A structural investigation on formal topology: coreflection of formal covers and exponentiability

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Abstract

We present and study the category of formal topologies and some of its variants. Two main results are proven. The first is that, for any inductively generated formal cover, there exists a formal topology whose cover extends in the minimal way the given one. This result is obtained by enhancing the method for the inductive generation of the cover relation by adding a coinductive generation of the positivity predicate. Categorically, this result can be rephrased by saying that inductively generated formal topologies are coreflective into inductively generated formal covers.

The second result is that unary formal covers are exponentiable in the category of inductively generated formal covers and hence, thanks to the coreflection, unary formal topologies are exponentiable in the category of inductively generated formal topologies.

From a localic point of view the exponentiability of unary formal topologies means that algebraic dcpos are exponentiable in the category of open locales. But, the coreflection theorem states that open locales are coreflective in locales and hence, as a consequence of well-known impredicative results on exponentiable locales, it allows to prove that locally compact open locales are exponentiable in the category of open locales.

Contents

1 Introduction

Formal topology is nowadays recognized as one of the main approaches to the development of constructive topology, where by constructive we mean both intuitionistic and predicative. Many results of classical and impredicative topology have been already studied, and found their place in a predicative framework, by using formal topology (see [Sam03] for an updated overview on formal topology). Formal topology is noticeable also from a structural point of view. Indeed, the category FTop of formal topologies and continuous relations is a predicative presentation of the category OpLoc of open locales (see [JT84]) and the category FTop^- of formal covers, namely, formal topologies without the positivity predicate, is a predicative presentation of the category Loc of locales (see [Joh82]).

In this paper, we begin to study a full sub-category of FTop, that is, the category $\mathsf{FTop_i}$ of inductively generated formal topologies (see [CSSV03]). We consider such a category instead of FTop because it is predicatively known to be cartesian while FTop is not, even if these categories are equivalent from an impredicative point of view.

A main result of the paper is a proof that $\mathsf{FTop_i}$ coreflects into the category $\mathsf{FTop_i}^-$ of inductively generated formal covers, that is, the inclusion functor of $\mathsf{FTop_i}$ into $\mathsf{FTop_i}^-$ has a right adjoint. To obtain this result, we show how to enhance the method for the inductive generation of a cover relation presented in [CSSV03] by adding a new method for a co-inductive generation of a positivity predicate; in this way we are able to construct, for any inductively generated cover $\lhd$, the formal topology whose cover extends $\lhd$ in the minimal way. Such a new method is directly inspired by Martin-Löf's idea of defining a binary positivity predicate in the framework of *basic topologies* by using coinduction (see [Sam03, Val04]). It can be easily justified by using Tarski fix-point theorem, but, after the work in [Coq96, Pal02], it can be fully justified also from a predicative point of view.

Then, we give a direct proof that unary formal covers are exponentiable in the category $\mathsf{FTop_i}^-$. Hence, as a corollary of the coreflection theorem above, we obtain that unary formal topologies are exponentiable within $\mathsf{FTop_i}$.

The question of characterizing exponentiable topologies has a long history in the development of topology. It is well known that the category Top of topological spaces and continuous functions is not cartesian closed. In fact, the topological spaces that can be exponentiated in Top are only those whose frames of open sets are continuous, corresponding to locally compact locales (for an overview on the topic see [EH02]). This result was reproduced by Hyland in the context of the intuitionistic but impredicative theory of locales by showing that in Loc only the locally compact locales can be exponentiated [Hyl81]. Later, his proof of exponentiability was adapted to the language of formal topology, but still working within an impredicative setting (see [Sig95]). More recently, Vickers reproduced most of Hyland's results by using geometric reasoning (see [Vic01]).

We think that a main contribution in proving exponentiability of unary topologies is a detailed analysis of the conditions characterizing continuous relations between a unary formal topology and an inductively generated one. In fact, after such an analysis, the axioms defining the cover of the exponent topology emerge naturally.

All the proofs in the paper are developed within Martin-Löf predicative type theory. However, both our main results have a precise meaning also in an impredicative setting. Indeed the coreflection theorem states that open locales are coreflective in locales and the exponentiability theorem states that algebraic dcpos are exponentiable in OpLoc. Moreover, an immediate consequence of our coreflection theorem and the result by Hyland above, is that any open locally compact local is exponentiable in the category of open locales.

2 Formal topologies and their morphisms

In this section the basic definitions of formal topology will be quickly recalled. The reader interested in having more details on formal topology and a deeper analysis of the foundational motivations for the formal development of topology within Martin-Löf's constructive type theory [NPS90, Mar84] is invited to look, for instance, at the updated overview in [Sam03].

2.1 Concrete topological spaces and formal topologies

We start by recalling how to describe predicatively a topological space. Let X be a set. Then $(X, \Omega(X))$ is a topological space if $\Omega(X)$ is a subset of $\mathcal{P}(X)$ which contains $\emptyset$ and X and is closed under finite intersection and under arbitrary union. The quantification implicitly used in this last condition is of the third order, since it says that, for all $F \subseteq \Omega(X)$, $\bigcup F \in \Omega(X)$. We can "go down" one step by thinking of $\Omega(X)$ as a family of subsets indexed by a set A through a map $\mathsf{ext} : A \to \mathcal{P}(X)$. Indeed, we can now quantify on A rather than on $\Omega(X)$. But, we have to say that, for all $U \in \mathcal{P}(A)$ there exists $c \in A$ such that $\cup_{a \varepsilon U} \mathsf{ext}(a) = \mathsf{ext}(c)$, which is still impredicative[1]. We can "go down" another step by defining opens to be of the form $\mathsf{Ext}(U) \equiv \cup_{a \varepsilon U} \mathsf{ext}(a)$ for an arbitrary subset U of A. In this way $\emptyset$ is open, because $\mathsf{Ext}(\emptyset) = \emptyset$, and closure under union is automatic, because obviously $\cup_{i \in I} \mathsf{Ext}(U_i) = \mathsf{Ext}(\cup_{i \in I} U_i)$. So, all we have to do is to require that $\mathsf{Ext}(A)$ is the whole X and closure under finite intersections, that is,

$$
\begin{aligned}
(*) \quad & (\forall a, b \in A)(\forall x \in X)\ (x \varepsilon \mathsf{ext}(a) \cap \mathsf{ext}(b) \to \\
& \qquad (\exists c \in A)\ (x \varepsilon \mathsf{ext}(c)\ \&\ \mathsf{ext}(c) \subseteq \mathsf{ext}(a) \cap \mathsf{ext}(b)))
\end{aligned}
$$

It is not difficult to realize that this amounts to the standard definition saying that $\{\mathsf{ext}(a) \subseteq X \mid a \in A\}$ is a base (see for instance [Eng77]). We can make $(*)$ a bit shorter by introducing the abbreviation

$$a \downarrow b \equiv \{c \in A \mid \mathsf{ext}(c) \subseteq \mathsf{ext}(a)\ \&\ \mathsf{ext}(c) \subseteq \mathsf{ext}(b)\}$$

so that it becomes $(\forall a, b \in A)\ \mathsf{ext}(a) \cap \mathsf{ext}(b) \subseteq \mathsf{Ext}(a \downarrow b)$. Now, note that $c \varepsilon a \downarrow b$ implies that $\mathsf{ext}(c) \subseteq \mathsf{ext}(a) \cap \mathsf{ext}(b)$, so that $\mathsf{Ext}(a \downarrow b) \equiv \cup_{c \varepsilon a \downarrow b} \mathsf{ext}(c) \subseteq \mathsf{ext}(a) \cap \mathsf{ext}(b)$. Thus we arrived at the definition of concrete topological space.

Definition 2.1 (Concrete topological space) *A concrete topological space is a triple $\mathcal{X} \equiv (X, A, \mathsf{ext})$ where X and A are sets and ext is a map from A to $\mathcal{P}(X)$ satisfying:*

$$(B_1) \quad X = \mathsf{Ext}(A)$$

$$(B_2) \quad (\forall a, b \in A)\ \mathsf{ext}(a) \cap \mathsf{ext}(b) = \mathsf{Ext}(a \downarrow b)$$

The notion of formal topology arises by describing, as well as possible, the structure induced by a concrete topological space (X, A, ext) on the set A, and then by taking the result as an axiomatic definition. The reason for such a move is that the definition of concrete topological space is too restrictive given

[1] All the set-theoretical notions that we use conform to the subset theory for Martin-Löf's type theory as presented in [SV97]. In particular, we use the symbol $\in$ for the membership relation between an element and a set or a collection and ε for the membership relation between an element and a subset, which is never a set but a propositional function, so that $a \varepsilon U$ holds if and only if $U(a)$ holds.

that in the most interesting cases of topological space we do not have, from a constructive point of view, a *set* of points to start with[2].

Since the elements in A are *names* for the basic opens of the topology on X, and any open set is the union of basic opens, we can specify an open set O by using the subset U_O of all the (names of the) basic opens which are used to form it, that is, $O = \mathsf{Ext}(U_O)$. However, it is possible that two different subsets of A have the same extension. Thus, we don't have a bijective correspondence between concrete opens and subsets of A and we need to introduce an equivalence relation if we want to obtain it. What we need is a relation which identifies the subsets U and V when $\mathsf{Ext}(U) = \mathsf{Ext}(V)$. The following lemma gives the correct hint.

Lemma 2.2 *Let U and V be subsets of A. Then $\mathsf{Ext}(U) = \mathsf{Ext}(V)$ if and only if, for all $a \in A$, $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U) \leftrightarrow \mathsf{ext}(a) \subseteq \mathsf{Ext}(V)$.*

Thus, in order to define the equivalence relation among subsets of A that we are looking for, we need to introduce a new proposition $a \lhd U$ between an element a and a subset U of A whose intended meaning is that $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U)$.

Now, we can simply state that a *formal open* is the "fullest" among the subsets which have the same extension, that is, for any subset U, we choose

$$\lhd(U) \equiv \{a \in A \mid a \lhd U\}$$

In fact, it is possible to prove that $\lhd(U) =_{\lhd} U$ by using the following valid conditions on $\lhd$:

$$\text{(reflexivity)}\ \frac{a \varepsilon U}{a \lhd U} \qquad \text{(transitivity)}\ \frac{a \lhd U \quad U \lhd V}{a \lhd V}$$

where $U \lhd V$ is a shorthand for a derivation of $u \lhd V$ under the assumption that $u \varepsilon U$.

Thus, we found a relation, that is, $\lhd$, and two conditions over it, that is, *reflexivity* and *transitivity*, which allow to deal with concrete open subsets by using only the subsets of A. But these conditions are not sufficient to describe completely the concrete situation; for instance there is no condition which describes formally the conditions (B_1) and (B_2).

While there is no easy way to formulate (B_1) within the formal side since this condition connects too deeply the concrete and the formal sides, to formulate (B_2) we can use the fact that

$$\mathsf{Ext}(U) \cap \mathsf{Ext}(V) \subseteq \mathsf{Ext}(U \downarrow V)$$

where $U \downarrow V \equiv \{a \in A \mid ((\exists u \varepsilon U)\ \mathsf{ext}(a) \subseteq \mathsf{ext}(u))\ \&\ ((\exists v \varepsilon V)\ \mathsf{ext}(a) \subseteq \mathsf{ext}(v))\}$. Now, let us suppose $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U)$ and $\mathsf{ext}(a) \subseteq \mathsf{Ext}(V)$, then we immediately

[2]Here we commit ourselves to Martin-Löf's constructive set theory; hence we distinguish between sets, which are inductively generated, and collections.

obtain $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U) \cap \mathsf{Ext}(V)$ and hence $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U \downarrow V)$. Its formal counterpart is

$$(\downarrow\text{-right}) \qquad \frac{a \lhd U \qquad a \lhd V}{a \lhd U \downarrow V}$$

where $U \downarrow V \equiv \{c \in A \mid (\exists u \varepsilon U)\ c \lhd \{u\}\ \&\ (\exists v \varepsilon V)\ c \lhd \{v\}\}$.

Thus, we arrived at the definition of formal cover.

Definition 2.3 (Formal cover) *A* formal cover *is a structure* $\mathcal{A} \equiv (A, \lhd)$ *where* A *is a set and* $\lhd$ *is an infinitary relation, called* cover *relation, between elements and subsets of* A *satisfying* reflexivity, transitivity *and* $\downarrow$-right.

However, to express constructively the fact that a basic open subset is inhabited it is convenient to introduce also a second primitive predicate $\mathsf{Pos}(-)$ on the elements of A. Its intended meaning is that, for any $a \in A$, $\mathsf{Pos}(a)$ holds if and only if there exists $x \in X$ such that $x \varepsilon \mathsf{ext}(a)$. We require the following conditions on this predicate.

$$(\text{monotonicity}) \ \frac{\mathsf{Pos}(a) \qquad a \lhd U}{(\exists u \varepsilon U)\ \mathsf{Pos}(u)} \qquad (\text{positivity}) \ \frac{\mathsf{Pos}(a) \to a \lhd U}{a \lhd U}$$

While the meaning of *monotonicity* is obvious and the proof of its validity in any concrete topological space is immediate, *positivity* may require some explanation. It states two things in one condition: first, that a not-inhabited basic open is covered by any subset, second that proof by cases on the positivity of a are valid when the conclusion is $a \lhd U$ (see [SVV96]). The proof of validity of *positivity* is straightforward and it uses only intuitionistic logic.

We thus arrived at the main definition.

Definition 2.4 (Formal topology) *A* formal topology *is a structure* $\mathcal{A} \equiv (A, \lhd, \mathsf{Pos})$ *where* $(A, \lhd)$ *is a formal cover and* Pos *is a predicate over* A*, called* positivity predicate*, satisfying* monotonicity *and* positivity.

It is useful to recall the following equivalent formulations of the *positivity* condition that we will often use in the next sections (see [Sam87]). To state them, given any predicate $\mathsf{Pos}(-)$ over elements of A and any subset U of A, we write U^+ to mean the subset $\{x \in A \mid x \varepsilon U\ \&\ \mathsf{Pos}(x)\}$.

Proposition 2.5 *Let* A *be a set,* $\lhd$ *be a relation between elements and subsets of* A *which satisfies* reflexivity *and* transitivity *and* Pos *be a predicate on elements of* A*. Then, the following conditions are equivalent:*

1. (positivity) *for any* $a \in A$ *and* $U \subseteq A$, $\mathsf{Pos}(a) \to a \lhd U$ *yields* $a \lhd U$*;*
2. (axiom positivity) *for any* $a \in A$, $a \lhd \{a\}^+$*;*
3. (cover positivity) *for any* $a \in A$ *and* $U \subseteq A$, $a \lhd U$ *yields* $a \lhd U^+$.

The first consequence of the previous proposition is the following theorem which shows that the cover relation uniquely determines the positivity predicate.

Theorem 2.6 *Let* $- \lhd -$ *be a cover relation over a set* A *and* $\mathsf{Pos}_1(-)$ *and* $\mathsf{Pos}_2(-)$ *be two positivity predicates with respect to such a cover. Then* $\mathsf{Pos}_1(-)$ *and* $\mathsf{Pos}_2(-)$ *are equivalent, namely, for any* $a \in A$, $\mathsf{Pos}_1(a)$ *if and only if* $\mathsf{Pos}_2(a)$.

Proof. By the *positivity axiom*, for every $a \in A$, $a \lhd a^{+_2}$, where a^{+_2} is a shorthand for the subset $\{x \in A \mid x = a \;\&\; \mathsf{Pos}_2(x)\}$. Now, let us assume $\mathsf{Pos}_1(a)$; then, by *monotonicity*, there exists $x \varepsilon a^{+_2}$ such that $\mathsf{Pos}_1(x)$ holds. But $x \varepsilon a^{+_2}$ means that both $x = a$ and $\mathsf{Pos}_2(x)$ hold and hence $\mathsf{Pos}_2(a)$ follows. Thus, by discharging the assumption $\mathsf{Pos}_1(a)$, we proved that $\mathsf{Pos}_1(a)$ yields $\mathsf{Pos}_2(a)$. In a completely analogous way we can prove the other implication.

2.2 Formal points

When working in formal topology one is in general interested in those properties of a concrete topological space (X, A, ext) which make no reference to the elements of X. Thus, one can dispense with the collection X and it is possible to work by using the set A only. But this does not mean that points are out of reach. In fact, a point $x \in X$ can be identified with the filter of the basic opens that, in the concrete case, contain x itself. So, we can associate to any $x \in X$, the following subset of A

$$\alpha_x \equiv \{a \in A \mid x \varepsilon \mathsf{ext}(a)\}$$

However, from a topological point of view, we can "see" only those points which can be distinguished by using the open sets and hence we are led to identify a concrete point x with the subset α_x.

If we want to move to the formal side, we have to find those properties which characterize such subsets and are expressible in our language. Here we point out the following ones:

(point inhabitance)	$(\exists a \in A)\ a \varepsilon \alpha_x$
(point convergence)	$\dfrac{a \varepsilon \alpha_x \quad b \varepsilon \alpha_x}{(\exists c \varepsilon a \downarrow b)\ c \varepsilon \alpha_x}$
(point splitness)	$\dfrac{a \varepsilon \alpha_x \quad \mathsf{ext}(a) \subseteq \mathsf{Ext}(U)}{(\exists u \varepsilon U)\ u \varepsilon \alpha_x}$
(point positivity)	$\dfrac{a \varepsilon \alpha_x}{(\exists x \in X)\ x \varepsilon \mathsf{ext}(a)}$

In fact, *point inhabitance* is an obvious corollary of the condition B_1, *point convergence* is an immediate consequence of the condition B_2, and *point splitness* and *point positivity* follows by logic. Thus, we are led to the following definition.

Definition 2.7 (Formal point) *Let* $(A, \lhd, \mathsf{Pos})$ *be a formal topology. Then an inhabited subset* α *of* A *is a* formal point *if, for any* $a, b \in A$ *and any* $U \subseteq A$, *it satisfies the following conditions:*

$$\text{(point convergence)}\ \frac{a \varepsilon \alpha \quad b \varepsilon \alpha}{(\exists c \varepsilon a \downarrow b)\ c \varepsilon \alpha} \qquad \text{(point splitness)}\ \frac{a \varepsilon \alpha \quad a \lhd U}{(\exists u \varepsilon U)\ u \varepsilon \alpha}$$

As observed by Peter Aczel, we can avoid to require the condition of *point positivity*, namely, that $\mathsf{Pos}(a)$ is a consequence of $a\varepsilon\alpha$, since it can be proved by using *point splitness*. In fact, we know that $a \lhd a^+$ and hence if, for some point α, $a\varepsilon\alpha$ then *point splitness* shows that there exists an element x in a^+ such that $x\varepsilon\alpha$. Then $x = a$ and $\mathsf{Pos}(x)$ hold and hence $\mathsf{Pos}(a)$ follows. So, no condition on the positivity predicate is necessary in the definition of formal point and hence the same definition can be used for formal covers as well.

In the following we call $\mathsf{Pt}(\mathcal{A})$ the collection of formal points of the formal topology $\mathcal{A}$. We can give $\mathsf{Pt}(\mathcal{A})$ the structure of a topological space if we mimic the situation of a concrete topological space even if $\mathsf{Pt}(\mathcal{A})$ is a collection and not a set. So, let us set, for any $a \in A$,

$$\mathsf{ext}^{\mathsf{Pt}}(a) \equiv \{\alpha \in \mathsf{Pt}(\mathcal{A}) \mid a\varepsilon\alpha\}$$

and use the set-indexed family $(\mathsf{ext}^{\mathsf{Pt}}(a))_{a\in A}$ as a base for a topology on $\mathsf{Pt}(\mathcal{A})$.

2.3 Continuous relations

In this section we report and explain the conditions defining continuous relations between formal topologies. The notion of continuous relation essentially goes back to the notion of frame morphism in [Joh82]. The conditions on the continuous relations that we present here and the explanations motivating them are the result of joint work by S.Valentini and P.Virgili [Vir90] in collaboration with G.Sambin and, later, with S.Gebellato.

A map from the topological space $\mathcal{X} = (X, \Omega(X))$ to the topological space $\mathcal{Y} = (Y, \Omega(Y))$ is a function $\phi : X \to Y$ such that, for any basic open $\mathcal{B}$ of $\mathcal{Y}$, the subset $\phi^{-1}(\mathcal{B}) \equiv \{x \in X \mid \phi(x) \in \mathcal{B}\}$ is an open set of $\mathcal{X}$. If we write this condition for the concrete topological spaces (X, A, ext_1) and (Y, B, ext_2) we obtain that the condition for a function $\phi : X \to Y$ to be continuous becomes

$$(\forall b \in B)(\exists U \subseteq A)\ \phi^{-1}(\mathsf{ext}_2(b)) = \mathsf{Ext}_1(U)$$

There is only one possible constructive meaning for this sentence, that is, there exists a map $\overleftarrow{F}: B \to \mathcal{P}(A)$ such that, for any $b \in B$, $\mathsf{Ext}_1(\overleftarrow{F}(b))$ is equal to $\phi^{-1}(\mathsf{ext}_2(b))$. Since $\mathsf{Ext}_1(\{a \in A \mid \mathsf{ext}_1(a) \subseteq \phi^{-1}(\mathsf{ext}_2(b))\})$ is always contained in $\phi^{-1}(\mathsf{ext}_2(b))$, the continuity requirement rests in the fact that $\phi^{-1}(\mathsf{ext}_2(b))$ is contained in $\mathsf{Ext}_1(\{a \in A \mid \mathsf{ext}_1(a) \subseteq \phi^{-1}(\mathsf{ext}_2(b))\})$. Hence, the best possible definition is to state that $\overleftarrow{F}(b)$ is the subset of all the basic opens $a \in A$ such that $\mathsf{ext}_1(a)$ is contained in $\phi^{-1}(\mathsf{ext}_2(b))$, that is, the image through ϕ of any point in the basic open $\mathsf{ext}_1(a)$ is in the basic open $\mathsf{ext}_2(b)$. Thus, the formal counterpart of a continuous function ϕ from X to Y is a relation F between elements of A and elements of B such that $a\ F\ b$ holds if and only if $a\varepsilon \overleftarrow{F}(b)$. So, to find a completely formal characterization of the notion of continuous function between topological spaces we have to express the condition above with no reference to the elements of X and Y.

In solving this problem we will use also an equivalent formulation of continuity, namely, that a function ϕ from the concrete topological spaces (X, A, ext_1) to (Y, B, ext_2) is continuous if and only if

$$(\forall b \in B)(\forall x \in X)\ \phi(x)\varepsilon\mathsf{ext}_2(b) \rightarrow \\ (\exists a \in A)\ x\varepsilon\mathsf{ext}_1(a)\ \&\ (\forall z \in X)\ z\varepsilon\mathsf{ext}_1(a) \rightarrow \phi(z)\varepsilon\mathsf{ext}_2(b)$$

that can by simplified in

$$(\forall b \in B)(\forall x \in X)\ \phi(x)\varepsilon\mathsf{ext}_2(b) \rightarrow (\exists a \in A)\ x\varepsilon\mathsf{ext}_1(a)\ \&\ aFb$$

provided that $-F-$ is the relation associated to ϕ that we want to characterize.

Now we look for suitable conditions, that do not rely on the presence of the set of concrete points in order to be formulated, and express that the relation F is the formal counterpart of a continuous function. To achieve this result we will proceed as follows. First, we will define a function ϕ_F from $\mathsf{Pt}(\mathcal{A})$ to $\mathsf{Pt}(\mathcal{B})$ associated with the relation F. Then, we will look for the conditions on F which are both expressible in the language of formal topologies and allow to prove that ϕ_F is a continuous function from $\mathsf{Pt}(\mathcal{A})$ to $\mathsf{Pt}(\mathcal{B})$. And finally, we will check the validity of such conditions in every concrete topological space.

So, let us suppose that F is a relation between two formal topologies. Then we want to define a continuous map ϕ_F from $\mathsf{Pt}(\mathcal{A})$ to $\mathsf{Pt}(\mathcal{B})$ such that $a\ F\ b$ holds if and only if, for any formal point $\alpha \in \mathsf{Pt}(\mathcal{A})$, if $\alpha \in \mathsf{ext}_1^{\mathsf{Pt}}(a)$ then $\phi_F(\alpha) \in \mathsf{ext}_2^{\mathsf{Pt}}(b)$.

An immediate consequence of this requirement is that if aFb and $a\varepsilon\alpha$ then $\phi_F(\alpha) \in \mathsf{ext}_2^{\mathsf{Pt}}(b)$. Now, $a\varepsilon\alpha$ means that $\alpha \in \mathsf{ext}_1^{\mathsf{Pt}}(a)$ and $\phi_F(\alpha) \in \mathsf{ext}_2^{\mathsf{Pt}}(b)$ means that $b\varepsilon\phi_F(\alpha)$. Hence, provided that we write $\overrightarrow{F}(a)$ to mean the subset $\{b \in B \mid aFb\}$, we have that

$$\bigcup_{a\varepsilon\alpha} \overrightarrow{F}(a) \subseteq \phi_F(\alpha)$$

On the other hand, continuity of ϕ_F means that

$$(\forall b \in B)(\forall \alpha \in \mathsf{Pt}(\mathcal{A}))\ \phi_F(\alpha)\varepsilon\mathsf{ext}_2^{\mathsf{Pt}}(b) \rightarrow (\exists a \in A)\ \alpha\varepsilon\mathsf{ext}_1^{\mathsf{Pt}}(a)\ \&\ aFb$$

and hence if $b\varepsilon\phi_F(\alpha)$ then there exists $a\varepsilon\alpha$ such that aFb, that is,

$$\phi_F(\alpha) \subseteq \bigcup_{a\varepsilon\alpha} \overrightarrow{F}(a)$$

Thus, we are forced to the following definition

$$\phi_F(\alpha) \equiv \bigcup_{a\varepsilon\alpha} \overrightarrow{F}(a)$$

Note that this definition guarantees that, if ϕ_F is a function from $\mathsf{Pt}(\mathcal{A})$ to $\mathsf{Pt}(\mathcal{B})$, then it is continuous. Hence, we only have to look for the conditions which make ϕ_F be a function between formal points, that is, the image $\phi_F(\alpha)$ of a formal point α of $\mathcal{A}$ is a formal point of $\mathcal{B}$.

To begin with, we have to prove that $\phi_F(\alpha)$ is inhabited, namely, that there exists $b \in B$ such that, for some $a\varepsilon\alpha$, $a\,F\,b$ holds. Now, we know that the point α is inhabited and hence in order to obtain the result it is sufficient to require

$$\text{(function totality)} \quad A \lhd_{\mathcal{A}} F^{-}(B)$$

where, for any subset V of B, $F^{-}(V) \equiv \{c \in A \mid (\exists v\varepsilon V)\ c\,F\,v\}$. Indeed, suppose $a\varepsilon\alpha$. Then $A \lhd_{\mathcal{A}} F^{-}(B)$ yields $a \lhd_{\mathcal{A}} F^{-}(B)$ and hence, by *point splitness*, there exists $c\varepsilon F^{-}(B)$ such that $c\varepsilon\alpha$. Thus, there exists $b \in B$ such that $c\,F\,b$ and $c\varepsilon\alpha$. Now, we have to check that *function totality* is valid for any concrete topological space. So, let us assume that (X, A, ext_1) and (Y, B, ext_2) are two concrete topological spaces, ϕ is a continuous map from X to Y and F is a relation between A and B such that $a\,F\,b$ if and only in, for all $x \in X$, $x\varepsilon\mathsf{ext}_1(a)$ yields $\phi(x)\varepsilon\mathsf{ext}_2(b)$. Then, we have to show that, for all $a \in A$ and all $x \in X$, if $x\varepsilon\mathsf{ext}_1(a)$ then there exists $u \in A$ such that both $x\varepsilon\mathsf{ext}_1(u)$ and $u\varepsilon F^{-}(B)$, that is, there exists $t \in B$ such that $u\,F\,t$. Now, by the condition (B_1), $x\varepsilon\mathsf{ext}_1(a)$ yields that there exists some element $t \in B$ such that $\phi(x)\varepsilon t$ and hence, by continuity of ϕ, there exists $u \in A$ such that both $x\varepsilon\mathsf{ext}_1(u)$ and $u\,F\,t$ hold.

The second condition that we have to verify is that, supposing $b\varepsilon\phi_F(\alpha)$ and $d\varepsilon\phi_F(\alpha)$, there exists $k\varepsilon b \downarrow d$ such that $k\varepsilon\phi_F(\alpha)$. To obtain this result it is sufficient to require the following two conditions:

$$\text{(function weak-saturation)} \quad \frac{a \lhd_{\mathcal{A}} c \qquad cFb}{aFb}$$

$$\text{(function convergence)} \quad \frac{aFb \qquad aFd}{a \lhd_{\mathcal{A}} F^{-}(b \downarrow d)}$$

In fact $b\varepsilon\phi_F(\alpha)$ and $d\varepsilon\phi_F(\alpha)$ yield that there are $a\varepsilon\alpha$ and $c\varepsilon\alpha$ such that aFb and cFd, and hence by *point convergence* there is also $e\varepsilon a \downarrow c$, namely, $e \lhd_{\mathcal{A}} a$ and $e \lhd_{\mathcal{A}} c$, such that $e\varepsilon\alpha$. So, by using *function weak-saturation*, we obtain both $e\,F\,b$ and $e\,F\,d$, which, by *function convergence*, yield $e \lhd_{\mathcal{A}} F^{-}(b \downarrow d)$. Then, by *point splitness*, $(\exists h\varepsilon F^{-}(b \downarrow d))\ h\varepsilon\alpha$, that is, there exists $k\varepsilon b \downarrow d$ such that $k\varepsilon\phi_F(\alpha)$. Also in this case it is necessary to check that the two required conditions are valid. In fact, it is easy to check that the following generalization of *function weak-saturation*

$$\text{(function saturation)} \quad \frac{a \lhd_{\mathcal{A}} W \qquad (\forall w\varepsilon W)\ wFb}{aFb}$$

is an immediate consequence, by intuitionistic logic, of the condition linking F and ϕ_F. Thus, let us prove the validity of *function convergence*. Suppose $x \in X$ and $x\varepsilon\mathsf{ext}_1(a)$, then aFb yields $\phi(x)\varepsilon\mathsf{ext}_2(b)$ and aFd yields $\phi(x)\varepsilon\mathsf{ext}_2(d)$; then, by the condition (B_2), there exists $k\varepsilon b \downarrow d$ such that $\phi(x)\varepsilon\mathsf{ext}_2(k)$. Finally, continuity of ϕ yields that there exists $h \in A$ such that $x\varepsilon\mathsf{ext}_1(h)$ and hFk, that is, $h\varepsilon F^{-}(b \downarrow d)$.

The third condition for $\phi_F(\alpha)$ being a formal point is that, if $b \varepsilon \phi_F(\alpha)$ and $b \lhd_{\mathcal{B}} V$, then there exists $v \varepsilon V$ such that $v \varepsilon \phi_F(\alpha)$. The necessary condition is

$$\text{(function continuity)} \quad \frac{aFb \qquad b \lhd_{\mathcal{B}} V}{a \lhd_{\mathcal{A}} F^{-}(V)}$$

Indeed, $b \varepsilon \phi_F(\alpha)$ yields that there is $a \varepsilon \alpha$ such that aFb and hence *function continuity*, together with *point splitness*, yields that there exists $c \varepsilon F^{-}(V)$ that is also an element of α, namely, there is $v \varepsilon V$ such that cFv and $c \varepsilon \alpha$, that is, $v \varepsilon \phi_F(\alpha)$. The proof of validity of this condition is immediate. Indeed, suppose that both aFb and $b \lhd_{\mathcal{B}} V$ hold. Then, for all $x \in X$, $x \varepsilon \mathsf{ext}_1(a)$ yields $\phi(x) \varepsilon \mathsf{ext}_2(b)$ and, for all $y \in Y$, $y \varepsilon \mathsf{ext}_2(b)$ yields that there exists $v \varepsilon V$ such that $y \varepsilon \mathsf{ext}_2(v)$. Thus, for any $x \varepsilon \mathsf{ext}_1(a)$, there is $v \varepsilon V$ such that $\phi(x) \varepsilon \mathsf{ext}_2(v)$ and hence, by continuity of ϕ, there is $c \in A$ such that $x \varepsilon \mathsf{ext}_1(c)$ and cFv.

So, we have finished to look for the conditions that make ϕ_F a well-defined function from $\mathsf{Pt}(\mathcal{A})$ to $\mathsf{Pt}(\mathcal{B})$. Hence, we can give the following definition of continuous relation between formal topologies.

Definition 2.8 (Continuous relation) *Suppose that* $\mathcal{A} = (A, \lhd_{\mathcal{A}}, \mathsf{Pos}_{\mathcal{A}})$ *and* $\mathcal{B} = (B, \lhd_{\mathcal{B}}, \mathsf{Pos}_{\mathcal{B}})$ *are two formal topologies. Then a* continuous relation *from* $\mathcal{A}$ *to* $\mathcal{B}$ *is a binary proposition* $a\,F\,b$, *for* $a \in A$ *and* $b \in B$, *which satisfies* function totality, function convergence, function saturation *and* function continuity.

Note that no condition in the definition of continuous relation involves the positivity predicate; hence, the same definition can be used for formal covers as well. As a consequence, all the following lemmas on basic properties of a continuous relation are valid both for formal topologies and formal covers.

It is worth observing that the definition of continuous relation above is obtained from the definition of frame morphism expressed in terms of relation in [Sam87] by taking the opposite relation and adapting the conditions of *function totality* and *function convergence* to our setting.

The following lemma is an immediate consequence of the definition.

Lemma 2.9 *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be formal topologies (covers) and* F *be a continuous relation between them. Then,*

- (cover anti-image) *if* $V \lhd_{\mathcal{B}} W$ *then* $F^{-}(V) \lhd_{\mathcal{A}} F^{-}(W)$*;*
- (weak-continuity) *if* $a\,F\,b$ *and* $b \lhd_{\mathcal{B}} d$ *then* $a\,F\,d$.

We want to prove now that formal topologies (covers) form a category with respect to continuous relations. The main problem is to define a suitable operation of composition between continuous relations. The first and naive idea is defining composition of continuous relations as relation composition but unfortunately relation composition of two continuous relations is not continuous because in general it does not satisfy *function saturation*. Indeed, one can prove only the following lemma.

Lemma 2.10 *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be formal topologies (covers) and let F be a continuous relation from $\mathcal{A}$ to $\mathcal{B}$ and G be a continuous relation from $\mathcal{B}$ to $\mathcal{C}$. Then, $G \circ F$, namely relation composition of F and G, satisfies* function totality, function convergence *and* function continuity.

The following proposition can be used to fix the problem of the missing condition.

Proposition 2.11 *Let $\mathcal{A}$ and $\mathcal{B}$ be two formal topologies (covers) and suppose that F is a relation which satisfies all of the conditions for a continuous relation except for* function saturation *which is replaced by* function weak-saturation. *Then*

$$aF^{\lhd}b \equiv a \lhd_{\mathcal{A}} \{c \in A \mid cFb\}$$

is the minimal continuous relation which extends F.

Proof. First note that, for any $W \subseteq B$, $F^{-}(W) \subseteq (F^{\lhd})^{-}(W)$. Now, it is easy to check that all of the conditions for $F^{\lhd}$ being a continuous relation hold and that it is the smallest continuous relation containing F. It can be useful to observe that *function weak-saturation* is necessary to prove the validity of *function convergence.*

Corollary 2.12 *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be formal topologies (covers) and F be a continuous relation from $\mathcal{A}$ to $\mathcal{B}$ and G be a continuous relation from $\mathcal{B}$ to $\mathcal{C}$. Then the relation $G * F$ defined by setting, for any $a \in A$ and $c \in C$,*

$$a\; G * F\; c \text{ if and only if } a\; (G \circ F)^{\lhd}\; c$$

is a continuous relation from $\mathcal{A}$ to $\mathcal{C}$.

Proof. After lemma 2.10 and proposition 2.11, one has only to prove that relation composition satisfies *function weak-saturation* which follows easily.

The next lemmas will be useful in the following.

Lemma 2.13 *Let $\mathcal{A}$, $\mathcal{B}$ be formal topologies (covers) and F be a continuous relation from $\mathcal{A}$ to $\mathcal{B}$. Then*

$$F^{\lhd} = F$$

Lemma 2.14 *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be formal topologies (covers), F be a relation between $\mathcal{A}$ and $\mathcal{B}$ which satisfies* function continuity *and G be a relation between $\mathcal{B}$ and $\mathcal{C}$. Then*

$$(G \circ F^{\lhd})^{\lhd} = (G \circ F)^{\lhd} \text{ and } (G^{\lhd} \circ F)^{\lhd} = (G \circ F)^{\lhd}$$

Proof. The proof only requires to expand the definitions. Let us only observe that *function continuity* of F is necessary to prove that $(G^{\lhd} \circ F)^{\lhd} \subseteq (G \circ F)^{\lhd}$.

We can now prove the main theorem of this section.

Theorem 2.15 *Formal topologies (covers) and continuous relations form a category* FTop *(*FTop^-*) where the operation of composition between continuous relations is* $- * -$ *and the cover relation is its unit.*

Proof. We only need to show that the operation $- * -$ between continuous relations is associative which is an immediate corollary of the previous lemma.

It is now trivial to realize that the cover relation is a continuous relation and we can use the previous corollary 2.12 to shorten the proof that the cover relation is the identity with respect to the operation $- * -$.

The category FTop of formal topologies is impredicatively equivalent to the category OpLoc of open locales [JT84] (for a recent proof see [Neg02]) while the category FTop^- of formal covers is impredicatively equivalent to the category Loc of locales as defined in [Joh82] (for a proof see [BS01]).

2.3.1 Continuous relations and the positivity predicate

In the definition of continuous relation the positivity predicate is not involved. However, there are specific properties that depend on its presence.

Lemma 2.16 *Let* F *be a continuous relation from the formal topology* $\mathcal{A}$ *to* $\mathcal{B}$*. Then, for any* $a \in A$ *and* $b \in B$*,* F *satisfies the following condition*

$$\text{(function monotonicity)} \quad \frac{\mathsf{Pos}_{\mathcal{A}}(a) \qquad aFb}{\mathsf{Pos}_{\mathcal{B}}(b)}$$

Proof. Let us suppose aFb. Then, the *positivity axiom* $b \lhd_{\mathcal{B}} b^+$ yields, by *function continuity*, that $a \lhd_{\mathcal{A}} F^-(b^+)$. Hence, by *monotonicity* of the cover relation, $\mathsf{Pos}_{\mathcal{A}}(a)$ yields that there exists some element $c \varepsilon F^-(b^+)$ such that $\mathsf{Pos}_{\mathcal{A}}(c)$ holds. Therefore, there exists $y \varepsilon b^+$ such that cFy. But $y \varepsilon b^+$ yields that $y = b$ and $\mathsf{Pos}_{\mathcal{B}}(y)$ hold and thus $\mathsf{Pos}_{\mathcal{B}}(b)$ follows.

The condition of *function monotonicity* above was firstly part of the original definition of continuous relation in [Vir90] as a consequence of its presence in the definition of frame morphisms between formal topology in [Sam87], but it was later recognized to be derivable in [Neg02].

Another important consequence of the conditions defining a continuous relation is that two relations are equal if they are equal on positive elements. Before proving this fact, let us observe that the following lemma holds.

Lemma 2.17 *Let* F *be a continuous relation from the formal topology* $\mathcal{A}$ *to* $\mathcal{B}$*. Then, for any* $a \in A$ *and* $b \in B$*,* $\mathsf{Pos}_{\mathcal{A}}(a) \to a\,F\,b$ *if and only if* $(\forall x \varepsilon a^+)\; x\,F\,b$*.*

Now, the following lemma is immediate.

Lemma 2.18 *Let* F *be a continuous relation from the formal topology* $\mathcal{A}$ *to* $\mathcal{B}$*. Then, for any* $a \in A$ *and* $b \in B$*,* F *satisfies the following condition*

$$\text{(function positivity)} \quad \frac{\mathsf{Pos}_{\mathcal{A}}(a) \to aFb}{aFb}$$

Proof. After the previous lemma we know that $\mathsf{Pos}_{\mathcal{A}}(a) \to aFb$ yields that, for all $x \varepsilon a^+$, xFb. Hence $a \lhd_{\mathcal{A}} a^+$ yields aFb by *function saturation.*

The condition of *function positivity* was introduced to force the faithfulness of the functor $\mathsf{Pt}(-)$ when working with Information Bases (see [SVV96]). In that context *function saturation* cannot be used and hence *function positivity* is part of the definition of continuous relation together with *function weak-saturation.*

At the end of this section, let us recall that in the literature there are also alternative presentations of the category of formal topologies where a continuous relation is defined by requiring all of the conditions in definition 2.8 except for *function saturation* (see for instance [GS02]). In this case one is forced to state that two continuous relations F and G are equal if $F^{\lhd}$ and $G^{\lhd}$ are equal. We prefer the approach presented here because we think that being able to use an equality between continuous relations not depending on the cover relation is more natural and allows a simpler technical treatment which becomes crucial in dealing with exponentiability.

2.3.2 Continuous relations and formal points

In this section we show that there is a bijective correspondence between the collection of the global elements of a formal topology (cover) $\mathcal{A}$ and the collection $\mathsf{Pt}(\mathcal{A})$ of the formal points of $\mathcal{A}$. First of all, let us recall how to define a terminal object $\mathcal{T}$ in FTop.

Lemma 2.19 *Let $\mathcal{T} \equiv (T, \lhd_T, \mathsf{Pos}_T)$ be the formal topology such that $T \equiv \{\top\}$ is a one element set, the cover relation is defined by setting, for any $a \in \{\top\}$ and any subset U of $\{\top\}$,*

$$a \lhd_T U \equiv a \varepsilon U$$

and the positivity predicate is defined by setting, for any $a \in \{\top\}$,

$$\mathsf{Pos}_T(a) \equiv \mathsf{True}$$

Then, $\mathcal{T}$ is a terminal object in FTop, that is, for any formal topology $\mathcal{A}$, the total relation $!_A$ defined by setting, for any $a \in A$, $a\ !_A \top$, is the only continuous relation from $\mathcal{A}$ to $\mathcal{T}$.

It is trivial to see that $(T, \lhd_T)$, where the set T and the cover relation $\lhd_T$ are defined as above, is a terminal object, that we will continue to call $\mathcal{T}$, in the category FTop^-.

We can now state the following theorem.

Theorem 2.20 *Let $\mathcal{A}$ be a formal topology (cover). Then there is a bijective correspondence between the collection $\mathsf{Pt}(\mathcal{A})$ of the formal points of $\mathcal{A}$ and the continuous relations from $\mathcal{T}$ to $\mathcal{A}$.*

Proof. Let us suppose that α is a formal point of the formal topology (cover) $\mathcal{A}$. Then the continuous relation from $\mathcal{T}$ to $\mathcal{A}$ associated with α is defined by setting, for any $u \in \{\top\}$ and any $a \in A$,

$$uF_\alpha a \equiv a\varepsilon\alpha$$

On the other hand, given any continuous relation F from $\mathcal{T}$ to $\mathcal{A}$, we can define a formal point α_F of $\mathcal{A}$ by setting, for any $a \in A$,

$$a\varepsilon\alpha_F \equiv \top Fa$$

It is now completely trivial to see that the two constructions are one the inverse of the other.

3 From topologies to covers and back

In this section we study the relations between the categories FTop of formal topologies and the category FTop^- of formal covers.

3.1 From formal topologies to formal covers

It is obvious that formal covers and formal topologies are closely connected structures. Indeed, any formal topology is a formal cover and any continuous relation between formal topologies is a continuous relation between formal covers as well. Thus, an obvious way to move from a formal topology to a formal cover is a forgetful functor $\mathcal{I}$ from the category FTop to the category FTop^- which just forgets the positivity predicate. It is worth noting that $\mathcal{I}$ is an embedding, that is, it is full and faithful.

The way back from FTop^- to FTop is not as easy: indeed we do not have a general solution and we know what to do only for inductively generated formal covers and inductively generated formal topologies that we recall in the next section.

3.2 Inductively generated formal topologies

One of the main tools in formal topology is the inductive generation of the cover since this allows to develop proofs by induction. The problem of generating inductively formal covers has been dealt with and solved in [CSSV03].

An inductive definition of a cover starts from some axioms, which at the moment we assume to be given by means of any relation $R(a, U)$, for $a \in A$ and $U \subseteq A$. We thus want to generate the least cover $\lhd_R$ such that $R(a, U)$ yields $a \lhd_R U$.

The first naive idea for an inductive generation of a cover relation is to use the conditions appearing in the definition of formal cover like rules. But such conditions, though written in the shape of rules, must be understood as requirements of validity, that is, if the premises hold then also the conclusion must hold. As they stand, they are by no means acceptable rules to generate inductively a cover relation. For instance, the operation $U \downarrow V$ on subsets, which occurs in the conclusion of $\downarrow$-*right*, is not even well defined unless we already have a complete knowledge of the cover.

Another problem is that admitting *transitivity* as acceptable rule for an inductive definition is equivalent to a fix-point principle, which does not have a predicative justification (see [CSSV03] for a detailed discussion of this topic). Thus we cannot accept all the possible infinitary propositions $R(a, U)$ and we have to impose some constraints.

We will solve the problem of generating the minimal formal topology which satisfies some given axioms in three steps. First we will show how to generate an infinitary relation which satisfies only *reflexivity* and *transitivity*, then we will extend this result in order to generate a cover relation and finally we will generate a formal topology.

As regard to the first step, the solution proposed in [CSSV03] for the impredicativity problem due to the *transitivity* condition is generating an infinitary relation satisfying *reflexivity* and *transitivity* only when the condition $R(a, U)$ can be expressed by using an *axiom-set*, that is, a family $I(a)$ of sets for $a \in A$ and a family $C(a, i)$ of subsets of A for $a \in A$ and $i \in I(a)$, whose intended meaning is to state that, for all $i \in I(a)$, a is covered by $C(a, i)$. Indeed, in this case an infinitary relation satisfying such an axiom-set, *reflexivity* and *transitivity* can be inductively generated by using the following rules:

$$\text{(reflexivity)}\ \frac{a \,\varepsilon\, U}{a \lhd U} \qquad \text{(infinity)}\ \frac{i \in I(a) \qquad C(a, i) \lhd U}{a \lhd U}$$

After solving the problem of generating the minimal infinitary relation satisfying a given axiom-set, *reflexivity* and *transitivity* one can strengthen the previous rules to new ones which allow to generate a cover relation, that is, an infinitary relation which satisfies *reflexivity*, *transitivity* and also $\downarrow$-*right*. In fact, in order to satisfy $\downarrow$-*right*, a possibility is to add a pre-order in the definition of formal cover expressing what, in the concrete case, is the inclusion between two basic open subsets.

Definition 3.1 ($\leq$-formal cover) *A* $\leq$-formal cover *is a structure* $(A, \leq, \lhd)$ *where* A *is a set,* $\leq$ *is a pre-order relation between elements of* A*, that is,* $\leq$ *is reflexive and transitive, and* $\lhd$ *is a relation between elements and subsets of* A *which satisfies* reflexivity*,* transitivity *and the following two conditions*

$$(\leq\text{-left})\ \frac{a \leq b \quad b \lhd U}{a \lhd U} \qquad (\leq\text{-right})\ \frac{a \lhd U \quad a \lhd V}{a \lhd U \downarrow_{\leq} V}$$

where $U \downarrow_{\leq} V \equiv \{c \in A \mid (\exists u \varepsilon U)\ c \leq u\ \&\ (\exists v \varepsilon V)\ c \leq v\}$.

It is straightforward to verify that $\leq$-*left* and $\leq$-*right* are valid in any concrete topological space under the intended interpretation. And only a bit more work is required to prove that any $\leq$-formal cover is a formal cover.

The proof that any formal cover is equivalent to a suitable $\leq$-formal cover is even more trivial. Indeed, it is sufficient to define an order relation between elements of A by setting $a \leq b$ if and only if $a \lhd \{b\}$ and it is obvious that all of the required conditions are satisfied.

Thus, in order to be able to generate inductively a formal cover we need only to be able to generate inductively a $\leq$-formal cover. So, let us suppose that we have a set A, an order relation $\leq$ between elements of A and a given axiom-set $I(-)$ and $C(-,-)$ and that we want to generate a $\leq$-formal cover over A. To this aim we can use the method proposed in [CSSV03] and generate by induction a cover relation which respects the given axiom-set, *reflexivity*, *transitivity*, $\leq$-*left* and $\leq$-*right* by using *reflexivity*, $\leq$-*left* and *infinity* as inductive rules. The only constraint for its applicability is the validity of the following localization condition which guarantees that $\leq$-*right* holds.

Definition 3.2 (Localization condition) *Let A be any set, $\leq$ be a pre-order relation on A and $I(-)$ and $C(-,-)$ be an axiom-set for a cover relation over A. Then such an axiom-set is* localized *if, for any $a \leq c$ and any $i \in I(c)$, there exists $j \in I(a)$ such that $C(a,j) \subseteq \{a\} \downarrow_{\leq} C(c,i)$.*

However, if we want to obtain a formal topology also a positivity predicate has to be provided. To this aim, let us say that a predicate $\mathsf{Pos}(-)$ satisfies $\leq$-*monotonicity* if, for any $a, b \in A$,

$$(\leq\text{-monotonicity}) \quad \frac{\mathsf{Pos}(a) \qquad a \leq b}{\mathsf{Pos}(b)}$$

holds and that it satisfies *axiom monotonicity* if, for all the axioms in the axiom-set $I(-)$ and $C(-,-)$ and for any $a \in A$,

$$(\text{axiom monotonicity}) \quad \frac{\mathsf{Pos}(a) \quad i \in I(a)}{(\exists y \varepsilon C(a,i))\ \mathsf{Pos}(y)}$$

holds.

Now, given any axiom-set $I(-)$, $C(-,-)$, it is possible to define a predicate $\mathsf{Pos}(-)$ which satisfies both $\leq$-*monotonicity* and *axiom monotonicity* by simply considering these two conditions as co-inductive rules (see the appendix). Hence, given any axiom-set, we will use such a predicate $\mathsf{Pos}(-)$ as a positivity predicate.

Finally, given any axiom-set $I(-)$ and $C(-,-)$ and any predicate enjoying $\leq$-*monotonicity* and *axiom monotonicity*, after lemma 2.5, we can always force the validity of the *positivity* condition by adding a single axiom schema stating that, for any $a \in A$, a is covered by the set a^+, namely, we can define a new axiom-set $I'(-)$ and $C'(-,-)$ by setting, for any $a \in A$,

$$I'(a) \equiv I(a) \cup \{\star\} \qquad C'(a,i) \equiv \begin{cases} a^+ & \text{if } i = \star \\ C(a,i) & \text{otherwise} \end{cases}$$

Note that if *axiom monotonicity* holds for $I(-)$ and $C(-,-)$ then it continues to hold also for $I'(-)$ and $C'(-,-)$ since $\mathsf{Pos}(a)$ clearly yields that there exists an element $c \varepsilon a^+$ such that $\mathsf{Pos}(c)$ holds.

We are finally ready to use the method in [CSSV03] to generate a $\leq$-formal topology, namely, a $\leq$-formal cover with a positivity predicate which enjoys *monotonicity* and *positivity*.

Theorem 3.3 *Let A be a set and $I(-)$ and $C(-,-)$ be a localized axiom-set for a cover over A. Then a $\leq$-formal topology on A can be defined by using* reflexivity, $\leq$-left *and* infinity *as inductive rules on the axiom-set $I'(-)$ and $C'(-,-)$ defined as above.*

Proof. In order to apply the method in [CSSV03], it is necessary to show that the axiom-set $I'(-)$ and $C'(-,-)$ is localized. So, suppose that $a \leq c$ and that we are considering the *positivity axiom* for c. Then, we have to show that there exists an index $j \in I'(a)$ such that $C(a,j) \subseteq \{a\} \downarrow_{\leq} c^{+}$. The correct choice for j is now the index for a^{+} since we can prove that $a^{+} \subseteq \{a\} \downarrow_{\leq} c^{+}$. Indeed, let us assume that $e \varepsilon a^{+}$. Then, both $e = a$ and $\mathsf{Pos}(e)$ follow and hence we first obtain $\mathsf{Pos}(a)$ by logic and then $\mathsf{Pos}(c)$ by $\leq$-*monotonicity.* So $c^{+} = \{c\}$ and hence both $e \leq a$ and $e \leq c$ hold since $e = a$, that is, $e \varepsilon \{a\} \downarrow_{\leq} c^{+}$.

Moreover, given any predicate which satisfies $\leq$-*monotonicity* and *axiom monotonicity* it is immediate to prove by induction on the length of the proof of $a \lhd U$ that, if $\mathsf{Pos}(a)$ and $a \lhd U$ hold, then there exists an element $u \varepsilon U$ such that $\mathsf{Pos}(u)$ holds, namely, that *monotonicity* holds.

Finally, the *positivity* condition clearly holds for such a cover relation and such a predicate since it is built in the axioms from which the cover relation is generated.

Note that this theorem is a corollary of a more general result in [Val04] which shows how to generate by co-induction a binary positivity predicate with proper axioms.

In the end, let us introduce the following definition.

Definition 3.4 (Inductively generated formal topology) *A formal cover $\mathcal{A} = (A, \lhd_{\mathcal{A}})$ is inductively generated if and only if there exist a pre-order $\leq$ and a localized axiom-set $I(-)$ and $C(-,-)$ such that $\lhd_{\mathcal{A}}$ coincides with the cover inductively generated using* reflexivity, $\leq$-left *and* infinity *as inductive rules.*

A formal topology $\mathcal{A} = (A, \lhd_{\mathcal{A}}, \mathsf{Pos}_{\mathcal{A}})$ is inductively generated if and only if the formal cover $\mathcal{I}(\mathcal{A}) = (A, \lhd_{\mathcal{A}})$ is inductively generated.

3.2.1 Points and maps of inductively generated formal topologies

If we restrict our attention to inductively generated $\leq$-formal covers or inductively generated $\leq$-formal topologies we can simplify many of the definitions given in the previous sections. To begin with, the definition of formal point can be simplified as follows.

Definition 3.5 (Formal point) *Let $\mathcal{A}$ be an inductively generated $\leq$-formal topology (cover) with axiom-set $I(-)$,$C(-,-)$. Then, an inhabited subset α of*

A is a formal point *if, for any $a, b \in A$ and any $U \subseteq A$, it satisfies the following conditions:*

$$\text{(point $\leq$-convergence)} \qquad \frac{a\varepsilon\alpha \quad b\varepsilon\alpha}{(\exists c\varepsilon a \downarrow_{\leq_{\mathcal{A}}} b)\ c\varepsilon\alpha}$$

$$\text{(point left-closure)} \qquad \frac{a\varepsilon\alpha \quad a \leq_{\mathcal{A}} b}{b\varepsilon\alpha}$$

$$\text{(point inductive splitness)} \qquad \frac{a\varepsilon\alpha \quad i \in I(a)}{(\exists y\varepsilon C(a,i))\ y\varepsilon\alpha}$$

Indeed, after observing that in the case of an inductively generated $\leq$-formal topology (cover), $a \downarrow_{\leq_{\mathcal{A}}} b \lhd_{\mathcal{A}} a \downarrow b$ and $a \downarrow b \lhd_{\mathcal{A}} a \downarrow_{\leq_{\mathcal{A}}} b$, it is clear that the conditions in the definition above are consequences of the ones in definition 2.7. On the other hand, it is possible to prove that a subset α which satisfies the conditions stated here satisfies also *point splitness*, namely, $a\varepsilon\alpha$ and $a \lhd_{\mathcal{A}} U$ yield $(\exists u\varepsilon U)\ u\varepsilon\alpha$, by developing a proof by induction on the length of the derivation of $a \lhd_{\mathcal{A}} U$.

The general conditions on a continuous relation that we presented in section 2.3 can be simplified when we are dealing with morphisms between inductively generated formal topologies (covers).

Lemma 3.6 *Let $\mathcal{A}$ and $\mathcal{B}$ be an inductively generated formal topologies (covers) and suppose that $I(-)$,$C(-,-)$ and $J(-)$, $D(-,-)$ are the axiom-sets for $\mathcal{A}$ and $\mathcal{B}$ respectively. Then, a relation F between A and B is continuous if and only if it satisfies the following conditions:*

$$\text{(function totality)} \qquad A \lhd_{\mathcal{A}} F^{-}(B)$$

$$\text{(function $\leq$-convergence)} \qquad \frac{aFb \qquad aFd}{a \lhd_{\mathcal{A}} F^{-}(b \downarrow_{\leq_{\mathcal{B}}} d)}$$

$$\text{(function $\leq$-saturation)} \qquad \frac{a \leq_{\mathcal{A}} c \quad c\,F\,b}{a\,F\,b}$$

$$\text{(function axiom-saturation)} \qquad \frac{i \in I(a) \quad (\forall x\varepsilon C(a,i))\ x\,F\,b}{a\,F\,b}$$

$$\text{(function $\leq$-continuity)} \qquad \frac{a\,F\,b \quad b \leq_{\mathcal{B}} d}{a\,F\,d}$$

$$\text{(function axiom-continuity)} \qquad \frac{a\,F\,b \quad j \in J(b)}{a \lhd_{\mathcal{A}} F^{-}(D(b,j))}$$

Proof. *Function $\leq$-saturation* and *function axiom-saturation* are obvious consequences of *reflexivity*, *$\leq$-left* and *function saturation*. On the other hand, let us suppose that $a \lhd_{\mathcal{A}} W$ and $(\forall x\varepsilon W)\ xFb$; then one can prove aFb by induction on the length of the derivation of $a \lhd_{\mathcal{A}} W$.

Moreover, it is obvious that *function axiom-continuity* is an instance of *function continuity* and *function $\leq$-continuity* is an immediate consequences of *reflexivity*, *$\leq$-left*, *function continuity* and *function saturation*. On the other hand, let us suppose aFb and $b \lhd_{\mathcal{B}} V$. Then, *function continuity* can be derived from the conditions stated here by reasoning by induction on the length of the derivation of $b \lhd_{\mathcal{B}} V$.

Finally, recall that for any inductively generated formal topology (cover) $\mathcal{B}$, if $V_1, V_2 \subseteq B$ then $V_1 \downarrow V_2 \lhd_{\mathcal{B}} V_1 \downarrow_{\leq_{\mathcal{B}}} V_2$ and $V_1 \downarrow_{\leq_{\mathcal{B}}} V_2 \lhd_{\mathcal{B}} V_1 \downarrow V_2$. Thus both $F^-(V_1 \downarrow V_2) \lhd_{\mathcal{A}} F^-(V_1 \downarrow_{\leq_{\mathcal{B}}} V_2)$ and $F^-(V_1 \downarrow_{\leq_{\mathcal{B}}} V_2) \lhd_{\mathcal{A}} F^-(V_1 \downarrow V_2)$ follows by lemma 2.9 which uses only *function continuity*. Hence the equivalence between *function convergence* and *function $\leq$-convergence* follows immediately by *transitivity*.

As a consequence of this lemma we get a slightly modified version of lemma 2.11 that we will use in the following.

Lemma 3.7 *Let $\mathcal{A}$ and $\mathcal{B}$ be two inductively generated formal topologies (covers) and suppose that F is a relation which satisfies all of the conditions for a continuous relation except for* function saturation *which is replaced by* function $\leq$-saturation. *Then*

$$aF^{\lhd}b \equiv a \lhd_{\mathcal{A}} \{c \in A \mid cFb\}$$

is the minimal continuous relation which extends F.

Proof. The proof goes on exactly as the one for lemma 2.11 except that here *function $\leq$-saturation* can be used to prove *function $\leq$-convergence* where there *function weak-saturation* was used to prove *function convergence*.

Finally, let us name the sub-categories of FTop and FTop^- whose objects are inductively generated formal topologies and inductively generated formal covers.

Definition 3.8 *We call $\mathsf{FTop_i}$ ($\mathsf{FTop_i}^-$) the full subcategory of FTop (FTop^-) whose objects are inductively generated formal topologies (covers).*

Note that from the impredicative point of view $\mathsf{FTop_i}$ is equivalent to FTop and $\mathsf{FTop_i}^-$ is equivalent to FTop^-; indeed, in this case, every formal topology (cover) $\mathcal{A}$ is inductively generated by the axiom-set obtained by considering the whole cover relation as an axiom-set indexed on $\mathcal{P}(A)$. By contrast, from a predicative point of view, $\mathsf{FTop_i}$ and FTop are not equivalent because there are formal topologies which can not be generated by induction (see the last section of [CSSV03]).

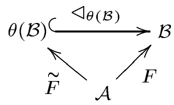
3.3 From formal covers to formal topologies

We are now ready to go back from $\mathsf{FTop_i}^-$ to $\mathsf{FTop_i}$, that is, to recover a suitable formal topology from a given inductively generated formal cover. Indeed, we can prove the following theorem.

Theorem 3.9 *Let $\mathcal{A}$ be an inductively generated formal cover. Then there exists a formal topology $\theta(\mathcal{A})$ such that*

1. *(Identity on formal topologies) if $\mathcal{A}$ is already an inductively generated formal topology then*
$$\theta(\mathcal{A}) = \mathcal{A}$$

2. *(Universal property) for any formal cover $\mathcal{B}$ of $\mathsf{FTop_i}^-$, $\theta(\mathcal{B})$ is the finest formal topology which approximates $\mathcal{B}$, namely, the cover relation $\lhd_{\theta(\mathcal{B})}$ of $\theta(\mathcal{B})$ is a continuous relation from $\theta(\mathcal{B})$ to $\mathcal{B}$ and for any inductively generated formal topology $\mathcal{A}$ and any continuous relation F from $\mathcal{A}$ to $\mathcal{B}$ there exists a unique continuous relation $\widetilde{F}$ from $\mathcal{A}$ to $\theta(\mathcal{B})$ such that the following diagram in $\mathsf{FTop_i}^-$ commutes:*



The rest of this section is devoted to the proof of this theorem.

After section 3.2, the definition of $\theta(\mathcal{A})$ on the inductively generated formal cover $\mathcal{A}$ is almost immediate. Indeed, let us suppose that $I(-)$ and $C(-,-)$ is the axiom-set for $\mathcal{A}$. Then, $\theta(\mathcal{A})$ is the formal topology that we obtain according to theorem 3.3.

Lemma 3.10 *Let $\mathcal{A}$ be an inductively generated formal cover. Then, for every $a \in A$ and $U \subseteq A$, if $a \lhd_{\mathcal{A}} U$ then $a \lhd_{\theta(\mathcal{A})} U$.*

Proof. The result follows immediately by induction on the length of the generation of $a \lhd_{\mathcal{A}} U$. Indeed, any rule for $\lhd_{\mathcal{A}}$ can be applied as well in order to generate $\lhd_{\theta(\mathcal{A})}$ and all of the axioms for $\lhd_{\mathcal{A}}$ are also axioms for $\lhd_{\theta(\mathcal{A})}$.

Lemma 3.10 proves the key step to show that the formal topology $\theta(\mathcal{A})$ embeds into the formal cover $\mathcal{A}$. An immediate corollary of this fact is the following lemma which is part of the proof of point (2) of theorem 3.9.

Lemma 3.11 *Let $\mathcal{B}$ be any inductively generated formal cover. Then the cover relation $\lhd_{\theta(\mathcal{B})}$ of the formal topology $\theta(\mathcal{B})$ is a continuous relation from $\theta(\mathcal{B})$ to $\mathcal{B}$.*

Now, we show how to define a continuous relation $\theta(F)$ from $\theta(\mathcal{A})$ to $\theta(\mathcal{B})$ for any continuous relation F from the inductively generated formal cover $\mathcal{A}$ to the inductively generated formal cover $\mathcal{B}$. Let us first prove the following lemma.

Lemma 3.12 *Let F be a continuous relation from the inductively generated formal cover $\mathcal{A}$ to the inductively generated formal cover $\mathcal{B}$. Then F is a relation from $\theta(\mathcal{A})$ to $\theta(\mathcal{B})$ which satisfies* function totality, function convergence, function continuity *and* function $\leq$-saturation.

Proof. Let us check that the various conditions hold. Note that, thanks to lemma 3.6, it is sufficient to prove *function $\leq$-convergence*, *function $\leq$-continuity* and *function axiom-continuity* instead that *function convergence* and *function continuity*.

1. (function totality) Immediate consequence of *function totality* for F.
2. (function $\leq$-convergence) Let us suppose that $a\,F\,b$ and $a\,F\,d$ hold. Then $a \lhd_{\mathcal{A}} F^{-}(b \downarrow_{\leq_{\mathcal{B}}} d)$ follows by *function $\leq$-convergence* for F and hence we obtain $a \lhd_{\theta(\mathcal{A})} F^{-}(b \downarrow_{\leq_{\mathcal{B}}} d)$ by lemma 3.10.
3. (function $\leq$-continuity) Let us suppose that $a\,F\,b$ and $b \leq_{\mathcal{B}} d$. Then $a\,F\,d$ follows by *function $\leq$-continuity* for F.
4. (function axiom-continuity) Let us suppose that $a\,F\,b$ holds and $j \in J(b)$ is an index for an axiom for the inductively generated formal topology $\theta(\mathcal{B})$. Then we have to show that $a \lhd_{\theta(\mathcal{A})} F^{-}(C(b,j))$. We argue according to the shape of the axiom for $\lhd_{\theta(\mathcal{B})}$ that we are considering.

   - Let j be an index for an axiom for the cover $\lhd_{\mathcal{B}}$. Then, $a\,F\,b$ yields immediately that $a \lhd_{\mathcal{A}} F^{-}(C(b,j))$ by *function axiom-continuity* for F and hence $a \lhd_{\theta(\mathcal{A})} F^{-}(C(b,j))$ follows by lemma 3.10.
   - Let $C(b,j)$ be $b^{+} \equiv \{y \in B \mid y = b \;\&\; \mathsf{Pos}_{\theta(\mathcal{B})}(b)\}$. Then we have to show that

     $$a \lhd_{\theta(\mathcal{A})} F^{-}(b^{+})$$

     In order to obtain this result it is sufficient to show that

     $$\text{(function monotonicity)} \quad \frac{\mathsf{Pos}_{\theta(\mathcal{A})}(a) \quad a\,F\,b}{\mathsf{Pos}_{\theta(\mathcal{B})}(b)}$$

     holds. Indeed, suppose that *function monotonicity* holds and assume that also $\mathsf{Pos}_{\theta(\mathcal{A})}(a)$ holds. Then $a\,F\,b$ yields $\mathsf{Pos}_{\theta(\mathcal{B})}(b)$ and hence $b \varepsilon b^{+}$ holds. Thus, $(\exists y \varepsilon b^{+})\, a\,F\,y$ holds and hence $a \varepsilon F^{-}(b^{+})$ follows. Then $a \lhd_{\theta(\mathcal{A})} F^{-}(b^{+})$ follows by *reflexivity* for $\lhd_{\theta(\mathcal{A})}$ and hence we can discharge the assumption $\mathsf{Pos}_{\theta(\mathcal{A})}(a)$ by *positivity*.
     So, let us prove that *function monotonicity* holds. We argue by co-induction. Indeed, let us set

     $$Q(y) \equiv (\exists x \in A)\, \mathsf{Pos}_{\theta(\mathcal{A})}(x) \;\&\; x\,F\,y$$

     Then $\mathsf{Pos}_{\theta(\mathcal{A})}(a)$ and $a\,F\,b$ yield $Q(b)$ and hence in order to conclude $\mathsf{Pos}_{\theta(\mathcal{B})}(b)$ by co-induction it is sufficient to show that $Q(-)$ satisfies *$\leq$-monotonicity* and *axiom-monotonicity* for the axioms of the cover $\lhd_{\mathcal{B}}$.

     - ($\leq$-monotonicity) Suppose that $Q(y)$ and $y \leq z$ hold. Then there exists $x \in A$ such that $\mathsf{Pos}_{\theta(\mathcal{A})}(x)$ and $x\,F\,y$ hold; hence $x\,F\,z$ follows by *function $\leq$-continuity* for F.

– (axiom monotonicity) Suppose that $Q(y)$ holds and that $j \in J(y)$ is an index for an axiom for the cover relation $\lhd_{\mathcal{B}}$. Then there exists $x \in A$ such that $\mathsf{Pos}_{\theta(\mathcal{A})}(x)$ and $x\,F\,y$ hold. Since we are considering an axiom for the cover relation $\lhd_{\mathcal{B}}$, we know that if $x\,F\,y$ holds then $x \lhd_{\mathcal{A}} F^{-}(C(y,j))$ follows by *function axiom-continuity* of F. Hence, $x \lhd_{\theta(\mathcal{A})} F^{-}(C(y,j))$ follows by lemma 3.10 and thus $\mathsf{Pos}_{\theta(\mathcal{A})}(x)$ yields by *monotonicity* that there exists $x' \varepsilon F^{-}(C(y,j))$ such that $\mathsf{Pos}_{\theta(\mathcal{A})}(x')$. Then there exists $v \varepsilon C(y,j)$ such that $x'\,F\,v$, that is, $(\exists v \varepsilon C(y,j))\, Q(v)$ holds.

5. (function $\leq$-saturation) Suppose that $a \leq c$ and $c\,F\,b$. Then $a\,F\,b$ immediately follows by *function $\leq$-saturation* for F.

After this result, it is clear that the correct definition for $\theta(F)$ is

$$\theta(F) \equiv F^{\lhd_{\theta(\mathcal{A})}}$$

Indeed, lemma 3.7 states that $F^{\lhd_{\theta(\mathcal{A})}}$ is a continuous relation from $\theta(\mathcal{A})$ to $\theta(\mathcal{B})$.

Let us now prove point (1) of theorem 3.9, namely, the fact that θ is the identity on formal topologies.

Lemma 3.13 *For any inductively generated formal topology $\mathcal{A}$, $\theta(\mathcal{A}) = \mathcal{A}$.*

Proof. Let us suppose that $I(-)$ and $C(-,-)$ is the axiom-set for $\mathcal{A}$. In order to obtain the result it is sufficient to prove that the positivity predicate $\mathsf{Pos}_{\theta(\mathcal{A})}$, generated by co-induction according to section 3.2, coincides with $\mathsf{Pos}_{\mathcal{A}}$. Now, note that, for every $a \in A$, $a \lhd_{\mathcal{A}} \{x \in A \mid x = a \;\&\; \mathsf{Pos}_{\mathcal{A}}(x)\}$ and hence $a \lhd_{\theta(\mathcal{A})} \{x \in A \mid x = a \;\&\; \mathsf{Pos}_{\mathcal{A}}(x)\}$ follows by lemma 3.10. Hence, by *monotonicity* for $\lhd_{\theta(\mathcal{A})}$, $\mathsf{Pos}_{\theta(\mathcal{A})}(a)$ yields that there exists $x \in A$ such that both $x = a$ and $\mathsf{Pos}_{\mathcal{A}}(x)$ hold; thus $\mathsf{Pos}_{\mathcal{A}}(a)$ follows by logic, that is, we proved that $\mathsf{Pos}_{\theta(\mathcal{A})}(a)$ yields $\mathsf{Pos}_{\mathcal{A}}(a)$.

Due to maximality of $\mathsf{Pos}_{\theta(\mathcal{A})}(-)$, to prove the other implication it is sufficient to show that $\mathsf{Pos}_{\mathcal{A}}(-)$ satisfies all of the conditions in the co-inductive generation of $\mathsf{Pos}_{\theta(\mathcal{A})}(-)$; and these conditions are valid because $\mathsf{Pos}_{\mathcal{A}}(-)$ is clearly monotone on the axioms of the axiom-set $I(-)$ and $C(-,-)$ and it trivially enjoys *$\leq$-monotonicity.*

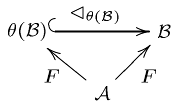
A useful consequence of this lemma is that $\theta(F) = F$ whenever F is a continuous relation whose domain is a formal topology.

Lemma 3.14 *Let $\mathcal{A}$ be an inductively generated formal topology, $\mathcal{B}$ be an inductively generated formal cover and F be a continuous relation from $\mathcal{A}$ to $\mathcal{B}$. Then $\theta(F) = F$.*

Proof. Note that $F \subseteq F^{\lhd_{\theta(\mathcal{A})}} = \theta(F)$ is trivial and hence we have only to prove the other inclusion. So, suppose that $a\,\theta(F)\,b$. Then $a \lhd_{\theta(\mathcal{A})} \{c \in A \mid c\,F\,b\}$ and hence $a \lhd_{\mathcal{A}} \{c \in A \mid c\,F\,b\}$ since $\theta(\mathcal{A}) = \mathcal{A}$ being $\mathcal{A}$ an inductively generated formal topology. Therefore, $a\,F\,b$ follows by *function saturation* for F.

Now, let us prove point (2) of theorem 3.9 expressing the fact that for any inductively generated formal cover $\mathcal{B}$ of $\mathsf{FTop_i}^-$, the cover of the formal topology $\theta(\mathcal{B})$ is the one that best approximates the cover of $\mathcal{B}$. To prove this result let us first prove the following lemma.

Lemma 3.15 *Let $\mathcal{A}$ be an inductively generated formal topology, $\mathcal{B}$ be an inductively generated formal cover and F be a continuous relation from $\mathcal{A}$ to $\theta(\mathcal{B})$. Then $\lhd_{\theta(\mathcal{B})} * F = F$, that is, the following diagram*



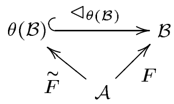
commutes.

Proof. Suppose $a \in A$ and $b \in B$. Then

$$\begin{array}{rll} a \lhd_{\theta(\mathcal{B})} * F\,b & \text{iff} & a \lhd_{\mathcal{A}} \{c \in A \mid c \lhd_{\theta(\mathcal{B})} \circ F\,b\} \\ & \text{iff} & a \lhd_{\mathcal{A}} \{c \in A \mid (\exists y \in B)\, c\,F\,y \;\&\; y \lhd_{\theta(\mathcal{B})} b\} \\ \text{by } \textit{weak-continuity} \text{ of } F & \text{iff} & a \lhd_{\mathcal{A}} \{c \in A \mid c\,F\,b\} \\ \text{by } \textit{fun. saturation} \text{ of } F & \text{iff} & a\,F\,b \end{array}$$

Now, we are ready to prove the following proposition.

Proposition 3.16 *For any inductively generated formal topology $\mathcal{A}$, any inductively generated formal cover $\mathcal{B}$ and any continuous relation F from $\mathcal{A}$ to $\mathcal{B}$ there exists a unique continuous relation $\widetilde{F}$ from $\mathcal{A}$ to $\theta(\mathcal{B})$ such that the following diagram in $\mathsf{FTop_i}^-$ commutes*



Proof. To prove such a universal property let us set

$$\widetilde{F} \equiv F$$

Then $\widetilde{F}$ is well defined since it is a continuous relation from $\mathcal{A}$ to $\theta(\mathcal{B})$. Indeed, we already know that, for any continuous relation F from $\mathcal{A}$ to $\mathcal{B}$, $\theta(F)$ is a continuous relation from $\theta(\mathcal{A})$ to $\theta(\mathcal{B})$ and hence, by lemma 3.14, $\theta(F) = F$ since, by lemma 3.13, $\theta(\mathcal{A}) = \mathcal{A}$. Now, by lemma 3.15, $\lhd_{\theta(\mathcal{B})} * F = F$ is immediate. In order to prove uniqueness, let us suppose that G is a continuous relation from $\mathcal{A}$ to $\theta(\mathcal{B})$ such that $\lhd_{\theta(\mathcal{B})} * G = F$. Then we obtain immediately $F = G$ since $\lhd_{\theta(\mathcal{B})} * G = G$ holds by lemma 3.15.

3.3.1 The formal points of $\theta(\mathcal{A})$

An interesting consequence of the previous results is the fact that the formal points of an inductively generated formal cover $\mathcal{A}$ coincide with the formal points of the formal topology $\theta(\mathcal{A})$, namely, the addiction of a positivity predicate has no effect on the spatial side.

Even if a direct proof is possible, the quickest way to show this result is to use the fact that for any formal cover (with or without positivity predicate) there is a correspondence between the continuous relations from the terminal formal topology $\mathcal{T}$ to $\mathcal{A}$ and the formal points of $\mathcal{A}$ (see section 2.3.2).

Proposition 3.17 *For any inductively generated formal cover $\mathcal{A}$, the collection of formal points of $\theta(\mathcal{A})$ is equal to that one of $\mathcal{A}$, that is,* $\mathsf{Pt}(\theta(\mathcal{A})) = \mathsf{Pt}(\mathcal{A})$.

Proof. It is obvious that any formal point of $\theta(\mathcal{A})$ is also a formal point of $\mathcal{A}$. So, let us prove that every formal point of $\mathcal{A}$ is also a formal point of $\theta(\mathcal{A})$. Let us suppose that α is a formal point of $\mathcal{A}$. Then, by lemma 2.20, which holds also for $\mathsf{FTop_i}^-$ since $\mathsf{FTop_i}^-$ is a full subcategory of FTop^-, there is a continuous relation F_α from $\mathcal{T}$ to $\mathcal{A}$ associated with α. Thus, $\theta(F_\alpha)$ is a continuous relation from $\theta(\mathcal{T})$ to $\theta(\mathcal{A})$. Observe now that, by lemma 3.13, $\theta(\mathcal{T})$ coincide with $\mathcal{T}$, since $\mathcal{T}$ is a formal topology, and hence, by lemma 3.14, $\theta(F_\alpha)$ coincides with F_α. Thus, the formal point $\alpha_{\theta(F_\alpha)}$ of $\theta(\mathcal{A})$ coincides with the formal point α.

3.3.2 Categorical content of previous results: the coreflection

In this section we express in categorical terms the second point in the statement of theorem 3.9. In fact, it states the existence of a coreflection of inductively generated formal covers into inductively generated formal topologies. Let us recall that a coreflection is an adjunction such that the left adjoint is the embedding functor of a subcategory into a category.

Theorem 3.18 (Coreflection) *The embedding functor $\mathcal{I} : \mathsf{FTop_i} \to \mathsf{FTop_i}^-$ has a right adjoint defined as $\theta(\mathcal{A})$ on an inductively generated formal cover $\mathcal{A}$ and as $\theta(F)$ on a continuous relation F between formal covers.*

Proof. For any inductively generated formal cover $\mathcal{B}$ of $\mathsf{FTop_i}^-$, we define the counit component

$$\epsilon_B : \mathcal{I}(\theta(\mathcal{B})) \to B$$

by setting $\epsilon_B \equiv \lhd_{\theta(\mathcal{B})}$. Then, proving the statement above is equivalent to proving the following universal property: for any inductively generated formal topology $\mathcal{A}$ of $\mathsf{FTop_i}$ and any continuous relation F from $\mathcal{I}(\mathcal{A})$ to $\mathcal{B}$ in $\mathsf{FTop_i}^-$ there exists a unique continuous relation $\widetilde{F}$ from $\mathcal{A}$ to $\theta(\mathcal{B})$ in $\mathsf{FTop_i}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{I}(\theta(\mathcal{B})) & \xhookrightarrow{\epsilon_B} & \mathcal{B} \\ {\scriptstyle \mathcal{I}(\widetilde{F})} \nwarrow & & \nearrow {\scriptstyle F} \\ & \mathcal{I}(\mathcal{A}) & \end{array}$$

After recalling that the functor $\mathcal{I}$ is the identity on the objects and morphisms it is clear that this result is implied by proposition 3.16.

Of course, a similar coreflection works for the corresponding categories in locale theory.

Corollary 3.19 (Local Coreflection) *The embedding functor*

$$\mathcal{I} : \mathsf{OpLoc} \to \mathsf{Loc}$$

has a right adjoint.

Proof. The coreflection is obtained by composing the functor θ from $\mathsf{FTop_i}^-$ to $\mathsf{FTop_i}$ with the functors used to prove the impredicative equivalences between OpLoc and $\mathsf{FTop_i}$ and between Loc and $\mathsf{FTop_i}^-$.

4 Structural analysis of $\mathsf{FTop_i}^-$ and $\mathsf{FTop_i}$

In this section we will prove that unary formal covers and unary formal topologies are exponentiable in $\mathsf{FTop_i}^-$ and $\mathsf{FTop_i}$ respectively. To obtain such a result, we first recall the definition of categorical product in $\mathsf{FTop_i}^-$ and $\mathsf{FTop_i}$, then we introduce unary formal covers and unary formal topologies and finally we show how to build exponential objects.

4.1 Categorical product of formal topologies

In this section we recall some basic definitions about the categorical product of two inductively generated formal topologies (covers). First of all, it is immediate to see that the terminal formal topology $\mathcal{T}$ that we introduced in section 2.3.2 is inductively generated.

Lemma 4.1 *Let $\mathcal{T}$ be the terminal formal topology defined in lemma 2.19. Then $\mathcal{T}$ can be generated inductively by using the empty set of axioms and the total order relation.*

Now, let us recall that at present it is still open the question whether FTop is cartesian. Indeed, we are able to define the binary product of formal topologies only by means of an inductive definition and thus only $\mathsf{FTop_i}$ and $\mathsf{FTop_i}^-$ are known to be cartesian (see [CSSV03]). Since no proof of this result appeared in [CSSV03] we show here some details of the proof.

Definition 4.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be two inductively generated formal topologies whose axiom-sets are respectively $I_A(-)$, $C_A(-,-)$ and $I_B(-)$, $C_B(-,-)$. Then we call* binary product *of $\mathcal{A}$ and $\mathcal{B}$ the formal topology $\mathcal{A}\times\mathcal{B}$ over the set $A\times B$, with order relation*

$$(a_1, b_1) \leq_{\mathcal{A}\times\mathcal{B}} (a_2, b_2) \equiv (a_1 \leq_{\mathcal{A}} a_2)\ \&\ (b_1 \leq_{\mathcal{B}} b_2)$$

and positivity predicate

$$\mathsf{Pos}_{\mathcal{A}\times\mathcal{B}}((a,b)) \equiv \mathsf{Pos}_{\mathcal{A}}(a) \ \& \ \mathsf{Pos}_{\mathcal{B}}(b),$$

whose cover relation is inductively generated from the axiom-set

$$\begin{array}{lcl} I((a,b)) & \equiv & I_A(a) + I_B(b) \\ C((a,b),i) & \equiv & \begin{cases} C_A(a,i_a) \times \{b\} & \text{if } i \equiv \mathsf{inl}(i_a) \\ \{a\} \times C_B(b,i_b) & \text{if } i \equiv \mathsf{inr}(i_b) \end{cases} \end{array}$$

One should note that in the previous definition we did not add the *positivity axiom.* In fact, we will prove that it is not necessary. Let us state first the following useful lemma.

Lemma 4.3 *Let $\mathcal{A}$ and $\mathcal{B}$ be inductively generated formal topologies, a be an element of A, b be an element of B, U be a subset of A and V be a subset of B. Then the following conditions are valid:*

$$(1)\ \frac{a \lhd_{\mathcal{A}} U}{(a,b) \lhd_{\mathcal{A}\times\mathcal{B}} U \times \{b\}} \quad (2)\ \frac{b \lhd_{\mathcal{B}} V}{(a,b) \lhd_{\mathcal{A}\times\mathcal{B}} \{a\} \times V} \quad (3)\ \frac{a \lhd_{\mathcal{A}} U \qquad b \lhd_{\mathcal{B}} V}{(a,b) \lhd_{\mathcal{A}\times\mathcal{B}} U \times V}$$

As an immediate corollary of this lemma, the product of two inductively generated formal topologies defined above is a formal topology.

Corollary 4.4 *Let $\mathcal{A}$ and $\mathcal{B}$ be inductively generated formal topologies. Then $\mathcal{A} \times \mathcal{B}$ is a formal topology.*

Proof. First of all note that it is immediate to check that the positivity predicate enjoys both $\leq$- *monotonicity* and *axiom monotonicity* and hence it is monotone with respect to the inductively generated cover.

Moreover, by using the previous lemma it is not difficult to show that the *positivity* condition is satisfied for the product topology as a consequence of its validity in the component topologies.

The pairing and the projection maps are now defined.

Lemma 4.5 *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be inductively generated formal topologies and suppose that F is a continuous relation from $\mathcal{C}$ to $\mathcal{A}$ and G is a continuous relation from $\mathcal{C}$ to $\mathcal{B}$. Then the following relations*

$$\begin{array}{lll} \text{(pairing)} & c\,\langle F,G\rangle\,(a,b) & \equiv \ c\,F\,a \ \&\ c\,G\,b \\ \text{(first proj.)} & (a,b)\,\Pi_1\,c & \equiv \ (a,b) \lhd_{\mathcal{A}\times\mathcal{B}} \{(x,y) \in A \times B \mid x \lhd_{\mathcal{A}} c\} \\ \text{(second proj.)} & (a,b)\,\Pi_2\,d & \equiv \ (a,b) \lhd_{\mathcal{A}\times\mathcal{B}} \{(x,y) \in A \times B \mid y \lhd_{\mathcal{B}} d\} \end{array}$$

are continuous and the following equations hold

$$\begin{array}{llcl} \text{(first projecting equation)} & \Pi_1 * \langle F,G\rangle & = & F \\ \text{(second projecting equation)} & \Pi_2 * \langle F,G\rangle & = & G \\ \text{(surjective pairing)} & \langle \Pi_1 * H, \Pi_2 * H\rangle & = & H \end{array}$$

for any continuous relation H from $\mathcal{C}$ to $\mathcal{A} \times \mathcal{B}$.

Proof. The proof that *pairing* and the *projections* are continuous relation is straightforward and hence we prove here only that the required equations hold.

Let us first notice that, for any $c \in C$, $a \in A$ and $b \in B$ and for any continuous relation H from $\mathcal{C}$ to $\mathcal{A} \times \mathcal{B}$,

$$\begin{array}{lll} (1) \quad c\,\Pi_1 * H\; a & \Rightarrow & c \lhd_{\mathcal{C}} H^{-}(\{(x,y) \in A \times B \mid x \lhd_{\mathcal{A}} a\}) \\ (2) \quad c\,\Pi_2 * H\; b & \Rightarrow & c \lhd_{\mathcal{C}} H^{-}(\{(x,y) \in A \times B \mid y \lhd_{\mathcal{B}} b\}) \end{array}$$

We can now proceed with the proof of validity of the equations.

- (first projecting equation) We have to prove that $\Pi_1 * \langle F, G\rangle = F$, namely, for any $c \in C$ and $a \in A$, $c\,\Pi_1 * \langle F, G\rangle\; a$ if and only if $c\,F\,a$. The right to left implication can be proved as follows. Suppose that $c\,F\,a$ holds. Then $c \lhd_{\mathcal{C}} F^{-}(a)$ and hence $c \lhd_{\mathcal{C}} F^{-}(a) \downarrow_{\leq} G^{-}(B)$ since $c \lhd_{\mathcal{C}} G^{-}(B)$ holds by *function totality*. Observe now that

  $$F^{-}(a) \downarrow_{\leq} G^{-}(B) \lhd_{\mathcal{C}} (\Pi_1 * \langle F, G\rangle)^{-}(a)$$

  Indeed, if $x \varepsilon F^{-}(a) \downarrow_{\leq} G^{-}(B)$ then $x\,F\,a$ and $x\,G\,b$, for some $b \in B$, and hence $x\langle F, G\rangle(a,b)$ and $(a,b)\,\Pi_1\,a$, that is, $x \lhd_{\mathcal{C}} (\Pi_1 * \langle F, G\rangle)^{-}(a)$. Therefore, $c \lhd_{\mathcal{C}} (\Pi_1 * \langle F, G\rangle)^{-}(a)$ and hence $c\,\Pi_1 * \langle F, G\rangle\,a$ follows by *function saturation*.

  Now, let us prove the other implication. Suppose that $c\;\Pi_1 * \langle F, G\rangle\; a$. Then, the observation (1) above shows that

  $$c \lhd_{\mathcal{C}} \langle F, G\rangle^{-}(\{(x,y) \in A \times B \mid x \lhd_{\mathcal{A}} a\})$$

  and hence we obtain cFa by *function saturation* since $c \lhd_{\mathcal{C}} F^{-}(\{a\})$ follows by *reflexivity* and *transitivity* because, by *function weak-continuity*, $\langle F, G\rangle^{-}(\{(x,y) \mid x \lhd_{\mathcal{A}} a\}) \subseteq F^{-}(\{a\})$.

- (second projecting equation) Completely analogous to the previous point.

- (surjective pairing) We have to prove that for any continuous relation H from $\mathcal{C}$ to $\mathcal{A} \times \mathcal{B}$, $\langle \Pi_1 * H, \Pi_2 * H\rangle = H$. Now, it is immediate to check that if $cH(a,b)$ then $c\;\langle \Pi_1 * H, \Pi_2 * H\rangle\;(a,b)$.

  To prove the other implication let us assume that $c\;\langle \Pi_1 * H, \Pi_2 * H\rangle\;(a,b)$ holds. Then both $c\,\Pi_1 * H\; a$ and $c\,\Pi_2 * H\; b$ follows and hence we obtain both $c \lhd_{\mathcal{C}} H^{-}(\{(x,y) \mid x \lhd_{\mathcal{A}} a\})$ and $c \lhd_{\mathcal{C}} H^{-}(\{(x,y) \mid y \lhd_{\mathcal{B}} b\})$. Thus

  $$c \lhd_{\mathcal{C}} H^{-}(\{(x,y) \mid x \lhd_{\mathcal{A}} a\}) \downarrow_{\leq} H^{-}(\{(x,y) \mid y \lhd_{\mathcal{B}} b\})$$

  follows by $\leq$*-right*. Now we can conclude $c\,H\,(a,b)$ by *function saturation* since $c \lhd_{\mathcal{C}} H^{-}(\{(a,b)\})$ follows by *transitivity* because

  $$H^{-}(\{(x,y) \mid x \lhd_{\mathcal{A}} a\}) \downarrow_{\leq} H^{-}(\{(x,y) \mid y \lhd_{\mathcal{B}} b\}) \lhd_{\mathcal{C}} H^{-}(\{(a,b)\})$$

Thus, we proved the main theorem of this section.

Proposition 4.6 $\mathsf{FTop_i}$ *and* $\mathsf{FTop_i}^-$ *are cartesian.*

Proof. We already provided all the necessary definitions and proofs in $\mathsf{FTop_i}$ and it is immediate to check that the positivity predicate of $\mathcal{A} \times \mathcal{B}$ was never used.

The next lemma and its corollary will be useful in the following.

Lemma 4.7 *Let $\mathcal{A}$ and $\mathcal{B}$ be inductively generated formal topologies (covers) and suppose that $(a, b) \in A \times B$, $c \in A$ and $d \in B$. Then, if $(a, b)\, \Pi_1\, c$ and $(a, b)\, \Pi_2\, d$ then $(a, b) \lhd_{\mathcal{A}\times\mathcal{B}} (c, d)$.*

Proof. Since $(a, b)\, \Pi_1\, c$ and $(a, b)\, \Pi_2\, d$ then

$$(a, b) \lhd_{\mathcal{A}\times\mathcal{B}} \{(x, y) \in A \times B \mid x \lhd_{\mathcal{A}} c\}$$
$$(a, b) \lhd_{\mathcal{A}\times\mathcal{B}} \{(x, y) \in A \times B \mid y \lhd_{\mathcal{B}} d\}$$

Therefore, by $\downarrow$-*right*,

$$(a, b) \lhd_{\mathcal{A}\times\mathcal{B}} \{(x, y) \in A \times B \mid x \lhd_{\mathcal{A}} c\} \downarrow_{\leq} \{(x, y) \in A \times B \mid y \lhd_{\mathcal{B}} d\}$$

and hence $(a, b) \lhd_{\mathcal{A}\times\mathcal{B}} (c, d)$ follows since

$$\{(x, y) \in A \times B \mid x \lhd_{\mathcal{A}} c\} \downarrow_{\leq} \{(x, y) \in A \times B \mid y \lhd_{\mathcal{B}} d\} \lhd_{\mathcal{A}\times\mathcal{B}} (c, d)$$

Indeed, if $(w_1, w_2)\varepsilon\{(x, y) \in A \times B \mid x \lhd_{\mathcal{A}} c\} \downarrow_{\leq} \{(x, y) \in A \times B \mid y \lhd_{\mathcal{B}} d\}$ then $w_1 \lhd_{\mathcal{A}} c$ and $w_2 \lhd_{\mathcal{B}} d$ and hence we get $(w_1, w_2) \lhd_{\mathcal{A}\times\mathcal{B}} (c, d)$ by lemma 4.3.

We can now provide an equivalent formulation of the definition of the projection maps.

Corollary 4.8 *Let $\mathcal{A}$ and $\mathcal{B}$ be inductively generated formal topologies (covers) and suppose that $(a, b) \in A \times B$, $c \in A$ and $d \in B$. Then, $(a, b)\, \Pi_1\, c$ if and only if $(a, b) \lhd_{\mathcal{A}\times\mathcal{B}} (c, b)$ and $(a, b)\, \Pi_2\, d$ if and only if $(a, b) \lhd_{\mathcal{A}\times\mathcal{B}} (a, d)$.*

Proof. We show the proof of the implication from right to left only for Π_1 since the one for Π_2 is completely similar. Note that $(c, b)\varepsilon\{(x, y) \in A \times B \mid x \lhd_{\mathcal{A}} c\}$ and hence $(a, b) \lhd_{\mathcal{A}\times\mathcal{B}} \{(x, y) \in A \times B \mid x \lhd_{\mathcal{A}} c\}$ follows from $(a, b) \lhd_{\mathcal{A}\times\mathcal{B}} (c, b)$.

To prove the other implication we can use the previous lemma 4.7 since $(a, b)\, \Pi_1\, a$ and $(a, b)\, \Pi_2\, b$ clearly hold.

4.2 Unary formal covers and unary formal topologies

Let us recall the definition of unary formal topology and unary formal cover.

Definition 4.9 (Unary formal topology) *A formal topology $(A, \lhd_{\mathcal{A}}, \mathsf{Pos}_{\mathcal{A}})$ is called* unary *if, for any $a \in A$ and $U \subseteq A$,*

$$a \lhd_{\mathcal{A}} U \textit{ if and only if } \mathsf{Pos}_{\mathcal{A}}(a) \to (\exists b \varepsilon U)\ a \lhd_{\mathcal{A}} \{b\}$$

It is trivial to see that unary formal topologies form a full sub-category of FTop that we will call unFTop.

The definition of unary formal topology needs to be slightly modified when we want to move to formal covers.

Definition 4.10 (Unary formal cover) *A formal cover* $(A, \lhd_A)$ *is called* unary *if, for any* $a \in A$ *and* $U \subseteq A$,

$$a \lhd_A U \text{ if and only if } (\exists b \varepsilon U)\ a \lhd_A \{b\}$$

It is trivial to turn a unary formal cover into a unary formal topology by simply considering the always true positivity predicate. On the other hand, it is well possible that the formal topology $\mathcal{A} = (A, \lhd, \mathsf{Pos})$ is unary while the formal cover $\mathcal{I}(\mathcal{A})$, obtained by forgetting the positivity predicate, is not a unary formal cover; for instance, consider that any non-positive element of A is covered by the empty set. However, we can solve this lack of uniformity. Let us first recall the following theorem from [Cur04].

Theorem 4.11 *Let* $\mathcal{A} \equiv (A, \lhd, \mathsf{Pos})$ *be a formal topology. Then*[3]

$$\mathcal{A}^{\mathsf{Pos}} \equiv (\mathsf{Pos}_A, \lhd^{\mathsf{Pos}}, \mathsf{Pos}^{\mathsf{Pos}})$$

where, for any $\langle a, \pi_a \rangle \in \mathsf{Pos}_A$ *and* $U \subseteq \mathsf{Pos}_A$,

$$\begin{array}{rcl} \langle a, \pi_a \rangle \lhd^{\mathsf{Pos}} U & \equiv & a \lhd \{u \in A \mid \langle u, \pi_u \rangle \varepsilon U\} \\ \mathsf{Pos}^{\mathsf{Pos}}(\langle a, \pi_a \rangle) & \equiv & \mathsf{True} \end{array}$$

is a formal topology isomorphic to $\mathcal{A}$.

Proof. Let us first prove that $\mathcal{A}^{\mathsf{Pos}}$ is a formal topology. To check the validity of *reflexivity* and *transitivity* is straightforward while the proof of the validity of $\downarrow$*-right* requires to use the *positivity* condition for $\lhd$. Moreover *monotonicity* and *positivity* for $\lhd^{\mathsf{Pos}}$ are trivially valid.

Finally, to check that the formal topologies $\mathcal{A}$ and $\mathcal{A}^{\mathsf{Pos}}$ are isomorphic one has just to check that the continuous relations R_1 from $\mathcal{A}$ to $\mathcal{A}^{\mathsf{Pos}}$ defined by setting, for any $a \in A$ and $\langle c, \pi_c \rangle \in \mathsf{Pos}_A$,

$$a\ R_1 \langle c, \pi_c \rangle \equiv a \lhd c$$

and R_2 from $\mathcal{A}^{\mathsf{Pos}}$ to $\mathcal{A}$ defined by setting, for any $\langle a, \pi_a \rangle \in \mathsf{Pos}_A$ and $c \in A$,

$$\langle a, \pi_a \rangle\ R_2\ c \equiv a \lhd c$$

are one the inverse of the other (to prove this result the *positivity* condition is required again).

[3] Recall that, provided A is a set and B is a property over elements of A, by $\Sigma(A, B)$ we mean the set whose elements are pairs $\langle a, b \rangle$ whose first element a is an element of A and whose second element is a proof that a enjoys the property B. In this work we often use the notation B_A to mean the set $\Sigma(A, B)$.

So, given a unary formal topology $\mathcal{A}$ we can first restrict ourselves to the formal topology $\mathcal{A}^{\mathsf{Pos}}$, which is isomorphic to $\mathcal{A}$ and still unary, and then consider $\mathcal{I}(\mathcal{A}^{\mathsf{Pos}})$ which is a unary formal cover.

Unary topologies are distinguishable among formal topologies because the collection of their formal points forms an algebraic dcpo and any algebraic dcpo is (isomorphic to) the collection of the formal points of a suitable unary formal topology, at least from an impredicative point of view (see [Sig90], [SVV96] or [Sam00]). In fact, it is easy to check that unary topologies form a full subcategory of the category UnFtop which is impredicatively equivalent to the category Alg of algebraic dcpos and Scott-continuous functions [AJ94].

Let us recall now the following theorem in [CSSV03].

Theorem 4.12 *Let $\mathcal{A}$ be a unary formal topology. Then $\mathcal{A}$ is inductively generated.*

While it is obvious that this result trivially holds from an impredicative point of view, it is interesting to note that a predicative proof requires the use of the axiom of choice which is an immediate consequence of the definition of Σ-type in Martin-Löf's Type theory [Mar84].

Theorem 4.12 concerns unary formal topologies, but it is trivial to see that completely similar result can be proved for unary formal covers. In the rest of this section we will present our results for $\mathsf{FTop_i}^-$ since in the following we will need to use them for such a category. However, it is not difficult to check that all what we do can be re-done within $\mathsf{FTop_i}$; the only difference is that one has to adapt the various proofs to the presence of the positivity predicate.

We present now a lemma that characterizes the topological product of formal covers in the case one of them is unary.

Lemma 4.13 *Consider the topological product of an inductively generated formal cover $\mathcal{C} \equiv (C, \leq_{\mathcal{C}}, \lhd_{\mathcal{C}})$ and a unary formal cover $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}})$ and suppose that $c \in C$, $a \in A$ and $W \subseteq C \times A$. Then, if $(c, a) \lhd_{\mathcal{C}\times\mathcal{A}} W$ then there exists a subset W_1 of C such that $c \lhd_{\mathcal{C}} W_1$ and, for every $w_1 \varepsilon W_1$ there exists $w_2 \in A$ such that $a \lhd_{\mathcal{A}} w_2$ and $(w_1, w_2) \varepsilon W$.*

Proof. The statement is proved by a straightforward induction on the length of the derivation of $(c, a) \lhd_{\mathcal{C}\times\mathcal{A}} W$. Let us only note that if $(c, a) \lhd_{\mathcal{C}\times\mathcal{A}} W$ has been obtained by *infinity* from $C_C(c, j) \times \{a\} \lhd_{\mathcal{C}\times\mathcal{A}} W$ then the required subset W_1 is $\bigcup_{y \varepsilon C_C(c,j)} W_y$ where W_y is the subset obtained by inductive hypothesis.

The following is a useful corollary of the previous lemma.

Corollary 4.14 *Consider the topological product of an inductively generated formal cover $\mathcal{C} \equiv (C, \leq_{\mathcal{C}}, \lhd_{\mathcal{C}})$ and a unary formal cover $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}})$ and suppose that $c, c_1 \in C$, $a \in A$. Then if $(c, a)\, \Pi_1\, c_1$ then $c \lhd_{\mathcal{C}} c_1$.*

Proof. Let us suppose that $(c,a)\,\Pi_1\,c_1$. Then, by corollary 4.8, we get that $(c,a) \lhd_{\mathcal{C}\times\mathcal{A}} (c_1,a)$ and hence, by the previous lemma, there exists a subset W_1 of C such that $c \lhd_{\mathcal{C}} W_1$ and, for any $w_1 \varepsilon W_1$, there exists $w_2 \in A$ such that $a \lhd_{\mathcal{A}} w_2$ and $(w_1,w_2)\varepsilon\{(c_1,a)\}$. But then, for any $w_1\varepsilon W_1$, $w_1 = c_1$ and so $w_1 \lhd_{\mathcal{C}} c_1$. Hence $c \lhd_{\mathcal{C}} W_1$ yields $c \lhd_{\mathcal{C}} c_1$ by *transitivity*.

The definition of continuous relation between formal covers can be substantially simplified if we restrict our attention to the case of continuous relations between a unary formal cover and a generic one. This simplification is the key for the possibility to define the exponential of a unary cover over an inductively generated one (see section 4.3.1).

Proposition 4.15 *Let $\mathcal{A} = (A, \lhd_{\mathcal{A}})$ be a unary formal cover and $\mathcal{B} = (B, \lhd_{\mathcal{B}})$ be an inductively generated formal cover whose axiom-set is $J(-)$, $D(-,-)$. Then a binary proposition $a\,F\,b$ between A and B is continuous if and only if it satisfies the following conditions:*

(unary function totality)	$(\forall a \in A)(\exists b \in B)\ a\,F\,b$
(unary function convergence)	$\dfrac{aFb \quad aFd}{(\exists y\varepsilon b \downarrow_{\leq_{\mathcal{B}}} d)\ aFy}$
(function weak-saturation)	$\dfrac{a \lhd_{\mathcal{A}} c \quad c\,F\,b}{a\,F\,b}$
(unary $\leq$-continuity)	$\dfrac{a\,F\,b \quad b \leq_{\mathcal{B}} d}{a\,F\,d}$
(unary axiom continuity)	$\dfrac{a\,F\,b \quad j \in J(b)}{(\exists y\varepsilon D(b,j))\ a\,F\,y}$

Proof. We show that the conditions here are equivalent to the standard ones when working with unary formal covers. Indeed, we already observed that *function weak-saturation* is a consequence of *function saturation*.

Now, *function totality* is obviously a consequence of *unary function totality* and, on the other hand, if $a \lhd_{\mathcal{A}} F^-(B)$ and $\mathcal{A}$ is a unary formal cover then there exists $c \in A$ such that $a \lhd_{\mathcal{A}} c$ and $c\varepsilon F^-(B)$, that is, there exists $b \in B$ such that $a \lhd_{\mathcal{A}} c$ and $c\,F\,b$ and hence $a\,F\,b$ follows by *function weak-saturation*.

Moreover, the validity of *unary function convergence* is a consequence of *function $\leq$-convergence* and *function weak-saturation* while the validity of *unary function continuity* is a consequence of *function continuity* and *function weak-saturation*. On the other hand, *function saturation* can be proved as follows: suppose that $a \lhd_{\mathcal{A}} W$; then, there exists $w\varepsilon W$ such that $a \lhd_{\mathcal{A}} w$, since $\mathcal{A}$ is a unary formal topology, and so $(\forall w\varepsilon W)\ wFb$ yields wFb and hence aFb follows by *function weak-saturation*. In the same way, *function convergence* and *function continuity* follow respectively by *unary function convergence* and by *unary function continuity*.

Finally, it is obvious that *unary axiom continuity* and *unary $\leq$-continuity* are immediate consequences of *function continuity*, *reflexivity* and *$\leq$-left*. Vice-versa, we can prove the validity of an instance of *function continuity* whose premises are $a\,F\,b$ and $b \lhd_{\mathcal{B}} V$ by using the conditions above and reasoning by induction on the length of the derivation of $b \lhd_{\mathcal{B}} V$.

4.3 The construction of the exponential object

We are now ready to prove exponentiability of unary formal covers in $\mathsf{FTop_i}^-$ and exponentiability of unary formal topologies in $\mathsf{FTop_i}$. The proof goes on as follows: we first show that unary formal covers are exponentiable over inductively generated formal covers and then we use the coreflection of $\mathsf{FTop_i}^-$ into $\mathsf{FTop_i}$ to move such a result to formal topologies.

4.3.1 The exponential formal cover

In this section, given a unary formal cover $\mathcal{A}$ and an inductively generated one $\mathcal{B}$, we show how to build an inductively generated formal cover, that we indicate by $\mathcal{A} \to \mathcal{B}$, whose formal points are (in bijective correspondence with) the continuous relations from $\mathcal{A}$ to $\mathcal{B}$.

The basic neighbourhoods of $\mathcal{A} \to \mathcal{B}$ are lists whose elements are pairs in the cartesian product $A \times B$ of A and B. The intended meaning of a list $l \in \mathsf{List}(A \times B)$ is to give a partial information on a continuous relation F from $\mathcal{A}$ to $\mathcal{B}$. To indicate that the list l approximates the continuous relation F we introduce the following definition

$$F \Vdash l \equiv (\forall (a,b) \epsilon l)\ aFb$$

where the proposition $x \epsilon l$ is defined by induction on the construction of l by setting $x \epsilon \mathsf{nil} \equiv \mathsf{False}$ and $x \epsilon (a,b) \cdot l \equiv (x = (a,b)) \vee x \epsilon l$.

Since we want to obtain an inductively generated formal cover, in order to apply the method in section 3.2, we have to introduce also a pre-order relation among lists. The obvious choice is to set

$$l \preceq m \equiv (\forall (a,b) \in A \times B)\ (a,b) \epsilon m \to (a,b) \epsilon l$$

stating that the list l is more precise, that is, it approximates fewer continuous relations, than the list m. This order relation is a refinement of the reverse sub-list relation, which states that m is a sub-list of l, because it does not consider the order among the elements in a list and their repetitions.

According to the explanation in section 3.2 we have now to specify an axiom-set from which the cover relation will be generated by induction.

Now, the inspiring idea for the axiom-set is to look for those axioms which will force a point of the exponential formal topology to be a continuous relation. Thus, each axiom has to explain how an information l on a continuous relation can be made more precise and still be part of a continuous relation. So, we add a new axiom schema in correspondence with each of the conditions defining a continuous relation. Meanwhile, we need to justify such axioms. According to the intended meaning of the cover relation in section 2.1, given an axiom stating that l is covered by U, such a justification amounts to show that $\mathsf{ext}(l) \subseteq \mathsf{Ext}(U)$, that is, every formal point containing l also contains a basic neighbourhood of U. Recalling that formal points are expected to be continuous relations, this means that we have to prove that, for any continuous relation F, if $F \Vdash l$ then there exists $m \varepsilon U$ such that $F \Vdash m$.

So, we have now a clear plan for finding our axiom-set. In order to keep the exposition clear, we are not going to formalize the axiom-set completely by specifying a set $I(-)$ of indexes for every list l and a family $C(-,-)$ of subsets defining all of the subsets which cover l by axiom. In fact, we are just going to write down which subsets have to appear in the family $C(-,-)$. We hope that it will be clear how such a formalization can be actually performed.

The first axiom schema that we require is the formalization of *unary function totality*, namely, for every $a \in A$ there exists $b \in B$ such that $a\,F\,b$. It is expressed by stating that, for any $l \in \mathsf{List}(A \times B)$ and any element $a \in A$, there is an index $k \in I(l)$ such that

$$\text{(unary totality axiom)} \quad C(l,k) \equiv \{(a,b) \cdot l \mid b \in B\}$$

Now, if F is any continuous relation which contains l, that is, such that $(x,y)\epsilon l$ yields $x\,F\,y$, then it also contains $(a,b) \cdot l$ for some $b \in B$ because of *unary function totality*.

The second axiom schema is a formalization of *unary function convergence*. This condition states that if $a\,F\,b$ and $a\,F\,d$ hold then there exists $y \varepsilon b \downarrow_{\leq_{\mathcal{B}}} d$ such that $a\,F\,y$. The corresponding axiom states that, provided that $(a,b)\epsilon l$ and $(a,d)\epsilon l$, there is an index $k \in I(l)$ such that

$$\text{(unary convergence axiom)} \quad C(l,k) \equiv \{(a,y) \cdot l \mid y \varepsilon b \downarrow_{\leq_{\mathcal{B}}} d\}$$

Now, if F is a continuous relation which contains l and $(a,b)\epsilon l$ and $(a,d)\epsilon l$ then we get $a\,F\,b$ and $a\,F\,d$; hence by *unary function convergence* there exists $y \varepsilon b \downarrow_{\leq_{\mathcal{B}}} d$ such that $a\,F\,y$; so F contains $(a,y) \cdot l$.

The third required condition is *function weak-saturation*, that is, if $a \lhd_{\mathcal{A}} c$ and $c\,F\,b$ then $a\,F\,b$. The corresponding axiom states that, provided that $(c,b)\epsilon l$ and $a \lhd_{\mathcal{A}} c$, there is an index $k \in I(l)$ such that

$$\text{(weak-saturation axiom)} \quad C(l,k) \equiv \{(a,b) \cdot l\}$$

Now, suppose that $(c,b)\epsilon l$ and $a \lhd_{\mathcal{A}} c$ and that F is any continuous relation containing l. Then $c\,F\,b$ holds and hence $a \lhd_{\mathcal{A}} c$ yields $a\,F\,b$ by *function weak-saturation*; so F contains $(a,b) \cdot l$.

We have to consider now *unary axiom continuity* and *unary $\leq$-continuity*. The first condition states that if $a\,F\,b$ and $j \in J(b)$, where $J(b)$ is the axiom-indexing set for $\mathcal{B}$, then there exists $y \varepsilon C(b,j)$ such that $a\,F\,y$. The corresponding axiom states that, provided that $(a,b)\epsilon l$ and $j \in J(b)$, there is an index $k \in I(l)$ such that

$$\text{(unary continuity axiom)} \quad C(l,k) \equiv \{(a,y) \cdot l \mid y \varepsilon C(b,j)\}$$

Now, if F is any continuous relation which contains l then $(a,b)\epsilon l$ yields $a\,F\,b$ and hence $j \in J(b)$ yields that there exists $y\varepsilon C(b,j)$ such that $a\,F\,y$; so F contains $(a,y) \cdot l$.

Finally *unary* $\leq$*-continuity* states that $a\,F\,b$ and $b \leq_{\mathcal{B}} d$ yield $a\,F\,d$. The corresponding axiom states that, provided $(a,b)\epsilon l$ and $b \leq_{\mathcal{B}} d$, there is an index $k \in I(l)$ such that

$$(\leq\text{-continuity axiom}) \quad C(l,k) \equiv \{(a,d) \cdot l\}$$

Now, if F is any continuous relation which contains l, then from $(a,b)\epsilon l$ we get $a\,F\,b$ and hence we conclude $a\,F\,d$ by *unary* $\leq$*-continuity* since $b \leq_{\mathcal{B}} d$. So F contains $(a,d) \cdot l$.

It is not too difficult to show that the axioms above form an axiom-set. However, it is interesting to note that to obtain this result it is necessary that the formal topology $\mathcal{B}$ is inductively generated; indeed, the continuity axiom for a general topology would have required that, provided that $(a,b)\epsilon l$ and $b \lhd_{\mathcal{B}} V$, there is an index in $I(l)$ for any subset $\{(a,v) \cdot l \mid v\varepsilon V\}$. But, in general, this cannot be possible since it would be necessary to quantify over the collection of all the subsets of B.

Thus, we have completed the definition of the axiom-set for the formal cover $\mathcal{A} \to \mathcal{B}$ and it is not too difficult to verify that such an axiom-set satisfies the localization condition of section 3.2. So we can finally generate by induction the formal cover $\mathcal{A} \to \mathcal{B}$.

The following lemma is an immediate consequence of the definition of exponential topology.

Lemma 4.16 *Let $l \in \mathsf{List}(A \times B)$, $(c,b)\epsilon l$, $a \lhd_{\mathcal{A}} c$ and $b \leq_{\mathcal{B}} d$. Then*

$$l \lhd_{\mathcal{A}\to\mathcal{B}} (a,b) \cdot \mathsf{nil} \text{ and } l \lhd_{\mathcal{A}\to\mathcal{B}} (c,d) \cdot \mathsf{nil}$$

Let us recall now the operation of appending two lists that we will use in the next lemmas. Given two lists m_1 and m_2 in $\mathsf{List}(A \times B)$ we will write $m_1 \cdot m_2$ to mean the result of appending the list m_1 to the list m_2 and, given two subsets $U_1, U_2 \subseteq \mathsf{List}(A \times B)$, we will write $U_1 \cdot U_2$ to mean the subset $U_1 \cdot U_2 \equiv \{m_1 \cdot m_2 \mid m_1\varepsilon U_1 \;\&\; m_2\varepsilon U_2\}$.

Lemma 4.17 *Let $l \in \mathsf{List}(A\times B)$ and $U_1, U_2 \subseteq \mathsf{List}(A\times B)$. Then the following condition holds:*

$$(\cdot\text{-right}) \quad \frac{l \lhd_{\mathcal{A}\to\mathcal{B}} U_1 \qquad l \lhd_{\mathcal{A}\to\mathcal{B}} U_2}{l \lhd_{\mathcal{A}\to\mathcal{B}} U_1 \cdot U_2}$$

Lemma 4.18 *Let F be a continuous relation from $\mathcal{C}$ to $\mathcal{A} \to \mathcal{B}$. Then the following condition holds, for any $c \in C$ and any $l_1, l_2 \in \mathsf{List}(A \times B)$,*

$$\frac{c\,F\,l_1 \qquad c\,F\,l_2}{c\,F\,l_1 \cdot l_2}$$

Even if the axioms for the exponential cover $\mathcal{A} \to \mathcal{B}$ that we introduced use directly the particular axiom-set used to generate the formal cover $\mathcal{B}$, the next lemma shows that the resulting cover does not depend on this particular axiom-set but on the cover of $\mathcal{B}$.

Lemma 4.19 *Let $\mathcal{A}$ be a unary formal cover and $\mathcal{B}$ be an inductively generated one. Then, for any list $l \in \mathsf{List}(A \times B)$ and any $V \subseteq B$, if $(a,b)\epsilon l$ and $b \lhd_{\mathcal{B}} V$ then $l \lhd_{\mathcal{A}\to\mathcal{B}} \{(a,y) \cdot l \mid y\varepsilon V\}$.*

4.3.2 Bijection between formal points and continuous relations

We can prove now that there is a bijective correspondence between the continuous relations from a unary formal cover $\mathcal{A}$ to an inductively generated one $\mathcal{B}$ and the formal points of the formal cover $\mathcal{A} \to \mathcal{B}$. It is clear that this result is an immediate consequence of the bijective correspondence between the collection of the formal points of the formal cover $\mathcal{A}$ and the morphisms from the terminal formal cover $\mathcal{T}$ to $\mathcal{A}$, and the fact that, for any unary formal cover $\mathcal{A}$ and any inductively generated formal cover $\mathcal{B}$, the formal cover $\mathcal{A} \to \mathcal{B}$ is the exponential of $\mathcal{A}$ and $\mathcal{B}$ that we will prove in the next section. However, we decided to insert here a direct proof since we think that it is more straight and perspicuous to understand how the axioms for the exponential have been found.

Theorem 4.20 *Let $\mathcal{A}$ be a unary formal cover and $\mathcal{B}$ be an inductively generated formal cover. Then there exists a bijective correspondence between the collection of the formal points of $\mathcal{A} \to \mathcal{B}$ and the collection of the continuous relations from $\mathcal{A}$ to $\mathcal{B}$.*

Proof. The bijective correspondence is defined as follows. To any formal point $\Phi \in \mathsf{Pt}(\mathcal{A} \to \mathcal{B})$ we associate

$$aF_\Phi b \equiv a \lhd_{\mathcal{A}} \{c \in A \mid (c,b) \cdot \mathsf{nil} \varepsilon \Phi\}$$

that can be proved to be a continuous relation, and to any continuous relation F from $\mathcal{A}$ to $\mathcal{B}$ we associate

$$l \varepsilon \Phi_F \text{ iff } F \Vdash l$$

which can be proved to be a formal point of $\mathcal{A} \to \mathcal{B}$.

It is easy to show that the two constructions are one the inverse of the other.

4.3.3 Application and abstraction

In this section we show that the formal cover defined in the previous sections is the exponential of a unary formal cover over an inductively generated formal cover. From a categorical point of view this means that, for any unary formal cover $\mathcal{A}$, the functor $- \times \mathcal{A} : \mathsf{FTop_i}^- \Rightarrow \mathsf{FTop_i}^-$ has a right adjoint $\mathcal{A} \to - : \mathsf{FTop_i}^- \Rightarrow \mathsf{FTop_i}^-$. Equivalently, this amounts to define, for any unary formal cover $\mathcal{A}$ and any inductively generated formal cover $\mathcal{B}$, a relation Ap from $(\mathcal{A} \to \mathcal{B}) \times \mathcal{A}$ to $\mathcal{B}$, called *application*, such that for any continuous relation F

from $\mathcal{C} \times \mathcal{A}$ to $\mathcal{B}$ there exists a continuous relation $\Lambda(F)$ from $\mathcal{C}$ to $\mathcal{A} \to \mathcal{B}$, called *abstraction* of F, such that, for any continuous relation G from $\mathcal{C}$ to $\mathcal{A} \to \mathcal{B}$, the following equations are satisfied

$$\begin{array}{rcl} \mathsf{Ap} * \langle \Lambda(F) * \Pi_1, \Pi_2 \rangle & = & F \\ \Lambda(\mathsf{Ap} * \langle G * \Pi_1, \Pi_2 \rangle) & = & G \end{array}$$

We propose the following definitions for the application and the abstraction:

$$\begin{array}{lcl} (l,a)\ \mathsf{Ap}\ b & \equiv & l \lhd_{\mathcal{A}\to\mathcal{B}} (a,b) \cdot \mathsf{nil} \\ c\ \Lambda(F)\ l & \equiv & (\forall (a,b) \epsilon l)\ (c,a)\, F\, b \end{array}$$

for any $l \in \mathsf{List}(A \times B)$, $a \in A$, $b \in B$ and $c \in C$,

The next lemma states that Ap is a continuous relation.

Lemma 4.21 *Let $\mathcal{A}$ be a unary formal cover and $\mathcal{B}$ be an inductively generated one. Then* Ap *is a continuous relation from $(\mathcal{A} \to \mathcal{B}) \times \mathcal{A}$ to $\mathcal{B}$.*

Proof. All of the required conditions have to be checked. We will show here only the non trivial cases.

- (function totality) Let $a \in A$ and $l \in \mathsf{List}(A \times B)$. We have to prove that $(l,a) \lhd_{(\mathcal{A}\to\mathcal{B})\times\mathcal{A}} \mathsf{Ap}^-(B)$. Now $l \lhd_{\mathcal{A}\to\mathcal{B}} \{(a,b)\cdot l \mid b \in B\}$, by *unary totality axiom*, and hence we conclude $(l,a) \lhd_{(\mathcal{A}\to\mathcal{B})\times\mathcal{A}} \mathsf{Ap}^-(B)$ by *transitivity* since $\{(a,b)\cdot l \mid b \in B\} \lhd_{\mathcal{A}\to\mathcal{B}} \{(a,b)\cdot \mathsf{nil} \mid b \in B\}$ and $\{(a,b)\cdot \mathsf{nil} \mid b \in B\} \times \{a\} \subseteq \mathsf{Ap}^-(B)$.

- (function convergence) Suppose $(l,a)\ \mathsf{Ap}\ b$ and $(l,a)\ \mathsf{Ap}\ d$. Then, we have to show that $(l,a) \lhd_{(\mathcal{A}\to\mathcal{B})\times\mathcal{A}} \mathsf{Ap}^-(b \downarrow_{\leq_{\mathcal{B}}} d)$. The assumptions yield $l \lhd_{\mathcal{A}\to\mathcal{B}} (a,b)\cdot\mathsf{nil}$ and $l \lhd_{\mathcal{A}\to\mathcal{B}} (a,d)\cdot\mathsf{nil}$. Hence, $l \lhd_{\mathcal{A}\to\mathcal{B}} (a,b)\cdot(a,d)\cdot\mathsf{nil}$ follows by lemma 4.17. Now $(a,b)\cdot(a,d)\cdot\mathsf{nil} \lhd_{\mathcal{A}\to\mathcal{B}} \{(a,y)\cdot\mathsf{nil} \mid y\varepsilon b \downarrow_{\leq_{\mathcal{B}}} d\}$ holds by *unary convergence axiom* and so $(l,a) \lhd_{(\mathcal{A}\to\mathcal{B})\times\mathcal{A}} \mathsf{Ap}^-(b \downarrow_{\leq_{\mathcal{B}}} d)$ follows by *transitivity* since $\{(a,y)\cdot\mathsf{nil} \mid y\varepsilon b \downarrow_{\leq_{\mathcal{B}}} d\} \times \{a\} \subseteq \mathsf{Ap}^-(b \downarrow_{\leq_{\mathcal{B}}} d)$.

- (function axiom-continuity) Suppose that $(l,a)\ \mathsf{Ap}\ b$ and $j \in J(b)$ hold in order to show that $(l,a) \lhd_{(\mathcal{A}\to\mathcal{B})\times\mathcal{A}} \mathsf{Ap}^-(C(b,j))$. Then, $l \lhd_{\mathcal{A}\to\mathcal{B}} (a,b)\cdot\mathsf{nil}$. But $(a,b)\cdot\mathsf{nil} \lhd_{\mathcal{A}\to\mathcal{B}} \{(a,y)\cdot(a,b)\cdot\mathsf{nil} \mid y\varepsilon C(b,j)\}$ by *unary continuity axiom*. Now, by *$\preceq$-left*, $(a,y)\cdot(a,b)\cdot\mathsf{nil} \lhd_{\mathcal{A}\to\mathcal{B}} (a,y)\cdot\mathsf{nil}$ holds for any $y\varepsilon C(b,j)$ and hence $(a,b)\cdot\mathsf{nil} \lhd_{\mathcal{A}\to\mathcal{B}} \{(a,y)\cdot\mathsf{nil} \mid y\varepsilon C(b,j)\}$ follows by *transitivity*. So, we obtain $l \lhd_{\mathcal{A}\to\mathcal{B}} \{(a,y)\cdot\mathsf{nil} \mid y\varepsilon C(b,j)\}$ by *transitivity* and hence $(l,a) \lhd_{(\mathcal{A}\to\mathcal{B})\times\mathcal{A}} \{((a,y)\cdot\mathsf{nil},a) \mid y\varepsilon C(b,j)\}$ follows by lemma 4.3. Thus $(l,a) \lhd_{(\mathcal{A}\to\mathcal{B})\times\mathcal{A}} \mathsf{Ap}^-(C(b,j))$ follows by *transitivity* since $\{((a,y)\cdot\mathsf{nil},a) \mid y\varepsilon C(b,j)\}$ is a subset of $\mathsf{Ap}^-(C(b,j))$.

- (function axiom-saturation) We have to show that if $k \in J((l,a))$ is an index for an axiom of the product topology and, for any $y\varepsilon C((l,a),k)$, $y\ \mathsf{Ap}\ b$ holds then also $(l,a)\ \mathsf{Ap}\ b$ holds. We will argue according to the shape of the considered axiom.

– Axioms whose shape is $(l, a) \lhd_{(\mathcal{A}\to\mathcal{B})\times\mathcal{A}} C(l, j) \times \{a\}$. We have to show that if $(\forall y \varepsilon C(l, j))\ (y, a)\ \mathsf{Ap}\ b$ then $(l, a)\ \mathsf{Ap}\ b$. Now, the assumption means that, for all $y \varepsilon C(l, j)$, $y \lhd_{\mathcal{A}\to\mathcal{B}} (a, b) \cdot \mathsf{nil}$. Then, by *transitivity*, we obtain $l \lhd_{\mathcal{A}\to\mathcal{B}} (a, b) \cdot \mathsf{nil}$, since $l \lhd_{\mathcal{A}\to\mathcal{B}} C(l, j)$, and hence $(l, a)\ \mathsf{Ap}\ b$.

– Axioms whose shape is $(l, a) \lhd_{(\mathcal{A}\to\mathcal{B})\times\mathcal{A}} \{l\} \times C(a, j)$. Since $\mathcal{A}$ is a unary formal cover, $a \lhd_{\mathcal{A}} C(a, j)$ yields that there exists an element $c \varepsilon C(a, j)$ such that $a \lhd_{\mathcal{A}} c$. Now, by hypothesis $(l, c)\ \mathsf{Ap}\ b$ holds and hence, by definition, $l \lhd_{\mathcal{A}\to\mathcal{B}} (c, b) \cdot \mathsf{nil}$. But, by lemma 4.16, $a \lhd_{\mathcal{A}} c$ yields $(c, b) \cdot \mathsf{nil} \lhd_{\mathcal{A}\to\mathcal{B}} (a, b) \cdot \mathsf{nil}$; hence $l \lhd_{\mathcal{A}\to\mathcal{B}} (a, b) \cdot \mathsf{nil}$ follows by *transitivity* and so $(l, a)\ \mathsf{Ap}\ b$.

Now we have to prove that the *abstraction* $\Lambda(F)$ of a continuous relation F is a continuous relation.

Lemma 4.22 *Let $\mathcal{A}$ be a unary formal cover and $\mathcal{C}$ and $\mathcal{B}$ be inductively generated formal covers. Suppose that F is any continuous relation from $\mathcal{C} \times \mathcal{A}$ to $\mathcal{B}$. Then $\Lambda(F)$ is a continuous relation from $\mathcal{C}$ to $\mathcal{A} \to \mathcal{B}$.*

Proof. It is necessary to check that all of the required conditions are satisfied. We will prove here only some of them.

- (function totality) Let $c \in C$. Then, $c \lhd_{\mathcal{C}} \Lambda(F)^{-}(\mathsf{List}(A \times B))$ is immediate since $c\,\Lambda(F)\,\mathsf{nil}$ holds by intuitionistic logic.

- (continuity axiom) We have to check that if $c\ \Lambda(F)\ l$ and $j \in J(l)$ then $c \lhd_{\mathcal{C}} \Lambda(F)^{-}(C(l, j))$ holds. The proof depends on the particular shape of the axiom indexed by j. We will show here only few of them.

  – (unary convergence axiom) We have to show that, provided that $(a, b)\epsilon l$ and $(a, d)\epsilon l$, then $c \lhd_{\mathcal{C}} \Lambda(F)^{-}(\{(a, y) \cdot l \mid y \varepsilon b \downarrow_{\leq_{\mathcal{B}}} d\})$ follows, that is, $c \lhd_{\mathcal{C}} \{w \in C \mid (\exists y \varepsilon b \downarrow_{\leq_{\mathcal{B}}} d)\ w\ \Lambda(F)\ (a, y) \cdot l\}$. Now, $c\,\Lambda(F)\,l$ yields $(c, a)\,F\,b$ and $(c, a)\,F\,d$; hence $(c, a) \lhd_{\mathcal{C}\times\mathcal{A}} F^{-}(b \downarrow_{\leq_{\mathcal{B}}} d)$ follows by *function convergence*. Thus, by lemma 4.13, we can find a subset W_1 of C such that $c \lhd_{\mathcal{C}} W_1$ and for any $w_1 \varepsilon W_1$ there exists an element $w_2 \in A$ such that $a \lhd_{\mathcal{A}} w_2$ and $(w_1, w_2) \varepsilon F^{-}(b \downarrow_{\leq_{\mathcal{B}}} d)$, that is, $(\exists y \varepsilon b \downarrow_{\leq_{\mathcal{B}}} d)\ (w_1, w_2) F y$. Then it is easy to see that W_1 is a subset of $\{w \in C \mid (\exists y \varepsilon b \downarrow_{\leq_{\mathcal{B}}} d)\ (w, a) F y\}$; indeed, $(w_1, w_2) F y$ yields $(w_1, a) F y$ by *function weak-saturation*, since $a \lhd_{\mathcal{A}} w_2$ yields $(w_1, a) \lhd_{\mathcal{C}\times\mathcal{A}} (w_1, w_2)$. Therefore, we know both that $c \lhd_{\mathcal{C}} W_1$ and that $W_1 \subseteq \{w \in C \mid (\exists y \varepsilon b \downarrow_{\leq_{\mathcal{B}}} d)\ (w, a) F y\}$. Hence, by *reflexivity* and *transitivity*, we get $c \lhd_{\mathcal{C}} \{w \in C \mid (\exists y \varepsilon b \downarrow_{\leq_{\mathcal{B}}} d)\ (w, a) F y\}$. Then, by $\downarrow$*-right*, we obtain $c \lhd_{\mathcal{C}} \{c\} \downarrow \{w \in C \mid (\exists y \varepsilon b \downarrow_{\leq_{\mathcal{B}}} d)\ (w, a) F y\}$.

  We will prove now that $\{c\} \downarrow \{w \in C \mid (\exists y \varepsilon b \downarrow_{\leq_{\mathcal{B}}} d)\,(w, a)\,F\,y\}$ is a subset of $\{w \in C \mid (\exists y \varepsilon b \downarrow_{\leq_{\mathcal{B}}} d)\,w\,\Lambda(F)\,(a, y) \cdot l\}$. Indeed, suppose that x is an element of C such that $x \lhd_{\mathcal{C}} c$ and $x \lhd_{\mathcal{C}} w$ for some $w \in C$ such that $(w, a)\,F\,y$ for some $y \varepsilon b \downarrow d$.

Then $(x, a)\, F\, y$ follows by *function weak-saturation* since $x \lhd_{\mathcal{C}} w$ yields $(x, a) \lhd_{\mathcal{C}\times\mathcal{A}} (w, a)$ by lemma 4.3. Moreover, for any $(s, t)\epsilon l$, $(c, s)\, F\, t$ holds, since by hypothesis $c\, \Lambda(F)\, l$. Hence, $(x, s)\, F\, t$ follows by *function weak-saturation* since $x \lhd_{\mathcal{C}} c$ yields $(x, s) \lhd_{\mathcal{C}\times\mathcal{A}} (c, s)$ by lemma 4.3. Thus, we proved that $x\, \Lambda(F)\, (a, y) \cdot l$, that is, we proved that $x\varepsilon\{w \in C \mid (\exists y\varepsilon b \downarrow_{\leq_{\mathcal{B}}} d)\, w\, \Lambda(F)\, (a, y) \cdot l\}$.

Now, we can finally conclude. Indeed, by *transitivity*, we get

$$c \lhd_{\mathcal{C}} \{w \in C \mid (\exists y\varepsilon b \downarrow_{\leq_{\mathcal{B}}} d)\, w\, \Lambda(F)\, (a, y) \cdot l\}$$

that is, $c \lhd_{\mathcal{C}} \Lambda(F)^{-}(\{(a, y) \cdot l \mid y\varepsilon b \downarrow_{\leq_{\mathcal{B}}} d\})$.

– (unary continuity axiom) Completely analogous to the previous one.

To finish the proof that the formal cover $\mathcal{A} \to \mathcal{B}$ is the exponential of $\mathcal{A}$ over $\mathcal{B}$ we have to show that the adjunction equations hold with respect to *application* and *abstraction*.

Proposition 4.23 *Let $\mathcal{A}$ be a unary formal cover and $\mathcal{C}$ and $\mathcal{B}$ be inductively generated formal covers. Then*

1. *for every continuous relation F from $\mathcal{C} \times \mathcal{A}$ to $\mathcal{B}$,*

$$\mathsf{Ap} * \langle \Lambda(F) * \Pi_1, \Pi_2 \rangle = F$$

2. *for every continuous relation G from $\mathcal{C}$ to $\mathcal{A} \to \mathcal{B}$,*

$$\Lambda(\mathsf{Ap} * \langle G * \Pi_1, \Pi_2 \rangle) = G$$

Proof. We prove the two implications of the considered equations one after the other.

- *(1. Right to left)* We have to prove that, for any $c \in C$, $a \in A$ and $b \in B$, if $(c, a)\, F\, b$ then $(c, a)\ \mathsf{Ap} * \langle \Lambda(F) * \Pi_1, \Pi_2 \rangle\ b$. Now, $(c, a)\, F\, b$ yields $c\ \Lambda(F)\ (a, b) \cdot \mathsf{nil}$ and hence $(c, a)\, \langle \Lambda(F) * \Pi_1, \Pi_2 \rangle\, ((a, b) \cdot \mathsf{nil}, a)$ follows. Then, we conclude $(c, a)\ \mathsf{Ap} * \langle \Lambda(F) * \Pi_1, \Pi_2 \rangle\ b$ since $((a, b) \cdot \mathsf{nil}, a)\ \mathsf{Ap}\ b$ obviously holds.

- *(1. Left to right)* We have to prove that $(c, a)\ \mathsf{Ap} * \langle \Lambda(F) * \Pi_1, \Pi_2 \rangle\ b$ yields $(c, a)\, F\, b$. Now, $(c, a)\ \mathsf{Ap} * \langle \Lambda(F) * \Pi_1, \Pi_2 \rangle\ b$ means that

$$(c, a) \lhd_{\mathcal{C}\times\mathcal{A}} \{(w_1, w_2) \in C \times A \mid (w_1, w_2)\, \mathsf{Ap} \circ \langle \Lambda(F) * \Pi_1, \Pi_2 \rangle\, b\}$$

  But $\mathcal{A}$ is a unary formal cover and hence, by lemma 4.13, there exists a subset W of C such that $c \lhd_{\mathcal{C}} W$ and for any $w_1 \varepsilon W$ there exists $w_2 \in A$ such that $a \lhd_{\mathcal{A}} w_2$ and $(w_1, w_2)\, \mathsf{Ap} \circ \langle \Lambda(F) * \Pi_1, \Pi_2 \rangle\, b$.

  Now, let us suppose that w_1 is an element of W and w_2 is the corresponding element in A. Then, there exist two elements $l_{(w_1,w_2)} \in \mathsf{List}(A \times B)$ and

$a_{(w_1,w_2)} \in A$ such that both $(w_1, w_2)\,\langle\Lambda(F) * \Pi_1, \Pi_2\rangle\,(l_{(w_1,w_2)}, a_{(w_1,w_2)})$ and $(l_{(w_1,w_2)}, a_{(w_1,w_2)})\,\mathsf{Ap}\,b$. So, it follows both $(w_1, w_2)\,\Lambda(F) * \Pi_1\, l_{(w_1,w_2)}$ and $(w_1, w_2)\,\Pi_2\, a_{(w_1,w_2)}$ and $l_{(w_1,w_2)} \lhd_{\mathcal{A}\to\mathcal{B}} (a_{(w_1,w_2)}, b) \cdot \mathsf{nil}$.

Hence, $(w_1, w_2) \lhd_{\mathcal{C}\times\mathcal{A}} \{(t_1, t_2) \in C \times A \mid (t_1, t_2)\,\Lambda(F) \circ \Pi_1\, l_{(w_1,w_2)}\}$. Consider now that $(t_1, t_2)\,\Lambda(F) \circ \Pi_1\, l_{(w_1,w_2)}$ means that there exists an element $u_1 \in C$ such that $(t_1, t_2)\,\Pi_1\, u_1$ and $u_1\,\Lambda(F)\, l_{(w_1,w_2)}$. But, by corollary 4.14, $(t_1, t_2)\,\Pi_1\, u_1$ yields $t_1 \lhd_{\mathcal{C}} u_1$ and hence, by *weak-saturation*, from $u_1\,\Lambda(F)\, l_{(w_1,w_2)}$ we get $t_1\,\Lambda(F)\, l_{(w_1,w_2)}$. So, we finally obtain by *transitivity* that $(w_1, w_2) \lhd_{\mathcal{C}\times\mathcal{A}} \{(t_1, t_2) \in C \times A \mid t_1\,\Lambda(F)\, l_{(w_1,w_2)}\}$.

Observe now that $(w_1, w_2)\,\Pi_2\, a_{(w_1,w_2)}$ yields $(w_1, w_2) \lhd_{\mathcal{C}\times\mathcal{A}} (w_1, a_{(w_1,w_2)})$ by corollary 4.8 and hence, by combining the latter with the previous result, we get

$$(w_1, w_2) \lhd_{\mathcal{C}\times\mathcal{A}} \{(t_1, t_2) \in C \times A \mid t_1\,\Lambda(F)\, l_{(w_1,w_2)}\} \downarrow_{\leq} \{(w_1, a_{(w_1,w_2)})\}$$

by $\downarrow_{\leq}$*-right*. So, by using again lemma 4.13, we get that there exists a subset $V_{(w_1,w_2)}$ of C such that $w_1 \lhd_{\mathcal{C}} V_{(w_1,w_2)}$ and, for all $v_1 \varepsilon V_{(w_1,w_2)}$, there exists $v_2 \in A$ such that $w_2 \lhd_{\mathcal{A}} v_2$ and (v_1, v_2) is en element of the subset $\{(t_1, t_2) \in C \times A \mid t_1\,\Lambda(F)\, l_{(w_1,w_2)}\} \downarrow_{\leq} \{(w_1, a_{(w_1,w_2)})\}$, that is, there exists $(t_1, t_2) \in C \times A$ such that $t_1\,\Lambda(F)\, l_{(w_1,w_2)}$ and $(v_1, v_2) \leq_{\mathcal{C}\times\mathcal{A}} (t_1, t_2)$ and $(v_1, v_2) \leq_{\mathcal{C}\times\mathcal{A}} (w_1, a_{(w_1,w_2)})$.

Suppose now that v_1 is an element of $V_{(w_1,w_2)}$ and v_2 is the corresponding element in A. Then, $(v_1, v_2) \leq_{\mathcal{C}\times\mathcal{A}} (t_1, t_2)$ yields $v_1 \leq_{\mathcal{C}} t_1$ and hence $t_1\,\Lambda(F)\, l_{(w_1,w_2)}$ yields $v_1\,\Lambda(F)\, l_{(w_1,w_2)}$ by $\leq$*-saturation*. Moreover $(v_1, v_2) \leq_{\mathcal{C}\times\mathcal{A}} (w_1, a_{(w_1,w_2)})$ yields $v_2 \leq_{\mathcal{A}} a_{(w_1,w_2)}$; so $v_2 \lhd_{\mathcal{A}} a_{(w_1,w_2)}$ and hence $a \lhd_{\mathcal{A}} w_2$ and $w_2 \lhd_{\mathcal{A}} v_2$ yield $a \lhd_{\mathcal{A}} a_{(w_1,w_2)}$ by *transitivity*. Thus, by lemma 4.16, $(a_{(w_1,w_2)}, b) \cdot \mathsf{nil} \lhd_{\mathcal{A}\to\mathcal{B}} (a, b) \cdot \mathsf{nil}$ and so $l_{(w_1,w_2)} \lhd_{\mathcal{A}\to\mathcal{B}} (a_{(w_1,w_2)}, b) \cdot \mathsf{nil}$ yields $l_{(w_1,w_2)} \lhd_{\mathcal{A}\to\mathcal{B}} (a, b) \cdot \mathsf{nil}$. Hence $v_1\,\Lambda(F)\,(a, b) \cdot \mathsf{nil}$ follows from $v_1\,\Lambda(F)\, l_{(w_1,w_2)}$ by *weak-continuity*.

Thus, we have that $w_1 \lhd_{\mathcal{C}} V_{(w_1,w_2)}$ and, we proved that for any $v_1 \varepsilon V_{(w_1,w_2)}$, $v_1\,\Lambda(F)\,(a, b) \cdot \mathsf{nil}$. Hence we get $w_1\,\Lambda(F)\,(a, b) \cdot \mathsf{nil}$ by *saturation*.

So we can finally conclude, again by *saturation*, that $c\,\Lambda(F)\,(a, b) \cdot \mathsf{nil}$, that is, $(c, a)\,F\,b$, since we have that $c \lhd_{\mathcal{C}} W$ and we proved that, for any $w_1 \varepsilon W$, $w_1\,\Lambda(F)\,(a, b) \cdot \mathsf{nil}$.

- *(2. Right to left)* We have to prove that, for any $c \in C$ and for any $l \in \mathsf{List}(A \times B)$, if $c\,G\,l$ holds then also $c\,\Lambda(\mathsf{Ap} * \langle G * \Pi_1, \Pi_2\rangle)\,l$ holds, that is, for any $(a, b) \epsilon l$, $(c, a)\ \mathsf{Ap} * \langle G * \Pi_1, \Pi_2\rangle\ b$. So, suppose $(a, b) \epsilon l$. Then, we get $c\,G\,\{(a, b) \cdot \mathsf{nil}\}$ by *weak-continuity* of G since $l \lhd_{\mathcal{A}\to\mathcal{B}} \{(a, b) \cdot \mathsf{nil}\}$ follows by $\leq$*-left* from $l \preceq \{(a, b) \cdot \mathsf{nil}\}$. Now, $(c, a)\ \Pi_1\ c$ and $(c, a)\ \Pi_2\ a$ clearly hold and hence we get $(c, a)\ \langle G * \Pi_1, \Pi_2\rangle\ ((a, b) \cdot \mathsf{nil}, a)$. Finally, we obtain $(c, a)\ \mathsf{Ap} * \langle G * \Pi_1, \Pi_2\rangle\ b$ since $((a, b) \cdot \mathsf{nil}, a)\ \mathsf{Ap}\ b$ obviously holds.

- *(2. Left to right)* We have to prove that, for every $c \in C$ and every $l \in \mathsf{List}(A \times B)$, $c\ \Lambda(\mathsf{Ap} * \langle G * \Pi_1, \Pi_2\rangle)\ l$ yields $c\ G\ l$. So, suppose that $c\ \Lambda(\mathsf{Ap} * \langle G * \Pi_1, \Pi_2\rangle)\ l$ holds. Then, for every $(a,b)\epsilon l$, we have $(c,a)\ \mathsf{Ap} * \langle G * \Pi_1, \Pi_2\rangle\ b$ and hence

$$(c,a) \lhd_{\mathcal{C}\times\mathcal{A}} \{(x,y) \in C \times A \mid (x,y)\ \mathsf{Ap} \circ \langle G * \Pi_1, \Pi_2\rangle\ b\}$$

follows by definition. At this point a proof completely analogous to the one for point *(1. Left to right)* can be developed in order to obtain that $c\,G\,(a,b)\cdot \mathsf{nil}$ holds for every $(a,b)\epsilon l$ and hence by successive applications of lemma 4.18 we conclude that $c\,G\,l$.

So, we are arrived at the main theorem of this section.

Theorem 4.24 *Unary formal covers are exponentiable in* $\mathsf{FTop_i}^-$*.*

Let us remark that the proof of this theorem is valid also intuitionistically since no use of the axiom of choice is required in an impredicative approach. So, theorem 4.24 constitutes a partial but completely predicative version of the results in [Hyl81, Sig95]. Indeed, in these papers it is shown that if $\mathcal{M}$ is a locally compact locale and $\mathcal{L}$ is any locale then the local $\mathcal{M} \to \mathcal{L}$ is generated from axioms on a proposition which represents the collection of frame morphisms f^* such that a is way-below $f^*(b)$. Now, when $\mathcal{M}$ is a locale representing an algebraic dcpo, such a proposition corresponds exactly to our $\mathsf{ext}^{Pt}((a,b)\cdot\mathsf{nil})$ since the latter represents the collection of all the continuous relations R such that $a\,R\,b$.

4.3.4 Exponentiability in $\mathsf{FTop_i}$

After the proof that formal topologies coreflect in $\mathsf{FTop_i}^-$, it is easy to turn the previous theorem 4.24 into a similar result within $\mathsf{FTop_i}$.

First of all, note that a full coreflection implies that the embedding functor "creates" colimits from $\mathsf{FTop_i}^-$ into its subcategory $\mathsf{FTop_i}$, that is, if a co-limit for a diagram in $\mathsf{FTop_i}$ exists in $\mathsf{FTop_i}^-$ then it exists also in $\mathsf{FTop_i}$ (see ex.7 on page 90 of [ML71]).

Moreover, it is also possible to show that even exponentiability is inherited in $\mathsf{FTop_i}$ from $\mathsf{FTop_i}^-$ because of the additional properties that the coreflection of $\mathsf{FTop_i}^-$ into $\mathsf{FTop_i}$ enjoys.

Proposition 4.25 *The functor* $\mathcal{I} : \mathsf{FTop_i} \to \mathsf{FTop_i}^-$ *preserves finite products.*

Proof. We have to prove that, for any formal topologies $\mathcal{A}$ and $\mathcal{B}$,

$$\mathcal{I}(\mathcal{A}\times\mathcal{B}) \simeq \mathcal{I}(\mathcal{A}) \times \mathcal{I}(\mathcal{B})$$

holds and that $\mathcal{I}$ preserves projections. The result follows because we showed that the cartesian product in $\mathsf{FTop_i}^-$ of two formal topologies inherits a positivity predicate.

Now we are ready to prove the inheritance of exponentiability from $\mathsf{FTop_i}^-$ into $\mathsf{FTop_i}$ as a corollary of the following categorical proposition.

Proposition 4.26 *Let $\mathcal{I} : \mathcal{S} \to \mathcal{C}$ be the embedding functor of the subcategory $\mathcal{S}$ of $\mathcal{C}$ and θ be a right adjoint of $\mathcal{I}$. Then, if*

1. *$\mathcal{C}$ and $\mathcal{S}$ are cartesian,*
2. *$\mathcal{I}$ preserves binary products,*
3. *the composition of θ with $\mathcal{I}$ is naturally isomorphic to the identity*

then, any object A of $\mathcal{S}$ such that $\mathcal{I}(A)$ is exponentiable in $\mathcal{C}$ is also exponentiable in $\mathcal{S}$.

Proof. Let A, B and C be objects of $\mathcal{S}$ and consider the following natural isomorphisms

$$\begin{array}{rcll} \mathcal{S}(C \times A, B) & \simeq & \mathcal{S}(C \times A, \theta(\mathcal{I}(B))) & \text{by fact (3)} \\ & \simeq & \mathcal{C}(\mathcal{I}(C \times A), \mathcal{I}(B)) & \text{by coreflection} \\ & \simeq & \mathcal{C}(\mathcal{I}(C) \times \mathcal{I}(A), \mathcal{I}(B)) & \text{by fact (2)} \\ & \simeq & \mathcal{C}(\mathcal{I}(C), \mathcal{I}(B)^{\mathcal{I}(A)}) & \text{by exponentiability in } \mathcal{C} \\ & \simeq & \mathcal{S}(C, \theta(\mathcal{I}(B)^{\mathcal{I}(A)})) & \text{by coreflection} \end{array}$$

Since the isomorphisms above yield an isomorphism between $\mathcal{S}(C \times A, B)$ and $\mathcal{S}(C, \theta(\mathcal{I}(B)^{\mathcal{I}(A)}))$, natural in C, then we conclude that $(-) \times A$ has a right adjoint $(-)^A : \mathcal{S} \to \mathcal{S}$ whose object part maps an object B of $\mathcal{S}$ to $\theta(\mathcal{I}(B)^{\mathcal{I}(A)})$ (see corollary 2. on page 83 of [ML71]).

Now, we apply this proposition to the coreflection of formal topologies into formal covers.

Corollary 4.27 *Given any inductively generated formal topology $\mathcal{A}$, if it is exponentiable in $\mathsf{FTop_i}^-$ then it is exponentiable also in $\mathsf{FTop_i}$. Thus, if $\mathcal{A}$ is a unary formal topology then $\mathcal{A}^{\mathsf{Pos}}$, which is isomorphic to $\mathcal{A}$, is exponentiable in $\mathsf{FTop_i}$.*

Proof. Thanks to theorem 3.18, lemmas 3.13 and 3.14 and prop. 4.25 the statement is an immediate corollary of prop. 4.26.

Moreover, if we apply prop. 4.26 to the category of locales we deduce the following exponentiability result.

Corollary 4.28 *Open locally compact locales are exponentiable in the category of open locales.*

Proof. First note that the coreflection in prop. 3.19 enjoys the conditions 2) and 3) of prop. 4.26 because it is obtained from the coreflection in prop. 3.18 via categorical equivalences. Hence, the result follows from prop. 4.26 since in [Hyl81] it is proved that locally compact locales are exponentiable in the category of locales.

5 Concluding remarks

We add here some observations that can be useful for a more complete understanding of the topic of the paper and which are immediate consequences of our work.

5.1 Why our result is limited to unary formal covers

We showed that all the conditions on a continuous relation F from a unary formal cover $\mathcal{A}$ to an inductively generated one $\mathcal{B}$ have in general one of the following shapes, for $a, a' \in A$, $b, b' \in B$ and $V \subseteq B$:

$$\frac{a\ R\ b \quad P(a,b,a',b')}{a'\ R\ b'} \qquad \frac{a\ R\ b \quad Q(a,b,V)}{(\exists y \varepsilon V)\ a\ R\ y}$$

Moreover, in section 4.3.1 we showed how to obtain an axiom out of each kind of condition. In fact, for any $l \in \mathsf{List}(A \times B)$, an axiom like

$$l \lhd (a', b') \cdot l$$

corresponds to a condition whose shape is

$$\frac{a\ R\ b \quad P(a,b,a',b')}{a'\ R\ b'}$$

if $(a, b)\epsilon l$ and $P(a, b, a', b')$ hold, and, for any $l \in \mathsf{List}(A \times B)$, an axiom like

$$l \lhd \{(a, y) \cdot l \mid y \varepsilon V\}$$

corresponds to a condition whose shape is

$$\frac{a\ R\ b \quad Q(a,b,V)}{(\exists y \varepsilon V)\ a\ R\ y}$$

if $(a, b)\epsilon l$ and $Q(a, b, V)$ hold.

Thus, we can define the exponential formal cover of an inductively generated formal cover over another one provided that we can express the general conditions on a continuous relation by using one of the shapes above. At present, we have obtained this result only in the case of having a unary formal cover as exponent.

5.2 Unary topologies are not closed under exponentiation

It is known that algebraic dcpos lack function spaces, that is, the category Alg is not cartesian closed [AJ94]. Our work suggests where the problem rests. Indeed, it is clear that all of the axioms of the exponential topology between unary topologies satisfy the unary condition except for the axiom on unary convergence because we can not limit it to a single element. This is to be contrasted with what happens in the case of the category of unary formal topologies equipped with a monoid operation on the elements of the base which expresses intersection of open subsets (see [MV03]). Indeed, this category turns out to be equivalent to the category of Scott Domains [SVV96] and it can be proved to be predicatively cartesian closed (see [Val03]).

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A On the definition of the positivity predicate

It is not difficult to realize that we can formalize the problem of defining the positivity predicate by expressing it as the problem of finding the maximal subset K of a set S satisfying the following conditions

$$\frac{x \varepsilon K \quad A(x,y)}{y \varepsilon K} \qquad \frac{x \varepsilon K \quad y \varepsilon B(x)}{(\exists v \varepsilon C(x,y))\ v \varepsilon K}$$

for some propositions $A(x,y)$, $B(x)$ and $C(x,y)$.

Now, it is easy to see that one can use Tarski fixed point theorem in order to solve such a problem in an impredicative way. Indeed, the map $\tau : \mathcal{P}(S) \to \mathcal{P}(S)$ defined by setting

$$\begin{aligned} \tau(X) \quad \equiv \quad & \{x \in S \mid (\forall y \in S)\ A(x,y) \to y \varepsilon X\} \\ & \cap \{x \in S \mid (\forall y \varepsilon B(x))(\exists v \varepsilon C(x,y))\ v \varepsilon X\} \end{aligned}$$

is clearly monotone and hence it admits a maximal fixed point which obviously satisfies the required conditions.

On the other hand, a completely predicative definition of the positivity predicate requires more attention. Here, we will adapt the approach proposed in [Coq96, Pal02] to our presentation of formal topologies.

Let A be a set and $I(a)$, for $a \in A$, and $C(a,i)$, for $a \in A$ and $i \in I(a)$, be an axiom-set for an inductively generated formal topology over A. In order to simplify the notation, let us suppose that the axiom-set contains, for any $a \in A$, an index i_b for an axiom of the form $C(a, i_b) = \{b\}$ for any $b \in A$ such that $a \leq b$ holds. It is clear that given any axiom-set we can extend it to a new axiom-set such that this condition is satisfied. In this way in the definition of the positivity predicate, we can dispense with the condition of *$\leq$-monotonicity* and we have to consider only *axiom monotonicity*. So, to define a positivity predicate means to define the biggest subset Pos of A such that if $x \varepsilon \mathsf{Pos}$ and $i \in I(x)$ then there exists $y \varepsilon C(x,i)$ such that $y \varepsilon \mathsf{Pos}$. Then, let us consider a map $\tau : \mathcal{P}(A) \to \mathcal{P}(A)$ defined by setting

$$\tau(X) = \{x \in S \mid (\forall i \in I(x))(\exists y \varepsilon C(x,i))\ y \varepsilon X\}$$

This definition is completely predicative; indeed, for any subset X of A, $\tau(X)$ is just the definition of another subset of A. It is trivial to realise that any subset X of A such that $X \subseteq \tau(X)$ satisfies *axiom monotonicity*, but it is not a positivity predicate since we need the biggest subset of A that satisfies such a condition.

On the other hand, it is straightforward to check that the map τ is monotone, since X has only a positive occurrence, and hence, for any family $(X_k)_{k \in K}$ of subsets of A such that $X_k \subseteq \tau(X_k)$, we have $\bigcup_{k \in K} X_k \subseteq \tau(\bigcup_{k \in K} X_k)$.

Thus, from an impredicative point of view, the easiest way to define a positivity predicate is just to consider the union of all the subsets Y of A such that $Y \subseteq \tau(Y)$ holds[4].

In order to transform the impredicative definition above into a predicative one we need to avoid making the union over *all* of the subsets Y of A such that $Y \subseteq \tau(Y)$ and limit such a union over a limited family of subsets of A.

Observe now that, since the axiom-set we are dealing with is fixed, there is some universe U^* such that A, $I(-)$ and $C(-,-)$ "live" within it, that is, A and $I(-)$ are elements of U^* and the family of subsets $C(-,-)$ of A is defined by using only propositions which are elements of U^*.

The idea is then to limit ourselves to consider the union over all the subsets X of A which "live" within U^* and satisfy the condition that $X \subseteq \tau(X)$. It can be useful to observe that also this family of subsets is not empty since $\emptyset$ surely "lives" in U^*.

Then the problem rests in showing that the union of all such subsets is still the biggest subset Z of A such that $Z \subseteq \tau(Z)$.

Theorem A.1 *Let Y be a subset of A such that $Y \subseteq \tau(Y)$ and suppose that $y \varepsilon Y$. Then, there exists a subset X of A which "lives" in U^* such that $X \subseteq \tau(X)$ and $y \varepsilon X$.*

Proof. Let us first consider what $Y \subseteq \tau(Y)$ means. We can simply rewrite this statement as follows

$$(\forall y \in A)\ (y \varepsilon Y) \rightarrow (\forall i \in I(y))(\exists w \in A)\ (w \varepsilon C(y,i))\ \&\ (w \varepsilon Y)$$

which, by an application of the so called *axiom of choice*, is equivalent to

$$(*)\ (\forall y \in A)\ (y \varepsilon Y) \rightarrow (\exists f_y \in I(y) \rightarrow A)(\forall i \in I(y))\ (f_y(i) \varepsilon C(y,i))\ \&\ (f_y(i) \varepsilon Y)$$

Consider now the following inductive definition of a sequence of subsets of A:

$$\begin{array}{rcl} X_0 & \equiv & \{y\} \\ X_{n+1} & \equiv & X_n \cup \{f_z(i) \in A \mid z \varepsilon X_n \text{ and } i \in I(z)\} \end{array}$$

In order to check that this sequence is well defined we have to show that, for any natural number n, $X_n \subseteq Y$. Indeed, in this way we can prove that the function f_z that we use in the inductive step is effectively known. The proof is by induction. The result is obviously true for n equal to 0 since $y \varepsilon Y$ holds by hypothesis. Assume now that $X_n \subseteq Y$ and consider any element $w \varepsilon X_{n+1}$. Then $w \varepsilon X_n$, and hence $w \varepsilon Y$ by inductive hypothesis, or $w = f_z(i)$ for some $z \varepsilon X_n$ and $i \in I(z)$. So, by inductive hypothesis, $z \varepsilon Y$ and hence $f_z(i) \varepsilon Y$ by $(*)$.

In order to conclude, let us set

$$X^* \equiv \bigcup_{n \in \mathsf{Nat}} X_n$$

[4] It can be useful to note that such a family of subsets is not empty since $\emptyset \subseteq \tau(\emptyset)$ holds trivially.

Then X^* is trivially a subset of A which contains the element y since $y \varepsilon X_0$.

Moreover $X^* \subseteq \tau(X^*)$ holds. Indeed, let us suppose that $w \varepsilon X^*$ and suppose that $i \in I(w)$. Then we have to show that there exists an element $z \varepsilon C(w, i)$ such that $z \varepsilon X^*$. Now, $w \varepsilon X^*$ means that there is a natural number n such that $w \varepsilon X_n$. Hence $w \varepsilon Y$ and thus $f_w(i) \varepsilon C(w, i)$, but, by definition, $f_w(i) \varepsilon X_{n+1}$ and hence $f_w(i) \varepsilon X^*$.

To conclude we have only to show that X^* "lives" in U^* and this amounts to showing that all of the subsets X_n "live" in U^*. This is obviously true for X_0 since $X_0 \equiv (x : A)\ x =_A y$. So, let us assume that X_n "lives" in U^* in order to show that X_{n+1} does. Now, the formal definition of X_{n+1} is

$$X_{n+1} \equiv (x : A)\ x \varepsilon X_n\ \vee\ (\exists z \in A)\ z \varepsilon X_n\ \&\ (\exists i \in I(z))\ x =_A f_z(i)$$

which is completely within U^* when X_n "lives" in U^*.[5]

[5]It can be useful to recall that the proposition $x =_A f_z(i)$ belongs to U^* whenever the set A belongs to U^* even if the definition of the function f_z requires a higher level universe.