Constructive version of Boolean algebra

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Abstract

The notion of overlap algebra introduced by G. Sambin provides a constructive version of complete Boolean algebra. Here we show that his notion of overlap morphism corresponds classically to that of map preserving arbitrary joins. Moreover we prove that the power collection of a set is the free overlap algebra join-generated from the set.

Then, we generalize the concept of overlap algebra and overlap morphism in various ways to provide constructive versions of the category of Boolean algebras with maps preserving arbitrary existing joins.

1 Introduction

The classical Tarski's representation theorem (see [14]), asserting that atomic complete Boolean algebras coincide with powersets, does not hold any longer if one works in a constructive foundation. The reason is simple: whenever one drops the law of excluded middle and works in a constructive set theory, powersets stop to be Boolean algebras.

To provide a constructive version of the mentioned representation theorem, we are faced at least with two questions. One is: what kind of representation theorem holds constructively for atomic complete Boolean algebras? The other is: what is the algebraic structure of constructive powersets?

The second question has been answered by G. Sambin in his forthcoming book [11]. To this purpose, he introduced the concept of overlap algebra in terms of a complete Heyting algebra equipped with a notion of *overlap* between elements of the algebra. The notion of overlap relation is a positive way to express when the meet of two elements is different from the bottom and it is crucial for giving the definition of atom. By using these notions, Sambin proved that power-collections coincide with atomic *set-based* overlap algebras. His statement is about power-collections and not powersets, because he developed his results in a predicative and constructive set theory, as the minimalist foundation in

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[9, 8], where a power-collection of a set is not generally a set, not even over a singleton. Moreover, predicatively, non-trivial examples of overlap algebras are necessarily collections and not sets. Of course, the notion of set-based algebra (i.e. an algebra which has a set of join-generators) is relevant only if one works predicatively, considering that in an impredicative setting with powersets all overlap algebras are sets and hence set-based.

Furthermore in [11], Sambin showed also that his overlap algebras are classically nothing but complete Boolean algebras: a further evidence of the fact that the overlap algebra representation theorem is an exhaustive constructive version of the classical one. Moreover, the first author in [2] gave a constructive version of the representation theorem of complete Boolean algebras in terms of regular opens of a topological space: he proved that overlap algebras coincide with the collection of regular opens of a pointfree topology, called formal topology, introduced by Sambin in [13]. Examples of non-atomic overlap algebras are given in [3].

In this paper, we show that Sambin's notion of overlap morphism in [11] corresponds classically to that of map preserving arbitrary joins. In categorical terms, the category of overlap algebras in [11] is classically equivalent to that of complete Boolean algebras with maps preserving arbitrary joins. Furthermore, by working in the minimalist foundation introduced in [8], we prove that the power-collection of subsets of a set is the free overlap algebra join-generated from the set.

Then, we generalize the notion of overlap algebra and overlap morphism to provide a constructive version of the category of (non necessarily complete) Boolean algebras and maps preserving existing joins. Basically we observe that join-completeness is not needed when proving the equivalence between the category of overlap algebras and that of complete Boolean algebras.

However, in generalizing the notion of overlap algebra to provide a constructive version of Boolean algebra, we are faced with various choices. Indeed we define different structures equipped with an overlap relation with the same properties as the one given by Sambin but related only to existing joins: we define a Boolean algebra with overlap, called *o-Boolean algebra*, a Heyting algebra with overlap, called *o-Heyting algebra*, and a lattice with an opposite (pseudocomplement) and overlap, called *oo-lattice*. We show that such structures with overlap, for short *o-structures*, classically (and impredicatively) all coincide with that of Boolean algebra. But constructively they do not. Moreover, we define a notion of morphism between o-structures as a generalization of Sambin's overlap morphism in [11]. We then show that the corresponding categories of o-structures are all classically (and impredicatively) equivalent to that of Boolean algebras and maps preserving existing arbitrary joins. In proving such an equivalence, one can see that *classically* (and impredicatively) our notion of morphism between o-structures has the sufficient and necessary properties that allow it to be lifted to an overlap morphism between the Dedekind-MacNeille completions of the given Boolean algebras.

As a future work we intend to provide a constructive explanation of such an insight by investigating the existence of constructive join-completions of our o-structures.

Moreover, as a further research goal, we would like to test whether our o-structures can be useful for providing constructive versions of classical Stone representation theorems for atomic Boolean algebras and possibly also for complete Boolean algebras or simply Boolean algebras.

2 Some remarks on foundations

When developing our theorems we assume to work in the extensional set theory of the two-level minimalist foundation in [8]. This was designed according to the principles given in [9].

The main characteristic is that our foundational set theory is *constructive* and *predicative*. The former characteristic says that our set theory is governed by intuitionistic logic, which does not validate excluded middle, and it enjoys a realizability model where to extract programs from proofs. The latter characteristic says that the power-collection of subsets of a set X, written $\mathcal{P}(X)$, is not a set, in general, but a proper collection. Hence in our set theory we have the notion of set and that of collection. To keep predicativity, a subset of a set X can *only* be defined by comprehension on a formula $\varphi(x)$, for $x \epsilon X$, with quantifiers restricted to sets; such a subset is written $\{x \epsilon X \mid \varphi(x)\}$.

It is worth mentioning that a join-complete semilattice that is a set is necessarily trivial in a predicative constructive foundation [5]. Therefore in a constructive predicative setting we are lead to define a join-complete semilattice as a collection closed under joins of *set-indexed* families (we cannot assume *arbitrary* joins to exist, otherwise we fall again into a trivial lattice).

As done in [12] and [1] we can make the definition of join-complete semilattice easier to handle by restricting ourselves to the notion of *set-based* join-complete semilattice, namely a semilattice that is join-generated from a set(-indexed family) of elements, called *join-generators*. In particular, this means that each element is the join of all the join-generators below it. For this to make sense (within our foundations) the collection of all join-generators below an element has to be a subset (equivalently, a set-indexed family). In order to achieve this, we need the order of the semilattice to be defined by a formula containing only quantifications over sets. For instance, the order in $\mathcal{P}(X)$ that makes it a set-based join-complete semilattice is written as follows: $A \subseteq B$ iff $(\forall x \epsilon X)(x \epsilon A \Rightarrow x \epsilon B)$.

In this paper we will deal with overlap algebras, which are in particular join-complete semilattices, and we assumed them to be all set-based.

Before starting, let us agree on some notation: X, Y, S and T will always denote sets with elements x, y, z,... a, b, c, ... and subsets A, B, C, D, E,... U, V, W, Z; on the contrary, $\mathcal{P}$ and Q will always stand for collections whose elements will be written as p, q, r, ... Accordingly, we use two different symbols to distinguish between the two kind of membership: ϵ for sets and subsets, : for collections.

3 Overlap algebras

Sambin introduced the notion of overlap algebra in [11] in order to give an algebraic description of the structure of the power-collection $\mathcal{P}(X)$ of a set X. He wanted to prove a constructive version of the Tarski's classical theorem in [14] stating that atomic complete Boolean algebras are powersets. Of course complete Boolean algebras are not apt to this purpose because, in a constructive foundation, power-collections are not Boolean algebras but only complete Heyting algebras. Sambin noticed also that the language of complete Heyting algebras does not seem to allow a positive definition of atom powerful enough to prove the desired representation theorem. To solve this problem he enriched the notion of complete Heyting algebra with an overlap relation, to form the so called *overlap algebra*. By using the overlap notion he then defined a notion of atom that allowed him to prove the desired representation theorem between power-collections and atomic (set-based) overlap algebras.

In the case of power-collections, the notion of overlap between two subsets $A, B \subseteq X$, denoted by $A \between B$, expresses *inhabitedness* of their intersection and is therefore defined as follows:

$$A \between B \stackrel{def}{\Longleftrightarrow} (\exists x \,\epsilon\, X)(x \,\epsilon\, A \cap B)\ . \tag{1}$$

Classically (and impredicatively), overlap algebras coincide with complete Boolean algebras (Proposition 3.8). Indeed, any complete Boolean algebra $\mathcal{B}$ is equipped with an overlap relation defined as $x \wedge y \neq 0$ for $x, y \,\epsilon\, \mathcal{B}$ (see [11]). Thus the notion of overlap is a constructive positive way to express *inhabitedness* of the meet of two elements.

3.1 Definition and basic properties

Definition 3.1 *An* overlap algebra *(*o-algebra *for short) is a triple* $(\mathcal{P}, \leq, \gtrless)$ *where* $(\mathcal{P}, \leq)$ *is a complete lattice and* $\gtrless$ *is a binary relation on* $\mathcal{P}$ *satisfying the following properties:*

- $p \gtrless q \ \Rightarrow\ q \gtrless p$ *(symmetry)*
- $p \gtrless q \ \Rightarrow\ p \gtrless (p \wedge q)$ *(meet closure)*
- $p \gtrless \bigvee_{i \epsilon I} q_i \iff (\exists i \,\epsilon\, I)(p \gtrless q_i)$ *(splitting of join)*
- $(\forall r : \mathcal{P})(r \gtrless p \ \Rightarrow\ r \gtrless q) \implies p \leq q$ *(density)*

(for any p and q in $\mathcal{P}$).

We say that an o-algebra $(\mathcal{P}, \leq, \gtrless)$ is *set-based* if the join-semilattice $(\mathcal{P}, \leq)$ admits a *base*, that is, a set-indexed family of generators (with respect to the operation of taking set-indexed joins), called *join-generators*. We agree to make no notational distinction between the base and its index set; thus S is a base for $\mathcal{P}$ if $p = \bigvee\{a \,\epsilon\, S \mid a \leq p\}$ for any $p : \mathcal{P}$. For the reasons mentioned in section 2, we shall assume each o-algebra to be set-based.

It is easily seen that all quantifications over the elements of an o-algebra $\mathcal{P}$ can be reduced to the base. For instance, the "density" axiom in definition 3.1 is equivalent to

$$(\forall a \,\epsilon\, S)(a \between p \;\Rightarrow\; a \between q) \;\Longrightarrow\; p \leq q \tag{2}$$

(S being a base). Just to get acquainted with the axioms, let us prove that "density" implies (2) (the other direction being trivial). It is enough to prove that $(\forall a \,\epsilon\, S)(a \between p \Rightarrow a \between q) \Longrightarrow (\forall r : \mathcal{P})(r \between p \Rightarrow r \between q)$. Take $r : \mathcal{P}$ such that $r \between p$. Since $r = \bigvee\{a \,\epsilon\, S \mid a \leq r\}$ and $\between$ splits $\bigvee$, there exists $a \,\epsilon\, S$ such that $a \leq r$ and $a \between p$. By hypothesis, we get $a \between q$ and hence $r \between q$ by the splitness property again.

For every set X, the structure $(\mathcal{P}(X), \subseteq, \between)$ (see equation (1) for the definition of the symbol $\between$) is an o-algebra with the singletons forming a base. In addition, we shall see in the following sections that $\mathcal{P}(X)$ is also atomistic and free over X. Here below, we list some useful properties of o-algebras. Detailed proofs can be found in [11], [15], [2] and [3].

Proposition 3.2 *Let $\mathcal{P}$ be an o-algebra and S a base for it; then the following hold:*

1. $p \between r \;\;\&\;\; r \leq q \;\Longrightarrow\; p \between q$
2. $p = q \;\Longleftrightarrow\; (\forall a \,\epsilon\, S)(a \between p \;\Leftrightarrow\; a \between q)$
3. $(p \wedge r) \between q \;\Longleftrightarrow\; p \between (r \wedge q)$
4. $p \between q \;\Leftrightarrow\; (p \wedge q) \between (p \wedge q) \;\Leftrightarrow\; (\exists a \,\epsilon\, S)(a \leq p \wedge q \;\;\&\;\; a \between a)$
5. $\neg(0 \between 0)$
6. $\neg(p \between q) \Longleftrightarrow p \wedge q = 0$

for every p, q, r in $\mathcal{P}$.

Proof: (1) From $p \between r$, it follows that $p \between (r \vee q)$ by splitness; but $r \vee q = q$ because $r \leq q$. (2) By density and item 1. (3) By meet closure and item 1. (4) By meet closure and splitness of join. (5) Because 0 is the join of the empty family. (6) If $p \wedge q = 0$, then $p \between q$ would contradict item 5 (by item 4). To prove the other direction we use the density axiom: for $a \,\epsilon\, S$, if $a \between (p \wedge q)$, then $p \between q$ (by items 4 and 1) which contradicts the assumption $\neg(p \between q)$; since *ex falso quodlibet*, we can derive $a \between 0$; so, by density, we have $p \wedge q \leq 0$. q.e.d.

As a corollary, we get that all the axioms in Definition 3.1 can be reversed. In particular, the order relation can be considered as a defined notion with the overlap relation as primitive (thanks to the axiom "density" and its converse, which is essentially the first item above). Not surprisingly, then, most of the times an inequality has to be proved, we shall apply "density" (as we have already done in the proof of item 6); similarly, for equalities we shall often use item 2 above.

The intuition underlying the relation $\between$ suggests there should be deep links between it and the positivity predicate in intuitionistic locale theory. Recall that a locale is a complete lattice in which binary meets distribute over arbitrary set-indexed joins, that is:

$$p \wedge \bigvee_{i \in I} q_i \quad = \quad \bigvee_{i \in I} (p \wedge q_i) \ . \tag{3}$$

Lemma 3.3 *Every set-based locale has an implication (hence it is a (set-based) complete Heyting algebra).*[1]

Proof: Define $p \to q$ as $\bigvee\{a \,\epsilon\, S \mid a \wedge p \leq q\}$, where S is a base. q.e.d.

So impredicatively "locale" is just another name for a complete Heyting algebra (a different name is justified by the different notion of morphism which is usually adopted; see [6] for further details). A locale is *overt* (or *open*) if it has a *positivity predicate*, that is, a unary predicate Pos such that: $\mathsf{Pos}(p)$ & $p \leq q \Longrightarrow \mathsf{Pos}(q)$, $\mathsf{Pos}\big(\bigvee_{i \in I} q_i\big) \Longrightarrow (\exists\, i \in I)\ \mathsf{Pos}(q_i)$ and $\big(\mathsf{Pos}(p) \Rightarrow p \leq q\big) \Longrightarrow p \leq q$ (the so-called "positivity axiom").

Proposition 3.4 *Every o-algebra $\mathcal{P}$ is an overt locale with $p \between p$ as the positivity predicate.*

Proof: Firstly, we claim that every o-algebra is, in fact, a locale; we need only to prove (3). For all $r : \mathcal{P}$ the following hold: $r \between (p \wedge \bigvee_{i \in I} q_i)$ iff $(r \wedge p) \between \bigvee_{i \in I} q_i$ iff $(r \wedge p) \between q_i$ for some $i \,\epsilon\, I$ iff $r \between (p \wedge q_i)$ for some $i \,\epsilon\, I$ iff $r \between \bigvee_{i \in I} (p \wedge q_i)$.

Let us put $\mathsf{Pos}(p) \Leftrightarrow (p \between p)$. We claim that Pos is a positivity predicate. All the requested properties are quite easy to prove. We only check validity of the "positivity axiom". Let $r : \mathcal{P}$ be such that $r \between p$; then, in particular, $p \between p$. By assumption, we get $p \leq q$ which, together with $r \between p$, gives $r \between q$. Summing up, we have proved that $r \between p$ implies $r \between q$; that is, $p \leq q$ (by density). q.e.d.

The notion of o-algebra is stronger than that of overt locale. For $\mathcal{P}$ an overt locale (with positivity predicate Pos) the binary predicate $\mathsf{Pos}(x \wedge y)$ satisfies all the axiom of an overlap relation but density. In other words, an overt locale $\mathcal{L}$ is an o-algebras if and only if its positivity predicate Pos satisfies the following:

$$(\forall r : \mathcal{L})\big(\mathsf{Pos}(r \wedge p) \Rightarrow \mathsf{Pos}(r \wedge q)\big) \Longrightarrow p \leq q$$

(for any $p, q : \mathcal{L}$).

Remark 3.5 *Since we are assuming in this paper that any o-algebra is set-based, it follows from the previous lemma and proposition that every o-algebra is a complete Heyting algebra.*[2]

[1] The validity of this statement is one of the advantages of working with set-based structures.

[2] The converse is generally not true because, as we are going to prove in the following paragraph, any o-algebra is classically a complete Boolean algebra (and there are examples of complete Heyting algebras which are not Boolean).

3.2 O-algebras classically

Building on item 6 of Proposition 3.2, we now state two lemmas, essentially due to Sambin (see [11]), which further clarify the relationship between the overlap relation $p \gtrless q$ and its negative counterpart $p \wedge q \neq 0$.

Lemma 3.6 *Classically, in any o-algebra, $p \gtrless q$ is tantamount to $p \wedge q \neq 0$.*

PROOF: From Proposition 3.2, item 6. q.e.d.

Lemma 3.7 *Let $(\mathcal{P}, \wedge, 0, -)$ be a $\wedge$-semilattice with bottom and with a pseudo-complement.*[3] *The following are equivalent:*

1. $(\forall p, q : \mathcal{P})\big((\forall r : \mathcal{P})(r \wedge p \neq 0 \Rightarrow r \wedge q \neq 0) \Longrightarrow p \leq q\big)$ *(*negative density*);*
2. $(\forall p : \mathcal{P})(p = - - p)$ & $(\forall p, q : \mathcal{P})\big(\neg(p \neq q) \Rightarrow p = q\big)$.

PROOF: Assume 1 and let r be such that $r \wedge - - p \neq 0$, that is, $r \nleq - - -p$. Since $- - -p = p$, this is tantamount to say that $r \nleq -p$, that is, $r \wedge p \neq 0$. Since r is arbitrary, we get $- - p \leq p$ by negative density. Hence $- - p = p$ for any $p : \mathcal{P}$. Assume now $\neg(p \neq q)$; we claim that $p = q$. It is enough to prove that $p \leq q$; so, by negative density, we must check that $r \wedge p \neq 0$ implies $r \wedge q \neq 0$: this is easy because if it were $r \wedge q = 0$, then it would be $p \neq q$ (since $r \wedge p \neq 0$), contrary to the assumption $\neg(p \neq q)$.

Vice versa, assume 2 and note that the implication $r \wedge p \neq 0 \Rightarrow r \wedge q \neq 0$ can be rewritten as $r \wedge q = 0 \Rightarrow \neg(r \wedge p \neq 0)$ which, by hypothesis, is equivalent to $r \wedge q = 0 \Rightarrow r \wedge p = 0$. Thus the antecedent of negative density becomes $(\forall r : \mathcal{P})(q \leq -r \Rightarrow p \leq -r)$; the latter gives in particular (even is equivalent to) $p \leq q$ (choose $r = -q$ and use $- - q = q$). q.e.d.

We think it is illuminating to compare o-algebras with complete Boolean algebras (this result was first suggested by Steve Vickers).

Proposition 3.8 *Assuming the law of excluded middle, every complete Boolean algebra (with $0 \neq 1$) is an o-algebra (with $1 \gtrless 1$), where $p \gtrless q \Leftrightarrow p \wedge q \neq 0$.*

Assuming the law of excluded middle and the powerset axiom, every o-algebra (with $1 \gtrless 1$) is a complete Boolean algebra (with $0 \neq 1$).

PROOF: (See [11] and [2]). Start with a complete Boolean algebra and (consider Lemma 3.6) define $p \gtrless q$ as $p \wedge q \neq 0$. This relation trivially satisfies symmetry and meet closure. Density follows from the previous Lemma. Finally, splitting of join can be easily reduced to $\neg(\forall i \in I)(p \wedge q_i = 0) \Leftrightarrow (\exists i \in I)(p \wedge q_i \neq 0)$ which is classically valid.

Conversely, every set-based o-algebra is a complete Heyting algebra, as we know. Moreover, by Lemma 3.6 and Lemma 3.7, we get that $- - q = q$, for any q. Finally, the powerset axiom allows considering the carrier of an o-algebra as a set, as required by the usual definition of complete Boolean algebra. q.e.d.

[3] A pseudocomplement is a unary operation $-$ such that $p \leq -q$ if and only if $p \wedge q = 0$.

3.3 Morphisms between o-algebras

Definition 3.9 *Let $f : \mathcal{P} \to \mathcal{Q}$ and $g : \mathcal{Q} \to \mathcal{P}$ be two maps between o-algebras. We say that f and g are* symmetric[4] *and we write $f \cdot|\cdot g$ if*

$$f(p) \between q \iff p \between g(q)$$

for all $p : \mathcal{P}$ and $q : \mathcal{Q}$.

In [11] Sambin proposed and widely justified the following definition of morphism between o-algebras.

Definition 3.10 *An* overlap morphism *(*o-morphism*) from an o-algebra $\mathcal{P}$ to an o-algebra $\mathcal{Q}$ is a map $f : \mathcal{P} \to \mathcal{Q}$ such that there exist $f^-, f^* : \mathcal{Q} \to \mathcal{P}$ and $f^{-*} : \mathcal{P} \to \mathcal{Q}$ satisfying the following conditions:*

1. $f(p) \leq q \iff p \leq f^*(q)$ $\qquad$ *($f \dashv f^*$)*
2. $f^-(q) \leq p \iff q \leq f^{-*}(p)$ $\qquad$ *($f^- \dashv f^{-*}$)*
3. $f(p) \between q \iff p \between f^-(q)$ $\qquad$ *($f \cdot|\cdot f^-$)*

(for all $p : \mathcal{P}$ and $q : \mathcal{Q}$).

Easily, the identity map $id_{\mathcal{P}}$ on $\mathcal{P}$ is an o-morphism (with $id_{\mathcal{P}}^- = id_{\mathcal{P}}^* = id_{\mathcal{P}}^{-*} = id_{\mathcal{P}}$); moreover, the composition $f \circ g$ of two o-morphisms is an o-morphism too (with $(f \circ g)^- = g^- \circ f^-$, $(f \circ g)^* = g^* \circ f^*$ and $(f \circ g)^{-*} = f^{-*} \circ g^{-*}$).

Definition 3.11 *O-algebras and o-relations form a category, called* **OA**.

Here we present an example of o-morphism which is actually the motivating one. For X and Y sets, it is possible (see [11]) to characterise o-morphisms between the overlap algebras $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ in terms of binary relations between X and Y. For any relation R between X and Y, consider its *existential image* defined by

$$R(A) \stackrel{def}{=} \{y \,\epsilon\, Y \mid (\exists x \,\epsilon\, X)(x\, R\, y \;\;\&\;\; x \,\epsilon\, A)\}$$

(for $A \subseteq X$). It is easy to check that the operator R is an o-morphism from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ with R^-, R^* and R^{-*} defined by:

$$\begin{array}{rcl} R^-(B) & \stackrel{def}{=} & \{x \,\epsilon\, X \mid (\exists y \,\epsilon\, Y)(x\, R\, y \;\;\&\;\; y \,\epsilon\, B)\} \\ R^*(B) & \stackrel{def}{=} & \{x \,\epsilon\, X \mid (\forall y \,\epsilon\, Y)(x\, R\, y \;\Rightarrow\; y \,\epsilon\, B)\} \\ R^{-*}(A) & \stackrel{def}{=} & \{y \,\epsilon\, Y \mid (\forall x \,\epsilon\, X)(x\, R\, y \;\Rightarrow\; x \,\epsilon\, A)\} \end{array} \qquad (4)$$

(for any $A \subseteq X$ and $B \subseteq Y$). Vice versa, any o-morphism $f : \mathcal{P}(X) \to \mathcal{P}(Y)$ is of that kind: it is enough to define $x\, R\, y$ as $y \,\epsilon\, f(\{x\})$. This correspondence

[4]This notion is classically equivalent to that of "conjugate" functions studied in [7].

is biunivocal and defines a full embedding of the category of sets and relations into **OA** (see [15]).

The functions f^* and f^{-*} of Definition 3.10 are the right adjoints of f and f^-, respectively, when the latter are regarded as functors (they are monotone functions) between the posets $\mathcal{P}(X)$ and $\mathcal{P}(Y)$. We are now going to show how the conditions defining an o-morphism can be simplified provided that two bases are known of the domain and codomain. First of all, one should recall from category theory that f^* (respectively f^{-*}), if exists, is uniquely determined by f (respectively f^-). Moreover, in the case of poset categories, f^* (respectively f^{-*}) exists if and only if f (respectively f^-) preserves all joins. This is true in an impredicative setting, but also predicatively at least for set-based structures. In the latter case $f^*(q)$ can be defined as $\bigvee\{a \epsilon S \mid f(a) \leq q\}$ (and similarly for f^{-*}). Before going on, let us prove a few properties about symmetric functions.

Proposition 3.12 *Let f be a map on the o-algebra $\mathcal{P}$ to the o-algebra $\mathcal{Q}$ and assume there exists $g : \mathcal{Q} \to \mathcal{P}$ with $f \cdot|\cdot g$; then:*

1. *g is unique; that is, if $h : \mathcal{Q} \to \mathcal{P}$ satisfies $f \cdot|\cdot h$, then $h = g$;*
2. *g is determined by f, in the sense that for any $q : \mathcal{Q}$*

$$g(q) = \bigvee\{a \epsilon S \mid (\forall x \epsilon S)(x \between a \;\Rightarrow\; f(x) \between q)\} \tag{5}$$

(where S is a base for $\mathcal{P}$).

Proof: (1) For any x in (a base for) $\mathcal{P}$, we have: $x \between h(y)$ iff $f(x) \between y$ iff $x \between g(y)$, for any y in $\mathcal{Q}$; hence by density $h = g$. (2): $g(q) = \bigvee\{a \epsilon S \mid a \leq g(q)\} = \bigvee\{a \epsilon S \mid (\forall x \epsilon S)(x \between a \Rightarrow x \between g(q)\} = \bigvee\{a \epsilon S \mid (\forall x \epsilon S)(x \between a \Rightarrow f(x) \between q)\}$. q.e.d.

Definition 3.13 *We say that a map $f : \mathcal{P} \to \mathcal{Q}$ between o-algebras is* symmetrizable *if there exists a (necessarily unique) map $f^- : \mathcal{Q} \to \mathcal{P}$ such that $f \cdot|\cdot f^-$. In that case, we say that f^- is "the"* symmetric *of f.*

Remark 3.14 *Since $\between$ is a symmetric binary relation, if f is symmetrizable also f^- is and $(f^-)^- = f$. Note also that, if $f : \mathcal{P} \to \mathcal{Q}$ is an o-morphism, then also $f^- : \mathcal{Q} \to \mathcal{P}$ is an o-morphism.*[5]

Lemma 3.15 *Let f be a symmetrizable map on the o-algebra $\mathcal{P}$ to the o-algebra $\mathcal{Q}$; then f (and f^-) preserves all (set-indexed) joins.*

Proof: For any $y : \mathcal{P}$, we have: $y \between f(\bigvee_{i\epsilon I} p_i)$ iff $g(y) \between \bigvee_{i\epsilon I} p_i$ iff $(\exists i \epsilon I)(g(y) \between p_i)$ iff $(\exists i \epsilon I)(y \between f(p_i))$ iff $y \between \bigvee_{i\epsilon I} f(p_i)$. Hence by density we can conclude that $f(\bigvee_{i\epsilon I} p_i) = \bigvee_{i\epsilon I} f(p_i)$. q.e.d.

[5]It is also easy to check that f^- is in fact the inverse of f when the latter is an isomorphism.

Proposition 3.16 *Let $f : \mathcal{P} \to \mathcal{Q}$ be a map between two (set-based) o-algebras; then the following are equivalent:*

1. *f is an o-morphism;*
2. *f is symmetrizable;*
3. *f satisfies the following property:*

$$f(p) \gtrless q \iff (\exists a \,\epsilon\, S)\big(p \gtrless a \ \ \& \ \ (\forall x \,\epsilon\, S)(x \gtrless a \ \Rightarrow \ f(x) \gtrless q)\big) \qquad (6)$$

for all $p : \mathcal{P}$ and $q : \mathcal{Q}$ (where S is a base of $\mathcal{P}$).

Proof: $(3 \Rightarrow 2)$ We show that the function $g(q) = \bigvee\{a \,\epsilon\, S \mid (\forall x \,\epsilon\, S)(x \gtrless a \Rightarrow f(x) \gtrless q)\}$ of Proposition 3.12 is in fact the symmetric of f. As $\gtrless$ splits joins, we have $p \gtrless g(q)$ if and only if $p \gtrless a$ for some a satisfying $(\forall x \,\epsilon\, S)(x \gtrless a \Rightarrow f(x) \gtrless q)$ and this holds if and only if, by 3, $f(p) \gtrless q$.
$(2 \Rightarrow 1)$ By Proposition 3.15, both f and f^- preserve joins; hence their right adjoints f^* and f^{-*} exist.
$(1 \Rightarrow 3)$ Let f^- be the symmetric of f. Then $(\exists a \,\epsilon\, S)(p \gtrless a \,\&\, (\forall x \,\epsilon\, S)(x \gtrless a \Rightarrow f(x) \gtrless q))$ iff $(\exists a \,\epsilon\, S)(p \gtrless a \,\&\, (\forall x \,\epsilon\, S)(x \gtrless a \Rightarrow x \gtrless f^-(q)))$ iff $(\exists a \,\epsilon\, S)(p \gtrless a \,\&\, a \leq f^-(q))$ iff [6] $p \gtrless f^-(q)$ iff $f(p) \gtrless q$. q.e.d.

Here we want to spend some words about item 3. Firstly, it is surely of some interest because it characterises the notion of o-morphism by an intrinsic property of the map f itself. Moreover, it seems the right notion of morphism in the non-complete case, as we shall see in the last sections. Furthermore, we think it is worth mentioning that (6) is a form of continuity. This fact is better seen in the context of formal topology (see [2]). However, we can here give a suggestion: following equations (4) we write $(\gtrless q)$ for $\{p \mid p \gtrless q\}$; then condition (6) can be rewritten as $f^{-1}(\gtrless q) = \bigcup\{(\gtrless a) \mid (\gtrless a) \subseteq f^{-1}(\gtrless q)\}$. Thus, if the families $\{\gtrless p\}_{p:\mathcal{P}}$ and $\{\gtrless q\}_{q:\mathcal{Q}}$ are taken as sub-bases for two topologies on $\mathcal{P}$ and $\mathcal{Q}$, respectively, then (6) is a notion of continuity for f (in fact, this is stronger than usual continuity because the $(\gtrless p)$'s do not form a base, in general). By the way, note that reading $\gtrless$ as a unary operator allows to rewrite $f \cdot|\cdot f^-$ as $f^{-1} \circ \gtrless = \gtrless \circ f^-$.

Proposition 3.17 *Classically and impredicatively, o-morphisms are exactly the maps preserving all joins.*

Proof: In a classical setting, an o-algebra is exactly a cBa (Proposition 3.8). As we already know from Proposition 3.15, every o-morphism is join-preserving. Viceversa, if $f : \mathcal{P} \to \mathcal{Q}$ preserves all joins, then (by the powerset axiom) it admits a right adjoint f^*. We claim that f^- exists and it is $f^-(q) = -f^*(-q)$. Indeed, for p in $\mathcal{P}$ we have: $p \gtrless -f^*(-q)$ iff $p \wedge -f^*(-q) \neq 0$ iff $p \nleq f^*(-q)$ iff $f(p) \nleq -q$ iff $f(p) \wedge q \neq 0$ iff $f(p) \gtrless q$. q.e.d.

[6] Because $p \gtrless q \iff p \gtrless \bigvee\{a \,\epsilon\, S \mid a \leq q\} \iff (\exists a \,\epsilon\, S)(p \gtrless a \ \ \& \ \ a \leq q)$.

Definition 3.18 *Let* **cBa** *be the category of complete Boolean algebras and complete homomorphisms.*[7]

We write $\mathbf{cBa}_{\bigvee}$ *for the category (which is not a subcategory of* **cBa***) of complete Boolean algebras and* join-preserving maps *(maps which preserve arbitrary joins).*

Corollary 3.19 *Classically, the categories* **OA** *and* $\mathbf{cBa}_{\bigvee}$ *are equivalent.*

PROOF: By Propositions 3.8 and 3.17. q.e.d.

Symmetrically, it is not difficult to select a subcategory of **OA** which is isomorphic to the entire of **cBa**.

Definition 3.20 *Let* $\mathbf{OA}^{\wedge}$ *be the subcategory of* **OA** *with the same objects as* **OA** *and whose morphisms are the o-morphisms which preserve finite meets.*

Corollary 3.21 *The category* **cBa** *of complete Boolean algebras is classically equivalent to the category* $\mathbf{OA}^{\wedge}$.

3.4 Atomic and free o-algebras

In a poset with zero, every minimal non-zero element is usually called an "atom". In the language of o-algebras, it is possible to define a more convenient (from an intuitionistic point of view) notion, though obeying to the same intuition.

Definition 3.22 *Let* $\mathcal{P}$ *be an overlap algebra. We say that* $m : \mathcal{P}$ *is an* atom *if* $m \between m$ *and for every* $p : \mathcal{P}$*, if* $m \between p$ *then* $m \leq p$.

There are several useful characterization of this notion; among them, we list the following.

Lemma 3.23 *In any o-algebra* $\mathcal{P}$*, the following are equivalent:*

1. *m is an atom;*
2. *$m \between m$ and, for every $p : \mathcal{P}$, if $p \between p$ and $p \leq m$, then $p = m$;*
3. *for every $p : \mathcal{P}$, $m \between p$ if and only if $m \leq p$;*
4. *$m \between m$ and, for every $p, q : \mathcal{P}$, if $m \between p$ and $m \between q$, then $m \between p \wedge q$.*

PROOF: See [11] and [2]. q.e.d.

We say that an overlap algebra is *atomistic* if its atoms form a base (in particular, they form a set). Clearly $\mathcal{P}(X)$ is atomistic; and this is, essentially, the only example.

Proposition 3.24 *An o-algebra* $\mathcal{P}$ *is atomistic if and only if it is isomorphic to* $\mathcal{P}(S)$*, for some set* S.

[7] A complete homomorphisms is a map which preserves arbitrary joins and meets.

Proof: See [10]. q.e.d.

The following proposition shows that $\mathcal{P}(X)$ is also the free o-algebra on a set X of join-generators.[8]

Proposition 3.25 *For any o-algebra $\mathcal{Q}$, any set X and any map $f : X \to \mathcal{Q}$, there exists a unique o-morphism $\overline{f} : \mathcal{P}(X) \to \mathcal{Q}$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[rr, hook, "i"] \arrow[dr, "f"'] & & \mathcal{P}(X) \arrow[dl, "\overline{f}"] \\ & \mathcal{Q} & \end{tikzcd}$$

(where $i(x) = \{x\}$, for any $x \,\epsilon\, X$).

Proof: For $U \subseteq X$, let us put $\overline{f}(U) = \bigvee\{f(x) \mid x \,\epsilon\, U\}$. Clearly this definition is compulsory: $\overline{f}(U) = \overline{f}(\bigcup\{\{x\} \mid x \,\epsilon\, U\}) = \bigvee\{\overline{f}(\{x\}) \mid x \,\epsilon\, U\}$ (because $\overline{f}$ must be an o-morphism, hence has to preserves joins) $= \bigvee\{f(x) \mid x \,\epsilon\, U\}$ (because $\overline{f} \circ i$ must be f). We claim that $\overline{f}$ is symmetrizable. Let $g : \mathcal{Q} \to \mathcal{P}(X)$ be the map defined as in equation (5) (with respect to $\overline{f}$). Since $\mathcal{P}(X)$ is based on singletons and $\{x\} \between \{a\}$ simply means $x = a$, we can simplify the expression defining g and get $g(q) = \{x \,\epsilon\, X \mid f(x) \gtrless q\}$. For all $U \subseteq X$ and $q : \mathcal{Q}$, the following hold: $U \between g(q) \Leftrightarrow (\exists x \,\epsilon\, U)(x \,\epsilon\, g(q)) \Leftrightarrow (\exists x \,\epsilon\, U)(f(x) \gtrless q)$ $\Leftrightarrow \bigvee\{f(x) \mid x \,\epsilon\, U\} \gtrless q \Leftrightarrow \overline{f}(U) \gtrless q$. Thus, $\overline{f}$ is an o-morphism. Moreover, for all $x \,\epsilon\, X$, $(\overline{f} \circ i)(x) = \overline{f}(\{x\}) = \bigvee\{f(y) \mid y \,\epsilon\, \{x\}\} = f(x)$. q.e.d.

4 Overlap lattices with opposite

Definition 4.1 *An* overlap lattice with opposite *(*oo-lattice *for short) is a quadruple $(\mathcal{P}, \leq, \gtrless, -)$ where $(\mathcal{P}, \leq)$ is a bounded lattice (with 0 and 1 as the bottom and top elements, respectively), $-$ is a pseudo-complement operation (that is, $p \wedge q = 0$ if and only $p \leq -q$) and $\gtrless$ is a binary relation on $\mathcal{L}$ satisfying the following properties:*

- $p \gtrless q \implies q \gtrless p$ *(symmetry)*
- $p \gtrless q \implies p \gtrless (p \wedge q)$ *(meet closure)*
- *if $\bigvee_{i \epsilon I} q_i$ exists, then: $p \gtrless \bigvee_{i \epsilon I} q_i \iff (\exists i \,\epsilon\, I)(p \gtrless q_i)$* *(splitting of existing joins)*
- $(\forall r : \mathcal{L})(r \gtrless p \Rightarrow r \gtrless q) \implies p \leq q$ *(density)*

(for any p and q in $\mathcal{P}$).

[8] Though **OA** and **cBa** share the same objects, they are very different as categories. For instance, free complete Boolean algebras generally do not exist (see [6]).

A set S of elements of an oo-lattice $\mathcal{P}$ is a base for $\mathcal{P}$ if for any p in $\mathcal{P}$ the join of the family $\{a \epsilon S \mid a \leq p\}$ exists and is p. From now on, as usual, we shall always work with set-based structures. It is easy to check that all properties stated in Proposition 3.2 still hold for oo-lattices. As perhaps expected, any (set-based) o-algebra is an example of oo-lattice: it is enough to define the opposite of an element p as $\bigvee\{a \epsilon S \mid a \wedge p \leq 0\}$. Vice versa, any oo-lattice which is complete (as a lattice) is automatically an o-algebra. In the final section of the paper we shall present several examples of oo-lattices which are not o-algebras.

Like o-algebras are always complete Heyting algebras, so oo-lattices are always distributive.

Proposition 4.2 *Any oo-lattice is a distributive lattice.*

PROOF: This proof is essentially the finitary version of the first part of that of Proposition 3.4. Clearly, it is enough to check that $p \wedge (q_1 \vee q_2) \leq (p \wedge q_1) \vee (p \wedge q_2)$. As usual, we apply "density": for all $r : \mathcal{P}$, $r \gtrless (p \wedge (q_1 \vee q_2))$ iff $(r \wedge p) \gtrless (q_1 \vee q_2)$ iff $(r \wedge p) \gtrless q_i$ for some $i = 1, 2$ iff $r \gtrless (p \wedge q_i)$ for some $i = 1, 2$ iff $r \gtrless (p \wedge q_1) \vee (p \wedge q_2)$. q.e.d.

Remark 4.3 *It is easy (by adapting the proof of Proposition 3.4) to prove the following strengthening of the previous Proposition.*

If $\bigvee_{i \epsilon I} q_i$ exists in an oo-lattice, then also $\bigvee_{i \epsilon I}(p \wedge q_i)$ exists and is equal to $p \wedge \bigvee_{i \epsilon I} q_i$.

Proposition 4.4 *Classically, the notion of oo-lattice and that of Boolean algebra coincide.*

PROOF: The proof is analogous to that of Proposition 3.8. Given a Boolean algebra, define $p \gtrless q$ as $p \wedge q \neq 0$ and use Lemma 3.7 to prove density. Conversely, suppose to have an oo-lattice. By the previous Proposition, an oo-lattice is a distributive lattice. Moreover, by Lemmas 3.6 and 3.7 we get that $-$ is an involution. Summing up, from a classical point of view, an oo-lattice is a complemented distributive lattice, that is, a Boolean algebra. q.e.d.

4.1 Morphisms between oo-lattices

Definition 4.5 *A morphism f between two o-lattices with opposite $(\mathcal{P}, \leq, -, \gtrless)$ and $(\mathcal{Q}, \leq, -, \gtrless)$ is a map $f : \mathcal{P} \to \mathcal{Q}$ satisfying*

$$f(p) \gtrless q \iff (\exists a \epsilon S)\big(p \gtrless a \ \ \& \ \ (\forall x \epsilon S)(x \gtrless a \ \Rightarrow \ f(x) \gtrless q)\big)$$

(condition (6) of Proposition 3.16), for all $p : \mathcal{P}$ and $q : \mathcal{Q}$.

Lemma 4.6 *A map f is a morphism of oo-lattices if and only if the following conditions hold:*

1. *f is monotone;*

2. $f(p) \gtrless q \implies (\exists a \,\epsilon\, S)\big(p \gtrless a \;\;\&\;\; (\forall x \,\epsilon\, S)(x \gtrless a \;\Rightarrow\; f(x) \gtrless q)\big)$.

Proof: Since f is a morphism if both 2. and its converse hold, we must prove that monotonicity of f is equivalent to the converse of 2. under the assumption 2. itself.

We firstly prove that each morphism f is monotone. Let $p, r : \mathcal{P}$ be such that $p \le r$. We prove that $f(p) \le f(r)$ (by using density in $\mathcal{Q}$). Let $q : \mathcal{Q}$ be such that $f(p) \gtrless q$. Then, there exists $a \,\epsilon\, S$ such that $p \gtrless a$ and $(\forall x \,\epsilon\, S)(x \gtrless a \Rightarrow f(x) \gtrless q)$. Since $p \le r$, then $r \gtrless a$. Thus, $f(r) \gtrless q$.

Assume now monotonicity. Let $p, q : \mathcal{P}$ be such that $(\exists a \,\epsilon\, S)(p \gtrless a \;\&\; (\forall x \,\epsilon\, S)(x \gtrless a \Rightarrow f(x) \gtrless q))$. Since S is a base for $\mathcal{P}$, from $p \gtrless a$ it follows that there exists $b \,\epsilon\, S$ such that $b \gtrless a$ (hence $f(b) \gtrless q$) and $b \le p$. By monotonicity of f, we get $f(b) \le f(p)$, which together with $f(b) \gtrless q$ gives $f(p) \gtrless q$. q.e.d.

Proposition 4.7 *Let $f : \mathcal{P} \to \mathcal{Q}$ be a morphism between two oo-lattices. If $\bigvee_{i \epsilon I} p_i$ exists, then also $\bigvee_{i \epsilon I} f(p_i)$ exists and $f(\bigvee_{i \epsilon I} p_i) = \bigvee_{i \epsilon I} f(p_i)$.*

Proof: We claim that $f(\bigvee_{i \epsilon I} p_i)$ is the least upper bound of the family $\{f(p_i)\}_{i \epsilon I}$. Clearly it is an upper bound since f is monotone. Let r be another upper bound (that is, $f(p_i) \le r$ for any $i \,\epsilon\, I$); we must show that $f(\bigvee_{i \epsilon I} p_i) \le r$. Let $q : \mathcal{Q}$ be such that $f(\bigvee_{i \epsilon I} p_i) \gtrless q$. Then, since f is a morphism and $\gtrless$ splits all existing joins, there exist $a \,\epsilon\, S$ and $i \,\epsilon\, I$ such that $p_i \gtrless a$ and $(\forall x \,\epsilon\, S)(x \gtrless a \Rightarrow f(x) \gtrless q)$. In particular, $f(p_i) \gtrless q$ (take $b \le p_i$ such that $b \gtrless a$ and use monotonicity of f); together with $f(p_i) \le r$, this gives $r \gtrless q$. The claim follows by density in $\mathcal{Q}$. q.e.d.

Proposition 4.8 *The following hold:*

1. *for any oo-lattice $\mathcal{P}$, the identity map $id_{\mathcal{P}} : \mathcal{P} \to \mathcal{P}$ is an oo-lattice morphism;*
2. *oo-lattice morphisms are closed under composition.*

Proof: (1) Let $p, q : \mathcal{P}$ be such that $p \gtrless q$. We must show that $(\exists a \,\epsilon\, S)$ $(p \gtrless a \;\&\; (\forall x \,\epsilon\, S)(x \gtrless a \Rightarrow x \gtrless q))$, that is, $(\exists a \,\epsilon\, S)(p \gtrless a \;\&\; a \le q)$; the latter holds because S is a base.

(2) Let $f : \mathcal{P} \to \mathcal{Q}$ and $g : \mathcal{Q} \to \mathcal{R}$ be two oo-lattice morphisms and assume $g(f(p)) \gtrless r$. Provided that T is a base for $\mathcal{Q}$, since g is a morphism we can find an element $b \,\epsilon\, T$ such that $f(p) \gtrless b$ and $(\forall y \,\epsilon\, T)(y \gtrless b \Rightarrow g(y) \gtrless r)$. Since $f(p) \gtrless b$ and f is a morphism, there exists $a \,\epsilon\, S$ (where S is a base for $\mathcal{P}$) such that $p \gtrless a$ and $(\forall x \,\epsilon\, S)(x \gtrless a \Rightarrow f(x) \gtrless b)$. We claim that this same element $a \,\epsilon\, S$ works for $g \circ f$ (that is, $p \gtrless a$ and $(g \circ f)(x) \gtrless r$ whenever $x \gtrless a$); indeed, if $x \gtrless a$, then $f(x) \gtrless b$; even if, in general, $f(x)$ does not belong to T, we can find $c \,\epsilon\, T$ such that $c \le f(x)$ and $c \gtrless b$ (since T is a base); thus $g(c) \gtrless r$ (thanks to the properties of the element b) and, also, $g(c) \le g(f(x))$ (since g is, in particular, monotone); hence $g(f(x)) \gtrless r$ and we are done. q.e.d.

Definition 4.9 *Let **OLat** be the category of oo-lattices as objects and oo-lattice morphisms as arrows.*

Proposition 4.10 *Classically and impredicatively, a map between Boolean algebras is a morphism of oo-lattices if and only if it preserves all joins which exist in the domain.*

PROOF: From Proposition 4.7 it follows that every oo-lattice morphism is a join-preserving map. Conversely, let $f : \mathcal{B} \to \mathcal{B}'$ be a join-preserving map between two Boolean algebras. Then f extends uniquely to a join-preserving map $\overline{f} : DM(\mathcal{B}) \to DM(\mathcal{B}')$ between the Dedekind-MacNeille completions (see [6]) of $\mathcal{B}$ and $\mathcal{B}'$, respectively. It follows from Propositions 3.17 and 3.16 that $\overline{f}$ satisfies equation (6) for any $p : DM(\mathcal{B})$ and $q : DM(\mathcal{B}')$ and for any base S of $DM(\mathcal{B})$. In particular, (6) holds for f once we note that every base S for $\mathcal{B}$ is also a base for $DM(\mathcal{B})$. q.e.d.

Definition 4.11 *We write $\boldsymbol{Ba}_{\bigvee}$ for the category of Boolean algebras and maps preserving all existing joins.*

Corollary 4.12 *Classically, the categories $\boldsymbol{Ba}_{\bigvee}$ and **OLat** are equivalent.*

PROOF: By Propositions 4.4 and 4.10. q.e.d.

4.2 Richer overlap structures

As we have seen in the previous pages, the notion of oo-lattice turns out to be classically equivalent to that of Boolean algebra; so it is a constructive version of the latter. Even if it seems the minimal structure enjoying such a property, it is by no means the only one. For instance, it is quite natural to consider also Heyting and Boolean algebras with overlap (o-Heyting and o-Boolean algebras). The idea is simply to add an overlap relation (satisfying all the axioms for $\between$ listed in Definition 4.1) to the usual algebraic structures.

Definition 4.13 *An* overlap Heyting algebra *(* o-Ha *for short) is an oo-lattice whose underlying lattice is a Heyting algebra. An* overlap Boolean algebra *(* o-Ba *for short) is an o-Ha whose underlying lattice is a Boolean algebra.*

Here is an example of o-Ba (examples of other o-structures are being given below): the family of all *recursive* subsets of $\mathbb{N}$ (the set of natural numbers). It works since recursive subsets are closed under union, intersection and complement; moreover, all singletons are recursive, hence density holds. Note also, that this o-Ba is an example of oo-lattice which is not an o-algebra (that is, it is not closed under arbitrary joins), otherwise all subsets of $\mathbb{N}$ would be recursive (each subset being a union of singletons).

Proposition 4.14 *Classically, the notions of oo-lattice, o-Ha, o-Ba and that of Boolean algebra all coincide.*

Proof: By the proof of Proposition 4.4 q.e.d.

We adopt for all o-structures the same notion of morphism we used for oo-lattices.

Definition 4.15 *Let* **OHa** *be the full subcategory of* **OLat** *whose objects are the o-Heyting algebras. We write* **OBa** *for the full subcategory of* **OHa** *with o-Boolean algebras as objects.*

Proposition 4.16 *Classically, the categories* $\boldsymbol{Ba}_{\bigvee}$*,* ***OLat****,* ***OHa*** *and* ***OBa*** *are all equivalent.*

Proof: By Propositions 4.14 and 4.10. q.e.d.

On the contrary, from a constructive point of view, the situation is completely different and can be summarized by the following picture:

$$\begin{tikzcd}
\text{Heyting algebras} & \text{oo-lattices} \\
\text{Boolean algebras} \arrow[u, Rightarrow] & \text{o-Heyting algebras} \arrow[u, Rightarrow] \arrow[ul, Rightarrow] \\
 & \text{o-Boolean algebras} \arrow[u, Rightarrow] \arrow[ul, Rightarrow]
\end{tikzcd} \qquad (7)$$

(all "implications" are strict). All this is shown by the following examples and remarks.

The following is an example of o-Ha which, in general, cannot be constructively proved to be a Boolean algebra. Classically, this will be nothing else than the Boolean algebra of finite-cofinite subsets. Let X be a set. We say that a subset $K \subseteq X$ is *finite* if either $K = \emptyset$ or $K = \{x_1, \ldots, x_n\}$ for some $x_1 \ldots x_n \,\epsilon\, X$. Clearly, the union of two finite subsets is finite, while the intersection is not (unless the equality of X is decidable: see [4]).

Definition 4.17 *For any set X, let $\mathcal{F}(X)$ be the sub-family of $\mathcal{P}(X)$ defined by the following condition:*

$$A : \mathcal{F}(X) \quad \overset{def}{\Longleftrightarrow} \quad (\ \exists K \subseteq X,\ K \textit{ finite })(\ A \subseteq --K \ \vee\ -K \subseteq A\ ) \qquad (8)$$

(for $A \subseteq X$).

Proposition 4.18 *For any set X, the collection $\mathcal{F}(X)$ is an o-Heyting algebra (but it is neither a Boolean algebra nor an o-algebra).*

Proof: $\mathcal{F}(X)$ contains both $\emptyset$ (which is finite) and X (because $X = -\emptyset$).

$\mathcal{F}(X)$ is closed under union: let $A, B : \mathcal{F}(X)$; if either A or B contains some cofinite subset, then so does $A \cup B$; otherwise, there exist two finite subsets K and L such that $A \subseteq --K$ and $B \subseteq --L$; so $A \cup B \subseteq --K \cup --L \subseteq --(K \cup L)$.

$\mathcal{F}(X)$ is closed under intersection: let $A, B : \mathcal{F}(X)$; if either A or B is contained in some *co*cofinite subset, then so is $A \cap B$; otherwise, there exist two finite subsets K and L such that $-K \subseteq A$ and $-L \subseteq B$; so $-(K \cup L) = -K \cap -L \subseteq A \cap B$.

This proves that $\mathcal{F}(X)$ is a sublattice of $\mathcal{P}(X)$. Since $\mathcal{P}(X)$ is an o-algebra, its overlap relation $\between$, when restricted to a sublattice, automatically inherits all properties required by Definition 4.1 but density. However, density hold for $\mathcal{F}(X)$ as well as for any other sublattice of $\mathcal{P}(X)$ which contains all singletons (as $\mathcal{F}(X)$ clearly does).

$\mathcal{F}(X)$ is closed under implication[9]: if $-K \subseteq B$, then $-K \subseteq A \to B$ (because $B \subseteq A \to B$); if $A \subseteq --K$, then $-K \subseteq -A$, hence $-K \subseteq A \to B$ (because $-A \subseteq A \to B$); finally, if $B \subseteq --K$ and $-L \subseteq A$, then $-(L \cup K) = -L \cap -K \subseteq A \cap -B \subseteq -(A \to B)$; hence $A \to B \subseteq --(K \cup L)$. In any case, $A \to B : \mathcal{F}(X)$ whenever $A, B : \mathcal{F}(X)$.

Thus $\mathcal{F}(X)$ is an o-Ha. Clearly, it is not an o-Ba, in general; for instance, $\{x\} \cup -\{x\}$ need not be equal to X (unless X has a decidable equality).

Finally, $\mathcal{F}(X)$ is clearly not complete (it is not an o-algebra): think of $X = \mathbb{N}$ and consider the elements $\{2n\} : \mathcal{F}(\mathbb{N})$, for $n \in \mathbb{N}$; their union is the set of all even numbers which, of course, does not belong to $\mathcal{F}(\mathbb{N})$. q.e.d.

The notion of Boolean algebra seems constructively weaker than its overlap version: at least, the relation $p \wedge q \neq 0$ (which seems to be the only possible candidate) fails to be an overlap relation constructively. In fact, if that were the case, any Boolean algebra would have a stable equality (by Lemma 3.7).

A fortiori, one cannot hope to find a general method for endowing a Heyting algebra with an overlap relation.

Finally, we give an example of oo-lattice which does not seem to be an o-Heyting algebra. Consider a pure first-order language with equality and define the smallest class of formulae which contains atomic formulae and is closed under disjunction, conjunction and negation. Given a set X, the family of all its subsets that can be obtained by comprehension on those formulae is an oo-lattice,[10] but there seems to be no constructive way to define implication.

Conclusions and future work

In this paper we have done the following:

- we have shown that classically the category of overlap algebras is equivalent to category of complete Boolean algebras and join-preserving maps;
- we have shown constructively that the power-collection of a set is the free overlap algebra generated from the set;

[9] Remember that $\mathcal{P}(X)$ is a Heyting algebra with $A \to B = \{x \in X \mid A \cap \{x\} \subseteq B\}$.

[10] In other words, this is the smallest sub-family of $\mathcal{P}(X)$ that contains all singletons and is closed under finite unions, finite intersections and pseudo-complementation. Classically, this is nothing else than another description of the Boolean algebra of finite-cofinite subsets.

- we have introduced various structures equipped with an overlap relation, called *o-structures*, that generalize Sambin's notion of overlap algebra;

- we have introduced corresponding overlap morphisms between our o-structures generalizing Sambin's notion of overlap morphism;

- we have shown that the corresponding categories of our o-structures are all equivalent to the category of Boolean algebras with maps preserving existing joins.

In the future, we aim to test whether our o-structures can be used to give a constructive version of Stone representation for (not necessarily complete) Boolean algebras as Sambin gave for atomic set-based overlap algebras. Moreover, we intend to investigate the existence of constructive join-completions of our o-structures.

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