Subspaces of an arithmetic universe via type theory

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Abstract

We define the notion of subspace of an arithmetic universe by using its internal dependent type theory.

1 Introduction

In the recent submitted paper with Steve Vickers [MV10] we defined the notion of subspace of an arithmetic universe as a free categorical structure built by means of the partial logic in [PV07].

Here we show how we can define subspaces of an arithmetic universe by using its internal type theory in [Mai05].

In the following we use the abbreviation AU for "arithmetic universe" as defined in [Mai10]. There we gave a general notion of the instance of arithmetic universes built by André Joyal [Joy05] in the seventies. By an AU functor between arithmetic universes we mean a functor preserving the AU structure up to isomorphisms.

By a subspace of $\mathcal{A}$ we mean an AU with extra structure S, expressed in terms of new arrows and commutativities, to be added to $\mathcal{A}$ and we call it $\mathcal{A}[S]_t$ (where t stands for a type-theoretic description of the free structure). We take as its universal property the following one: we have an AU embedding functor $\mathcal{I} : \mathcal{A} \to \mathcal{A}[S]_t$ and for any AU $\mathcal{B}$, the category of AU functors $\mathbf{AU}(\mathcal{A}[S]_t, \mathcal{B})$ is equivalent to the category of pairs (F, α) where $F : \mathcal{A} \to \mathcal{B}$ is an AU functor and α interprets the structure in S with respect to F as in [Mai05].

To show the existence of such subspaces we first define the internal language of an arithmetic universe $\mathcal{A}$ as the free arithmetic universe generated from $\mathcal{A}$, as defined in [Mai05], to which we add coherent isomorphisms making the free AU-structure added to $\mathcal{A}$ isomorphic to the existing AU structure in $\mathcal{A}$. We call $T_{iso}(\mathcal{A})$ the internal type theory of $\mathcal{A}$ with such coherent isomorphisms. This internal type theory of $\mathcal{A}$ differs from that defined in [Mai05], and called $T(\mathcal{A})$, because the first has coherent isomorphisms. This difference becomes clear when we look at the embedding of $\mathcal{A}$ in the syntactic categories $\mathcal{C}_{T(\mathcal{A})}$ and $\mathcal{C}_{T_{iso}(\mathcal{A})}$ built out of $T(\mathcal{A})$ and $T_{iso}(\mathcal{A})$ respectively: while the category $\mathcal{A}$ embeds into $\mathcal{C}_{T(\mathcal{A})}$ via a functor preserving the AU structure strictly, it embeds in $\mathcal{C}_{T_{iso}(\mathcal{A})}$ only via an AU functor.

Then we show that the category of AU functors from the AU-category $\mathcal{A}$ to the AU $\mathcal{B}$ is in equivalence with that of translations from the internal type theory with coherent isomorphisms of $\mathcal{A}$ to that of $\mathcal{B}$.

Then we can define a subspace $\mathcal{A}[S]_t$ of an arithmetic universe $\mathcal{A}$ with extra structure S, expressed in terms of new arrows and commutativities between them, as the syntactic category of the extension of $T_{iso}(\mathcal{A})$ with the extra structure. In this way we can prove the desired universal property that AU functors from $\mathcal{A}$ in an AU $\mathcal{B}$ with the necessary structure to interpret the extra structure S lift to AU functors from $\mathcal{A}[S]_t$ in an uniquely up to iso way.

If we define subspaces with extra structure by using $T(\mathcal{A})$ instead of $T_{iso}(\mathcal{A})$ we just get a subspace satisfying a lifting property only for functors preserving the AU structure strictly.

2 Arithmetic universes

Arithmetic universes are very much the creation of André Joyal, in unpublished work from the 1970s. The general notion was not clearly defined, and we shall follow [Mai10] (which also discusses their background in some detail) in defining them as *list arithmetic pretoposes.*

Definition 1 *A* pretopos *is a category equipped with finite limits, stable finite disjoint coproducts and stable effective quotients of equivalence relations. (For more detailed discussion, see, e.g., [Joh02, A1.4.8].)*

A finitely complete category has parameterized list objects *(see [Mai10]; also [Coc90]) if for any object A there is an object $\mathsf{List}(A)$ with maps $\mathsf{r}_0^A : 1 \to \mathsf{List}(A)$ and $\mathsf{r}_1^A : \mathsf{List}(A) \times A \to \mathsf{List}(A)$ such that for every $b : B \to Y$ and $g : Y \times A \to Y$ there is a unique $\mathsf{rec}(b, g)$ making the following diagrams commute*

$$\begin{tikzcd}
B \arrow[r, "<id{,}r_o^A\cdot !_B>"] \arrow[dr, "b"'] & B\times \mathtt{List}(A) \arrow[d, "rec_l(b{,}g)"] & B\times(\mathtt{List}(A)\times A) \arrow[l, "id\times r_1^A"'] \arrow[d, "(rec_l(b{,}g)\times id_A)\cdot\alpha"] \\
& Y & Y\times A \arrow[l, "g"]
\end{tikzcd}$$

where $\alpha : B \times (\mathsf{List}(A) \times A) \to (B \times \mathsf{List}(A)) \times A$ is the associativity isomorphism.

An arithmetic universe *(or* AU*) [Mai10] is a pretopos with parameterized list objects. We assume that each arithmetic universe is equipped with a choice of its structure. For example, given two objects A, B we can choose their product and the pairing morphisms of two morphisms. Note that an AU has all coequalizers, not just the quotients of equivalence relations as shown in [Mai10].*

This is because the list objects allow one to construct the transitive closure of any relation.

A functor between AUs is an AU functor *if it preserves the AU structure (finite limits, finite colimits, list objects) non-strictly, i.e. up to isomorphism. We write* **AU** *for the category of AUs and AU functors. (We shall sometimes refer to a strict AU functor, preserving structure on the nose, as an* AU homomorphism.*)*

2.1 Free structures via type theory

In order to adjoin structure freely to an AU we can use its internal type theory devised in [Mai05].

We start by recalling the necessary notions from [Mai05].

Definition 2 ($\mathcal{T}_{au}$-theory) *We write $\mathcal{T}_{au}$ for the typed calculus that provides the internal language of arithmetic universes in [Mai05, section 3].*

*We call a theory T of the typed calculus of arithmetic universes $\mathcal{T}_{au}$, (in short: a $\mathcal{T}_{au}$-*theory*), a typed calculus extended with judgements of the form*

$$B\ [\Gamma] \qquad B = C\ [\Gamma] \qquad c \in C\ [\Gamma] \qquad c = d \in C\ [\Gamma]$$

i.e. new types, new elements of types, and new equalities between them.

Definition 3 (syntactic category) *For a given $\mathcal{T}_{au}$-theory T, let $\mathcal{C}_T$ be the syntactic category built out of T as in [Mai05, section 5.2].*

Definition 4 (internal theory of an AU as an AU) *Given an arithmetic universe $\mathcal{A}$, let $T(\mathcal{A})$ be the $\mathcal{T}_{au}$-theory that is the internal language of $\mathcal{A}$. It is defined by the method exemplified with pretoposes in [Mai05, section 5.4].*

Let us call $\mathtt{Em}^T : \mathcal{A} \to T(\mathcal{A})$ the embedding of an object in $\mathcal{A}$ as a proper type and of a morphism in $\mathcal{A}$ as a proper term in its internal type theory. Then, let us simply call $\mathtt{Em} : \mathcal{A} \to \mathcal{C}_{T(\mathcal{A})}$ the embedding of an object X and a morphism f to their copy in the syntactic category $\mathcal{C}_{T(\mathcal{A})}$ defined on page 1119 of [Mai05]. Finally, let us call $\mathtt{V} : \mathcal{C}_{T(\mathcal{A})} \to \mathcal{A}$ the functor establishing an equivalence with $\mathtt{Em}$ *(this called $\epsilon_{\mathcal{A}}^{-1}$ in [Mai98]).*

Definition 5 (theory defining the free AU) *Given an AU $\mathcal{A}$, let $T_{cat}(\mathcal{A})$ be the free $\mathcal{T}_{au}$-*theory *generated from $\mathcal{A}$ as a category, i.e. the extension of the typed calculus $\mathcal{T}_{au}$ with the axioms arising from $\mathcal{A}$ considered as a category according to definition 5.30 of [Mai05].*

Its syntactic category $\mathcal{C}_{T_{cat}(\mathcal{A})}$ is the free AU generated from $\mathcal{A}$ as a category, as shown in [Mai05, section 5.5].

Definition 6 *Given an AU $\mathcal{A}$, let $\mathcal{Y} : \mathcal{A} \to \mathcal{C}_{T_{cat}(\mathcal{A})}$ be the functor embedding of theorem 5.31 in [Mai05], sending an object X and a morphism f to their copy in $\mathcal{C}_{T_{cat}(\mathcal{A})}$. For easiness we keep the same notation here.*

Then, let $\mathtt{Tr}_{\mathcal{A}} : T_{cat}(\mathcal{A}) \longrightarrow T(\mathcal{A})$ be the interpretation functor defined as follows: it sends proper types and terms arising respectively from objects and *morphisms of $\mathcal{A}$ to the corresponding ones in $T(\mathcal{A})$ and types and terms constructors of $\mathcal{T}_{au}$ to their copy in $T(\mathcal{A})$ according to the interpretation exemplified for pretopoi in section 5 of [Mai05].*

Then, the functor $\mathcal{C}(\mathtt{Tr}_{\mathcal{A}}) : \mathcal{C}_{T_{cat}(\mathcal{A})} \to \mathcal{C}_{T(\mathcal{A})}$ sends each closed type and term in $\mathcal{C}_{T_{cat}(\mathcal{A})}$ to their translation via $\mathtt{Tr}_{\mathcal{A}}$ in $T(\mathcal{A})$.

Now we intend to define the internal type theory of an AU $\mathcal{A}$ as the extension of $T_{cat}(\mathcal{A})$ with *coherent isomorphisms* connecting the free AU-structure with the chosen AU-structure in $\mathcal{A}$. We will call such an internal type theory $T_{iso}(\mathcal{A})$.

Categorically this means that we require the existence of a natural isomorphism between the identity functor $\mathtt{Id} : \mathcal{C}_{T_{cat}(\mathcal{A})} \to \mathcal{C}_{T_{cat}(\mathcal{A})}$ and the functor

$$(\,\mathcal{Y} \cdot \mathtt{V}\,) \cdot \mathcal{C}(\mathtt{Tr}_{\mathcal{A}}) : \mathcal{C}_{T_{cat}(\mathcal{A})} \to \mathcal{C}_{T(\mathcal{A})} \to \mathcal{A} \to \mathcal{C}_{T_{cat}(\mathcal{A})}$$

Given the importance of this functor we give it a new name:

Definition 7 ($\mathcal{A}$-reflection) *Let the functor*

$$\mathtt{R}^{\mathcal{A}} : \mathcal{C}_{T_{cat}(\mathcal{A})} \to \mathcal{C}_{T_{cat}(\mathcal{A})}$$

be defined as $\mathtt{R}^{\mathcal{A}} \equiv (\,\mathcal{Y} \cdot \mathtt{V}\,) \cdot \mathcal{C}(\mathtt{Tr}_{\mathcal{A}})$ and called the $\mathcal{A}$-reflector functor.

Note also that the $\mathcal{A}$-*reflector* functor restricted to $\mathcal{A}$ is essentially the identity:

Lemma 8 *For any given AU $\mathcal{A}$ the functor $\mathcal{Y} : \mathcal{A} \to \mathcal{C}_{T_{cat}(\mathcal{A})}$ is naturally isomorphic to $\mathtt{R}^{\mathcal{A}} \cdot \mathcal{Y}$, that is the restriction of the $\mathcal{A}$-reflector functor on $\mathcal{A}$.*

Proof. This follows from the fact that proper types and terms via $Tr_{\mathcal{A}}$ are interpreted in objects and terms isomorphic to the interpreted ones. Indeed, for a given object X in $\mathcal{A}$ then $(\,(\,\mathcal{Y} \cdot \mathtt{V}\,) \cdot \mathcal{C}(Tr_{\mathcal{A}})\,)(\,\mathtt{Y}(X)\,)$ is $\mathtt{V}(X^{\mathtt{Em}})$, which is only isomorphic to $\mathcal{Y}(X) \equiv X$ (indeed $\mathcal{C}_{T(\mathcal{A})}$ is only equivalent to $\mathcal{A}$ and not isomorphic to it!). $\blacksquare$

In order to define $T_{iso}(\mathcal{A})$ in an explicit way we need to define the natural isomorphism as a family of isomorphisms indexed on the objects of $\mathcal{C}_{T_{cat}(\mathcal{A})}$. Since such objects are closed types in $T_{cat}(\mathcal{A})$ that are defined inductively out of the whole collection of types in $T_{cat}(\mathcal{A})$, we thought of describing the desired natural isomorphism as a consequence of an isomorphism between suitable interpretations of $T_{cat}(\mathcal{A})$ in $\mathcal{C}_{T_{cat}(\mathcal{A})}$. This means that we will defined a family of suitable isomorphisms indexed on the whole types of $T_{cat}(\mathcal{A})$. These isomorphisms will be called *coherent isomorphisms*.

Before proceeding we review some key aspects of how to interpret a dependent typed calculus, like $\mathcal{T}_{au}$, into a category $\mathcal{C}$ as defined in [Mai05]. In particular we review how types, terms with their equalities are interpreted together with the interpretation of substitution and weakening in types, in order to fix the notation of morphisms that will be involved in the notion of morphism between interpretations.

First of all the interpretation of a typed calculus in a category $\mathcal{A}$ according to [Mai05] is *actually* given in the category $Pgr(\mathcal{A})$ defined as follows:

Definition 9 *Given a category $\mathcal{A}$ with terminal object 1, the objects of the category* $\boldsymbol{Pgr(\mathcal{A})}$ *are finite sequences $b_1, b_2, ..., b_n$ of morphisms of $\mathcal{A}$*

$$\begin{tikzcd} 1 & B_1 \arrow[l, "b_1"'] & B_2 \arrow[l, "b_2"'] \cdots\cdots & & B_n \arrow[l, "b_n"'] \end{tikzcd}$$

and a morphism from $b_1, b_2, ..., b_n$ to $c_1, c_2, ..., c_m$ is a morphism d of $\mathcal{A}$ such that $c_n \cdot d = b_n$ in $\mathcal{A}$

$$\begin{tikzcd} & & & B_n \arrow[rr, "d"] \arrow[dr, "b_n"'] & & C_n \arrow[dl, "b_n"] \\ 1 & B_1 \arrow[l, "!_{B_1}"'] \cdots\cdots & & & B_{n-1} \arrow[ll, "b_{n-1}"'] & \end{tikzcd}$$

provided that $n = m$ and $b_i = c_i$ for $i = 1, ..., n-1$. Equality, composition and identity is that induced from $\mathcal{A}$.

Now, given an arithmetic universe $\mathcal{A}$, the interpretation of a dependent type $B\ [x \in C_1, .., x_n \in C_n]$ is given by an object in of $Pgr(\mathcal{A})$

$$\begin{tikzcd} & & & & B_\Sigma \arrow[dl, "B^I"] \\ 1 & C_{1\Sigma} \arrow[l, "C_1^I"'] \cdots & & C_{n\Sigma} \arrow[ll, "C_n^I"'] & \end{tikzcd}$$

The interpretation of a term judgement $b \in B\ [\Gamma]$ is a section in $\mathcal{A}$ of the last morphism B^I of the sequence interpreting the dependent type B under the context

$$\begin{tikzcd} & & & C_{n\Sigma} \arrow[rr, "b^I"] \arrow[dr, "id"'] & & B_\Sigma \arrow[dl, "B^I"] \\ 1 & C_{1\Sigma} \arrow[l, "C_1^I"'] \cdots\cdots & & & C_{n\Sigma} \arrow[ll, "C_n^I"'] & \end{tikzcd}$$

The equality between types under context is interpreted as equality of the objects interpreting them in $Pgr(\mathcal{A})$. The equality between typed terms under context is interpreted as equality between the sections interpreting them in $Pgr(\mathcal{A})$.

Now we pass to show how substitution of terms in types and weakening of assumptions in types are interpreted in $Pgr(\mathcal{A})$.

The notion of interpretation requires to be able to interpret substitution and weakening as follows. Given a dependent type $B(x_1, x_2)\ [x_1 \in C_1, x_2 \in C_2]$ and a term $c_2 \in C_2\ [x_1 \in C_1]$ interpreted as

$$\begin{tikzcd} & & & B_\Sigma \arrow[dl, "B^I"] \\ 1 & C_{\Sigma 1} \arrow[l, "C_1^I"'] & C_{\Sigma 2} \arrow[l, "C_2^I"'] & \end{tikzcd} \qquad \begin{tikzcd} & C_{\Sigma 1} \arrow[rr, "c_2^I"] \arrow[dr, "id"'] & & C_{2\Sigma} \arrow[dl, "C_2^I"] \\ 1 & & C_{1\Sigma} \arrow[ll, "C_1^I"'] & \end{tikzcd}$$

we interpret $B(x_1, x_2)[x_2/c_2] \equiv B(x_1, c_2)\ [x_1 \in C_1]$ as

$$\begin{tikzcd} & & B[x_2/c_2]_\Sigma \arrow[dl, "{B[x_2/c_2]^I}"] \\ 1 & C_{1\Sigma} \arrow[l, "C_1^I"'] & \end{tikzcd}$$

where the last morphism $B[x_2/c_2]^I$ is the first projection of the following substitution diagram:

$$\begin{tikzcd}
B[x_/c_2]_\Sigma \arrow[r, "q^I_{B[x/c_2]}"] \arrow[d, "B[x_2/c_2]^I"'] & B_\Sigma \arrow[d, "B^I"] \\
C_{1\Sigma} \arrow[r, "c_2^I"'] & C_{2\Sigma}
\end{tikzcd}$$

Moreover, the type $B\ [x_1 \in C_1, y \in D]$ obtained by weakening the dependent type $B\ [x_1 \in C_1]$ with the type $D\ [x_1 \in C_1]$ interpreted as

$$\begin{tikzcd}
 & & B_\Sigma \arrow[dl, "B^I"] \\
1 & C_{1\Sigma} \arrow[l, "C_1^I"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & D_\Sigma \arrow[dl, "D^I"] \\
1 & C_{1\Sigma} \arrow[l, "C_1^I"] &
\end{tikzcd}$$

is interpreted as

$$\begin{tikzcd}
 & & & w(B,D)_\Sigma \arrow[dl, "w(B,D)^I"] \\
1 & C_{1\Sigma} \arrow[l, "C_1^I"] & D_\Sigma \arrow[l, "D^I"] &
\end{tikzcd}$$

where the last morphism $w(B,D)^I$ is the first projection of the following weakening diagram:

$$\begin{tikzcd}
w(B,D)^I_\Sigma \arrow[r, "q^I_{w(B,D)}"] \arrow[d, "w(B,D)^I"'] & B_\Sigma \arrow[d, "B^I"] \\
D_\Sigma \arrow[r, "D^I"'] & C_{1\Sigma}
\end{tikzcd}$$

where if the substitution or weakening is performed in the middle of the context we still use the same notation as follows.
Recall that $\Gamma^n_{j+1} \equiv x_{j+1} \in C_{j+1}, ..., x_n \in C_n$ for a given context Γ_n denoting with Γ_o the empty context, then the type judgement

$$B[x_j/c_j]\ [\Gamma_{j-1}, \Gamma^{n'}_{j+1}]$$

obtained by substitution with $c_j \in C_J\ [\Gamma_{j-1}]$ where $\Gamma^{n'}_{j+1} \equiv x'_{j+1} \in C_{j+1}[x_j/c_j], ..., x'_n \in C_n[x_j/c_j]$ is interpreted as a morphism of $Pgr(\mathcal{A})$

$$\begin{tikzcd}
 & & & & B[x_j/c_j]_\Sigma \arrow[dl, "B[x_j/c_j]^I"] \\
1 & C_{1\Sigma} \arrow[l, "C_1^I"] & \cdots & C_n[x_j/c_j]_\Sigma \arrow[l, "C_n[x_j/c_j]^I"] &
\end{tikzcd}$$

where, if $\Gamma_{j+1->n}$ is not empty, the last morphism is the first projection of the substitution diagram:

$$\begin{tikzcd}
B[x_j/c_j]_\Sigma \arrow[r, "{q_{B[x_j/c_j]}}^I"] \arrow[d, "{B[x_j/c_j]^I}"'] & B_\Sigma \arrow[d, "B^I"] \\
C_n[x_j/c_j]_\Sigma \arrow[r, "{q_{C_n[x_j/c_j]}}^I"'] & C_{n\Sigma}
\end{tikzcd}$$

Moreover the type judgement $B\ [\Gamma_j, y \in D, \Gamma^n_{j+1}]$ obtained by weakening with a variable in the middle of the context is interpreted as

$$\begin{tikzcd}
 & & & w(B,D)_\Sigma \arrow[dl, "w(B,D)^I"] \\
1 & C_{1\Sigma} \arrow[l, "C_1^I"'] & w(C_n,D)_\Sigma \arrow[l, dotted, "w(C_n,D)^I"'] &
\end{tikzcd}$$

If Γ^n_{j+1} is not empty, then its last morphism is the first projection of the following weakening diagram:

$$\begin{tikzcd}
w(B,D)_\Sigma \arrow[r, "{q_{w(B,D)}}^I"] \arrow[d, "w(B,D)^I"'] & B_\Sigma \arrow[d, "B^I"] \\
w(C_n,D)_\Sigma \arrow[r, "{q_{w(C_n,D)}}^I"'] & C_{n\Sigma}
\end{tikzcd}$$

Then we give the following definition of *generic interpretation*:

Definition 10 *A* generic *interpretation of a typed calculus T is one that validates all judgements of the typed calculus T according to the above notion of judgement interpretations including substitution and weakening.*

Note that to interpret substitution of terms in types correctly we need a functorial choice of the above substitution diagrams in $\mathcal{A}$.

If we require the substitution and weakening diagrams to be *pullbacks* in $\mathcal{A}$, as done in [Mai05], then we need to provide a functorial choice of pullbacks in $\mathcal{A}$.

In order to build an interpretation of the type calculus $\mathcal{T}_{au}$ in an arithmetic universe with an arbitrary fixed choice of its structure (and hence with a choice of pullbacks that is not necessarily functorial), one possibility is to define it via a preinterpretation of types and terms into fibred functors and natural transformations as described in [Mai05]. Here we refer to this interpretation defined via fibred functors as a *canonical interpretation* of the typed calculus $\mathcal{T}_{au}$. We do not recall the definition of such an interpretation here and we refer the reader to [Mai05]. We just remind that this canonical interpretation is crucial to describe the internal type theory of an arithmetic universe.

Here we will mention two interpretations of the typed calculus $T_{cat}(\mathcal{A})$ in $\mathcal{C}_{T_{cat}(\mathcal{A})}$ that are not defined via fibred functors (hence they are not canonical) and are only *generic* ones. These are those interpretations whose action on the syntactic category $\mathcal{C}_{T_{cat}(\mathcal{A})}$ gives rises respectively to the identity functor and the $\mathcal{A}$-reflector. This implies that to meet our purpose of building a natural isomorphism between the identity functor and the $\mathcal{A}$-reflector is enough to build an isomorphism between the corresponding interpretations.

We now pass to describe the interpretation corresponding to the identity functor on $\mathcal{C}_{T_{cat}(\mathcal{A})}$:

Definition 11 (The $(-)^{\mathcal{H}}$ interpretation) *Let us call $(-)^{\mathcal{H}}$ the interpretation of $T_{cat}(\mathcal{A})$ into the category $Pgr(\mathcal{C}_{T_{cat}(\mathcal{A})})$ via indexed sums as defined on page 1138 in [Mai05] with the warning of interpreting the closed type X as $X \to \top$ (and not as $\Sigma_{z\in\top}X \xrightarrow{\pi_1} \top$). For example a type $B(x)$ $[x \in C]$ is interpreted as*

$$\Sigma_{z\in C}B(z) \xrightarrow{\ \pi_1\ } C \xrightarrow{\ !^C\ } \top$$

and a term $b(x) \in B(x)$ $[x \in C]$ is interpreted as a section $< id, b(x) >$ of the interpretation π_1 of its type, namely $\pi_1\cdot < id, b(x) >= id$ in $\mathcal{C}_{T_{cat}(\mathcal{A})}$.

In essence $(-)^{\mathcal{H}}$ interprets types and terms in themselves as indexed sum types and sections. Hence it corresponds on $\mathcal{C}_{T_{cat}(\mathcal{A})}$ to the identity functor.

Then, the interpretation of $T_{cat}(\mathcal{A})$ in $Pgr(\mathcal{C}_{T_{cat}(\mathcal{A})})$ corresponding to the $\mathcal{A}$-reflector functor is obtained by turning the translation $\mathtt{Tr}_{\mathcal{A}} : T_{cat}(\mathcal{A}) \longrightarrow T(\mathcal{A})$, used to built the reflector, into an interpretation $Int_{\mathcal{A}} : T_{cat}(\mathcal{A}) \longmapsto Pgr(\mathcal{A})$ by composing $\mathtt{Tr}_{\mathcal{A}}$ with the semantic denotation of $T(\mathcal{A})$-types and terms. Then, by using the embedding functor $\mathcal{Y} : \mathcal{A} \to \mathcal{C}_{T_{cat}(\mathcal{A})}$ we can think of $Int_{\mathcal{A}}$ in $Pgr(\mathcal{C}_{T_{cat}(\mathcal{A})})$ by keeping the same name

$$Int_{\mathcal{A}} : T_{cat}(\mathcal{A}) \longmapsto Pgr(\mathcal{C}_{T_{cat}(\mathcal{A})})$$

Note that this interpretation induces the $\mathcal{A}$-reflector functor on $\mathcal{C}_{T_{cat}(\mathcal{A})}$.

Now our task is to build an isomorphism between the interpretations $(-)^{\mathcal{H}}$ and $Int_{\mathcal{A}}$. In order to do so we need to first define the notion of morphism between interpretations. To this purpose we can not work in $Pgr(\mathcal{C}_{T_{cat}(\mathcal{A})})$ but we pass to consider the category of arrow lists $\mathcal{C}^{\to fin}_{T_{cat}(\mathcal{A})}$ defined in [Mai05] as follows:

Definition 12 (Category of arrow lists) *Given a category $\mathcal{A}$, we define the category $\mathcal{A}^{\to fin}$ as follows: its objects are sequences*

$$C_0 \xleftarrow[c_1]{} C_1 \xleftarrow[c_2]{} C_2 \cdots\cdots\cdots \xleftarrow[c_n]{} C_n$$

of composable $\mathcal{A}$-morphisms and a morphism from c_1 , c_2 ,..., c_n to b_1 , b_2 ,..., b_n is a sequence ϕ_0 , ϕ_1 ,..., ϕ_n of $\mathcal{A}$-morphisms such that all the following squares *commute in $\mathcal{A}$*

$$\begin{tikzcd}
C_n \arrow[r, "\phi_n"] \arrow[d, "c_n"'] & B_n \arrow[d, "b_n"] \\
C_{n-1} \arrow[r, "\phi_{n-1}"] \arrow[d, "c_{n-1}"'] & B_{n-1} \arrow[d, "b_{n-1}"] \\
\vdots \arrow[d, dotted, no head] & \vdots \arrow[d, dotted, no head] \\
C_2 \arrow[r, "\phi_2"] \arrow[d, "c_2"'] & B_2 \arrow[d, "b_2"] \\
C_1 \arrow[r, "\phi_1"] \arrow[d, "c_1"'] & B_1 \arrow[d, "b_1"] \\
C_0 \arrow[r, "\phi_0"] & B_0
\end{tikzcd}$$

Two morphisms are equal if their n-th components are equal for each n in the list. Composition of morphisms is a morphism whose n-th component is the composition of the n-th components of the given morphisms, and the identity is the morphism whose components are all identities.

Observe that $Pgr(\mathcal{A})$ is a subcategory of $\mathcal{A}^{\rightarrow_{fin}}$ with the same objects and where morphisms are all identities except for the last component.

Then we define the notion of *interpretation morphism*, and the associated one of *interpretation isomorphism*, between interpretations of a generic $\mathcal{T}_{au}$-theory T into a generic arithmetic universe $\mathcal{B}$, or better in $Pgr(\mathcal{B})$, by relating them in $\mathcal{B}^{\rightarrow fin}$ as follows.

Definition 13 *Given a $\mathcal{T}_{au}$-theory T and an arithmetic universe $\mathcal{B}$ and two interpretation $Int_1 : T \longmapsto Pgr(\mathcal{B})$ and $Int_2 : T \longmapsto Pgr(\mathcal{B})$ of T, we say that there is an* morphism of interpretation *from Int_1 to Int_2*

$$\sigma(-) : Int_1 \longrightarrow Int_2$$

if for each type judgement $B\ [\Gamma]$ of T there exists an morphism in $\mathcal{B}^{\rightarrow_{fin}}$

$$\sigma_{B\ [\Gamma]} : (\,B\ [\Gamma]\,)^{Int_1} \to (\,B\ [\Gamma]\,)^{Int_2}$$

Moreover, supposed to represent

$$(\,B\ [\Gamma]\,)^{Int_1} \;\equiv\; B_{\Sigma}{}^{Int_1} \xrightarrow{B^{Int_1}} C_{n\Sigma}{}^{Int_1} \xrightarrow{C_n^{Int_1}} \cdots\cdots\cdots\; C_{1\Sigma}{}^{Int_1} \xrightarrow{C_1^{Int_1}} 1^{Int_1}$$

$$(\,B\ [\Gamma]\,)^{Int_2} \;\equiv\; B_{\Sigma}{}^{Int_2} \xrightarrow{B^{Int_2}} C_{n\Sigma}{}^{Int_2} \xrightarrow{C_n^{Int_2}} \cdots\cdots\cdots\; C_{1\Sigma}{}^{Int_2} \xrightarrow{C_1^{Int_2}} 1^{Int_2}$$

we represent $\sigma_{B\ [\Gamma]}$ in $\mathcal{B}^{\rightarrow_{fin}}$ as follows:

$$
\begin{tikzcd}
{B_{\Sigma}}^{Int_1} \arrow[r, "\sigma_B"] \arrow[d, "B^{Int_1}"'] & {B_{\Sigma}}^{Int_2} \arrow[d, "B^{Int_2}"] \\
{C_{n\Sigma}}^{Int_1} \arrow[r, "\sigma_{C_n}"] \arrow[d, "C_n^{Int_1}"'] & {C_{n\Sigma}}^{Int_2} \arrow[d, "C_n^{Int_2}"] \\
\vdots & \vdots \\
{C_{2\Sigma}}^{Int_1} \arrow[r, "\sigma_{C_2}"] \arrow[d, "{C_2}^{Int_1}"'] & {C_{2\Sigma}}^{Int_2} \arrow[d, "C_2^{Int_2}"] \\
{C_{1\Sigma}}^{Int_1} \arrow[r, "\sigma_{C_1}"] \arrow[d, "C_1^{Int_1}"'] & {C_{1\Sigma}}^{Int_2} \arrow[d, "C_1^{Int_2}"] \\
1^{Int_1} \arrow[r, "\sigma_{C_1}"] & 1^{Int_2}
\end{tikzcd}
$$

Then we require that each component σ_B satisfy the following conditions:

- `naturality condition` *the last component of $\sigma_{B\ [\Gamma]}$ commutes with the interpretation of its terms: for every term judgement $b \in B\ [\Gamma]$ of T*

$$\sigma_B \cdot b^{Int_1} = b^{Int_2} \cdot \sigma_{C_n}$$

  in $\mathcal{B}$, supposed $\Gamma \ \equiv\ x_1 \in C_1, \ldots, x_n \in C_n$

- `weakening condition` *the last component of $\sigma_{B\ [\Gamma]}$ commutes with the interpretation of weakening: for every judgement D type $[\Gamma_j]$ in T with Γ_j sublist of Γ*

$$\sigma_B \cdot q^{Int_1}_{w(B,D)} = q^{Int_2}_{w(B,D)} \cdot \sigma_{w(B,D)}$$

  where $w(B,D)$ is the type B weakened on D and $q^{Int_1}_{w(B,D)}$ is the second projection of the weakening diagram of the last morphism interpreting B according to Int_1 along the context weakened with D.

- `substitution condition` *the last component of $\sigma_{B\ [\Gamma]}$ commutes with the interpretation of substitution: for every term judgement $c_j \in C_j\ [\Gamma_{j-1}]$ in T with Γ_{j-1} sublist of Γ*

$$\sigma_B \cdot q^{Int_1}_{B[x/c_j]} = q^{Int_2}_{B[x/c_j]} \cdot \sigma_{B[x_j/c_j]}$$

  where $q^{Int_1}_{B[x/c_j]}$ is the second projection of the substitution diagram of the last morphism interpreting B according to Int_1 along the morphism expressing the substitution with c_j.

The interpretation morphism is an interpretation isomorphism *if each*

$$\sigma_{B\ [\Gamma]} : (\,B\ [\Gamma]\,)^{Int_1} \to (\,B\ [\Gamma]\,)^{Int_2}$$

is an isomorphism in $\mathcal{B}^{\to_{fin}}$, i.e. it has an inverse $\sigma^{-1}_{B\ [\Gamma]}$ in $\mathcal{B}^{\to_{fin}}$ i.e. such that

$$\sigma_{B\ [\Gamma]} \cdot \sigma^{-1}_{B\ [\Gamma]} = id \qquad \sigma_{B\ [\Gamma]^{-1}} \cdot \sigma_{B\ [\Gamma]} = id$$

Now, we are ready to give the definition of $T_{iso}(\mathcal{A})$ as the extension of $T_{cat}(\mathcal{A})$ with a natural isomorphism between the interpretations $(-)^{\mathcal{H}}$ and $(-)^{\mathcal{A}}$ via *coherent isomorphisms*:

Definition 14 ($T_{iso}(\mathcal{A})$=$T_{cat}(\mathcal{A})$+ coherent isos) *Given an AU $\mathcal{A}$, let us consider the above interpretations $(-)^{\mathcal{H}}$ and $(-)^{\mathcal{A}}$ of $T_{cat}(\mathcal{A})$ in $Pgr(\mathcal{C}_{T_{cat}(\mathcal{A})})$.*

*Then we define $T_{iso}(\mathcal{A})$ as the $\mathcal{T}_{au}$-*theory *extending $T_{cat}(\mathcal{A})$ with new terms and equalities formalizing the existence of an isomorphism of interpretation*

$$\sigma_{-} : (-)^{\mathcal{H}} \longrightarrow (-)^{\mathcal{A}}$$

in $Pgr(\mathcal{C}_{T_{iso}(\mathcal{A})})$.

Such an isomorphism of interpretation is given by a family of coherent isomorphisms

$$\sigma_{B\ [\Gamma]} : (B\ [\Gamma])^{\mathcal{H}}{}_{\Sigma} \longrightarrow (B\ [\Gamma])^{\mathcal{A}}{}_{\Sigma}$$

indexed on any type under context $B\ [\Gamma]$ of $T_{cat}(\mathcal{A})$ satisfying all the naturality, weakening and substitution conditions of an isomorphism of interpretation as in definition 13 with respect to types and terms in $T_{cat}(\mathcal{A})$. Now we proceed to define such coherent isomorphisms by induction on types and terms of $T_{cat}(\mathcal{A})$.

In order to define a coherent isomorphism $\sigma_{B\ [\Gamma]}$ indexed on a type $B\ [\Gamma]$ interpreted by a limit (as the terminal type, the equality type), we actually define its inverse $\sigma^{-1}_{B\ [\Gamma]}$ as the induced morphism from the universal property of the limit. Instead we define a coherent isomorphism $\sigma_{B\ [\Gamma]}$ indexed on a type $B\ [\Gamma]$ interpreted by a colimit (as the false type, the sum type, the quotient type) or by an initial algebra (the list type) directly as the induced morphism from the universal property of the colimit (or of the initial algebra).

Hence, the coherent isomorphism $\sigma_{\top}$ indexed on the terminal type is defined as the inverse of $\sigma^{-1}_{\top} : \top^{\mathcal{A}}_{\Sigma} \to \top$, where $\top^{\mathcal{A}}_{\Sigma}$ is a terminal object in $\mathcal{A}$, (that is the domain interpretation of the terminal type). In turn $\sigma^{-1}_{\top}$ is defined as the unique morphism in $\mathcal{C}_{T_{cat}(\mathcal{A})}$ to the terminal object $\top$ of $\mathcal{C}_{T_{cat}(\mathcal{A})}$.

The coherent isomorphism indexed on $\top\ [\Gamma]$ weakened on a context is defined in a way as to satisfy the weakening condition.

The coherent isomorphism indexed on any proper type C coming from $\mathcal{A}$

$$\sigma_C : C^{\mathcal{H}}{}_{\Sigma} \to C^{\mathcal{A}}{}_{\Sigma}$$

is the isomorphism coming from the natural isomorphism of $\mathcal{V} \cdot \mathtt{Em} : \mathcal{A} \to \mathcal{C}_{T(\mathcal{A})} \to \mathcal{A}$ with the identity (recall that $\mathcal{V}$ and $\mathtt{Em}$ gives an equivalence between $\mathcal{A}$ and $\mathcal{C}_{T(\mathcal{A})}$), since $C^{\mathcal{H}}{}_{\Sigma} \equiv C$ while $C^{\mathcal{A}}{}_{\Sigma} \equiv \mathcal{V} \cdot \mathtt{Em}(C)$. Note that this isomorphism is in $\mathcal{A}$.

In the next to simplify the notation, given a context $\Gamma \equiv x_1 \in C_1, \ldots, x_n \in C_n]$, we simply indicate the component σ_{C_n} of the last context assumption with σ_{Γ} as the context consisted of one single assumption.

We define the coherent isomorphism indexed on the Indexed Sum type

$$\sigma_{\Sigma_{x\in D}\, B(x)[\Gamma]} : (\Sigma_{x\in D}\, B(x))^{\mathcal{H}}{}_{\Sigma} \to (\Sigma_{x\in D}\, B(x))^{\mathcal{A}}{}_{\Sigma}$$

as the inverse of $\sigma^{-1}_{\Sigma_{x\in D}\, B(x)[\Gamma]}$ defined in turn as follows. Observe that the last morphism interpreting $\Sigma_{x\in D}\, B(x)\ [\Gamma]$ according to $(-)^{\mathcal{A}}$ is $(\Sigma_{x\in D}\, B(x)\ [\Gamma])^{\mathcal{A}} \equiv D^{\mathcal{A}} \cdot B^{\mathcal{A}}$. Moreover observe that the last morphism interpreting $\Sigma_{x\in D}\, B(x)$ according to $(-)^{\mathcal{H}}$ is isomorphic to $B^{\mathcal{H}} \cdot D^{\mathcal{H}}$ in $\mathcal{C}_{T_{cat}(\mathcal{A})}/\Gamma^{\mathcal{H}}_{\Sigma}$. Hence we define $\sigma^{-1}_{\Sigma_{x\in D}\, B(x)[\Gamma]} \equiv \nu \cdot \sigma^{-1}_{B(x)[\Gamma, x\in D]}$ where $\nu : B^{\mathcal{H}} \cdot D^{\mathcal{H}} \to (\,\Sigma_{x\in D}\, B(x)\,)^{\mathcal{H}}$ is the isomorphism between the two object in $\mathcal{C}_{T_{cat}(\mathcal{A})}/\Gamma^{\mathcal{H}}_{\Sigma}$ (that is defined as $< \pi_1 \cdot \pi_1, < \pi_2 \cdot \pi_1, \pi_2 \cdot \pi_2 >$ by using the projections π_1, π_2 of the Indexed Sum type).

We define the isomorphism indexed on the Equality type

$$\sigma_{\mathtt{Eq}(C,c,d)[\Gamma]} : \mathtt{Eq}(C,c,d)^{\mathcal{H}}{}_{\Sigma} \to \mathtt{Eq}(C,c,d)^{\mathcal{A}}{}_{\Sigma}$$

as follows. Recall that $\mathtt{Eq}(C,c,d)^{\mathcal{A}} \equiv eq(c^{\mathcal{A}}, d^{\mathcal{A}})$ is the equalizer of $c^{\mathcal{A}}$ and $d^{\mathcal{A}}$ in $\mathcal{A}$, as well as $\mathtt{Eq}(C,c,d)^{\mathcal{H}}$ is an equalizer of $c^{\mathcal{H}}$ and $d^{\mathcal{H}}$ in $\mathcal{C}_{T_{cat}(\mathcal{A})}$. Hence we define $\sigma^{-1}_{\mathtt{Eq}(C,c,d)[\Gamma]}$ as the unique morphism toward the equalizer $\mathtt{Eq}(C,c,d)^{\mathcal{H}}$ induced by $\sigma_{\Gamma}^{-1} \cdot eq(c^{\mathcal{A}}, d^{\mathcal{A}})$. This is well defined since by hypothesis and naturality of the coherent isomorphisms we have (recall that equality of morphisms in $\mathcal{A}$ is preserved in $\mathcal{C}_{T_{cat}(\mathcal{A})}$)

$$\begin{aligned} &c^{\mathcal{H}} \cdot (\,\sigma_{\Gamma}^{-1} \cdot eq(c^{\mathcal{A}}, d^{\mathcal{A}})\,) = \sigma^{-1}_{C\ [\Gamma]} \cdot (\,c^{\mathcal{A}} \cdot eq(c^{\mathcal{A}}, d^{\mathcal{A}})\,) = \\ &= \sigma^{-1}_{C\ [\Gamma]} \cdot (\,d^{\mathcal{A}} \cdot eq(c^{\mathcal{A}}, d^{\mathcal{A}})\,)\,) = d^{\mathcal{H}} \cdot (\,\sigma_{\Gamma}^{-1} \cdot eq(c^{\mathcal{A}}, d^{\mathcal{A}})\,) \end{aligned}$$

We define the coherent isomorphism indexed on the empty set $\perp$

$$\sigma_{\perp} : \perp \to \perp^{\mathcal{A}}$$

where $\perp^{\mathcal{A}}$ is the name of the initial object in $\mathcal{A}$, as the unique morphism in $\mathcal{C}_{T_{cat}(\mathcal{A})}$ from $\perp$ to $\perp^{\mathcal{A}}$.

Moreover, we define the coherent isomorphism indexed on the empty set weakened on a context in a way as to satisfy the weakening condition.

The isomorphisms for the quotient type, disjoint sums and lists are defined analogously.

Note that the described isomorphism of interpretation is indeed uniquely determined from the isomorphisms indexed on proper types (because of the naturality, weakening, substitution conditions).

Definition 15 *Let $\mathcal{Y}_{iso} : \mathcal{A} \to \mathcal{C}_{T_{iso}(\mathcal{A})}$ be the functor defined as the embedding of an object X and a morphism f to their copy as they were in $\mathcal{C}_{T_{cat}(\mathcal{A})}$.*

Observe that the embedding functor $\mathcal{Y}_{iso}$ preserves the AU structure up to isomorphisms:

Lemma 16 *The functor $\mathcal{Y}_{iso} : \mathcal{A} \to \mathcal{C}_{T_{iso}(\mathcal{A})}$ is an AU functor.*

Proof. This follows thanks to the presence of coherent isomorphisms. ■

We can prove that the synctactic category associated to $T_{iso}(\mathcal{A})$ is equivalent to $\mathcal{A}$. To this purpose we define a translation of $T_{iso}(\mathcal{A})$ in $T(\mathcal{A})$:

Definition 17 *Let* $\mathtt{St} : T_{iso}(\mathcal{A}) \to T(\mathcal{A})$ *be the functor sending any type and term arising respectively from objects and morphisms of* $\mathcal{A}$ *to the corresponding one in* $T(\mathcal{A})$ *and sending types and terms constructors of* $\mathcal{T}_{au}$ *to their copy in* $T(\mathcal{A})$*. Finally coherent isomorphisms get interpreted as parts of the natural isomorphism between* $\mathcal{V} \cdot \mathtt{Em} : \mathcal{A} \to \mathcal{C}_{T(\mathcal{A})} \to \mathcal{A}$ *and the identity. Indeed,* $(B\ [\Gamma])^{\mathcal{A}} = \mathcal{V} \cdot \mathtt{Em}(\,(B\ [\Gamma])^{\mathcal{H}}\,)$

Let $\mathcal{C}(\mathtt{St}) : \mathcal{C}_{T_{iso}(\mathcal{A})} \to \mathcal{C}_{T(\mathcal{A})}$ *be the syntactic functor induced by* $\mathtt{St}$.

Lemma 18 *The functor* $\mathcal{Y}_{iso} : \mathcal{A} \to \mathcal{C}_{T_{iso}(\mathcal{A})}$ *gives rise to an equivalence of category with the functor* $\mathtt{V}_{iso} \equiv \mathtt{V} \cdot \mathcal{C}(\mathtt{St}) : \mathcal{C}_{T_{iso}(\mathcal{A})} \to \mathcal{C}_{T(\mathcal{A})} \to \mathcal{A}$.

Proof. Clearly $(\,\mathtt{V} \cdot \mathcal{C}(\mathtt{St})\,) \cdot \mathcal{Y}_{iso}$ is naturally isomorphic to the identity. Instead we prove that $\mathcal{Y}_{iso} \cdot (\,\mathtt{V} \cdot \mathcal{C}(\mathtt{St})\,)$ is isomorphic to the identity thanks to coherent isomorphisms when the functor is applied to $\mathcal{T}_{au}$-constructors. ■

This means that we can speak of $T_{iso}(\mathcal{A})$ as *the internal theory of* $\mathcal{A}$ *with coherent isomorphisms.*

Now our purpose is to prove that given two arithmetic universes $\mathcal{A}$ and $\mathcal{B}$, the AU functors from $\mathcal{A}$ to $\mathcal{B}$ correspond to translations between their internal theories with coherent isomorphisms, i.e. to translations from $T_{iso}(\mathcal{A})$ to $T_{iso}(\mathcal{B})$. To this purpose we first lift an AU functor to a translation between the corresponding free theories generated from the arithmetic universes:

Definition 19 *Given the arithmetic universes* $\mathcal{A}$ *and* $\mathcal{B}$ *with an AU functor* $F : \mathcal{A} \to \mathcal{B}$*, we can define a translation between the free* $\mathcal{T}_{au}$*-theories generated from them*

$$(-)^F : T_{cat}(\mathcal{A}) \to T_{cat}(\mathcal{B})$$

as follows: $(-)^F$ *translates types and terms arising from* $\mathcal{A}$ *via* F*, i.e. each proper type arising from an object* C *of* $\mathcal{A}$ *is translated into* $F(C)$ *and each proper term arising from a morphism* c *is translated into* $F(c)$*; moreover* $\mathcal{T}_{au}$*-constructors are interpreted as the corresponding ones in* $T_{cat}(\mathcal{B})$.

Lemma 20 *Given an AU functor* $F : \mathcal{A} \to \mathcal{B}$*, the translation* $(-)^F : T_{cat}(\mathcal{A}) \to T_{cat}(\mathcal{B})$ *induced between the corresponding free theories satisfies the following: for any judgement* $B\ [\Gamma]$ *then*

$$(\,(\,B\ [\Gamma]\,)^{\mathcal{H}}\,)^F \;\equiv\; (\,B\ [\Gamma]\,)^F\,)^{\mathcal{H}}$$

Proof. It follows from the fact that $(-)^F$ is a translation and hence it preserves indexed sums strictly. ■

Lemma 21 *Given an AU functor* $F : \mathcal{A} \to \mathcal{B}$*, the translation* $(-)^F : T_{cat}(\mathcal{A}) \to T_{cat}(\mathcal{B})$ *induced between the corresponding free theories allows to define the following interpretations of* $T_{cat}(\mathcal{A})$ *in* $Pgr(\mathcal{B})$

$$(-)^{F^{\mathcal{B}}} : T_{cat}(\mathcal{A}) \longrightarrow Pgr(\mathcal{B}) \qquad (-)^{\mathcal{A}^F} : T_{cat}(\mathcal{A}) \longrightarrow Pgr(\mathcal{B})$$

(by precomposing $(-)^F$ with $(-)^{\mathcal{B}}$ and postcomposing it with $(-)^{\mathcal{A}}$) between which there exists an isomorphism of interpretation

$$\tau(-) : (-)^{F^{\mathcal{B}}} \longrightarrow (-)^{\mathcal{A}^F}$$

Proof. We define the required isomorphism of interpretation by using the coherent isomorphisms of F needed to preserve the AU structure.

For example the isomorphism indexed on the terminal type $\tau_\top : \top^{\mathcal{B}}_\Sigma \to F(\top^{\mathcal{A}})_\Sigma$ is the part of the coherent isomorphism of F preserving the terminal object of $\mathcal{A}$ represented by $\top^{\mathcal{A}}$ from the terminal object of $\mathcal{B}$ given by $\top^{\mathcal{B}}_\Sigma$.

Moreover, for any proper type C coming from $\mathcal{A}$ we define

$$\tau_C : F(C)^{\mathcal{B}}{}_\Sigma \to F(C^{\mathcal{A}}{}_\Sigma)$$

as the composition of the following isomorphisms $F(C)^{\mathcal{B}}{}_\Sigma \simeq F(C) \simeq F(C^{\mathcal{A}}{}_\Sigma)$ all derived from the natural isomorphism of $\mathcal{V} \cdot \mathtt{Em}$ with the identity both for $\mathcal{A}$ and $\mathcal{B}$: indeed $C^{\mathcal{A}}{}_\Sigma \equiv \mathcal{V} \cdot \mathtt{Em}(C)$ is isomorphic to C in $\mathcal{A}$, hence in $\mathcal{C}_{T_{cat}(\mathcal{A})}$ which gives an isomorphism $F(C) \simeq F(C^{\mathcal{A}}{}_\Sigma)$, as well as $F(C)^{\mathcal{B}}{}_\Sigma \equiv \mathcal{V} \cdot \mathtt{Em}(F(C))$ is isomorphic to $F(C)$ for the analogous reason.

The coherent isomorphism indexed on the Indexed Sum type

$$\tau_{\Sigma_{x\in D}\, B(x)[\Gamma]} : (\Sigma_{x\in D}\, B(x))^{F^{\mathcal{B}}}{}_\Sigma \to (\Sigma_{x\in D}\, B(x))^{\mathcal{A}^F}{}_\Sigma$$

is defined as follows. Observe that the last morphism interpreting $(\Sigma_{x\in D}\, B(x))^F$ according to $(-)^{\mathcal{B}}$ is $(\Sigma_{x\in D}\, B(x))^{F^{\mathcal{B}}} \equiv D^{F^{\mathcal{B}}} \cdot B^{F^{\mathcal{B}}}$. Moreover for the same reason $(\,\Sigma_{x\in D}\, B(x)\,)^{\mathcal{A}} \equiv D^{\mathcal{A}} \cdot B^{\mathcal{A}}$ and hence $(\,\Sigma_{x\in D}\, B(x)\,)^{\mathcal{A}^F} \equiv F(D^{\mathcal{A}}) \cdot F(B^{\mathcal{A}})$. Therefore we define $\tau_{\Sigma_{x\in D}\, B(x)\,[\Gamma]} \equiv \tau_{B(x)\,[\Gamma, x\in D]}$.

The coherent isomorphism indexed on the Equality type

$$\tau_{\mathtt{Eq}(C,c,d)[\Gamma]} : \mathtt{Eq}(C,c,d)^{F^{\mathcal{B}}}{}_\Sigma \to \mathtt{Eq}(C,c,d)^{\mathcal{A}^F}{}_\Sigma$$

is defined as the inverse of $\tau^{-1}_{\mathtt{Eq}(C,c,d)[\Gamma]}$ defined in turn as follows. Recall that $\mathtt{Eq}(C,c,d)^{\mathcal{A}} \equiv eq(c^{\mathcal{A}}, d^{\mathcal{A}})$ is the equalizer of $c^{\mathcal{A}}$ and $d^{\mathcal{A}}$ in $\mathcal{A}$. Then, by coherent isomorphisms of F preserving the AU structure we know that $F(\mathtt{Eq}(C,c,d)^{\mathcal{A}})$ is an equalizer of $F(c^{\mathcal{A}})$ and $F(d^{\mathcal{A}})$ in $\mathcal{B}$. Moreover, also $\mathtt{Eq}(C,c,d)^{F^{\mathcal{B}}} \equiv \mathtt{Eq}(C^F, c^F, d^F)^{\mathcal{B}}$ is an equalizer of $c^{F^{\mathcal{B}}}$ and $d^{F^{\mathcal{B}}}$ in $\mathcal{B}$. Therefore we define $\tau^{-1}_{\mathtt{Eq}(C,c,d)[\Gamma]}$ as the unique morphism toward the equalizer $\mathtt{Eq}(C^F, c^F, d^F)^{\mathcal{B}}$ induced by $\tau^{-1}_\Gamma \cdot F(eq(c^{\mathcal{A}}, d^{\mathcal{A}}))$. This is well defined with an argument analogous to that in definition 14.

The isomorphisms on the other types are defined analogously. ■

Now recall from page 1143 of [Mai05] that we can view a theory as a category and a translation as a functor. Hence, given AU's $\mathcal{A}$ and $\mathcal{B}$, we can think of the collection of translations from $T_{iso}(\mathcal{A})$ to $T_{iso}(\mathcal{B})$ as a category $Th(T_{iso}(\mathcal{A}), T_{iso}(\mathcal{B}))$ with translations as objects and natural transformations as morphisms. Hence we state the following correspondence between AU functors and translations between internal theories with coherent isomorphisms:

Theorem 22 *For any AU's $\mathcal{A}$ and $\mathcal{B}$, there is an equivalence between the category $AU(\mathcal{A},\mathcal{B})$ of AU functors and natural transformations and the category $Th(T_{iso}(\mathcal{A}), T_{iso}(\mathcal{B}))$ of translations and natural transformations.*

Proof. Given an AU functor $F : \mathcal{A} \to \mathcal{B}$ we define the translation $T(F) : T_{iso}(\mathcal{A}) \to T_{iso}(\mathcal{B})$ as follows: $T(F)$ interprets types and terms arising from $\mathcal{A}$ via F, i.e. each proper type arising from an object C of $\mathcal{A}$ is translated into the specific type of $T_{iso}(\mathcal{B})$ arising from $F(C)$, and each specific term arising from a morphism c of $\mathcal{A}$ is translated into the term arising from $F(c)$; moreover $\mathcal{T}_{au}$-constructors are interpreted as the corresponding ones in $T_{iso}(\mathcal{B})$; lastly the interpretation of a coherent isomorphism $\sigma^F_{B\ [\Gamma]}$ is given as the composition of a suitable coherent isomorphism of $T_{iso}(\mathcal{B})$ with $\tau_{B\ [\Gamma]}$ in lemma 21: more in detail

$$\sigma^F_{B\ [\Gamma]} : (\,B\ [\Gamma]\,)^{\mathcal{H}^F_\Sigma} \longrightarrow (\,B\ [\Gamma]\,)^{\mathcal{A}^F_\Sigma}$$

gets interpreted as

$$\sigma^F_{B\ [\Gamma]} \equiv \tau_{B\ [\Gamma]} \cdot \sigma_{B\ [\Gamma]^F}$$

where $B\ [\Gamma]^F$ is the translation in $T_{cat}(\mathcal{B})$ of the judgement $B\ [\Gamma]$. Note that the domain of $\sigma_{B\ [\Gamma]^F}$ can be taken to be $(\,B\ [\Gamma]\,)^{\mathcal{H}^F_\Sigma}$ thanks to lemma 20.

The translation $T(F)$ is uniquely determined by F up to a natural isomorphism because the interpretation of coherent isomorphisms, given that they commute with terms, substitution and weakening, is uniquely determined by interpretation of proper types and terms given by F.

Conversely any translation $L : T_{iso}(\mathcal{A}) \to T_{iso}(\mathcal{B})$ gives rise to an AU functor $\mathcal{C}(L) : \mathcal{C}_{T_{iso}(\mathcal{A})} \to \mathcal{C}_{T_{iso}(\mathcal{B})}$ defined on objects and morphisms in $\mathcal{C}_{T_{iso}(\mathcal{A})}$ as their translations in $T_{iso}(\mathcal{B})$. Finally $\mathtt{V}_{iso} \cdot (\,\mathcal{C}(L) \cdot \mathcal{Y}_{iso}\,) : \mathcal{A} \longrightarrow \mathcal{C}_{T_{iso}(\mathcal{A})} \to \mathcal{C}_{T_{iso}(\mathcal{B})} \to \mathcal{B}$ gives an AU functor as desired.

The given correspondence establishes an equivalence of categories. ■

From this we can deduce the following:

Corollary 23 *Given the AU's $\mathcal{A}$ and $\mathcal{B}$, the category of interpretations of $T_{iso}(\mathcal{A})$ into $\mathcal{B}$ as in section 5 of [Mai05] with interpretation morphisms is in equivalence with the category of AU functors from $\mathcal{A}$ to $\mathcal{B}$.*

Proof. Giving an interpretation $\mathcal{J}$ as in section 5 of [Mai05] means to give a translation $Tr_{\mathcal{J}}$ from $T_{iso}(\mathcal{A})$ to $T(\mathcal{B})$ (because types and terms of $T(\mathcal{B})$ are defined together with their interpretation in $\mathcal{B}$). Hence, from [Mai05] we know that $Tr_{\mathcal{J}}$ provides an AU homomorphism between the corresponding syntactic categories $\mathcal{C}(Tr_{\mathcal{J}}) : \mathcal{C}_{T_{iso}(\mathcal{A})} \to \mathcal{C}_{T(\mathcal{B})}$. This composed with the suitable parts of the equivalence of the syntactic categories, respectively with $\mathcal{A}$ and $\mathcal{B}$, gives an AU functor

$$(\,\mathtt{V} \cdot (\,\mathcal{C}(L)\,) \cdot \mathcal{Y}_{iso} : \mathcal{A} \longrightarrow \mathcal{B}$$

Conversely, given an AU functor F, by theorem 22 we get a translation $T(F) : T_{iso}(\mathcal{A}) \longrightarrow T_{iso}(\mathcal{B})$ which composed with the translation $\mathtt{St}$ in definition 17 gives a translation $\mathtt{St} \cdot T(F) : T_{iso}(\mathcal{A}) \longrightarrow T(\mathcal{B})$. This translation corresponds to an interpretation of $T_{iso}(\mathcal{A})$ in $\mathcal{B}$ because types and terms of

$T(\mathcal{B})$ are defined with their interpretation in $\mathcal{B}$ (i.e. the translation of types and terms of $T_{iso}(\mathcal{A})$ in $T(\mathcal{B})$ comes by definition with the interpretation of them in $\mathcal{B}$). ∎

Definition 24 *Given an AU $\mathcal{A}$, let $T_{iso}(\mathcal{A})[S]$ be the $\mathcal{T}_{au}$-*theory *extending the typed calculus $\mathcal{T}_{au}$ with $T_{iso}(\mathcal{A})$ and with some extra AU axioms S of the form*

$$c \in C \ [x \in B] \qquad c = d \in C \ [x \in B]$$

i.e. we add extra morphisms and equalities based on $\mathcal{A}$. Then we write $\mathcal{A}[S]_t$ for the syntactic category $\mathcal{C}_{T_{iso}(\mathcal{A})[S]}$.

We then call $\mathcal{I} : \mathcal{A} \to \mathcal{A}[S]_t$ the functor embedding an object into its type naming it in $\mathcal{A}[S]_t$ and a morphism into the term naming it in $\mathcal{A}[S]_t$.

Theorem 25 *Let $\mathcal{A}$ and S be as in the above definition. Then $\mathcal{A}[S]_t$ is universal with respect to being equipped with an AU functor $\mathcal{I} : \mathcal{A} \to \mathcal{A}[S]_t$ and an interpretation of the extra structure in S according to the notion of interpretation of a morphism in section 5.31 of [Mai05]: for any AU $\mathcal{B}$, the category $\mathbf{AU}(\mathcal{A}[S]_t, \mathcal{B})$ is equivalent to the category of pairs (F, α) where $F : \mathcal{A} \to \mathcal{B}$ is an functor and α interprets the structure in S with respect to F.*

Proof. Given an AU functor $F : \mathcal{A} \to \mathcal{B}$ we lift it to an interpretation L^F of $T_{iso}(\mathcal{A})$ in $\mathcal{B}$ by corollary 23 and we extend it to interpret $T_{iso}(\mathcal{A})[S]$ by interpreting the new added structure as assigned.

Then the interpretation L^F seen as a translation from $T_{iso}(\mathcal{A})[S]$ to $T(\mathcal{B})$ gives rise to a functor $\mathcal{C}(L^F) : \mathcal{C}_{T_{iso}(\mathcal{A})[S]} \to \mathcal{C}_{T(\mathcal{B})}$ and one from $\mathcal{C}_{T_{iso}(\mathcal{A})[S]}$ to $\mathcal{B}$ defined as $\widetilde{F} \equiv \mathtt{V} \cdot \mathcal{C}(L^F) : \mathcal{C}_{T_{iso}(\mathcal{A})[S]} \to \mathcal{C}_{T(\mathcal{B})} \to \mathcal{B}$.

Any other functor extending F can be proved to be naturally isomorphic to $\widetilde{F}$ by induction on the type in $T(\mathcal{A})[S]_t$ as done in theorem 5.31 of [Mai05] (note that also the interpretation of coherent isomorphisms is determined by F and the interpretation of S). ∎

Now considering that the universal property defining our subspace $\mathcal{A}[S]_t$ is the same as that in [MV10] we conclude that the two notions are equivalent:

Corollary 26 *Let $\mathcal{A}$ and S be as in definition 24. Then $\mathcal{A}[S]_t$ is equivalent the notion of subspace $\mathcal{A}[S]$ in [MV10].*

Remark 27 *From [MV10], we recall that examples of subspaces of an AU $\mathcal{A}$ are the following: the subspace $\mathcal{A}[c : 1 \to U]$, called open, with the addition of a global element $n : 1 \to U$ for an object U in $\mathcal{A}$, is equivalent to the slice category $\mathcal{A}/U$; the subspace $\mathcal{A}[c : \phi \to \perp]$, called closed, with the addition of an element from ϕ, subobject of the terminal object in $\mathcal{A}$, to the interpretation of falsum in $\mathcal{A}$, is equivalent to a suitable category of sheaves.*

2.2 Classifying category

Here we prove that the syntactic category $\mathcal{C}_T$ of a T_{au}-theory T classifies suitable generic interpretations of T in an arithmetic universe $\mathcal{B}$.

Definition 28 *A* standard interpretation $\mathcal{J}$ *of a* T_{au}*-theory* T *in an arithmetic universe* $\mathcal{B}$ *is a generic interpretation where the substitution and weakening diagrams are pullbacks and the induced functor on the syntactic category*

$$\mathcal{C}(\mathcal{J}) : \mathcal{C}_T \longrightarrow \mathcal{B}$$

is an AU functor. *We recall that* $\mathcal{C}(\mathcal{J})$ *is defined as follows: on closed types* C *as* $dom(\mathcal{J}(C))$ *and on terms* $f(x) \in B\ [x \in C]$ *as* $q(\,\mathcal{J}(B)\,,\,\mathcal{J}(C)\,) \cdot \mathcal{J}(\,b \in B\ [x \in C]\,)$.

Definition 29 (standard interpretation functor) *Given an arithmetic universe* $\mathcal{A}$ *and a* T_{au}*-theory* T*, there exists a* standard interpretation functor *from the category of arithmetic universes and AU functors to the category of small categories* `Cat`*:*

$$Int_T : AU \longrightarrow \texttt{Cat}$$

assigning to an arithmetic universe $\mathcal{B}$ *the category of standard interpretations with interpretation morphisms* $Int(T, \mathcal{B})$*, and to an AU functor* $F : \mathcal{A} \to \mathcal{B}$ *the functor*

$$Int_T(F) : Int(T, \mathcal{A}) \longrightarrow Int(T, \mathcal{B})$$

assigning to a standard interpretation $\mathcal{J}$ *the interpretation* $\mathcal{J}_{\mathcal{F}}$ *obtained as follows: if* $\mathcal{J}$ *interprets a type* $B\ [\Gamma]$ *as* $b_1, b_2, ..., b_n$ *with* $b_1 : C \to 1$*, then* $\mathcal{J}_{\mathcal{F}}$ *interprets the same type as* $!^F(C), F(b_2), ..., F(b_n)$*; and if* $\mathcal{J}$ *interprets a term* $b \in B\ [\Gamma]$ *as the section* $b^{\mathcal{J}}$*, then* $\mathcal{J}_{\mathcal{F}}$ *interprets the same term as* $F(b^{\mathcal{J}})$*. The pullback and weakening diagrams are the value under* F *of those induced by* $\mathcal{J}$*. This is a standard interpretation because* F *is an AU functor.*

We can show that the syntactic category of a theory represents the interpretation functor $Int_T : AU \longrightarrow \texttt{Cat}$:

Theorem 30 *Given a* T_{au}*-theory* T*, its interpretation functor* $Int_T : AU \longrightarrow$ `Cat` *is natural isomorphic to the covariant functor* $AU(\,\mathcal{C}_T\,,\,-\,)$*, and hence for every AU* $\mathcal{A}$ *the category of standard interpretations of* T *in* $\mathcal{A}$ *is isomorphic to that of AU functors and natural transformations* $AU(\,\mathcal{C}_T\,,\,\mathcal{A}\,)$*.*

Proof. By definition a standard interpretation $\mathcal{J}$ of T in $\mathcal{A}$ induces an AU functor $\mathcal{C}(\mathcal{J}) : \mathcal{C}_T \longrightarrow \mathcal{A}$. Conversely given an AU functor $F : \mathcal{C}_T \longrightarrow \mathcal{A}$ we define the interpretation $\mathcal{I}_F$ of T in $\mathcal{A}$ as $Int_T(F)((-)^{\mathcal{H}})$ since the $(-)^{\mathcal{H}}$ intepretation is indeed standard in $\mathcal{C}_T$. ∎

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