Predicative exponentiation of locally compact formal topologies over inductively generated ones

Maria Emilia Maietti
Dipartimento di Matematica Pura ed Applicata
Università di Padova

CLARENDON PRESS · OXFORD
2005

Abstract

We provide an intuitionistic predicative proof of the well known result proved by M. Hyland that locally compact locales are the exponentiable objects in the category of locales.

More precisely, by adapting Hyland's proofs to the context of formal topology, we prove that locally compact formal covers are the exponentiable objects in the category of inductively generated formal covers. Then, we deduce the analogous result for inductively generated formal topologies.

1 Introduction

It is well known that the category of topological spaces is not closed and that the exponentiable objects are exactly those spaces whose frame is continuous (see for example the overview [8]).

M. Hyland in [9] proved the analogous result for locales, namely that locally compact locales are the exponentiable objects in the category of locales. Later, I. Sigstam in [21] adapted his proof to show that locally compact locales are exponentiable in the category of (effectively given) formal spaces, but still working within a (classical) impredicative setting. [1] Moreover, Vickers reproduced most of Hyland's results using geometric reasoning (see [22]).

Here, we want to give an intuitionistic predicative proof of the whole characterization in [9].

To this purpose we consider the category of inductively generated formal covers [6] and continuous relations [17] which constitutes a predicative presentation of the category of locales (see [10]), and also of its opposite, namely that of frames, by taking the opposite relations. Then, we also consider the full subcategory of inductively generated formal topologies which provides a predicative presentation of open locales (see [11]). In both categories, we address the question of characterizing the exponentiable objects predicatively.

A predicative study of exponentiation in the context of formal topology has been started in [13] where it is proved predicatively that unary formal topologies are exponentiable in the category of inductively generated formal topologies. This result was deduced after proving the corresponding result for inductively generated formal covers and the existence of a coreflection of inductively generated formal covers into inductively generated formal topologies thanks to which exponentiation within the subcategory of formal topologies is inherited from the category of inductively generated formal covers.

Here, we show predicatively that locally compact formal covers are the exponentiable objects in the category of inductively generated formal covers by adapting Hyland's proofs in [9]. In particular, the construction of the exponential topology of a locally compact formal cover over an inductively generated one follows the original idea in [9] of generating the exponential topology by means

[1] Actually, she proves exponentiation of a locally compact formal space only over formal spaces admitting generators made of directed subsets.

of the "way-below" component of a locale morphism, since this construction fits perfectly with the use of the predicative method for the inductive generation of a formal cover in [6], provided that we can define predicatively the 'way-below" relation between open subsets, which was shown to be possible for inductively generated formal covers in [7]. Hence, inspired by Hyland's construction and thanks to the analysis in [13] of the conditions defining a continuous relation, we characterize a continuous relation via its way-below component in terms of conditions that can be put as basic axioms from which to generate the exponential topology candidate as in [9,10,21]. Then, the proof of exponentiation mostly consists of inductions over the inductive generation of the covers involved.

Afterwards, we prove that any exponentiable inductively generated formal cover is locally compact following very closely the original proof in [9] that, being almost completely based on categorical properties related to exponentiation and on properties of the Sierpinski locale, is already essentially predicative.

Finally, thanks to the coreflection proved in [13], we deduce the analogous characterization for inductively generated formal topologies, namely that locally compact formal topologies are the exponentiable objects in the category of inductively generated formal topologies. [2]

2 Preliminaries on formal covers

There are at least two different equivalent presentations of formal topology: one based on a monoid [17] or on a preorder [6]. Here, we deal with formal covers and formal topologies based on a preorder, called $\leq$-formal covers and $\leq$-formal topologies, that are inductively generated [6].
Note that when we speak of "set" we mean a set in Martin-Löf constructive set theory [16] and we adopt the set-theoretic notation in [19].

We start with recalling the notion of formal cover based on a preorder [6] corresponding predicatively to that of locale [10]:

Definition 2.1. ($\leq$-Formal cover) *A $\leq$-formal cover is a structure*

$$\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}})$$

where A is a set of basic opens (or base), $\leq_{\mathcal{A}}$, is a preorder between elements of A and $\lhd_{\mathcal{A}}$ is an infinitary relation, called cover relation, between elements and subsets of A satisfying the following conditions:

$$\textit{(reflexivity)} \quad \frac{a \varepsilon U}{a \lhd U} \qquad \textit{(transitivity)} \quad \frac{a \lhd U \quad U \lhd V}{a \lhd V}$$

$$(\leq\textit{-left}) \quad \frac{a \leq b \qquad b \lhd U}{a \lhd U} \qquad (\leq\textit{-right}) \quad \frac{a \lhd U \qquad a \lhd V}{a \lhd U \downarrow_{\leq} V}$$

where $U \lhd V$ is an abbreviation for $(\forall u \varepsilon U)\ u \lhd V$ and $U \downarrow_{\leq} V$ for $\{x \in A \mid \exists u \varepsilon U\ x \leq u\ \&\ \exists v \varepsilon V\ x \leq v\ \}$.

[2] More detailed proofs of main results in this paper can be found in [12].

Then, we give the associated notion of formal topology based on a preorder [6] corresponding predicatively to that of open locale [11]:

Definition 2.2. ($\leq$-Formal topology) *A $\leq$-formal topology is a structure*

$$\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}}, \mathsf{Pos}_{\mathcal{A}})$$

where $(A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}})$ is a formal cover and $\mathsf{Pos}_{\mathcal{A}}$ is a predicate over A, called positivity predicate, satisfying the following conditions:

$$\textit{(monotonicity)}\ \frac{\mathsf{Pos}_{\mathcal{A}}(a) \qquad a \lhd U}{(\exists u \varepsilon U)\ \mathsf{Pos}_{\mathcal{A}}(u)} \qquad \textit{(positivity axiom)}\ \ a \lhd a^{+}$$

where $a^{+} \equiv \{ x \in A \ \mid \ x = a\ \&\ \mathsf{Pos}_{\mathcal{A}}(x)\ \}$.

Now, we recall the notion of inductively generated formal cover [6], closely related to Aczel's notion of set-based closure operation in [2] and that of inductive definition in [1]:

Definition 2.3. (I.g. formal cover) *A $\leq$-formal cover $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}})$ is inductively generated (abbreviated with i.g.) if and only if there exists a localized axiom-set $I(-)$ and $C(-,-)$ (as defined in [6]) such that $\lhd_{\mathcal{A}}$ is inductively generated by using reflexivity, $\leq$-left and*

$$\textit{(infinity)}\ \frac{i \in I(a) \qquad C(a,i) \lhd U}{a \lhd U}.$$

We recall from [6] that in general the relation generated by an axiom-set by means of *reflexivity*, *$\leq$-left* and *infinity* does not necessarily satisfy *$\leq$-right* unless the axiom-set is localized. However, given any axiom-set we can always localize it as shown in [6].

Then, we give the definition of inductively generated formal topology, namely an inductively generated formal cover with a positivity predicate (further readings on this are [14, 5, 4]):

Definition 2.4. (I.g. formal topology) *$\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}}, Pos_{\mathcal{A}})$ is an inductively generated (abbreviated with i.g.) $\leq$-formal topology if and only if the formal cover $(A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}})$ is inductively generated.*

Then, we recall the definition of continuous relation between $\leq$-formal covers (as in [13] we take the opposite relation of that defining a frame morphism in [17]):

Definition 2.5. (Continuous relation) *Suppose that $\mathcal{A} = (A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}})$ and $\mathcal{B} = (B, \leq_{\mathcal{B}}, \lhd_{\mathcal{B}})$ are two formal covers. Then a continuous relation from $\mathcal{A}$ to $\mathcal{B}$ is a binary proposition $a\, F\, b$, for $a \in A$ and $b \in B$, satisfying the following conditions:*

$$\textit{(totality)}\qquad A \lhd_{\mathcal{A}} F^{-}(B) \qquad\qquad \textit{($\leq$-convergence)}\ \frac{aFb \qquad aFd}{a \lhd_{\mathcal{A}} F^{-}(b \downarrow_{\leq} d)}$$

$$\textit{(saturation)}\ \frac{a \lhd_{\mathcal{A}} W \quad (\forall w \varepsilon W)\ wFb}{aFb} \qquad\qquad \textit{(continuity)}\ \frac{aFb \qquad b \lhd_{\mathcal{B}} V}{a \lhd_{\mathcal{A}} F^{-}(V)}$$

The definition of continuous relation simplifies when we consider continuous relations between i.g. formal covers as proved in [13]:

Lemma 2.6 *Let $\mathcal{A}$ and $\mathcal{B}$ be i.g. formal covers having $I_{\mathcal{A}}(-)$,$C_{\mathcal{A}}(-,-)$ and $I_{\mathcal{B}}(-), C_{\mathcal{B}}(-,-)$ as axiom-sets respectively. Then, a relation F from A to B is continuous if and only if it satisfies the following conditions:*

$$\textit{(totality)} \quad A \lhd_{\mathcal{A}} F^{-}(B) \qquad \textit{($\leq$-convergence)} \quad \frac{aFb \qquad aFd}{a \lhd_{\mathcal{A}} F^{-}(b \downarrow_{\leq} d)}$$

$$\textit{($\leq$-saturation)} \ \frac{a \leq c \quad c\,F\,b}{a\,F\,b} \qquad \textit{(axiom-saturation)} \ \frac{i \in I_{\mathcal{A}}(a) \quad (\forall x \varepsilon C_{\mathcal{A}}(a,i))\ x\,F\,b}{a\,F\,b}$$

$$\textit{($\leq$-continuity)} \ \frac{a\,F\,b \quad b \leq d}{a\,F\,d} \qquad \textit{(axiom-continuity)} \quad \frac{a\,F\,b \quad j \in I_{\mathcal{B}}(b)}{a \lhd F^{-}(C_{\mathcal{B}}(b,j))}.$$

Note that to get a continuous relation it is sufficient to saturate a relation satisfying all the above conditions except of *axiom saturation*:

Proposition 2.7 *Let $\mathcal{A}$ and $\mathcal{B}$ be two i.g. formal covers and suppose that F is a relation satisfying totality, $\leq$-convergence, $\leq$-saturation, $\leq$-continuity and axiom-continuity. Then*

$$a\,F^{\lhd}\,b \equiv a \lhd_{\mathcal{A}} \{c \in A \mid cFb\}$$

is the minimal continuous relation extending F, called the saturation of F.

Moreover, it is sufficient to prove *axiom-continuity* of a relation just with respect to an axiom-set whose localization gives the proper axiom-set generating the cover:

Lemma 2.8 *Let $\mathcal{A}$ be an i.g. formal cover and $\mathcal{B}$ be an i.g. formal cover having $I'_{\mathcal{B}}(-)$, $C'_{\mathcal{B}}(-,-)$ as the axiom-set localizing $I_{\mathcal{B}}(-)$, $C_{\mathcal{B}}(-,-)$ according to [6]. Let F be a relation from $\mathcal{A}$ to $\mathcal{B}$ satisfying totality, $\leq$-continuity, $\leq$-saturation, $\leq$-convergence and*

$$\textit{(non-loc-axiom-continuity)} \ \frac{aFb \quad j \in I_{\mathcal{B}}(b)}{a \lhd_{\mathcal{A}} F^{-}(C_{\mathcal{B}}(b,j))}$$

then F satisfies axiom-continuity with respect to $I'_{\mathcal{B}}(-)$, $C'_{\mathcal{B}}(-,-)$. And, hence its saturation $F^{\lhd}$ is a continuous relation from $\mathcal{A}$ to $\mathcal{B}$.

We denote with $\mathsf{FTop}_{\mathsf{i}}{}^{-}$ the category of inductively generated $\leq$-formal covers and continuous relations. We recall that the composition $G * F$ of a continuous relation G from $\mathcal{B}$ to $\mathcal{C}$ with a continuous relation F from $\mathcal{A}$ to $\mathcal{B}$ is defined as $G * F \ \equiv \ (G \circ F)^{\lhd}$ that is

$$a\,G * F\,b \ \equiv \ a \lhd_{\mathcal{A}} \{x \in A \mid \ \exists y \in B \ x\,F\,y \ \&\ y\,G\,b\}$$

with A the base of $\mathcal{A}$, and two continuous relations are equal if they are equivalent relations. This category provides a constructive presentation of the category of locales [3]. Moreover, we denote with $\mathsf{FTop}_{\mathsf{i}}$ the full subcategory of $\mathsf{FTop}_{\mathsf{i}}{}^{-}$ with

inductively generated formal topologies. This category provides a constructive presentation of the category of open locales [15].

Here, we recall a lemma proved in [13] that it is useful when we need to compose a continuous relation with one obtained by saturation:

Lemma 2.9 *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be formal covers, F be a relation from $\mathcal{A}$ to $\mathcal{B}$ which satisfies continuity and G be a relation from $\mathcal{B}$ to $\mathcal{C}$. Then $(G \circ F^{\lhd})^{\lhd} = (G \circ F)^{\lhd}$ and $(G^{\lhd} \circ F)^{\lhd} = (G \circ F)^{\lhd}$.*

In [13], the reader may also find the construction of a binary product of two formal covers:

Lemma 2.10 *Let $\mathcal{A}$ and $\mathcal{B}$ be two i.g. formal covers whose axiom-sets are respectively $I_A(-)$, $C_A(-,-)$ and $I_B(-)$, $C_B(-,-)$. Then, a binary product of $\mathcal{A}$ and $\mathcal{B}$ in $\mathsf{FTop}_{\mathsf{i}}^{-}$ is the formal cover $\mathcal{A} \times \mathcal{B}$ over the set $A \times B$, with order relation*

$$(a_1, b_1) \leq_{\mathcal{A}\times\mathcal{B}} (a_2, b_2) \equiv (a_1 \leq_{\mathcal{A}} a_2) \;\&\; (b_1 \leq_{\mathcal{B}} b_2)$$

whose cover relation is inductively generated from the axiom-set

$$\begin{array}{ll} I((a,b)) & \equiv I_A(a) + I_B(b) \\ C((a,b),i) & \equiv \begin{cases} C_A(a, i_a) \times \{b\} & \text{if } i \equiv \mathsf{inl}(i_a) \\ \{a\} \times C_B(b, i_b) & \text{if } i \equiv \mathsf{inr}(i_b) \end{cases} \end{array}$$

and whose first and second projections are defined as follows: $\Pi_1 \equiv p_1^{\lhd}$ where $(a,b)\, p_1\, a' \equiv a \lhd_A a'$ and $\Pi_2 \equiv p_2^{\lhd}$ where $(a,b)\, p_2\, b' \equiv b \lhd_B b'$. Moreover, we indicate with $\langle F, G\rangle$ the relation from $\mathcal{C}$ to $\mathcal{A} \times \mathcal{B}$ that is the pairing of the relation F from $\mathcal{C}$ to $\mathcal{A}$ with the relation G from $\mathcal{C}$ to $\mathcal{B}$.

Then, from [13] we report the following useful lemma about the binary product:

Lemma 2.11 *Let $\mathcal{A}$ and $\mathcal{B}$ be i.g. formal covers and let $a \in A$, $b \in B$, $U \subseteq A$ and $V \subseteq B$.*

Then the following conditions are valid:

$$(1)\ \frac{a \lhd_{\mathcal{A}} U}{(a,b) \lhd_{A\times B} U \times \{b\}} \qquad (2)\ \frac{b \lhd_{\mathcal{B}} V}{(a,b) \lhd_{A\times B} \{a\} \times V} \qquad (3)\ \frac{a \lhd_{\mathcal{A}} U \qquad b \lhd_{\mathcal{B}} V}{(a,b) \lhd_{A\times B} U \times V}$$

About binary products, we recall that one reason to work with inductively generated formal covers is that we know how to build a binary product of two formal covers predicatively only if the formal covers are inductively generated.

2.1 *Join formal covers*

Observe that any inductively generated formal cover is isomorphic to a i.g. join-formal cover, that is a formal cover whose base is closed under a binary join operation and has a bottom element.

Definition 2.12 *We call $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \vee_{\mathcal{A}}, \bot_{\mathcal{A}}, \lhd_{\mathcal{A}})$ a join-formal cover (abbreviated $\vee$-formal cover) if $(A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}})$ is a formal cover whose base A is closed under the binary join operation $\vee_{\mathcal{A}}$ and has a bottom element $\bot_{\mathcal{A}}$ satisfying*

$$(\bot\text{-left}) \quad \bot_{\mathcal{A}} \lhd_{\mathcal{A}} U \qquad (\vee\text{-left}) \quad \frac{a \lhd_{\mathcal{A}} U \qquad b \lhd_{\mathcal{A}} U}{a \vee_{\mathcal{A}} b \lhd_{\mathcal{A}} U}$$

$$(\vee\text{-right}) \quad \frac{a \lhd_{\mathcal{A}} b}{a \lhd_{\mathcal{A}} \{b \vee_{\mathcal{A}} c\}} \qquad \frac{a \lhd_{\mathcal{A}} b}{a \lhd_{\mathcal{A}} \{c \vee_{\mathcal{A}} b\}}.$$

We call $\mathsf{FTop}_{\mathsf{i}\vee}^{-}$ the full subcategory of $\mathsf{FTop}_{\mathsf{i}}^{-}$ with $\vee$-formal covers. Actually, we can prove that these two categories are equivalent since the following holds:

Lemma 2.13 *Any i.g. formal cover $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}})$ with axiom-set $I_{\mathcal{A}}(-)$, $C_{\mathcal{A}}(-,-)$, is isomorphic to a $\vee$-formal cover $\mathcal{A}^{\vee} \equiv (A^{\vee}, \leq_{\mathcal{A}^{\vee}}, \vee_{\mathcal{A}^{\vee}}, \bot_{\mathcal{A}^{\vee}}, \lhd_{\mathcal{A}^{\vee}})$ inductively generated with an axiom-set $I^{\vee}(-)$, $C^{\vee}(-,-)$ generating the cover $\lhd_{\mathcal{A}^{\vee}}$ by means of the rules of* reflexivity, $\leq$-left *and* infinity.

We call $\phi_{\vee} : \mathcal{A} \to \mathcal{A}^{\vee}$ the isomorphism and $\phi_{\vee}^{-1} : \mathcal{A}^{\vee} \to \mathcal{A}$ its inverse.

Proof. We can build $\mathcal{A}^{\vee}$ by taking the set of lists of elements of A as the basic set, ordered by inclusion of elements. Then, we can define the join operation as the append operation of lists and the bottom element as the empty list.

Note that elements covered by the bottom element behaves like non positive elements:

Lemma 2.14 *Let $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \bot_{\mathcal{A}}, \lhd_{\mathcal{A}})$ and $\mathcal{B} \equiv (B, \leq_{\mathcal{B}}, \bot_{\mathcal{B}}, \lhd_{\mathcal{B}})$ be $\vee$-formal covers and let F be a continuous relation from $\mathcal{A}$ to $\mathcal{B}$. If $c \lhd_{\mathcal{A}} \bot_{\mathcal{A}}$, or $c \lhd_{\mathcal{A}} \emptyset$, then $c\, F\, b$ follows for every $b \in B$.*

3 Locally compact formal covers

Here, we give a definition of locally compact formal cover suggested in [7]. This definition is equivalent to the original one in [15] and it is based on the fact that the way-below relation between subsets is definable in an inductively generated formal cover as proved in [7]:

Proposition 3.1 *Let $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}})$ be an i.g. formal cover. We can predicatively define the way-below relation $U \ll V$ between subsets $U, V \subseteq A$, meaning that for all $W \subseteq A$, if $V \lhd W$ then there is a finite (possibly empty) list $w_1, .., w_n$ of elements in W such that $U \lhd \{w_1, .., w_n\}$.*

Proof. We can assume from [6] that the axiom-set $I_A(-), C_A(-,-)$ is set-based according to [6], that is

$$a \lhd W \;\equiv\; \exists j \in I(a) \quad C(a,j) \subseteq W$$

Then, following page.61 in [7], we can define the way-below relation between subsets $U \ll V$ as follows:

$$\forall f \in \Pi_{v \varepsilon V}\, I(v) \quad \exists c_1 \in A ... \exists c_n \in A\ \{c_1, .., c_n\} \subseteq \bigcup_{v \varepsilon V} C(v, f(v)) \;\&\; U \lhd \{c_1, .., c_n\}$$

where $\Pi_{v \varepsilon V}\, I(v)$ is an abbreviation for $\Pi_{z \in \Sigma_{x \in A} V(x)} I(\pi_1(z))$.

Note that the predicative definability of the way-below relation between subsets is a crucial point to build the exponential formal cover according to the idea in [9, 21].

Then, following [7] we give the definition of locally compact formal cover.

Definition 3.2 *A formal cover $\mathcal{A}$ is locally compact if it is inductively generated and $a \lhd \{b : b \ll a\}$, for all $a \in S$ – where $b \ll a$ stands for the predicative definition mentioned in prop. 3.1 –.*

The notion of locally compact formal cover is preserved by isomorphisms as shown in [7]:

Lemma 3.3 *Let be $\mathcal{A}$ and $\mathcal{A}'$ two formal covers isomorphic via continuous relations. If $\mathcal{A}$ is locally compact, then $\mathcal{A}'$ is locally compact, too.*

Moreover, the following properties about the way-below relation can be proved predicatively:

Lemma 3.4 *Let $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \vee_{\mathcal{A}}, \perp_{\mathcal{A}}, \lhd_{\mathcal{A}})$ be a locally compact $\vee$-formal cover.*
(1) If $U \ll V$ then $U \lhd_{\mathcal{A}} V$.
(2) If $U \ll V$ and $V \lhd_{\mathcal{A}} W$ then $U \ll W$.
(3) If $U \lhd_{\mathcal{A}} V$ and $V \ll W$ then $U \ll W$.
(4) (interpolation property) If $a \ll V$ there exists $a' \in A$ such that $a \ll a' \ll V$.

Remark 3.5 Note that we can express the above *interpolation property* because of the presence of joins in the base.

In absence of joins in the base, we need to express the interpolation property by saying that: if $a \ll V$ then there exists a finite list $a_1, .., a_n$, possibly empty, such that $a \ll \{a_1, .., a_n\}$ and $\{a_1, .., a_n\} \ll V$.

Now, we state a lemma which is auxiliary to prove the next lemma on binary products.

Lemma 3.6 *Let $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \vee_{\mathcal{A}}, \perp_{\mathcal{A}}, \lhd_{\mathcal{A}})$ be a locally compact $\vee$-formal cover. If $d \ll a$ and $a \lhd_{\mathcal{A}} U$, then there exist a list $u_1 \ll u'_1, .., u_n \ll u'_n$ such that $d \lhd_{\mathcal{A}} \{u_1, .., u_n\}$ and $u'_i \,\varepsilon\, U$ for $i = 1, .., n$, or $d \lhd_{\mathcal{A}} \perp_{\mathcal{A}}$.*

Then, we prove a lemma on the product of an i.g. formal cover with a locally compact $\vee$-formal cover that will be crucial to prove that the abstraction relation relative to the exponential formal cover is continuous. It expresses that, in some cases, if $(c, a) \lhd_{\mathcal{C}\times\mathcal{A}} W$ then it is possible to decompose W in an open U covering c and in a suitable finite subset covering any given d way-below a.

Lemma 3.7 *Consider the product of an i.g. formal cover $\mathcal{C} \equiv (C, \leq_{\mathcal{C}}, \lhd_{\mathcal{C}})$ having $I_{\mathcal{C}}(-)$, $C_{\mathcal{C}}(-,-)$ as axiom-set and a locally compact $\vee$-formal cover $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \vee_{\mathcal{A}}, \perp_{\mathcal{A}}, \lhd_{\mathcal{A}})$. Suppose also that $c \in C$, $a \in A$, $(c, a) \lhd_{\mathcal{C}\times\mathcal{A}} W$ where $W \subseteq C \times A$ satisfies the following properties:*
(1) if $(c', a') \,\varepsilon\, W$, then $(c'', a') \,\varepsilon\, W$ for every c'' such that $c'' \leq c'$;
(2) $(c, \perp_{\mathcal{A}}) \,\varepsilon\, W$.

Then, for any $d \ll a$ there exists a subset U of C such that $c \lhd_{\mathcal{C}} U$ and, for every $u \,\varepsilon\, U$ there exists $e_1 \ll e'_1, .., e_n \ll e'_n \in A$ with $d \lhd_{\mathcal{A}} \{e_1, .., e_n\}$ and $(u, e'_j) \,\varepsilon\, W$ for $j = 1, .., n$.

Proof. Suppose $d \ll a$. We prove the statement by induction on the derivation of $(c, a) \lhd_{C\times A} W$. If $(c, a) \lhd_{C\times A} W$ follows by *reflexivity* then $(c, a) \,\varepsilon\, W$. Hence,

take $U \equiv \{c\}$ and for $u \equiv c$ take $e_1 \equiv d$ and $e'_1 \equiv a$. Instead, if $(c,a) \lhd_{C\times A} W$ follows by *infinity* then we have to consider two cases, namely that it follows from $\{c\} \times C_{\mathcal{A}}(a,j) \lhd_{C\times A} W$ or from $C_{\mathcal{C}}(c,j) \times \{a\} \lhd_{C\times A} W$.

If it follows from $\{c\} \times C_{\mathcal{A}}(a,j) \lhd_{C\times A} W$, we apply lemma 3.6 to $d \ll a$ and $a \lhd_{\mathcal{A}} C_{\mathcal{A}}(a,j)$. If $d \lhd_{\mathcal{A}} \bot_{\mathcal{A}}$ we put $U \equiv \{c\}$ and $e_1 = e'_1 \equiv \bot_{\mathcal{A}}$, since $\bot_{\mathcal{A}} \ll \bot_{\mathcal{A}}$ and $(c, \bot_{\mathcal{A}}) \in W$. Instead, if we have $h_1 \ll h'_1, .., h_m \ll h'_m$ with $h'_i \,\varepsilon\, C_{\mathcal{A}}(a,j)$ for $i = 1, .., m$ such that $d \lhd_{\mathcal{A}} \{h_1, .., h_m\}$, then by inductive hypothesis for every $h'_i \,\varepsilon\, C_{\mathcal{A}}(a,j)$ with $i = 1, .., m$ there exists a subset $U_{h'_i}$ of C such that $c \lhd_{\mathcal{C}} U_{h'_i}$ and for every $u_{h'_i} \,\varepsilon\, U_{h'_i}$ there exist $e_1^{h'_i} \ll (e^{h'_i})'_1, .., e_n^{h'_i} \ll (e^{h'_i})'_{n_i}$ such that $h_i \lhd_{\mathcal{A}} \{e_1^{h'_i}, .., e_{n_i}^{h'_i}\}$ and $(u_{h'_i}, (e^{h'_i})'_j) \,\varepsilon\, W$ for $j = 1, .., n_i$. Therefore, take $U \equiv U_{h'_1} \downarrow_{\leq} .. \downarrow_{\leq} U_{h'_m}$. Indeed, for every $u \,\varepsilon\, U$ there exist $u_{h'_i} .., u_{h'_m}$ such that $u \leq u_{h'_1} .. u \leq u_{h'_m}$. So, we take all the $e_1^{h'_i} \ll (e^{h'_i})'_1, .. e_{n_i}^{h'_i} \ll (e^{h'_i})'_{n_i}$ associated to each $u_{h'_i}$ for $i = 1, .., m$ since $d \lhd_{\mathcal{A}} \cup_{i=1,..,m} \{e_1^{h'_i}, .., e_{n_i}^{h'_i}\}$. Moreover, observe also that for any $j = 1, .., n$ and $i = 1, .., m$ then $(u, (e^{h'_i})'_j) \,\varepsilon\, W$ follows from the hypothesis on W since $u \leq u_{h'_j}$ and $(u_{h'_i}, (e^{h'_i})'_j) \,\varepsilon\, W$.

Instead, if $(c,a) \lhd_{C\times A} W$ follows by using *infinity* on $C_{\mathcal{C}}(c,j) \times \{a\} \lhd_{C\times A} W$, then take $U \equiv \bigcup_{v \varepsilon C_{\mathcal{C}}(c,j)} U_v$ with U_v obtained by inductive hypothesis on $(v,a) \lhd_{C\times A} W$ for $v \varepsilon C_{\mathcal{C}}(c,j)$. Finally, if $(c,a) \lhd_{C\times A} W$ follows from $\leq$-*left* on $(c,a) \leq (c',a')$ and $(c',a') \lhd_{C\times A} W$, take U obtained by the inductive hypothesis on $(c',a') \lhd_{C\times A} W$.

4 Exponentiation

In this section we prove that any locally compact $\vee$-formal cover is exponentiable in $\mathsf{FTop}_{|\vee}^{-}$. This suffices to prove that any locally compact formal cover is exponentiable in $\mathsf{FTop}_{|}^{-}$ since by lemma 3.3 and lemma 2.13 we know that any locally compact formal cover is isomorphic to a locally compact $\vee$-formal cover and that $\mathsf{FTop}_{|\vee}^{-}$ and $\mathsf{FTop}_{|}^{-}$ are equivalent.

To exponentiate a locally compact $\vee$-formal cover $\mathcal{A}$ over any $\vee$-formal cover $\mathcal{B}$ in $\mathsf{FTop}_{|\vee}^{-}$, we need to define the exponential cover of $\mathcal{A}$ over $\mathcal{B}$ in such a way that its formal points are in bijection with the continuous relations from $\mathcal{A}$ to $\mathcal{B}$.

As seen in [13] we are able to perform such a correspondence by encoding a continuous relation via lists of pairs of basic opens (a,b) for $a \in A$ and $b \in B$ (where A is the set of basic opens for $\mathcal{A}$ and B the set of basic opens for $\mathcal{B}$) if the conditions defining a continuous relation can be formulated according to one of the forms

$$\frac{a \;R\; b \qquad P(a,b,a',b')}{a' \;R\; b'} \qquad\qquad \frac{a \;R\; b \qquad Q(a,b,V)}{(\exists y \,\varepsilon\, V)\, a \;R\; y}$$

But, note that *axiom-saturation* condition of a continuous relation from a locally compact formal cover to any other i.g. formal cover is not of the above forms – and it does not seem to be reducible to one of the above forms –, contrary to what happens with continuous relations from a unary topology (as we noticed in [13]). However, inspired by the fact that the exponential topology built in [9], [10], [21]

is generated from axioms expressing the properties of the relation $a \ll f^*(b)$ for $a \in A$ and $b \in B$ and f locale morphism, observe that to any continuous relation R, from a locally compact $\vee$-formal cover to any i.g. $\vee$-formal cover, we can associate its way-below component, defined as $a \ll R^-(b)$, whose properties can be expressed by conditions of the above forms and hence can be used to generate an exponential cover. Therefore, we give the definition of *wb-relation* in terms of the conditions satisfied by $a \ll R^-(b)$. Then, after proving that wb-relations from a locally compact $\vee$-formal cover to any i.g. $\vee$-formal cover are in bijection with the corresponding continuous relations, we build the exponential topology according to the semantics given in [13] with reference to wb-relations instead of referring to continuous relations.

Definition 4.1 *Let $\mathcal{A}$ be a locally compact $\vee$-formal cover and $\mathcal{B}$ be an i.g. $\vee$-formal cover with axiom-set $I_{\mathcal{B}}(-)$, $C_{\mathcal{B}}(-,-)$. Then, F^{wb} is a* wb-relation *from $\mathcal{A}$ to $\mathcal{B}$ if it is a relation from A to B and it satisfies the following conditions:*

(wb-bottom continuity)
$$\frac{c\, F^{wb}\ \bot_{\mathcal{B}}}{c \lhd_{\mathcal{A}} \bot_{\mathcal{A}}}$$

(wb-bottom saturation)
$$\frac{b \in B}{\bot_{\mathcal{A}}\ F^{wb}\, b}$$

(wb-totality)
$$\frac{d \ll a}{\exists\, y_1, .., y_n \in B \quad \exists\, x_1, .., x_n \in A \quad x_i\, F^{wb}\, y_i \quad i = 1, .., n \quad \& \quad d \lhd_{\mathcal{A}} \{x_1, .., x_n\}}$$

(wb-$\leq$-convergence)
$$\frac{a\, F^{wb}\, b \qquad a\, F^{wb}\, c \qquad d \ll a}{\exists\, y_1, .., y_n \in (b \downarrow_{\leq} c) \cup \{\bot_{\mathcal{B}}\}\ \exists\, x_1, .., x_n \in A\ x_i\, F^{wb}\, y_i\ i = 1, .., n\ \&\ d \lhd_{\mathcal{A}} \{x_1, .., x_n\}}$$

(wb-finite-saturation)
$$\frac{a \lhd_{\mathcal{A}} \{c_1, .., c_n\} \qquad \forall x\, \varepsilon\, \{c_1, .., c_n\} \qquad x\, F^{wb}\, b}{a\, F^{wb}\, b}$$

(wb-axiom-continuity)
$$\frac{a\, F^{wb}\, b \qquad j \in J(b) \qquad d \ll a}{\exists y_1, .., y_n\, \varepsilon\, C(b, j) \cup \{\bot_{\mathcal{A}}\}\ \exists\, x_1, .., x_n \in A\ x_i\, F^{wb}\, y_i\ i = 1, ., n \& d \lhd_{\mathcal{A}} \{x_1, .., x_n\}}$$

(wb-$\leq$-continuity)
$$\frac{a\, F^{wb}\, b \qquad b \leq d}{a\, F^{wb}\, d}$$

(interpolation)
$$\frac{a\, F^{wb}\, b}{\exists x \in A \quad x\, F^{wb}\, b \quad \& \quad a \ll x}$$

Lemma 4.2 *Let $\mathcal{A}$ be a locally compact $\vee$-formal cover and $\mathcal{B}$ any i.g. $\vee$-formal cover with axiom-set $I_{\mathcal{B}}(-)$, $C_{\mathcal{B}}(-,-)$. Then, the collection of continuous relations from $\mathcal{A}$ to $\mathcal{B}$ is in bijection with the collection of wb-relations from $\mathcal{A}$ to $\mathcal{B}$. Indeed,*
(1) Given any continuous relation F from $\mathcal{A}$ to $\mathcal{B}$, then $a\, F^{wb}\, b \equiv a \ll F^-(b)$, called the wb-component of F, is a wb-relation from $\mathcal{A}$ to $\mathcal{B}$.

(2) Given any wb-relation R from $\mathcal{A}$ to $\mathcal{B}$ then $a\ R^{\lhd}\ b \equiv a \lhd_{\mathcal{A}} \{x \in S \mid x\ R\ b\}$ is a continuous relation from $\mathcal{A}$ to $\mathcal{B}$.
(3) Given any continuous relation F from $\mathcal{A}$ to $\mathcal{B}$ then $F = (F^{wb})^{\lhd}$.
(4) Given any wb-relation R from $\mathcal{A}$ to $\mathcal{B}$ then $R = (R^{\lhd})^{wb}$.

4.0.1 *The exponential formal cover* For any locally compact $\vee$-formal cover $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \vee_{\mathcal{A}}, \bot_{\mathcal{A}}, \lhd_{\mathcal{A}})$ and any i.g. $\vee$-formal cover $\mathcal{B} \equiv (B, \leq_{\mathcal{B}}, \vee_{\mathcal{B}}, \bot_{\mathcal{B}}, \lhd_{\mathcal{B}})$ with axiom-set $I_{\mathcal{B}}(-)$, $C_{\mathcal{B}}(-,-)$, the exponential cover candidate $\mathcal{B}^{\mathcal{A}}$ of $\mathcal{A}$ over $\mathcal{B}$ is defined as the $\vee$-formal cover isomorphic to a formal cover $[\mathcal{A} \to \mathcal{B}]$ whose base does not have naturally joins.

As for the exponential formal cover in [13], the set of basic neighbourhoods of $[\mathcal{A} \to \mathcal{B}]$ is $List(A \times B)$, that is lists of pairs of elements in the cartesian product $A \times B$, where we indicate the empty list with $[\,]$ and the list of elements $(a_1, b_1), .., (a_n, b_n)$ with $[(a_1, b_1), .., (a_n, b_n)]$.

The intended meaning of a list $l \in List(A \times B)$ is to give a partial information on a wb-relation from $\mathcal{A}$ to $\mathcal{B}$. The order on $List(A \times B)$ is

$$l \leq_{[\mathcal{A}\to\mathcal{B}]} m \ \equiv\ (\forall (a,b) \in A \times B) \quad (a,b)\,\varepsilon\, m \to (a,b)\,\varepsilon\, l$$

stating that the list l is more precise, that is, it approximates fewer wb-relations than the list m. This order relation is a refinement of the reverse sub-list relation, which states that m is a sub-list of l, because it does not consider the order among the elements in a list and their repetitions.

We define the cover relation $\lhd_{[\mathcal{A}\to\mathcal{B}]}$ according to the usual semantics of formal topology (see [18]) as follows: $l \lhd_{[\mathcal{A}\to\mathcal{B}]} U$ *means that every formal point containing l also contains a basic neighbourhood of U*. Recalling that formal points are expected to be wb-relations, this means that we have to prove that *for any wb-relation R, if $a\,R\,b$ holds for each pair $(a,b)\,\varepsilon\, l$, then there exists $m\,\varepsilon\, U$ such that $a\ R\ b$ holds for each pair $(a,b)\,\varepsilon\, m$.*

Since we want to obtain an i.g. formal cover, we apply the method in [6] and we define the axiom-set $I_{[\mathcal{A}\to\mathcal{B}]}(-)$, $C_{[\mathcal{A}\to\mathcal{B}]}(-,-)$ as the localization of another set of axioms $I'_{[\mathcal{A}\to\mathcal{B}]}(-)$, $C'_{[\mathcal{A}\to\mathcal{B}]}(-,-)$ that we describe in the following: we express each axiom in the form $l \lhd_{[\mathcal{A}\to\mathcal{B}]} U$ by intending that for some index $j \in I'_{[\mathcal{A}\to\mathcal{B}]}(l)$ then $C'_{[\mathcal{A}\to\mathcal{B}]}(l,j) \equiv U$. We hope that it will be clear how such a formalization can be actually performed.

The general recipe is that each axiom is in correspondence with a condition defining a wb-relation in definition 4.1.

The set of axioms $I'_{[\mathcal{A}\to\mathcal{B}]}(-)$ and $C'_{[\mathcal{A}\to\mathcal{B}]}(-,-)$ include only:

(wb-bottom continuity)
$[(a, \bot_{\mathcal{B}})] \lhd_{[\mathcal{A}\to\mathcal{B}]} \{\,[(\bot_{\mathcal{A}}, \bot_{\mathcal{B}})] \mid a \lhd_{\mathcal{A}} \bot_{\mathcal{A}}\}$

(wb-bottom saturation)
$[\,] \lhd_{[\mathcal{A}\to\mathcal{B}]} [(\bot_{\mathcal{A}}, b)]$
provided that $b \in B$

(wb-totality)
$[\,] \lhd_{[\mathcal{A}\to\mathcal{B}]} \{\,[(x_1, y_1), .., (x_n, y_n)] \mid y_1, .., y_n \in B \quad \& \quad d \lhd_{\mathcal{A}} \{x_1, .., x_n\}\}$
provided that $d \ll a$ for some $a \in A$

(*wb-$\leq$-convergence*)
$[(a,b),(a,b')] \lhd_{[\mathcal{A}\to\mathcal{B}]} \{\, [(x_1,y_1),..,(x_n,y_n)] \mid y_i \,\varepsilon\, (b\downarrow_{\leq} b') \cup \{\bot_{\mathcal{B}}\} \text{ for } i=1,..,n \;\&\; d \lhd_{\mathcal{A}} \{x_1,..,x_n\}\}$
provided that $d \ll a$

(*wb-finite-saturation*)
$[(d_1,b),..,(d_n,b)] \lhd_{[\mathcal{A}\to\mathcal{B}]} [(a,b)]$
provided that $a \lhd_{\mathcal{A}} \{d_1,..,d_n\}$ for some $a \in A$

(*wb-axiom-continuity*)
$[(a,b)] \lhd_{[\mathcal{A}\to\mathcal{B}]} \{\, [(x_1,y_1),..,(x_n,y_n)] \mid y_i \,\varepsilon\, C(b,j) \cup \{\bot_{\mathcal{B}}\} \text{ for } i=1,..,n \;\&\; d \lhd_{\mathcal{A}} \{x_1,..,x_n\}\}$
provided that $d \ll a$ and $j \in J(b)$

(*wb-$\leq$-continuity*)
$[(a,b)] \lhd_{[\mathcal{A}\to\mathcal{B}]} [(a,b')]$
provided that $b \leq b'$

(*interpolation*)
$[(a,b)] \lhd_{[\mathcal{A}\to\mathcal{B}]} \{\, [(x,b)] \mid a \ll x\}$

Then, as said, we localize such an axiom-set $I'_{[\mathcal{A}\to\mathcal{B}]}(-)$ and $C'_{[\mathcal{A}\to\mathcal{B}]}(-,-)$ according to [6] and we obtain the axiom-set $I_{[\mathcal{A}\to\mathcal{B}]}(-)$, $C_{[\mathcal{A}\to\mathcal{B}]}(-,-)$ that generates the formal cover $[\mathcal{A}\to\mathcal{B}]$.

Note that the formal cover $[\mathcal{A}\to\mathcal{B}]$ lives in $\mathsf{FTop}_{\mathsf{i}}{}^{-}$ but not in $\mathsf{FTop}_{\mathsf{i}\vee}^{-}$. Hence, we take its isomorphic copy whose base is closed under joins $[\mathcal{A}\to\mathcal{B}]_{\vee}$ and we define

$$\mathcal{B}^{\mathcal{A}} \equiv [\mathcal{A}\to\mathcal{B}]_{\vee}.$$

Remark 4.3 Note that since the formal cover $[\mathcal{A}\to\mathcal{B}]$ lives in $\mathsf{FTop}_{\mathsf{i}}{}^{-}$ one could suggest to work within $\mathsf{FTop}_{\mathsf{i}}{}^{-}$. But then we would need to express the exponential cover axioms without the use of a symbol for a bottom element and for joins both for $\mathcal{A}$ and for $\mathcal{B}$. Now, it is immediate to observe that to express the exponential axioms we do not need joins in $\mathcal{B}$ but just a bottom element and this is why in [12] we worked in the full subcategory $\mathsf{FTop}_{\mathsf{i}\bot}^{-}$ of $\mathsf{FTop}_{\mathsf{i}}{}^{-}$ with formal covers having a bottom element. The bottom element in $\mathcal{B}$ seems necessary to express *wb-bottom continuity* and *wb-$\leq$-convergence* by including the case in which an element is cover by the emptyset, that is to circumvent the fact that we work with formal covers that do not necessarily have a positivity predicate.

In principle, we could work in $\mathsf{FTop}_{\mathsf{i}\bot}^{-}$ with a locally compact cover having a base without joins, but then to express the *interpolation* axiom, as remarked in 3.5 about the interpolation property, we would need to use finite lists in the first component of the pairs, namely we would need to change the base of the exponential to $List(List(A)\times B)$ making the various proofs more complicated.

Let us recall now useful lemmas on the exponential topology proved in [13]. Given $l_1, l_2 \in List(A\times B)$, we indicate with $l_1 \cdot l_2$ the append operation of the list l_1 with l_2.

Lemma 4.4 *Let $l \in List(A \times B)$ and $U_1, U_2 \subseteq List(A \times B)$. Then the following rule holds:*

$$(\cdot\text{-right}) \quad \frac{l \lhd_{[\mathcal{A}\to\mathcal{B}]} U_1 \qquad l \lhd_{[\mathcal{A}\to\mathcal{B}]} U_2}{l \lhd_{[\mathcal{A}\to\mathcal{B}]} U_1 \cdot U_2}$$

where $U_1 \cdot U_2 \equiv \{m_1 \cdot m_2 \mid m_1 \,\varepsilon\, U_1 \,\&\, m_2 \,\varepsilon\, U_2\}$.

Lemma 4.5 *Let F be a continuous relation from $\mathcal{C}$ to $[\mathcal{A} \to \mathcal{B}]$. Then, for any $c \in C$ and any $l_1, l_2 \in List(A \times B)$, the following rule holds*

$$\frac{c\, F\, l_1 \qquad c\, F\, l_2}{c\, F\, l_1 \cdot l_2}.$$

As proved in [13], the cover of $[\mathcal{A} \to \mathcal{B}]$ does not depend on the particular axiom-set generating the cover of $\mathcal{B}$ since we can show:

Lemma 4.6 *Let $\mathcal{A}$ be a locally compact $\vee$-formal cover and $\mathcal{B}$ be an i.g. $\vee$-formal cover. Then, for any list $l \in List(A \times B)$ and any $V \subseteq B$, if $(a, b) \,\varepsilon\, l$ and $b \lhd_{\mathcal{B}} V$ then $l \lhd_{[\mathcal{A}\to\mathcal{B}]} \{(a, y) \mid y \,\varepsilon\, V\}$.*

4.0.2 *Application and abstraction* Now we are going to show that the formal cover $\mathcal{B}^{\mathcal{A}} \equiv [\mathcal{A} \to \mathcal{B}]_\vee$ is the exponential of the locally compact $\vee$-formal cover $\mathcal{A}$ over the i.g. $\vee$-formal cover $\mathcal{B}$.

From a categorical point of view, we prove that for any locally compact $\vee$-formal cover $\mathcal{A}$ the functor $- \times \mathcal{A} : \mathsf{FTop}_{i\vee}^{-} \Rightarrow \mathsf{FTop}_{i\vee}^{-}$ has got a right adjoint $(-)^{\mathcal{A}} : \mathsf{FTop}_{i\vee}^{-} \Rightarrow \mathsf{FTop}_{i\vee}^{-}$. Equivalently, this amounts to define, for any locally compact $\vee$-formal cover $\mathcal{A}$ and any i.g. $\vee$-formal cover $\mathcal{B}$, an *application relation* Ap from $(\mathcal{B}^{\mathcal{A}}) \times \mathcal{A}$ to $\mathcal{B}$ such that for any continuous relation F from $\mathcal{C} \times \mathcal{A}$ to $\mathcal{B}$ there exists a continuous relation $\Lambda(F)$ from $\mathcal{C}$ to $\mathcal{B}^{\mathcal{A}}$, called *abstraction* of F, such that $\mathsf{Ap} * \langle \Lambda(F) * \Pi_1, \Pi_2 \rangle = F$ holds, and for any continuous relation G from $\mathcal{C}$ to $\mathcal{B}^{\mathcal{A}}$ then also $\Lambda(\mathsf{Ap} * \langle G * \Pi_1, \Pi_2 \rangle) = G$ holds.

Since we prefer to work with $[\mathcal{A} \to \mathcal{B}]$ than with $[\mathcal{A} \to \mathcal{B}]_\vee$, we give the definitions of application and abstraction with respect to $[\mathcal{A} \to \mathcal{B}]$ and we use such definitions to give those of application and abstraction for $[\mathcal{A} \to \mathcal{B}]_\vee$ by the isomorphisms $\phi_\vee : [\mathcal{A} \to \mathcal{B}] \longrightarrow [\mathcal{A} \to \mathcal{B}]_\vee$ and $\phi_\vee^- : [\mathcal{A} \to \mathcal{B}]_\vee \longrightarrow [\mathcal{A} \to \mathcal{B}]$.

Indeed, we define the application $\mathsf{Ap} : [\mathcal{A} \to \mathcal{B}]_\vee \times \mathcal{A} \longrightarrow \mathcal{B}$ as the composition of the application $\widetilde{\mathsf{Ap}}^{\lhd} : [\mathcal{A} \to \mathcal{B}] \times \mathcal{A} \longrightarrow \mathcal{B}$ relative to $[\mathcal{A} \to \mathcal{B}]$ with the morphism $\phi_\vee^- \times \mathsf{id} : [\mathcal{A} \to \mathcal{B}]_\vee \times \mathcal{A} \longrightarrow [\mathcal{A} \to \mathcal{B}] \times \mathcal{A}$, that is

$$\mathsf{Ap} \equiv \widetilde{\mathsf{Ap}}^{\lhd} * (\phi_\vee^- \times \mathsf{id})$$

where

$$(l, a)\, \widetilde{\mathsf{Ap}}\, b \equiv l \lhd_{[\mathcal{A}\to\mathcal{B}]} [(a, b)]$$

for any $l \in List(A \times B)$, $a \in A$ and $b \in B$.

Then, for any given relation $F : \mathcal{C} \times \mathcal{A} \longrightarrow \mathcal{B}$, we define the abstraction $\Lambda(F) : \mathcal{C} \longrightarrow [\mathcal{A} \to \mathcal{B}]_\vee$ as the composition of $\phi_\vee : [\mathcal{A} \to \mathcal{B}] \longrightarrow [\mathcal{A} \to \mathcal{B}]_\vee$ with the abstraction $\widetilde{\Lambda}(F)^{\lhd} : \mathcal{C} \longrightarrow [\mathcal{A} \to \mathcal{B}]$ relative to $[\mathcal{A} \to \mathcal{B}]$, that is

$$\Lambda(F) \equiv \phi_\vee * (\widetilde{\Lambda}(F)^\lhd)$$

where

$$c\,\widetilde{\Lambda}(F)\,l \;\equiv\; (\forall (a,b)\,\varepsilon\, l)\; a \ll \{x \in A \mid (c,x)\,F\,b\}$$

for any $l \in List(A \times B)$ and $c \in C$.

To prove that Ap is a continuous relation, we just need to show that $\widetilde{\mathsf{Ap}}^\lhd$ is a continuous relation. To this purpose it is sufficient to show that $\widetilde{\mathsf{Ap}}$ satisfies all the conditions in proposition 2.7. Before doing this we prove the following useful lemma.

Lemma 4.7 *Let $\mathcal{A}$ be a locally compact $\vee$-formal cover and $\mathcal{B}$ be an i.g. $\vee$-formal cover. Then, for any subset U of B*

$$\widetilde{\mathsf{Ap}}^{-}(U \cup \{\bot_\mathcal{B}\}) \lhd_{[\mathcal{A}\to\mathcal{B}]\times\mathcal{A}} \widetilde{\mathsf{Ap}}^{-}(U)$$

Proof. Suppose $(l,a)\,\varepsilon\,\widetilde{\mathsf{Ap}}^{-}(U\cup\{\bot_\mathcal{B}\})$. Then, either there exists $b \in U$ such that $(l,a)\,\widetilde{\mathsf{Ap}}\,b$, and hence $(l,a) \lhd_{(A\to B)_\vee \times A} \widetilde{\mathsf{Ap}}^{-}(U)$, or $(l,a)\,\widetilde{\mathsf{Ap}}\;\bot_\mathcal{B}$. In this latter case, by definition we get $l \lhd_{[\mathcal{A}\to\mathcal{B}]} [(a,\bot_\mathcal{B})]$ and hence by *transitivity* with *wb-bottom continuity* axiom we obtain $l \lhd_{[\mathcal{A}\to\mathcal{B}]} \{[(\bot_\mathcal{A},\bot_\mathcal{B})] \mid a \lhd_\mathcal{A} \bot_\mathcal{A}\}$, from which by lemma 2.11 $(l,a) \lhd_{[\mathcal{A}\to\mathcal{B}]\times\mathcal{A}} \{[(\bot_\mathcal{A},\bot_\mathcal{B})] \mid a \lhd_\mathcal{A} \bot_\mathcal{A}\} \times \{a\}$ follows. Then, we conclude $(l,a) \lhd_{[\mathcal{A}\to\mathcal{B}]\times\mathcal{A}} \widetilde{\mathsf{Ap}}^{-}(U)$ since $\{[(\bot_\mathcal{A},\bot_\mathcal{B})] \mid a \lhd_\mathcal{A} \bot_\mathcal{A}\} \times \{a\} \lhd_{[\mathcal{A}\to\mathcal{B}]\times\mathcal{A}} \emptyset$.

Lemma 4.8 *Let $\mathcal{A}$ be a locally compact $\vee$-formal cover and $\mathcal{B}$ be an i.g. $\vee$-formal cover. Then, $\widetilde{\mathsf{Ap}}$ is a relation from $List(A \times B) \times A$ to B which satisfies* totality, $\leq$-convergence, axiom continuity, $\leq$-continuity *and* $\leq$-saturation.

Proof. We just prove ($\leq$-*convergence*). Suppose $(l,a)\,\widetilde{\mathsf{Ap}}\,b$ and $(l,a)\,\widetilde{\mathsf{Ap}}\,b'$, that is $l \lhd_{[\mathcal{A}\to\mathcal{B}]} [(a,b)]$ and $l \lhd_{[\mathcal{A}\to\mathcal{B}]} [(a,b')]$. Then, $l \lhd_{[\mathcal{A}\to\mathcal{B}]} [(a,b),(a,b')]$ follows by lemma 4.4, and by *wb-*$\leq$*-convergence* axiom and *transitivity* we obtain for $d \ll a$

$$l \lhd_{[\mathcal{A}\to\mathcal{B}]} \{[(x_1,y_1),..,(x_n,y_n)] \mid y_i\,\varepsilon\,(b\downarrow_\leq b') \cup \{\bot_\mathcal{B}\} \text{ for } i=1,..,n \;\&\; d \lhd_\mathcal{A} \{x_1,..,x_n\}\}$$

and by lemma 2.11

$$(l,d) \lhd_{[\mathcal{A}\to\mathcal{B}]\times\mathcal{A}} \{[(x_1,y_1),..,(x_n,y_n)] \mid y_i\,\varepsilon\,(b\downarrow_\leq b') \cup \{\bot_\mathcal{B}\} \text{ for } i=1,..,n \;\&\; d \lhd_\mathcal{A} \{x_1,..,x_n\}\} \times \{d\}$$

Then, for every $[(x_1,y_1),..,(x_n,y_n)]$ in the above subset covering l we obtain

$$([(x_1,y_1),..,(x_n,y_n)],d) \lhd_{[\mathcal{A}\to\mathcal{B}]\times\mathcal{A}} \{[(x_1,y_1),..,(x_n,y_n)]\} \times \{x_1,..,x_n\}.$$

Since $[(x_1,y_1),..,(x_n,y_n)] \leq [(x_i,y_i)]$, we get $([(x_1,y_1),..,(x_n,y_n)],x_i) \lhd_{[\mathcal{A}\to\mathcal{B}]} ([(x_i,y_i)],x_i)$ and $([(x_i,y_i)],x_i) \lhd_{[\mathcal{A}\to\mathcal{B}]\times\mathcal{A}} \widetilde{\mathsf{Ap}}^{-}((b\downarrow b')\cup\{\bot_\mathcal{B}\})$ for $i=1,..,n$,

since $([(x_i,y_i)],x_i)\,\widetilde{\mathsf{Ap}}\,y_i$ holds with $y_i \varepsilon (b\downarrow_{\leq} b')\cup\{\bot_{\mathcal{B}}\}$. Then, we conclude $([(x_i,y_i)],x_i) \lhd_{[\mathcal{A}\to\mathcal{B}]\times\mathcal{A}} \widetilde{\mathsf{Ap}}^{-}((b\downarrow_{\leq} b')\cup\{\bot_{\mathcal{B}}\})$ and by *transitivity* and lemma 4.7 also that $([(x_1,y_1),..,(x_n,y_n)],d) \lhd_{[\mathcal{A}\to\mathcal{B}]\times\mathcal{A}} \widetilde{\mathsf{Ap}}^{-}(b\downarrow b')$.

Since this holds for any list $[(x_1,y_1),..,(x_n,y_n)]$ under the above hypothesis, we conclude $(l,d) \lhd_{[\mathcal{A}\to\mathcal{B}]\times\mathcal{A}} \widetilde{\mathsf{Ap}}^{-}(b\downarrow b')$. Lastly, since this holds for any $d \ll a$, from $(l,a) \lhd_{[\mathcal{A}\to\mathcal{B}]\times\mathcal{A}} \{\,(l,d)\mid d\ll a\}$, obtained by lemma 2.11 from $a \lhd_{\mathcal{A}} wb(a)$, we conclude $(l,a) \lhd_{[\mathcal{A}\to\mathcal{B}]\times\mathcal{A}} \widetilde{\mathsf{Ap}}^{-}(b\downarrow b')$.

Finally, by lemma 4.8 and lemma 2.7 we conclude:

Corollary 4.9 *Let $\mathcal{A}$ be a locally compact $\vee$-formal cover and $\mathcal{B}$ be an i.g. $\vee$-formal cover. Then, Ap is a continuous relation from $[\mathcal{A}\to\mathcal{B}]_{\vee}\times\mathcal{A}$ to $\mathcal{B}$.*

To prove that the *abstraction* $\Lambda(F)$ of a continuous relation F is a continuous relation, too, we just show that $\widetilde{\Lambda}(F)^{\lhd}$ is a continuous relation. To this purpose it is sufficient to show that $\widetilde{\Lambda}(F)$ satisfies all the conditions in lemma 2.8. Before showing this, note this useful lemma.

Lemma 4.10 *Let $\mathcal{A}$ be a locally compact $\vee$-formal cover and $\mathcal{C}$ and $\mathcal{B}$ be i.g. $\vee$-formal covers. Suppose that F is any continuous relation from $\mathcal{C}\times\mathcal{A}$ to $\mathcal{B}$. Then $a\ll\{x\mid (c,x)\,F\,b\}$ if and only if $\exists x\; a\ll x$ and $(c,x)\,F\,b$.*

Note that in the following lemma to prove *non-loc-axiom-continuity* for $\widetilde{\Lambda}(F)$ it is crucial to apply lemma 3.7 to decompose some $(c,a') \lhd_{\mathcal{C}\times\mathcal{A}} F^{-}(U)$ for some subset U into a component covering c and some other component covering any given way-below element of a'.

Lemma 4.11 *Let $\mathcal{A}$ be a locally compact $\vee$-formal cover and $\mathcal{C}$ and $\mathcal{B}$ be two i.g. $\vee$-formal covers. Suppose that F is any continuous relation from $\mathcal{C}\times\mathcal{A}$ to $\mathcal{B}$. Then $\widetilde{\Lambda}(F)$ is a relation from C to $List(A\times B)$ satisfying* totality, $\leq$-convergence, non-loc-axiom-continuity, $\leq$-continuity *and* $\leq$-saturation.

Proof. We just prove some conditions:

(*totality*) Let $c\in C$. Then, $c \lhd_{\mathcal{C}} \widetilde{\Lambda}(F)^{-}(List(A\times B))$ follows by *reflexivity* since $c\,\widetilde{\Lambda(F)}\,[\,]$ holds by intuitionistic logic.

($\leq$-*convergence*) If $c\,\widetilde{\Lambda}(F)\,l_1$ and $c\,\widetilde{\Lambda}(F)\,l_2$ then by definition we immediately get $c\,\widetilde{\Lambda}(F)\,l_1\cdot l_2$. and hence by *reflexivity* $c \lhd_{\mathcal{C}} \widetilde{\Lambda}(F)^{-}(l_1\downarrow l_2)$ since $l_1\cdot l_2 \in l_1\downarrow l_2$.

(*non-loc-axiom-continuity*) We must check that if $c\,\widetilde{\Lambda}(F)\,l$ and $j\in J_{[\mathcal{A}\to\mathcal{B}]}(l)$ then we obtain $c \lhd_{\mathcal{C}} \widetilde{\Lambda}(F)^{-}(C_{[\mathcal{A}\to\mathcal{B}]}(l,j))$. The proof depends on the particular shape of the axiom indexed by j. We just check the most relevant ones:

(*wb-bottom continuity*) Suppose $c\,\widetilde{\Lambda}(F)\,[(a,\bot_{\mathcal{B}})]$, that is $a\ll\{x\in A\mid (c,x)\,F\;\bot_{\mathcal{B}}\}$. Since, by *continuity* of F, $F^{-}(\bot_{\mathcal{B}}) \lhd_{\mathcal{C}\times\mathcal{A}} F^{-}(\emptyset)$, then for any $a'\in A$ such that $(c,a')\,F\;\bot_{\mathcal{B}}$ we get $(c,a') \lhd_{\mathcal{C}\times\mathcal{A}} \emptyset$, from which by *transitivity* $(c,a') \lhd_{\mathcal{C}\times\mathcal{A}} \{(x,\bot_{\mathcal{A}})\mid x\leq c\}$ and on this we apply lemma 3.7 since its hypothesis are satisfied. Therefore, for any $d\ll a'$ we get a subset U of C such that $c \lhd_{\mathcal{C}} U$ and, for every $u\,\varepsilon\,U$ there exists $e_1\ll e_1',..,e_n\ll e_n'\in A$ such that

$d \lhd_{\mathcal{A}} \{e_1,..,e_n\}$ and $(u,e'_j)\,\varepsilon\,\{(x,\bot_{\mathcal{A}}) \mid x \leq c\}$ for $j=1,..,n$. Now, note that $e'_j = \bot_{\mathcal{A}}$ for every $j=1,..,n$ and hence $d \lhd_{\mathcal{A}} \emptyset$. Since $a' \lhd_{\mathcal{A}} wb(a')$ and for any $d \ll a'$ we have proved $d \lhd_{\mathcal{A}} \emptyset$, we conclude $a' \lhd_{\mathcal{A}} \emptyset$. Since this holds for any $a'\,\varepsilon\,\{x\,\varepsilon\,A \mid (c,x)\,F\ \bot_{\mathcal{B}}\}$ we get $\{x \in A \mid (c,x)\,F\ \bot_{\mathcal{B}}\} \lhd_{\mathcal{A}} \emptyset$ and by lemma 3.4 we obtain $a \ll \emptyset$ and hence $a \lhd_{\mathcal{A}} \emptyset$. Therefore, from this, since $c\,\widetilde{\Lambda}(F)\,(\bot_{\mathcal{A}},\bot_{\mathcal{B}})$ trivially holds, we conclude $c \lhd_{\mathcal{C}} \widetilde{\Lambda}(F)^{-}(\{\,[(\bot_{\mathcal{A}},\bot_{\mathcal{B}})] \mid a \lhd_{\mathcal{A}} \bot_{\mathcal{A}}\})$.

(*wb-continuity axiom*) Suppose $c\,\widetilde{\Lambda}(F)\,[(a,b)]$ and $j \in J(b)$ and $d \ll a$. Then, by lemma 4.10 there exists a' such that $(c,a')\,F\,b$ and $a \ll a'$.

So, $(c,a') \lhd_{\mathcal{C}\times\mathcal{A}} F^{-}(C(b,j))$ follows by *axiom-continuity* of F from which we obtain $(c,a') \lhd_{\mathcal{C}\times\mathcal{A}} F^{-}(C(b,j) \cup \{\bot_{\mathcal{B}}\})$ by *continuity* of F considering that $C(b,j) \lhd_{\mathcal{B}} C(b,j) \cup \{\bot_{\mathcal{B}}\}$.

Now, note that $F^{-}(C(b,j) \cup \{\bot_{\mathcal{B}}\})$ satisfies the hypothesis of lemma 3.7: indeed the first condition is satisfied by $\leq$ *saturation*, while the second is satisfied since $(c,\bot_{\mathcal{A}}) \lhd_{\mathcal{C}\times\mathcal{A}} \emptyset$ follows by lemma 2.11 from $\bot_{\mathcal{A}} \lhd_{\mathcal{A}} \emptyset$ and by lemma 2.14 we get $(c,\bot_{\mathcal{A}})\,F\ \bot_{\mathcal{B}}$, that is $(c,\bot_{\mathcal{A}})\,\varepsilon\,F^{-}(C(b,j) \cup \{\bot_{\mathcal{B}}\})$.

Then, we apply lemma 3.7 on $d \ll a'$, which follows from $d \ll a$, and $(c,a') \lhd_{\mathcal{C}\times\mathcal{A}} F^{-}(C(b,j) \cup \{\bot_{\mathcal{B}}\})$ by getting a subset V of C satisfying $c \lhd_{\mathcal{C}} V$ and such that for every $v\,\varepsilon\,V$ there exists a finite list $e_1 \ll e'_1,..,e_n \ll e'_n \in A$ with $d \lhd_{\mathcal{A}} \{e_1,..,e_n\}$ and $(v,e'_i)\,\varepsilon\,F^{-}(C(b,j) \cup \{\bot_{\mathcal{B}}\})$ for $i=1,..,n$. This means that there exist $y_1,..,y_n\,\varepsilon\,C(b,j) \cup \{\bot_{\mathcal{B}}\}$ such that $(v,e'_i)\,F\,y_i$ for $i=1,..,n$, and hence $v\,\widetilde{\Lambda}(F)\,[(e_i,y_i)]$ follows by lemma 4.10 for $i=1,..,n$. By lemma 4.5 we get $v\,\widetilde{\Lambda}(F)\,[(e_1,y_1),..,(e_n,y_n)]$ and hence we obtain that each $v\,\varepsilon\,V$ is in $\widetilde{\Lambda}(F)^{-}\,\{[(x_1,v_1),..,(x_n,v_n)] \mid v_i\,\varepsilon\,C(b,j) \cup \{\bot_{\mathcal{B}}\}$ for $i=1,..,n\ \&\ d \lhd_{\mathcal{A}} \{x_1,..,x_n\}\}$ and hence we conclude $c \lhd_{\mathcal{C}} \widetilde{\Lambda}(F)^{-}\{[(x_1,v_1),..,(x_n,v_n)] \mid v_i\,\varepsilon\,C(b,j) \cup \{\bot_{\mathcal{B}}\}$ for $i=1,..,n\ \&\ d \lhd_{\mathcal{A}} \{x_1,..,x_n\}\}$ by *transitivity*.

Then, by lemma 4.11 and lemma 2.8 we conclude:

Corollary 4.12 *Let $\mathcal{A}$ be a locally compact $\vee$-formal cover and $\mathcal{C}$ and $\mathcal{B}$ be i.g. $\vee$-formal covers. Suppose that F is any continuous relation from $\mathcal{C} \times \mathcal{A}$ to $\mathcal{B}$. Then $\Lambda(F)$ is a continuous relation from $\mathcal{C}$ to $[\mathcal{A} \to \mathcal{B}]_{\vee}$.*

Note that $\widetilde{\Lambda}(F)^{\lhd}$ enjoys this useful property:

Lemma 4.13 *Let $\mathcal{A}$ be a locally compact $\vee$-formal cover and $\mathcal{C}$ and $\mathcal{B}$ be i.g. $\vee$-formal covers. Suppose that F is any continuous relation from $\mathcal{C} \times \mathcal{A}$ to $\mathcal{B}$. If $c\,\widetilde{\Lambda}(F)^{\lhd}\,[(a,b)]$ holds, then $(c,a)\,F\,b$ holds, too.*

To finish the proof that the formal cover $\mathcal{B}^{\mathcal{A}} \equiv [\mathcal{A} \to \mathcal{B}]_{\vee}$ is the exponential cover of $\mathcal{B}$ over $\mathcal{A}$ we need to show the validity of the adjunction equations about abstraction and application.

Note that the equations are reducible to the corresponding equations about abstraction and application relative to $[\mathcal{A} \to \mathcal{B}]$, because they are the same up to the isomorphisms $\phi_{\vee}$ and $\phi^{-}_{\vee}$.

Lemma 4.14 *Let $\mathcal{A}$ be a locally compact $\vee$-formal cover and $\mathcal{C}$ and $\mathcal{B}$ be i.g. $\vee$-formal covers. The following holds:*

1. *for every continuous relation F from $\mathcal{C} \times \mathcal{A}$ to $\mathcal{B}$ then*

$$\mathsf{Ap} * \langle \Lambda(F) * \Pi_1, \Pi_2 \rangle = F \qquad \textit{iff} \qquad (\widetilde{\mathsf{Ap}}^{\triangleleft}) * \langle \widetilde{\Lambda}(F)^{\triangleleft} * \Pi_1, \Pi_2 \rangle = F$$

2. *for every continuous relation G' from $\mathcal{C}$ to $\mathcal{B}^{\mathcal{A}} \equiv [\mathcal{A} \to \mathcal{B}]_{\vee}$ we have*

$$\Lambda(\mathsf{Ap} * \langle G' * \Pi_1, \Pi_2 \rangle) = G'$$

if and only if for every continuous relation G from $\mathcal{C}$ to $[\mathcal{A} \to \mathcal{B}]$ we have

$$\widetilde{\Lambda}(\widetilde{\mathsf{Ap}}^{\triangleleft} * \langle G * \Pi_1, \Pi_2 \rangle)^{\triangleleft} = G.$$

Now, in the following we prove the equations for abstraction and application relative to $[\mathcal{A} \to \mathcal{B}]$. But, first, we state a lemma useful to compute the compositions involved in the equations and that can be proved by means of lemma 2.9:

Lemma 4.15 *Let $\mathcal{A}$ be a locally compact $\vee$-formal cover and $\mathcal{C}$ and $\mathcal{B}$ be i.g. $\vee$-formal covers. For any continuous relation G from $\mathcal{C}$ to $[\mathcal{A} \to \mathcal{B}]$ we can prove:*
(1) $(\widetilde{\mathsf{Ap}}^{\triangleleft}) * \langle G * \Pi_1, \Pi_2 \rangle = (\widetilde{\mathsf{Ap}} \cdot \langle G \cdot p_1, p_2 \rangle)^{\triangleleft}$
(2) if $(c, a)\ (\widetilde{\mathsf{Ap}}^{\triangleleft}) * \langle G * \Pi_1, \Pi_2 \rangle\ b$ *then* $(c, a) \lhd_{\mathcal{C} \times \mathcal{A}} \{\, (x, y) \in C \times A \mid x\, G\, [(y, b)] \,\}$.

When checking the adjunction equations relative to $[\mathcal{A} \to \mathcal{B}]$, we make a crucial use of lemma 3.7 to unravel the saturation of a relation with domain $\mathcal{C} \times \mathcal{A}$:

Proposition 4.16 *Let $\mathcal{A}$ be a locally compact $\vee$-formal cover and $\mathcal{C}$ and $\mathcal{B}$ be i.g. $\vee$-formal covers. Then,*
(1) for every continuous relation F from $\mathcal{C} \times \mathcal{A}$ to $\mathcal{B}$ we have

$$(\widetilde{\mathsf{Ap}}^{\triangleleft}) * \langle (\widetilde{\Lambda}(F)^{\triangleleft}) * \Pi_1, \Pi_2 \rangle = F.$$

(2) for every continuous relation G from $\mathcal{C}$ to $[\mathcal{A} \to \mathcal{B}]$ we have

$$\widetilde{\Lambda}((\widetilde{\mathsf{Ap}}^{\triangleleft}) * \langle G * \Pi_1, \Pi_2 \rangle)^{\triangleleft} = G.$$

Proof. We prove the two implications of the considered equations one after the other.

(1. Right to left) Suppose $(c, a)\, F\, b$. Then, for any $d \ll a$ by lemma 4.10 we get $c\ \widetilde{\Lambda}(F)^{\triangleleft}\ [(d, b)]$ and hence $(c, d)\ (\widetilde{\mathsf{Ap}}^{\triangleleft}) * \langle (\widetilde{\Lambda}(F)^{\triangleleft}) * \Pi_1, \Pi_2 \rangle\ b$ since $([(d, b)], d)\ (\widetilde{\mathsf{Ap}}^{\triangleleft})\ b$ holds. By *saturation* on $(c, a) \lhd_{C \times A} \{c\} \times wb(a)$, we conclude $(c, a)\ (\widetilde{\mathsf{Ap}}^{\triangleleft}) * \langle (\widetilde{\Lambda}(F))^{\triangleleft} * \Pi_1, \Pi_2 \rangle\ b$.

(1. Left to right) Suppose $(c, a)\ (\widetilde{\mathsf{Ap}}^{\triangleleft}) * \langle (\widetilde{\Lambda}(F))^{\triangleleft} * \Pi_1, \Pi_2 \rangle\ b$. Then, by lemma 4.15 we get

$$(c, a) \lhd_{\mathcal{C} \times \mathcal{A}} \{\, (x, y) \in C \times A \mid x\, (\widetilde{\Lambda}(F))^{\triangleleft}\, [(y, b)] \,\}$$

To this we apply lemma 3.7. Indeed, note that the first condition of the lemma on W is satisfied by $\leq$-*saturation* of $(\widetilde{\Lambda}(F))^{\triangleleft}$, while the second condition is satisfied

since for any $b \in B$ then $c\,\widetilde{\Lambda}(F)\,[(\bot_{\mathcal{A}}, b)]$, that is $\bot_{\mathcal{A}} \ll \{\, x \in A \mid (c,x)\,F\,b \,\}$, trivially holds.

Therefore, by lemma 3.7 for any $d \ll a$ we find a subset U such that $c \lhd_{\mathcal{C}} U$ and for every $u \varepsilon U$ there exists $e_1 \ll e'_1, .. e_n \ll e'_n \in A$ such that $d \lhd_{\mathcal{A}} \{e_1, .., e_n\}$ and $(u, e'_j) \varepsilon \{\, (x,y) \in C \times A \mid x\,(\widetilde{\Lambda}(F))^{\lhd}\,[(y,b)] \,\}$, that is $u\,(\widetilde{\Lambda}(F))^{\lhd}\,[(e'_j, b)]$, for $j = 1, .., n$. Hence, by lemma 4.5 we deduce $u\,(\widetilde{\Lambda}(F))^{\lhd}\,[(e'_1, b), .., (e'_n, b)]$. Then, by *continuity* and *saturation* of $(\widetilde{\Lambda}(F))^{\lhd}$ on *wb-finite-saturation* axiom considering that $d \lhd_{\mathcal{A}} \{e'_1, .., e'_n\}$, which follows from $d \lhd_{\mathcal{A}} \{e_1, .., e_n\}$ and $e_j \ll e'_j$ for $j = 1, .., n$ by lemma 3.4 and *transitivity*, we get $u\,(\widetilde{\Lambda}(F))^{\lhd}\,[(d, b)]$.

Since this holds for each $u\,\varepsilon\,U$ we conclude $c\,(\widetilde{\Lambda}(F))^{\lhd}\,[(d,b)]$ by *saturation* of $(\widetilde{\Lambda}(F))^{\lhd}$. Then, we obtain $(c,d)\,F\,b$ by lemma 4.13. Finally, by *saturation* of F on $(c,a) \lhd_{C\times A} \{c\} \times wb(a)$, obtained by lemma 2.11 from $a \lhd_{\mathcal{A}} wb(a)$, we conclude $(c,a)\,F\,b$.

(2. Right to left) Suppose $c\,G\,l$. If $l \equiv [\,]$ then we conclude immediately $c\,\widetilde{\Lambda}((\widetilde{\mathsf{Ap}}^{\lhd}) * \langle G * \Pi_1, \Pi_2 \rangle)^{\lhd}\,l$ by definition of abstraction.

Otherwise, if l is not empty, then for every $(a,b)\epsilon l$ we get $c\;G\;[(a,b)]$ by *continuity* and *saturation* of G on $l \lhd_{[\mathcal{A}\to\mathcal{B}]} [(a,b)]$ which follows by $\leq$*-left* and *reflexivity* from $l \leq [(a,b)]$. By *continuity* of G on *interpolation* axiom we get $c \lhd_{\mathcal{C}} G^{-}(\{\, [(x,b)] \mid a \ll x \})$.

Now, for any $c' \varepsilon\, G^{-}(\{\, [(x,b)] \mid a \ll x \})$, that is $c'\,G\,[(a',b)]$ for some a' with $a \ll a'$, since $([(a',b)], a')\ (\widetilde{\mathsf{Ap}}^{\lhd})\ b$ holds, we get $(c,a')\ (\widetilde{\mathsf{Ap}}^{\lhd}) * \langle G * \Pi_1, \Pi_2 \rangle\ b$. Therefore, by lemma 4.10 we get $c\ \widetilde{\Lambda}((\widetilde{\mathsf{Ap}}^{\lhd}) * \langle G * \Pi_1, \Pi_2 \rangle)^{\lhd}\ [(a,b)]$ and by successive application of lemma 4.5 on each $(a,b)\epsilon l$ we conclude $c\,\widetilde{\Lambda}((\widetilde{\mathsf{Ap}}^{\lhd}) * \langle G * \Pi_1, \Pi_2 \rangle)^{\lhd}\,l$.

(2. Left to right) Suppose $c\ \widetilde{\Lambda}((\widetilde{\mathsf{Ap}}^{\lhd}) * \langle G * \Pi_1, \Pi_2 \rangle)^{\lhd}\ l$. If l is the empty list then we conclude $c\,G\,[\,]$ by *continuity* and *saturation* of G since by *totality* $c \lhd_{\mathcal{C}} G^{-}([A \to B])$ holds and $[A \to B] \lhd_{[\mathcal{A}\to\mathcal{B}]} [\,]$ also holds.

Instead, if l is not the empty list, the assumption $c\ \widetilde{\Lambda}((\widetilde{\mathsf{Ap}}^{\lhd}) * \langle G * \Pi_1, \Pi_2 \rangle)^{\lhd}\ l$ is rewritten as $c \lhd_{\mathcal{C}} \{\, y \in C \mid y\ \widetilde{\Lambda}((\widetilde{\mathsf{Ap}}^{\lhd}) * \langle G * \Pi_1, \Pi_2 \rangle)\ l \,\}$.

Now, for any c' such that $c'\ \widetilde{\Lambda}((\widetilde{\mathsf{Ap}}^{\lhd}) * \langle G * \Pi_1, \Pi_2 \rangle)\ l$, by definition and lemma 4.10 we get that for every $(a,b)\epsilon l$, there exists a' such that $a \ll a'$ and $(c', a')\ (\widetilde{\mathsf{Ap}}^{\lhd}) * \langle G * \Pi_1, \Pi_2 \rangle\ b$. Now, by lemma 4.15 we get

$$(c', a') \lhd_{C\times A} \{(x,y) \in C \times A \mid x\,G\,[(y,b)]\,\}$$

and here we proceed analogously to the proof of *(1. Left to right)* since we can apply lemma 3.7. Indeed, the first condition on W of the lemma is satisfied by $\leq$*-saturation* of G, while the second condition is satisfied since $c'\,G\,[(\bot_{\mathcal{A}}, b)]$ follows by *continuity* and *saturation* of G on the axiom $[\,] \lhd_{[\mathcal{A}\to\mathcal{B}]} [(\bot_{\mathcal{A}}, b)]$ being $c'\,G\,[\,]$ valid.

Then, analogously to the proof of *(1. Left to right)* we obtain $c'\,G\,[(a,b)]$ and, since this holds for every $(a,b)\epsilon l$, by successive applications of lemma 4.5 we

conclude $c'\ G\ l$.

Since c' was arbitrary, by *saturation* for G we conclude also $c\ G\ l$.

So, we are arrived at the main theorem of this section.

Theorem 4.17 *Locally compact $\vee$-formal covers are exponentiable in the category of i.g. $\vee$-formal covers $\mathsf{FTop}_{\mathsf{i}\vee}^{-}$.*

Since $\mathsf{FTop}_{\mathsf{i}\vee}^{-}$ is equivalent to $\mathsf{FTop}_{\mathsf{i}}{}^{-}$, by lemmas 2.13 and 3.3, we deduce the following corollary:

Corollary 4.18 *Locally compact formal covers are exponentiable in the category of i.g. formal covers $\mathsf{FTop}_{\mathsf{i}}{}^{-}$.*

Thanks to the predicative coreflection of inductively generated formal covers into formal topologies in [13] we conclude:

Corollary 4.19 *Locally compact formal topologies are exponentiable in the category of inductively generated formal topologies.*

Remark 4.20 After seeing such exponentiation results, a natural question to ask is whether the exponential formal cover between locally compact formal covers happens to be locally compact, too. Even classically this is not true and a counterexample is the formal cover of the Baire space (see [9, 21, 20]).

5 Any exponentiable formal cover is locally compact

Here, we show that any formal cover that is exponentiable in the category of inductively generated formal covers is locally compact, and we deduce the same result for inductively generated formal topologies. The proof is obtained by following closely the corresponding proof in [9], which is essentially based on categorical properties related to exponentiation and on properties of the Sierpinski locale and hence it is essentially predicative. We recall that $\top \equiv (\{1\}, \leq_1, \lhd_1)$ indicates the terminal formal cover in $\mathsf{FTop}_{\mathsf{i}}{}^{-}$.

Then, we recall the definition of the Sierpinski formal cover:

Definition 5.1 *Let $\mathcal{S} \equiv (\{0,1\}, \leq, \lhd_{\mathcal{S}})$ be the Sierpinski formal cover where $0 \leq 1$ and $a \lhd_{\mathcal{S}} U \equiv \exists u \varepsilon U\ a \leq u$.*

Clearly, the Sierpinski formal cover is inductively generated being a unary formal topology [6].

Then, we observe that the morphisms from any given formal cover $\mathcal{A}$ to the Sierpinski cover are in bijection with the opens of $\mathcal{A}$, where the opens of a formal cover are its saturated subsets, that is subsets $U \subseteq A$ such that $U = \mathcal{F}(U)$ where $\mathcal{F}(U) \equiv \{x \in A \mid x \lhd_{\mathcal{A}} U\}$:

Proposition 5.2 *The saturated subsets of a formal cover $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}})$ are in bijection with the continuous relations from $\mathcal{A}$ to $\mathcal{S}$, that is for any saturated subset U there exists a continuous relation $R_U : \mathcal{A} \to \mathcal{S}$ and conversely, given a relation $R : \mathcal{A} \to \mathcal{S}$ then $R^{-}(0)$ is a saturated subset of $\mathcal{A}$.*

Moreover, the bijection preserves the order associated to each collection as follows: if $R \leq H$*, that is* $\forall a \in A \quad \forall s \in \{0,1\} \quad a\,R\,s \;\rightarrow\; a\,H\,s$*, then* $R^{-}(0) \subseteq H^{-}(0)$*, and if* $U \subseteq V$ *then* $R_U \leq R_V$*.*

Proof. We just give the definition of R_U as $a\,R_U\,s \equiv s =_{\{0,1\}} 0 \rightarrow a \varepsilon U$.

The main idea of the proof in [9] that any exponentiable locale is locally compact is that we can use the exponential of a locally compact formal cover over the Sierpinski one to characterize the way-below relation between basic elements (see also [22]):

Lemma 5.3 *Let* $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \vee_{\mathcal{A}}, \perp_{\mathcal{A}}, \lhd_{\mathcal{A}})$ *be an i.g.* $\vee$*-formal cover which is exponentiable in the category of i.g formal covers. Then, for any* $a \in A$*:*

$$\textit{if} \quad a' \;\varepsilon\; wb(a) \equiv \{x \in A \;\mid\; \exists\, l \quad (l,x)\,\mathsf{Ap}\,0 \quad \& \quad 1\,P_{\mathcal{F}(a)}\,l\;\} \quad \textit{then} \quad a' \ll a$$

Moreover,

$$a \lhd wb(a)$$

where $P_{\mathcal{F}(a)} \equiv \Lambda(R_{\mathcal{F}(a)} * \Pi_{\mathcal{A}})$ *and* $\Pi_{\mathcal{A}} : \top \times \mathcal{A} \rightarrow \mathcal{A}$ *is the second projection of the cartesian product* $\top \times \mathcal{A}$*.*

The proof of this lemma makes essential use of the fact that the order between saturated subsets of an exponentiable formal cover $\mathcal{A}$ corresponds to the order between the corresponding global elements of $\mathcal{S}^{\mathcal{A}}$, that is, for any saturated subsets $U, V \subseteq A$, then $U \subseteq V$ if and only if $P_U \leq P_V$ where $P_U \equiv \Lambda(R_U * \Pi_{\mathcal{A}})$.

Then, from lemma 5.3 (whose detailed proof, following closely that in [9], can be found in [12]) we conclude:

Theorem 5.4 *Let* $\mathcal{A}$ *be an inductively generated formal cover that is exponentiable in the category of inductively generated formal covers. Then, it is locally compact.*

Considering that the Sierpinski cover and the other formal covers used in the proof are formal topologies (with $Pos(a)$ always true for every element of the corresponding base) we also conclude:

Corollary 5.5 *An inductively generated formal topology that is exponentiable in the category of inductively generated formal topologies is locally compact.*

Acknowledgements. I am very grateful to Giovanni Curi for various discussions on this topic and for pointing out to me the results in his PhD thesis [7] stating that we can represent predicatively the way-below relation on subsets thanks to which morphisms from a locally compact join-formal cover to any inductively generated join-formal cover could be put in bijection with formal points of an exponential topology candidate following [21]. I also thank Giovanni Sambin for general discussions on formal topology and Silvio Valentini for his collaboration on exponentiation in formal topology.

REFERENCES

[1] P. Aczel. An introduction to inductive definitions. In Y. N. Moschovakis Jon Barwise. With the cooperation of H. J. Keisler, K. Kunen and A. S. Troelstra., editors, *Handbook of mathematical logic.*, volume 90 of *Studies in Logic and the Foundations of Mathematics*, pages 739–782. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.

[2] P. Aczel and M. Rathjen. Notes on constructive set theory. Mittag-Leffler Technical Report No.40, 2000/2001.

[3] G. Battilotti and G. Sambin. Pretopologies and a uniform presentation of sup-lattice, quantales and frames. Draft, 2001.

[4] S. Berardi and S. Valentini. Between formal topology and game semantics: an explicit solution of the conditions for an inductive generation of formal topologies. Draft, 2004.

[5] T. Coquand. Formal topologies with posets. 1996, see http:\\www.cs.chalmers.se \ ∼coquand.

[6] T. Coquand, G. Sambin, J. Smith, and S. Valentini. Inductively generated formal topologies. *Annals of Pure and Applied Logic*, 124(1-3):71–106, 2003.

[7] G. Curi. Geometry of observations: some contributions to (constructive) point-free topology. Tesi di dottorato, University of Siena, 2004.

[8] M. Escardo and R. Heckmann. Topologies on spaces of continuous functions. *Topology proceedings.*, 26:545–564, 2001-2002.

[9] J.M.E. Hyland. Function spaces in the category of locales. *Lecture Notes in Mathematics*, 871:264–281, 1981.

[10] P.T. Johnstone. *Stone spaces.* Cambridge University Press, 1982.

[11] A. Joyal and M. Tierney. An extension of the Galois theory of Grothendieck. *Mem. Amer. Math. Soc.*, 51:no.309, 1984.

[12] M. E. Maietti. A predicative proof of exponentiation locally compact formal topologies. Technical Report - University of Padova, also in http:\\www.math.unipd.it \ ∼maietti\pubb.html, 2004.

[13] M.E. Maietti and S. Valentini. A structural investigation on formal topology: coreflection of formal covers and exponentiability. *Journal of Symbolic Logic*, 69(4):967–1005, 2004.

[14] P. Martin-Löf and G. Sambin. Generating positivity by coinduction. In "The Basic Picture", Preprint n. 08, Dipartimento di Matematica, Università di Padova, 2003.

[15] S. Negri. Continuous domains as formal spaces. *Mathematical Structures in Computer Science.*, 12:19–52, 2002.

[16] B. Nordström, K. Peterson, and J. Smith. *Programming in Martin Löf's Type Theory.* Clarendon Press, Oxford, 1990.

[17] G. Sambin. Intuitionistic formal spaces - a first communication. *Mathematical logic and its applications*, pages 187–204, 1987.

[18] G. Sambin. Some points in formal topology. *Theoretical Computer Science*, 305(1-3):347–408, 2003.

[19] G. Sambin and S. Valentini. Building up a tool-box for Martin-Löf's type theory. In J. Smith G. Sambin, editor, *Twenty five years of Constructive Type Theory, Venice 1995*, pages 221–244. Oxford Science Publications, 1998.

[20] I. Sigstam. *On formal spaces and their effective presentations.* PhD thesis, Department of Mathematics, University of Uppsala, 1990.

[21] I. Sigstam. Formal spaces and their effective presentations. *Arch. Math. Logic*, 34:211–246, 1995.

[22] S.J. Vickers. The double powerlocale and exponentiation: A case study in geometric reasoning. *Theory and Applications of Categories*, 12:372–422, 2004.