M}$ then $\mathrm{pr}_A f \in \mathcal{M}$.*

Proof. (*1*) ⇒ (*2*) It follows from [20, Prop. 5.3] and [14, Thm. 4.4.4 and Thm. 4.9.4].

(*2*) ⇒ (*3*) It follows from Corollary 7.16 and Remark 7.15 by taking $\mathcal{M}$ the class of all monomorphisms.

(*3*) ⇒ (*1*) Under these assumptions, in particular by (a), we have that P is an existential m-variational doctrine by Remark 7.15 and the arrows of $\mathcal{M}$ are exactly the comprehensions of P. Now, let us consider a functional, entire relation $\rho \in P(A \times B)$. First, notice that ρ functional implies $\mathrm{pr}_A \{\!|\rho|\!\}$ monic by Lemma 2.25. Hence, by our assumption (b), $\mathrm{pr}_A \{\!|\rho|\!\}$ is a comprehension. Moreover, we have that

$$\top_A \leq \exists_{\mathrm{pr}_A}(\rho) = \exists_{\mathrm{pr}_A \{\!|\rho|\!\}}(\top) \tag{11}$$

because ρ is entire and $\rho = \exists_{\{\!|\rho|\!\}}(\top)$ by Lemma 2.21. Therefore we can apply (3) of Proposition 7.10, because $\mathrm{pr}_A \{\!|\rho|\!\}$ is a comprehension. Hence, there exists an arrow $f\colon A \longrightarrow (A \times B)_\rho$, where $(A \times B)_\rho$ denotes the domain of $\{\!|\rho|\!\}$, such that

$$\top_A \leq P_f(\top)$$

and $\mathrm{pr}_A \{\!|\rho|\!\} f = \mathrm{id}_A$, namely $\{\!|\rho|\!\} f = \langle \mathrm{id}_A, \mathrm{pr}_B \{\!|\rho|\!\} f\rangle$. In particular, we have that

$$\top_A \leq P_f(\top) = P_{\{\!|\rho|\!\} f}(\rho) = P_{\langle \mathrm{id}, \mathrm{pr}_B \{\!|\rho|\!\} f\rangle}(\rho).$$

Therefore, P satisfies the RUC. □

7.3. The realizability hyperdoctrine

In this section we are going to prove that all the realizability triposes [13, 29, 41] are full generalized existential completions as shown independently in [7] and presented in the talk by the second author [35] in 2020. A different presentation of realizability triposes via free constructions (essentially acting on the representing object of a given representable indexed preorder) can be found in [9].

We start by recalling some related fundamental notions. A **partial combinatory algebra** (pca) is specified by a set $\mathbb{A}$ together with a partial binary operation $(-)\cdot(-)$: $\mathbb{A}\times\mathbb{A}\rightharpoonup\mathbb{A}$ for which there exist elements $k, s\in\mathbb{A}$ satisfying for all $a, a', a''\in\mathbb{A}$ that

$$k\cdot a\downarrow \text{ and } (k\cdot a)\cdot a'\equiv a$$

and

$$s\cdot a\downarrow,\ (s\cdot a)\cdot a'\downarrow,\ \text{and } ((s\cdot a)\cdot a')\cdot a''\equiv(a\cdot a'')\cdot(a'\cdot a'')$$

where $e\downarrow$ means "e is defined" and $e\equiv e'$ means "e is defined if and only e' is, and in that case they are equal". For more details we refer to [41].

Given a pca $\mathbb{A}$, we can consider the **realizability hyperdoctrine**

$$\mathcal{P}\colon \mathrm{Set}^{\mathrm{op}}\longrightarrow \mathbf{InfSL}$$

over Set introduced in [13], see also [29]. For each set X, the partial ordered set $(\mathcal{P}(X),\leq)$ is defined as follows: let $\mathrm{P}(\mathbb{A})^X$ denote the set of functions from X to the powerset $\mathrm{P}(\mathbb{A})$ of $\mathbb{A}$. Let $\leq$ be the binary relation on this set defined as: $\alpha\leq\beta$ if there exists an element $c\in\mathbb{A}$ such that for all $x\in X$ and all $a\in\alpha(x)$ we have that $c\cdot a$ is defined and $c\cdot a\in\beta(x)$. By standard properties of pcas this relation is reflexive and transitive, i.e. it is a preorder. Then $\mathcal{P}(X)$ is defined as the quotient of $\mathrm{P}(\mathbb{A})^X$ by the equivalence relation $\sim$ generated by $\leq$. The partial order on the equivalence classes $[\alpha]$ is that induced by $\leq$.

Given a function $f\colon X\longrightarrow Y$ of Set, the functor $\mathcal{P}_f\colon\mathcal{P}(Y)\longrightarrow\mathcal{P}(X)$ sends an element $[\alpha]\in\mathcal{P}(Y)$ to the element $[\alpha\circ f]\in\mathcal{P}(X)$ given by the composition of the two functions. With these assignments $\mathcal{P}\colon\mathrm{Set}^{\mathrm{op}}\longrightarrow\mathbf{InfSL}$ is a hyperdoctrine, see [29, Example 2.3] or [41].

Here, we just recall the conjunctive, elementary and existential structure of the hyperdoctrine $\mathcal{P}$ by employing the operations $\mathbf{p}$, $\mathbf{p_1}$ and $\mathbf{p_2}$, called *pairing* and *projections* operators respectively, defined in [41] by using the elements k and s. In each fibre $\mathcal{P}(X)$ we have:

- $\top_X := [\lambda x\in X.\mathbb{A}]$;
- $[\alpha]\wedge[\beta] := [\lambda x\in X.\{(\mathbf{p}\cdot a)\cdot b \mid a\in\alpha(x) \text{ and } b\in\beta(x)\}]$
- $\delta_X := [\lambda(x_1,x_2)\in X\times X.\mathbb{A} \text{ if } x_1=x_2, \emptyset \text{ otherwise}]$;
- for every projection $\mathrm{pr}_X\colon X\times Y\longrightarrow X$, the functor $\exists_{\mathrm{pr}_X}$ sends an element $[\gamma]\in\mathcal{P}(X\times Y)$ to the following element of the fibre $\mathcal{P}(X)$:

$$\exists_{\mathrm{pr}_X}([\gamma]) = [\lambda x\in X.\bigcup_{y\in Y}\gamma(x,y)].$$

It also follows that $\mathcal{P}$ has left adjoints along arbitrary functions, defined as in Remark 6.4, which satisfy BCC and FR, i.e. $\mathcal{P}$ is a full existential doctrine (see [33, Lem. 5.2] or [41]).

Now we show that the realizability hyperdoctrine is an instance of full existential completion. Thus, we fix the class Λ_{Set} to be the class of all functions of Set.

Hence, we need to understand what are the full-existential-free objects of the realizability hyperdoctrine.

Definition 7.18. Let $\mathbb{A}$ be a pca, and let $\mathcal{P}\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be the realizability tripos associated with the pca $\mathbb{A}$. An element $[\gamma]\colon X \longrightarrow \mathrm{P}(\mathbb{A})$ of the fibre $\mathcal{P}(X)$ is said to be a **singleton predicate** if there exists a singleton function $\alpha\colon X \longrightarrow \mathrm{P}(\mathbb{A})$, i.e. $\alpha(x) = \{a\}$ for some a in $\mathbb{A}$, such that $\gamma \sim \alpha$.

We claim that the singleton predicates are exactly the full-existential-free objects provided that we assume *the axiom of choice* in our meta-theory as we do.

Lemma 7.19. *Every singleton predicate is a full-existential-free object.*

Proof. Assume that $[\gamma]\colon X \longrightarrow \mathrm{P}(\mathbb{A})$ is a singleton predicate, i.e. γ assigns to every element x of X a singleton $\gamma(x) = \{a\}$ for some a in $\mathbb{A}$. In order to show that this is a full-existential-free object is enough to show that it is a full-existential splitting since the action of $\mathcal{P}_f$ preserves singletons, i.e. for every function $m\colon Z \longrightarrow X$ of Set, we have that $\mathcal{P}_m([\gamma]) = [\gamma \circ m]$ is again a singleton predicates. To this purpose, observe that if $[\gamma] \leq \exists_g([\beta])$ for some $[\beta] \in \mathcal{P}(Y)$ and $g\colon Y \longrightarrow X$, then there exists an element $\overline{b} \in \mathbb{A}$ such that for every $x \in X$ and every $a \in \gamma(x)$ then $\overline{b} \cdot a \in \exists_g \beta(x)$. By Remark 6.4 we have

$$\exists_g(\beta) = \exists_{\mathrm{pr}_X}(\mathcal{P}_{\mathrm{pr}_Y}(\beta) \wedge \mathcal{P}_{g\times \mathrm{id}_X}(\delta_X))$$

and since $\gamma(x)$ is a singleton $\{a\}$ for every $x \in X$, we have that $\overline{b} \cdot a \in \bigcup_{y\in Y}(\mathcal{P}_{\mathrm{pr}_Y}(\beta) \wedge \mathcal{P}_{g\times \mathrm{id}_X}(\delta_X))(y, x)$. Hence we have that $\overline{b} \cdot a = (\mathbf{p} \cdot c_1) \cdot c_2$ for some $c_1 \in \beta(y_a)$, $c_2 \in \mathbb{A}$, and for some $y_a \in Y$ such that $g(y_a) = x$. By the axiom of choice, we can define a function $f\colon X \longrightarrow Y$ such that $f(x) = y_a$. In particular, we have that $\lambda z.\mathbf{p_1}(\overline{b}z)$ realizes $\gamma \leq \mathcal{P}_f(\beta)$, and we have that $gf = \mathrm{id}$. This concludes the proof that singletons predicates are full-existential splitting (by Proposition 4.4). ☐

Corollary 7.20. *For every set X, $[\top_X] \in \mathcal{P}(X)$ is a full-existential-free object.*

Proof. Let us consider an element $c \in \mathbb{A}$ and the singleton function $\iota_c\colon X \longrightarrow \mathrm{P}(\mathbb{A})$ given by the constant assignment $x \mapsto \{c\}$. It is straightforward to check that $\top_X \sim \iota_c$, i.e. that $[\top_X]$ is a singleton predicate, and then a full-existential-free object by Lemma 7.19. ☐

Notice that from Corollary 7.20 it follows that:

Corollary 7.21. *The realizability hyperdoctrine $\mathcal{P}\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ satisfies the Extended Rule of Choice in 2.12.*

Remark 7.22. Notice that singleton predicates are closed under binary meet since if $[\alpha]$ and $[\beta]$ are singletons of $\mathcal{P}(X)$, by definition, we have that

$$([\alpha] \wedge [\beta]) = [\lambda x \in X.\{(p \cdot a) \cdot b \mid a \in \alpha(x) \text{ and } b \in \beta(x)\}].$$

is again a singleton function.

Employing Corollary 7.20 and Remark 7.22, we can define following primary doctrine:

Definition 7.23. Let $\mathbb{A}$ be a pca, and let $\mathcal{P}\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be the realizability tripos associated with the pca $\mathbb{A}$. We denote by $\mathcal{P}^{\mathrm{sing}}\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ the fibred sub-doctrine of $\mathcal{P}\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ whose elements of the fibre $\mathcal{P}^{\mathrm{sing}}(X)$ are only singleton predicates of $\mathcal{P}(X)$.

Now we ready to show the main result of this section.

Theorem 7.24. *The realizability hyperdoctrine $\mathcal{P}\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is isomorphic to the full existential completion of the primary doctrine $\mathcal{P}^{\mathrm{sing}}\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ of singletons.*

Proof. By Lemma 7.19 we have that every singleton predicate is a full-existential-free element, by Corollary 7.20 we have that also every top element is a full-existential-free element, and by Remark 7.22 we have that singleton predicates are closed under binary meet. Moreover, one can directly check that the singleton predicates *cover* the realizability hyperdoctrine, i.e. for every $[\beta] \in \mathcal{P}(X)$ there exists a singleton predicate $[\alpha] \in \mathcal{P}(Y)$ and a function $g\colon Y \longrightarrow X$ such that $[\beta] = \exists_g([\alpha])$. In particular, we can define $Y := \biguplus_{x\in X} \beta(x)$ the disjoint union of the fibres of β and $g\colon Y \longrightarrow X$ as its first projection π_1 sending (x, a) to x and $\alpha\colon Y \longrightarrow \mathrm{P}(\mathbb{A})$ as the singleton predicate sending (x, b) to $\{b\}$. Then, by Proposition 4.15 we have that singleton predicates are exactly the full-existential-free elements of $\mathcal{P}$. Therefore the realizability hyperdoctrine satisfies all the conditions of Theorem 4.16 (2) and then we can conclude that it is the full existential completion of the primary doctrine $\mathcal{P}^{\mathrm{sing}}$ of singletons. ☐

7.4. Localic doctrines

Let us consider a locale $\mathcal{A}$, i.e. $\mathcal{A}$ is a poset with finite meets and arbitrary joins, satisfying the *infinite distributive law* $x \wedge (\bigvee_i y_i) = \bigvee_i (x \wedge y_i)$. Recall from [13, 29] that, as we anticipated in Example 2.8(4), given a locale $\mathcal{A}$ we can define the *canonical localic doctrine* of $\mathcal{A}$:

$$\mathcal{A}^{(-)}\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

by assign $I \mapsto \mathcal{A}^I$, and the partial order is provided by the pointwise partial order on functions $f\colon I \longrightarrow \mathcal{A}$. Propositional connectives are defined pointwise. The existential quantifier along a given function $f\colon I \longrightarrow J$ maps a function $\phi \in \mathcal{A}^I$ to $\exists_f(\phi)$ given by $j \mapsto \bigvee_{\{i\in I \mid f(i)=j\}} \phi(i)$ and these are called existential since they satisfy the FR and BCC conditions.

Now we are going to consider localic doctrines [13, 29] whose locale is **supercoherent** as defined in [1]. For the reader's convenience we just recall a few related basic notions from [1].

Definition 7.25. An element c of a locale $\mathcal{A}$ is said to be **supercompact** if whenever $c \leq \bigvee_{k \in K} b_k$, there exists $\overline{k} \in K$ such that $c \leq b_{\overline{k}}$.

Remark 7.26. As a consequence of Definition 7.25, notice that a supercompact element of a non-trivial locale must be different from the bottom of the locale (since the bottom is the join on the empty set).

Definition 7.27. A locale $\mathcal{A}$ is called **supercoherent** if:

- each element $d \in \mathcal{A}$ is a join $d = \bigvee_I c_i$ of supercompact elements c_i;
- supercompact elements are closed under finite meets.

Remark 7.28. Recall that the category of locales is defined as the opposite of the category of *frames* and hence the notion of locale coincides with that of frame (see for example [15]). In particular, the category of supercoherent frames, denoted by **SCohFrm** in [1], is a coreflexive subcategory of the category of frames . This result together with the general notion of supercoherent frame was introduced in [1].

Following the notation introduced in [1], let **M** be the category of meet-semilattices, and let **Frm** be the category of frames. Recall from [15] that we can define a functor

$$D\colon \mathbf{M} \longrightarrow \mathbf{Frm}$$

sending a meet-semilattice $\mathcal{M}$ to the *down-set lattice* $D(\mathcal{M})$, i.e. the lattice of all the $X \subseteq \mathcal{M}$ such that $a \in X$ implies that for all $b \leq a$ we have $b \in X$ and the order is provided by the set-theoretical inclusion. This functor is left adjoint to the inclusion functor, see [1, Lem. 1] or [15, Thm. 1.2]. In particular, we have a natural injection $\eta\colon \mathcal{M} \longrightarrow D(\mathcal{M})$ sending $a \mapsto \downarrow(a)$. It is direct to see that sets of the form $\downarrow(a)$ are supercompact elements in $D(\mathcal{M})$.

Remark 7.29. Recall from [1, Rem. 3] that the functor $D\colon \mathbf{M} \longrightarrow \mathbf{Frm}$ induces an equivalence $\mathbf{M} \equiv \mathbf{SCohFrm}$. Essentially this means that supercoherent frames are the frame completion of inf-semilattices.

Definition 7.30. Let $\mathcal{A}$ be an arbitrary locale, and let $\mathcal{A}^{(-)}\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be the localic doctrine. We call **supercompact predicate** an element $\phi \in \mathcal{A}^J$ such that $\phi(j)$ is a supercompact element for every $j \in J$.

Lemma 7.31. *Every supercompact predicate of the localic doctrine* $\mathcal{A}^{(-)}\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *is a full-existential-free element.*

Proof. Let $\phi \in \mathcal{A}^J$ be a supercompact predicate. If we have

$$\phi \leq \exists_f(\psi)$$

for some $f\colon I \longrightarrow J$ and $\psi \in \mathcal{A}^I$, then in particular

$$\phi(j) \leq \bigvee_{\{i \in I \mid f(i)=j\}} \psi(i)$$

and, since $\phi(j)$ is supercompact, there exists $\bar{i}^j \in \{i \in I | f(i) = j\}$ such that $\phi(j) \leq \psi(\bar{i}^j)$. Hence, we can define a function $g\colon J \longrightarrow I$ sending $j \mapsto \bar{i}^j$. Hence, by definition, we have that $fg = \mathrm{id}$ and $\phi \leq \mathcal{A}_g^{(-)}(\psi)$, and then we can apply Proposition 4.4 and conclude that ϕ is full-existential-splitting. Moreover it is direct to see that supercompact predicates are stable under re-indexing, and hence supercompact predicates are full-existential-free.

□

Theorem 7.32. *Let $\mathcal{A}$ be a locale.*

1. *If $\mathcal{A}$ is a supercoherent locale then the localic doctrine $\mathcal{A}^{(-)}\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow$ **InfSL** is isomorphic to the full existential completion of the primary doctrine $\mathcal{N}^{(-)}\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow$ **InfSL** of supercompact predicates of $\mathcal{A}$.*

2. *If the localic doctrine $\mathcal{A}^{(-)}\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow$ **InfSL** is a full existential completion then its locale $\mathcal{A}$ is supercoherent.*

Proof. (*1*) Let $\mathcal{A}$ be a supercoherent locale. By Lemma 7.31 supercompact predicates are full-existential-free. Moreover since $\mathcal{A}$ is supercoherent, the supercompact predicates are closed under finite meets, and since every element of $a \in \mathcal{A}$ is a join of supercompact elements, then every element of every fibre of $\mathcal{A}^{(-)}$ is covered by a supercompact predicates. In particular, if we consider an element $\phi\colon I \longrightarrow \mathcal{A}$, we have that for every $i \in I$, $\phi(i) = \bigvee_{j \in J_i} c^i_j$ with c^i_j supercompact elements. So, we can define the disjoint sum $J = \oplus_{i \in I} J_i$ and a supercompact predicate $\psi\colon J \longrightarrow \mathcal{A}$ given by $\psi(i,j) = c^i_j$. Then it is direct to see that $\phi = \exists_f(\psi)$ where $f\colon J \longrightarrow I$ is the function mapping $f(i,j) = i$.

Hence, by Proposition 4.15 we have that supercompact elements are exactly the full-existential-free elements of $\mathcal{A}^{(-)}$ and by Theorem 4.16 we can conclude that $\mathcal{A}^{(-)}$ is the full existential completion of the primary doctrine $\mathcal{N}^{(-)}\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow$ **InfSL** such that $\mathcal{N}(I)$ is the inf-semilattice whose objects are the supercompact predicates of $\mathcal{A}^I$.

(*2*) Suppose that the localic doctrine $\mathcal{A}^{(-)}\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow$ **InfSL** is a full existential completion of a doctrine $P\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow$ **InfSL**, i.e. $\mathcal{A}^{(-)} \cong \mathsf{fEx}(P)$. We first show that P is a doctrine of supercompact predicates of $\mathcal{A}$ after recalling by Theorem 4.15 that fibres of P are made of all the full-existential-free objects of $\mathcal{A}^{(-)}$.

Now suppose that ϕ is a full-existential-free object. Let $\tilde{j}$ any index in J and suppose that $\phi(\tilde{j}) \leq \bigvee_{i \in I} b_i$. Then we can define an element $\psi\colon J \times I \longrightarrow \mathcal{A}$ such that $\psi(\tilde{j}, i) = b_i$ and $\psi(j,i) = \top$ for every $j \neq \tilde{j}$. Hence it follows that $\phi(j) \leq \bigvee_{i \in I} \psi(j,i)$ for each j in J which means $\phi \leq \exists_{\mathrm{pr}_I}(\psi)$. Then, since ϕ is a full-existential-free object, by Proposition 4.4 there exists a function $g\colon J \longrightarrow I$ such $\phi \leq \mathcal{A}^{(-)}_{\langle \mathrm{id}_J, g\rangle}(\psi)$, and then, in particular $\phi(\tilde{j}) \leq \psi(\tilde{j}, g(\tilde{j})) = b_{g(\tilde{j})}$. Hence every element $\phi(j)$ is supercompact. Finally, every element $a \in \mathcal{A}$ is the join of supercompact elements because we can define the function $\alpha\colon \{*\} \longrightarrow \mathcal{A}$ as $* \mapsto a$, and since the doctrine has enough-full-existential-free elements, we have $\alpha = \exists_f \phi$ for a full-existential-free object ϕ, i.e. a is the join of supercompact elements. □

8. Future work

A preliminary version of the characterization of generalized existential completion presented here has already been fruitfully employed in recent works [36, 37, 38] to give a categorical version to the Gödel Dialectica interpretation [8] in terms of quantifier-completions.

In the future, we intend to broaden the study of regular and exact completions of generalized existential completions initiated in [26] by including dialectica triposes in [2] and modified realizability triposes [9, 40].

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