Elementary quotient completion

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Abstract

We extend the notion of exact completion on a category with weak finite limits to Lawvere's elementary doctrines. We show how any such doctrine admits an elementary quotient completion, which is the universal solution to adding certain quotients. We note that the elementary quotient completion can be obtained as the composite of two other universal constructions: one adds effective quotients, the other forces extensionality of morphisms. We also prove that each construction preserves comprehension.

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1 Introduction

Constructions for completing a category by quotients have been widely studied in category theory. The main instance is the so-called exact completion, see [Carboni and Celia Magno, 1982, Carboni and Vitale, 1998], which is the universal construction of an exact category over a category with finite limits; it formally adds quotients of (pseudo-)equivalence relations. In general, the category-theoretic analysis of the properties of quotients provides a very robust, mathematically structured theory which can be applied in various situations: the contents of the present paper offers precisely this with respect to the study of foundational theories for constructive mathematics.

Indeed, the use of quotients is pervasive in interactive theorem proving where proofs are performed in appropriate systems of formalized mathematics in a computer-assisted way. Indeed some kind of quotient completion is compulsory when mathematics is formalized within an intensional type theory, such as the Calculus of (Co)Inductive Constructions [Coquand, 1990, Coquand and Paulin-Mohring, 1990] or Martin-Löf's type theory [Nordström et al., 1990]. In such a context, an abstract, finitary construction of quotient completion provides a

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formal framework where to combine the usual practice of (extensional) mathematics, with the need of formalizing it in an intensional theory with strong decidable properties (such as decidable type-checking) on which to perform the extraction of algorithmic contents from proofs.

To make explicit the use of quotient completion in the formalization of constructive mathematics, the paper [Maietti, 2009] included such notion as part of the very definition of constructive foundation which refines that originally given in [Maietti and Sambin, 2005] in terms of a two-level theory. According to [Maietti, 2009], a constructive foundation must be equipped with an intensional level, which can be represented by a suitable starting category $\mathcal{C}$, and an extensional level that can be seen as (a fragment of) the internal language of a suitable quotient completion of $\mathcal{C}$. As investigated in [Maietti and Rosolini, 2013a], some examples of quotient completion performed on intensional theories, such as the intensional level of the minimalist foundation in [Maietti, 2009], or the Calculus of Constructions, do not fall under the known constructions of exact completion given that the corresponding type theoretic categories closed under quotients are not exact.

In [Maietti and Rosolini, 2013a] we studied the abstract category-theoretical structure behind such quotient completions. To this purpose we introduced the notion of equivalence relation and quotient relative to a suitable fibered poset and produced a universal construction adding effective quotients—hence the name elementary quotient completion—to elementary doctrines.

In the present paper we isolate the basic components of the universal constructions in [Maietti and Rosolini, 2013a]. After recalling the basic notions required in the sequel, we show how to add effective quotients universally to an elementary doctrine in the sense of [Lawvere, 1970], a fibered inf-semilattice on a category with finite products, endowed with equality. Separately, we describe how to force extensional equality of morphisms to (the base of) an elementary doctrine. Then we prove that the two constructions can be combined to give the elementary quotient completion. Finally we check that the exact completion of a category with products and weak equalizers is an instance of the elementary quotient completion while the regular completion of a category is an instance of a rather different construction.

2 Doctrines

We analyse quotients within the general theory of fibrations, in particular, the basic fibrational concept that we shall employ is that of a doctrine. It was introduced, in a series of seminal papers, by F.W. Lawvere to synthetize the structural properties of logical systems, see [Lawvere, 1969a, Lawvere, 1969b, Lawvere, 1970], see also [Lawvere and Rosebrugh, 2003] for a unified survey. Lawvere's crucial intuition was to consider logical languages and theories as fibrations to study their 2-categorical properties, *e.g.* connectives and quantifiers are determined by structural adjunctions. That approach proved extremely fruitful, see [Makkai and Reyes, 1977, Carboni, 1982, Lambek and Scott, 1986, Jacobs, 1999, Taylor, 1999, van Oosten, 2008] and references therein.

Taking advantage of the category-theoretical presentation of logic by fibrations, we first introduce a general notion of elementary doctrine which we found appropriate to study the notion of quotient of an equivalence relation, see [Maietti and Rosolini, 2013a, Maietti and Rosolini, 2013b].

Denote by $\mathcal{InfSL}$ the category of inf-semilattice, *i.e.* posets with finite infima, and functions between them which preserves finite infima.

2.1 DEFINITION. Let $\mathcal{C}$ be a category with binary products. An ***elementary doctrine*** (on $\mathcal{C}$) is an indexed inf-semilattice $P: \mathcal{C}^{\text{op}} \longrightarrow \mathcal{InfSL}$ such that, for every object A in $\mathcal{C}$, there is an object δ_A in $P(A \times A)$ such that

(i) the assignment

$$\mathit{\exists}_{\langle \mathrm{id}_A, \mathrm{id}_A \rangle}(\alpha) := P_{\mathrm{pr}_1}(\alpha) \wedge_{A \times A} \delta_A$$

for α in $P(A)$ determines a left adjoint to $P_{\langle \mathrm{id}_A, \mathrm{id}_A \rangle}: P(A \times A) \to P(A)$—here and below we write P_f for the value of the indexing functor P on a morphism f;

(ii) for every morphism e of the form $\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_2 \rangle: X \times A \to X \times A \times A$ in $\mathcal{C}$, the assignment

$$\mathit{\exists}_e(\alpha) := P_{\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle}(\alpha) \wedge_{X \times A \times A} P_{\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle}(\delta_A)$$

for α in $P(X \times A)$ determines a left adjoint to $P_e: P(X \times A \times A) \to P(X \times A)$.

2.2 REMARK. (a) Condition (i) determines δ_A uniquely for every object A in $\mathcal{C}$.
(b) Since $\langle \mathrm{pr}_2, \mathrm{pr}_1 \rangle \circ \langle \mathrm{id}_A, \mathrm{id}_A \rangle = \langle \mathrm{id}_A, \mathrm{id}_A \rangle$, from (a) it follows that

$$\mathit{\exists}_{\langle \mathrm{id}_A, \mathrm{id}_A \rangle}(\alpha) = P_{\mathrm{pr}_2}(\alpha) \wedge_{A \times A} \delta_A$$

for every α in $P(A)$.
(c) In case $\mathcal{C}$ has a terminal object, conditions (ii) entails condition (i).
(d) One has that $\top_A \leq_A P_{\langle \mathrm{id}_A, \mathrm{id}_A \rangle}(\delta_A)$ and $\delta_A \leq_{A \times A} P_{f \times f}(\delta_B)$ when $f: A \to B$.

2.3 REMARK. For α_1 in $P(X_1 \times Y_1)$ and α_2 in $P(X_2 \times Y_2)$, it is useful to introduce a notation like $\alpha_1 \boxtimes \alpha_2$ for the object

$$P_{\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle}(\alpha_1) \wedge P_{\langle \mathrm{pr}_2, \mathrm{pr}_4 \rangle}(\alpha_2)$$

in $P(X_1 \times X_2 \times Y_1 \times Y_2)$ where $\mathrm{pr}_i, i = 1, 2, 3, 4$, are the projections from $X_1 \times X_2 \times Y_1 \times Y_2$ to each of the four factors—like we did above, we shall often drop the index in an infimum on in an inequality when it is clear in which fiber it is. Then condition 2.1(ii) is equivalent to the requirement that, for every pair of objects A and B in $\mathcal{C}$, one has $\delta_{A \times B} = \delta_A \boxtimes \delta_B$. We refer the reader to [Jacobs, 1999, Maietti and Rosolini, 2013a] for further details.

2.4 Examples. (a) The standard example of an indexed poset is the fibration of subobjects. Consider a category $\mathcal{C}$ with products and pullbacks. The functor $S: \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ assigns to any object A in $\mathcal{C}$ the poset $S(A)$ of subobjects of A in $\mathcal{C}$. For a morphism $f: B \to A$, the assignment that maps a subobject in $S(A)$ to that represented by the left-hand morphism in any pullback along f of its produces a functor $S_f: S(A) \to S(B)$ that preserves products.

The elementary structure is provided by the diagonal morphisms.

(b) (For logicians) The Lindenbaum-Tarski algebras of well-formed formulas of a theory $\mathscr{T}$ with equality in the first order language $\mathscr{L}$ provide another instance of elementary doctrine, in fact we believe it shows how elementary doctrines provide the appropriate abstract mathematical structure for that construction. The domain category is the category $\mathcal{V}$ of lists of variables and term substitutions:

object of $\mathcal{V}$ are lists of distinct variables $\vec{x} = (x_1, \ldots, x_n)$;

morphisms are lists of substitutions[1] for variables $[\vec{t}/\vec{y}]: \vec{x} \to \vec{y}$ where each term t_j in $\vec{t}$ is built in $\mathscr{L}$ on the variables $x_1, \ldots, x_n$;

composition $\vec{x} \xrightarrow{[\vec{t}/\vec{y}]} \vec{y} \xrightarrow{[\vec{s}/\vec{z}]} \vec{z}$ is given by simultaneous substitutions

$$\vec{x} \xrightarrow{\left[s_1[\vec{t}/\vec{y}]/z_1, \ldots, s_k[\vec{t}/\vec{y}]/z_k\right]} \vec{z}\ .$$

The product of two objects $\vec{x}$ and $\vec{y}$ is given by a(ny) list $\vec{w}$ of as many distinct variables as the sum of the number of variables in $\vec{x}$ and of that in $\vec{y}$. Projections are given by substitution of the variables in $\vec{x}$ with the first in $\vec{w}$ and of the variables in $\vec{y}$ with the last in $\vec{w}$.

The elementary doctrine $LT: \mathcal{V}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ on $\mathcal{V}$ is given as follows: for a list of distinct variables $\vec{x}$, the inf-semilattice $LT(\vec{x})$ has

objects equivalence classes of well-formed formulas of $\mathscr{L}$ with no more free variables than $x_1, \ldots, x_n$ with respect to provable reciprocal consequence $W \dashv\vdash_{\mathscr{T}} W'$ in $\mathscr{T}$;

morphisms $[W] \to [V]$ are the provable consequences $W \vdash_{\mathscr{T}} V$ in $\mathscr{T}$ for some pair of representatives (hence for any pair);

composition is given by the cut rule in the logical calculus;

identities $[W] \to [W]$ are given by the logical rules $W \vdash_{\mathscr{T}} W$.

For a list of distinct variables $\vec{x}$, the psoet $LT(\vec{x})$ has finite infima: the top element is $\vec{x} = \vec{x}$ and the infimum of a pair of formulas is obtained by conjunction.

(c) Consider a category $\mathcal{S}$ with binary products and weak pullbacks. Another example of elementary doctrine which appears *prima facie* very similar to previous example (a) is given by the functor of *weak subobjects* $\Psi: \mathcal{S}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$

[1] We shall employ a vector notation for lists of terms in the language as well as for simultaneous substitutions such as $[\vec{t}/\vec{y}]$ in place of $[t_1/y_1, \ldots, t_m/y_m]$.

which evaluates as the poset reflection of each comma category $\mathcal{S}/A$ at each object A of $\mathcal{S}$, introduced in [Grandis, 2000].

The apparently minor difference between the present example and example (a) depends though on the possibility of factoring an arbitrary morphism as a retraction followed by a monomorphism: for instance this can be achieved in the category $\mathcal{S}et$ of sets and functions thanks to the Axiom of Choice, see *loc.cit.*

It is possible to express precisely how the examples are related once we consider the 2-category **ED** of elementary doctrines:

the 1-morphisms are pairs (F, b) where $F\colon \mathcal{C} \to \mathcal{D}$ is a functor and $b\colon P \dot{\to} R \circ F^{\mathrm{op}}$ is a natural transformation as in the diagram

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \arrow[dd, "F^{\mathrm{op}}"'] \arrow[rrd, "P"] & & \\ & \arrow[r, phantom, "b\,\downarrow\cdot"] & \mathcal{I}nf\mathcal{SL} \\ \mathcal{D}^{\mathrm{op}} \arrow[rru, "R"'] & & \end{tikzcd}$$

where the functor F preserves products and, for every object A in $\mathcal{C}$, the functor $b_A\colon P(A) \to R(F(A))$ preserves all the structure. More explicitly, for every object A in $\mathcal{C}$, the function b_A preserves finite infima and

$$b_{A\times A}(\delta_A) = R_{\langle F(\mathrm{pr}_1), F(\mathrm{pr}_2)\rangle}(\delta_{F(A)}); \tag{1}$$

the 2-morphisms are natural transformations $\theta\colon F \dot{\to} G$ such that

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \arrow[dd, bend right, "F^{\mathrm{op}}"'] \arrow[dd, bend left, "G^{\mathrm{op}}"] \arrow[rrd, "P"] & & \\ \arrow[r, phantom, "\overset{\cdot}{\leftarrow}_{\theta^{\mathrm{op}}}"] & \arrow[r, phantom, "b\cdot \;\leq\; \cdot c"] & \mathcal{I}nf\mathcal{SL} \\ \mathcal{D}^{\mathrm{op}} \arrow[rru, "R"'] & & \end{tikzcd}$$

so that, for every A in $\mathcal{C}$ and every α in $P(A)$, one has $b_A(\alpha) \leq_{F(A)} R_{\theta_A}(c_A(\alpha))$.

2.5 EXAMPLES. (a) Given a theory $\mathscr{T}$ with equality in a first order language $\mathscr{L}$ (say with a single sort), a 1-morphism $(F, b)\colon LT \to S$ from the elementary doctrine $LT\colon \mathcal{V}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ as in 2.4(b) into $S\colon \mathcal{S}et^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$, the elementary doctrine in 2.4(a) with $\mathcal{C} = \mathcal{S}et$, determines a model $\mathfrak{M}$ of $\mathscr{T}$ where the set underlying the interpretation is $F(x)$. In fact, there is an equivalence between the category $\mathbf{ED}(LT, S)$ and the category of models of $\mathscr{T}$ and $\mathscr{L}$-homomorphisms.
(b) Given a category $\mathcal{C}$ with products and pullbacks, one can consider the two indexed posets: that of subobjects $S\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$, and the other $\Psi\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$, obtained by the poset reflection of each comma category $\mathcal{C}/A$, for A in $\mathcal{C}$. The inclusions of the poset $S(A)$ of subobjects over A into the poset reflection of $\mathcal{C}/A$ extend to a 1-morphism from S to Ψ which is an equivalence exactly when every morphism in $\mathcal{C}$ can be factored as a retraction followed by a monic.

3 Quotients in an elementary doctrine

The structure of elementary doctrine is suitable to describe the notions of an equivalence relation and of a quotient for such a relation.

3.1 Definition. Given an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{InfSL}$, an object A in $\mathcal{C}$ and an object ρ in $P(A \times A)$, we say that ρ is a ***P-equivalence relation on** A* if it satisfies

***reflexivity*:** $\delta_A \leq \rho$;

***symmetry*:** $\rho \leq P_{\langle \mathrm{pr}_2, \mathrm{pr}_1 \rangle}(\rho)$, for $\mathrm{pr}_1, \mathrm{pr}_2\colon A \times A \to A$ the first and second projection, respectively;

***transitivity*:** $P_{\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle}(\rho) \wedge P_{\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle}(\rho) \leq P_{\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle}(\rho)$, for $\mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_3\colon A \times A \times A \to A$ the projections to the first, second and third factor, respectively.

In elementary doctrines as those presented in 2.4, P-equivalence relations concide with the usual notion for those of the form (a) or (b); more interestingly, in cases like (c) a Ψ-equivalence relation is a pseudo-equivalence relation in $\mathcal{S}$ in the sense of [Carboni and Celia Magno, 1982].

For $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{InfSL}$ an elementary doctrine, the object δ_A is a P-equivalence relation on A. And for a morphism $f\colon A \to B$ in $\mathcal{C}$, the functor $P_{f \times f}\colon P(B \times B) \to P(A \times A)$ takes a P-equivalence relation σ on B to a P-equivalence relation on A. Hence, the ***P-kernel equivalence of*** $f\colon A \to B$, the object $P_{f \times f}(\delta_B)$ of $P_{A \times A}$ is a P-equivalence relation on A. In such a case, one speaks of $P_{f \times f}(\delta_B)$ as an ***effective*** P-equivalence relation.

3.2 Remark. A 1-morphism $(F, b)\colon P \to R$ in $\mathbf{ED}$ takes a P-equivalence relation on A to an R-equivalence relation on FA.

3.3 Remark. The notion of P-equivalence relation can be stated in any indexed inf-semilattice $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{InfSL}$ replacing the condition of reflexivity by

$$\top_A \leq P_{\langle \mathrm{id}_A, \mathrm{id}_A \rangle}\rho.$$

We refer the interested reader to [Pasquali, 2013].

3.4 Definition. Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{InfSL}$ be an elementary doctrine. Let ρ be a P-equivalence relation on A. A ***P-quotient of*** ρ (or simply a ***quotient*** when the doctrine is clear from the context) is a morphism $q\colon A \to C$ in $\mathcal{C}$ such that $\rho \leq P_{q \times q}(\delta_C)$ and, for every morphism $g\colon A \to Z$ such that $\rho \leq P_{g \times g}(\delta_Z)$, there is a unique morphism $h\colon C \to Z$ such that $g = h \circ q$.

We say that such a P-quotient is ***stable*** if, in every pullback

$$\begin{tikzcd} A' \arrow[r, "q'"] \arrow[d, "f'"'] & C' \arrow[d, "f"] \\ A \arrow[r, "q"'] & C \end{tikzcd}$$

in $\mathcal{C}$, the morphism $q'\colon A' \to C'$ is a P-quotient.

Note that the inequality $\rho \leq P_{q\times q}(\delta_C)$ in 3.4 becomes an equality exactly when ρ is effective.

3.5 Remark. We should pause briefly to point out that the previous requirement of stability differs slightly from the usual one, see [Janelidze and Tholen, 1994, Janelidze et al., 2004, Joyal and Moerdijk, 1995, Hyland et al., 1990], where *existence of any* pullback of a quotient would be enforced in order to declare it stable. But we must recall that the main intention of the present paper is to adopt the point of view of category theory to analyse foundational theories. All examples in that area suggest to look at indexed categories—as their syntactic presentation yields directly that structure and the induced fibration of points has a cleavage—, and very rarely the base category of indices has pullbacks. Also, the universal solution will appear *only* if one states stability as in 3.4.

In the elementary doctrine $S\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{InfSL}$ obtained from a category $\mathcal{C}$ with products and pullbacks as in 2.4(a), a quotient of the S-equivalence relation $[r\colon R \rightarrowtail A\times A]$ is precisely a coequalizer of the pair of

$$R \underset{\mathrm{pr}_2\circ r}{\overset{\mathrm{pr}_1\circ r}{\rightrightarrows}} A$$

In particular, all S-equivalence relations have stable, effective quotients if and only if the category $\mathcal{C}$ is exact.

Similarly, in the elementary doctrine $\Psi\colon \mathcal{S}^{\mathrm{op}} \longrightarrow \mathcal{InfSL}$ obtained from a category $\mathcal{C}$ with binary products and weak pullbacks as in 2.4(c), a quotient of the Ψ-equivalence relation $[r\colon R \to A\times A]$ is precisely a coequalizer of the pair of

$$R \underset{\mathrm{pr}_2\circ r}{\overset{\mathrm{pr}_1\circ r}{\rightrightarrows}} A$$

In particular, all Ψ-equivalence relations have quotients which are stable if and only if the category $\mathcal{C}$ is exact.

The abstract theory that captures the essential action of a quotient is that of descent. We recall some basic concepts from that in our particular case of interest of an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{InfSL}$, see [Janelidze et al., 2004, Janelidze and Tholen, 1994, Janelidze and Tholen, 1997] for an excellent survey on descent theory.

For a P-equivalence relation ρ on an object A in $\mathcal{C}$, the poset of descent data $\mathcal{Des}_\rho$ is the sub-poset of $P(A)$ on those α such that

$$P_{\mathrm{pr}_1}(\alpha) \wedge_{A\times A} \rho \leq P_{\mathrm{pr}_2}(\alpha),$$

where $\mathrm{pr}_1, \mathrm{pr}_2\colon A\times A \to A$ are the projections. It is easy to see that $\mathcal{Des}_\rho \subseteq P(A)$ is closed under infima.

It follows immediately from 2.2(b) that, for any object A in $\mathcal{C}$, one has that

$$\mathcal{Des}_{\delta_A} = P(A).$$

For $f\colon A \to B$ in $\mathcal{C}$, write ϕ for the P-kernel equivalence $P_{f\times f}(\delta_B)$. The functor $P_f\colon P(B) \to P(A)$ maps $P(B)$ into $\mathcal{D}es_\phi$—it is the usual *comparison* functor. The morphism f is ***descent*** if the (obviously faithful) functor $P_f\colon P(B) \to \mathcal{D}es_\phi$ is also full. The morphism f is ***effective descent*** if the functor $P_f\colon P(B) \to \mathcal{D}es_\phi$ is an equivalence.

Consider the 2-full 2-subcategory **QED** of **ED** whose objects are elementary doctrines $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ in which every P-equivalence relation has a stable P-quotient that is a descent morphism.

The 1-morphisms are those pairs (F, b) in **ED**

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[d, "F"'] \arrow[dr, "P"] \arrow[dr, phantom, shift right=3ex, "b\,\big\downarrow\cdot"] & \\
\mathcal{D}^{\mathrm{op}} \arrow[r, "R"'] & \mathcal{I}nf\mathcal{SL}
\end{tikzcd}$$

such that F preserves quotients in the sense that, if $q\colon A \to C$ is a quotient of a P-equivalence relation ρ on A, then $Fq\colon FA \to FC$ is a quotient of the R-equivalence relation $R_{\langle F(\mathrm{pr}_1), F(\mathrm{pr}_2)\rangle}(b_{A\times A}(\rho))$ on FA.

4 Completing with quotients as a universal construction

It is a simple construction that produces an elementary doctrine with quotients. We shall present it below and prove that it satisfies a suitable universal property.

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ denote an elementary doctrine for the rest of the section. Consider the category $\mathcal{R}_P$ of "equivalence relations of P":

an object of $\mathcal{R}_P$ is a pair (A, ρ) such that ρ is a P-equivalence relation on A;

a morphism $f\colon (A, \rho) \to (B, \sigma)$ is a morphism $f\colon A \to B$ in $\mathcal{C}$ such that $\rho \leq_{A\times A} P_{f\times f}(\sigma)$ in $P(A \times A)$.

Composition is given by that of $\mathcal{C}$, and identities are the identities of $\mathcal{C}$.

The indexed poset $(P)_{\mathrm{q}}\colon {\mathcal{R}_P}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ on $\mathcal{R}_P$ will be given by categories of descent data: on an object (A, ρ) it is defined as

$$(P)_{\mathrm{q}}(A, \rho) := \mathcal{D}es_\rho$$

and the following lemma is instrumental to give the assignment on morphisms using the action of P on morphisms.

4.1 Lemma. *With the notation used above, let (A, ρ) and (B, σ) be objects in $\mathcal{R}_P$, and let β be in $\mathcal{D}es_\sigma$. If $f\colon (A, \rho) \to (B, \sigma)$ is a morphism in $\mathcal{R}_P$, then $P_f(\beta)$ is in $\mathcal{D}es_\rho$.*

Proof. Let $\mathrm{pr}_1, \mathrm{pr}_2\colon A \times A \to A$ and $\mathrm{pr}'_1, \mathrm{pr}'_2\colon B \times B \to B$ be the product projections. Since β is in $\mathcal{D}es_\sigma$, it is

$$P_{\mathrm{pr}'_1}(\beta) \wedge \sigma \leq_{B\times B} P_{\mathrm{pr}'_2}(\beta).$$

Hence

$$P_{f\times f}(P_{\mathrm{pr}_1}(\beta)) \wedge P_{f\times f}(\sigma) \leq_{A\times A} P_{f\times f}(P_{\mathrm{pr}_2}(\beta))$$

as $P_{f\times f}$ preserves the structure. Since $\rho \leq_{A\times A} P_{f\times f}(\sigma)$, we have

$$P_{\mathrm{pr}_1}(P_f(\beta)) \wedge \rho \leq_{A\times A} P_{\mathrm{pr}_2}(P_f(\beta)).$$

$\square$

4.2 Lemma. *With the notation used above, the functor* $(P)_{\mathrm{q}}\colon \mathcal{R}_P{}^{op} \longrightarrow \mathcal{I}nf\mathcal{SL}$ *is an elementary doctrine.*

Proof. For (A,ρ) and (B,σ) in $\mathcal{R}_P$ let $\mathrm{pr}_1, \mathrm{pr}_3\colon A \times B \times A \times B \to A$ and $\mathrm{pr}_2, \mathrm{pr}_4\colon A \times B \times A \times B \to B$ be the four projections. As an infimum of two P-equivalence relations on $A \times B$, the P-equivalence relation

$$\rho \boxtimes \sigma := P_{\langle \mathrm{pr}_1, \mathrm{pr}_3\rangle}(\rho) \wedge_{A\times B\times A\times B} P_{\langle \mathrm{pr}_2, \mathrm{pr}_4\rangle}(\sigma)$$

provides an object $(A \times B, \rho \boxtimes \sigma)$ in $\mathcal{R}_P$ which, together with the morphisms determined by the two projections from $A\times B$, is a product of (A,ρ) and (B,σ) in $\mathcal{R}_P$.
For an object (A,ρ) in $\mathcal{R}_P$, one sees that $\rho \in P(A\times A)$ is in $\mathcal{D}es_{\rho\boxtimes\rho}$ using symmetry and transitivity. We check that the assignment $((\mathcal{H})_{\mathrm{q}})_{\langle \mathrm{id}_A, \mathrm{id}_A\rangle}(\alpha) := P_{\mathrm{pr}_1}(\alpha) \wedge_{A\times A} \rho$, for α in $\mathcal{D}es_\rho$, gives the left adjoint $((\mathcal{H})_{\mathrm{q}})_{\langle \mathrm{id}_A, \mathrm{id}_A\rangle}$ for $((P)_{\mathrm{q}})_{\langle \mathrm{id}_A, \mathrm{id}_A\rangle}$ and leave the proof of 2.1(ii) to the reader.
Consider β in $\mathcal{D}es_{\rho\boxtimes\rho}$ such that $\alpha \leq_{(A,\rho)} ((P)_{\mathrm{q}})_{\langle \mathrm{id}_A, \mathrm{id}_A\rangle}(\beta)$, *i.e.* $\alpha \leq_A P_{\langle \mathrm{id}_A, \mathrm{id}_A\rangle}(\beta)$. Therefore $\mathcal{H}_{\langle \mathrm{id}_A, \mathrm{id}_A\rangle}(\alpha) \leq_{A\times A} \beta$, which is the same as $P_{\mathrm{pr}_1}(\alpha) \wedge \delta_A \leq_{A\times A} \beta$ by 2.1(i). It follows that

$$\begin{aligned} P_{\mathrm{pr}'_1}(\alpha) \wedge P_{\langle \mathrm{pr}'_1, \mathrm{pr}'_2\rangle}(\delta_A) \wedge P_{\langle \mathrm{pr}'_2, \mathrm{pr}'_3\rangle}(\rho) &\leq_{A\times A\times A} P_{\langle \mathrm{pr}'_1, \mathrm{pr}'_2\rangle}(\beta) \wedge P_{\langle \mathrm{pr}'_2, \mathrm{pr}'_3\rangle}(\rho) \\ &\leq_{A\times A\times A} P_{\langle \mathrm{pr}'_1, \mathrm{pr}'_3\rangle}(\beta) \end{aligned}$$

for $\mathrm{pr}'_i\colon A\times A\times A \to A, \quad i = 1,2,3$, the projections. Reindexing the inequality along the morphism $\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_2\rangle\colon A\times A \to A\times A\times A$, one obtains that $P_{\mathrm{pr}_1}(\alpha) \wedge \rho \leq_{A\times A} \beta$, *i.e.*

$$((\mathcal{H})_{\mathrm{q}})_{\langle \mathrm{id}_A, \mathrm{id}_A\rangle}(\alpha) \leq_{(A\times A, \rho\boxtimes\rho)} \beta.$$

The reverse implication that $\alpha \leq ((P)_{\mathrm{q}})_{\langle \mathrm{id}_A, \mathrm{id}_A\rangle}(\beta)$ when $((\mathcal{H})_{\mathrm{q}})_{\langle \mathrm{id}_A, \mathrm{id}_A\rangle}(\alpha) \leq \beta$ follows immediately by reflexivity of ρ. $\square$

There is an obvious 1-morphism $(J,j)\colon P \to (P)_{\mathrm{q}}$ in **ED**, where $J\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{R}_P$ sends an object A in $\mathcal{C}$ to (A,δ_A) and a morphism $f\colon A \to B$ to $f\colon (A,\delta_A) \to (B,\delta_B)$ since $\delta_A \leq_{A\times A} P_{f\times f}(\delta_B)$, and $j_A\colon P(A) \to (P)_{\mathrm{q}}(A,\delta_A)$ is the identity since

$$(P)_{\mathrm{q}}(A,\delta_A) = \mathcal{D}es_{\delta_A} = P(A).$$

It is immediate to see that J is full and faithful and that (J,j) is a change of base.

4.3 Remark. Note that an object of the form (A,δ_A) in $\mathcal{R}_P$ is projective with respect to quotients of $(P)_{\mathrm{q}}$-equivalence relation, and that every object in $\mathcal{R}_P$ is a quotient of a $(P)_{\mathrm{q}}$-equivalence relation on such a projective.

4.4 Lemma. *With the notation used above, $(P)_{\mathrm{q}}\colon {\mathcal{R}_P}^{op} \longrightarrow \mathcal{I}nf\mathcal{SL}$ has descent quotients of $(P)_{\mathrm{q}}$-equivalence relations. Moreover, quotients are stable and effective descent, and P-equivalence relations are effective.*

Proof. Since the sub-poset $\mathcal{D}es_\rho \subseteq P(A)$ is closed under finite infima, a $(P)_{\mathrm{q}}$-equivalence relation τ on (A,ρ) is also a P-equivalence relation on A. It is easy to see that $\mathrm{id}_A\colon (A,\rho) \to (A,\tau)$ is a descent quotient since $\rho \leq_{A\times A} \tau$—actually, effectively so. It follows immediately that τ is the P-kernel equivalence of the quotient $\mathrm{id}_A\colon (A,\rho) \to (A,\tau)$. To see that it is also stable, suppose

$$\begin{tikzcd}
(B,\upsilon) \arrow[r, "f'"'] \arrow[d, "g"] & (A,\rho) \arrow[d, "\mathrm{id}_A"'] \\
(C,\sigma) \arrow[r, "f"] & (A,\tau)
\end{tikzcd}$$

is a pullback in $\mathcal{R}_P$. So in the commutative diagram

$$\begin{tikzcd}
(C,\sigma\wedge P_{f\times f}(\rho)) \arrow[rrr, bend left, "f"] \arrow[drr, "\mathrm{id}_C"'] & & (B,\upsilon) \arrow[r, "f'"'] \arrow[d, "g"] & (A,\rho) \arrow[d, "\mathrm{id}_A"'] \\
& & (C,\sigma) \arrow[r, "f"] & (A,\tau)
\end{tikzcd}$$

there is a fill-in morphism $h\colon (C,\sigma\wedge P_{f\times f}(\rho)) \to (B,\upsilon)$. It is now easy to see that $g\colon (B,\upsilon) \to (C,\sigma)$ is a quotient. $\square$

We can now prove that there is a left biadjoint to the forgetful 2-functor $U\colon \mathbf{QED} \to \mathbf{ED}$.

4.5 Theorem. *For every elementary doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathcal{I}nf\mathcal{SL}$, pre-composition with the 1-morphism*

$$\begin{tikzcd}
\mathcal{C}^{op} \arrow[dr, "P"] \arrow[dd, "J"'] & \\
\arrow[r, phantom, "j\,\downarrow\cdot"] & \mathcal{I}nf\mathcal{SL} \\
{\mathcal{R}_P}^{op} \arrow[ur, "(P)_q"'] &
\end{tikzcd}$$

in **ED** *induces an essential equivalence of categories*

$$- \circ (J, j) \colon \mathbf{QED}((P)_{\mathrm{q}}, Z) \equiv \mathbf{ED}(P, Z) \tag{2}$$

for every Z in **QED**.

Proof. Suppose Z is a doctrine in **QED**. As to full faithfulness of the functor in (2), consider two pairs (F, b) and (G, c) of 1-morphisms from $(P)_{\mathrm{q}}$ to Z. By 4.3, the natural transformation $\theta \colon F \dot{\to} G$ in a 2-morphism from (F, b) to (G, c) in **QED** is completely determined by its action on objects in the image of J and $(P)_{\mathrm{q}}$-equivalence relations on these. And, since a quotient $q \colon U \to V$ of an Z-equivalence relation r on U is descent, $Z(V)$ is a full sub-poset of $Z(U)$. Thus essential surjectivity of the functor in (2) follows from 4.3. $\square$

Recall that, for an elementary doctrine $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$, and for an object α in some $P(A)$, a ***comprehension*** of α is a morphism $\{\!|\alpha|\!\} \colon X \to A$ in $\mathcal{C}$ such that $P_{\{\!|\alpha|\!\}}(\alpha) = \top_X$ and, for every $f \colon Z \to A$ such that $P_f(\alpha) = \top_Z$ there is a unique morphism $g \colon Z \to X$ such that $f = \{\!|\alpha|\!\} \circ g$. Hence a comprehension is necessarily monic.

One says that P ***has comprehensions*** if every α has a comprehension, and that P ***has full comprehensions*** if, moreover, $\alpha \leq \beta$ in $P(A)$ whenever $\{\!|\alpha|\!\}$ factors through $\{\!|\beta|\!\}$.

4.6 Lemma. *Let $P \colon \mathcal{C}^{op} \longrightarrow \mathcal{I}nf\mathcal{SL}$ be an elementary doctrine. If P has comprehensions, then $(P)_{\mathrm{q}}$ has comprehensions. Moreover, given a comprehension $\{\!|\alpha|\!\} \colon X \to A$ of α in $P(A)$, the morphism $J(\{\!|\alpha|\!\}) \colon JX \to JA$ is a comprehension of $j_A(\alpha)$ if and only if $\delta_X = P_{\{\!|\alpha|\!\} \times \{\!|\alpha|\!\}}(\delta_A)$.*

Proof. Suppose (A, ρ) is in $\mathcal{R}_P$ and α in $(P)_{\mathrm{q}}(A, \rho) = \mathcal{D}es_\rho \subseteq P(A)$. Let $\{\!|\alpha|\!\} \colon X \to A$ be a comprehension in $\mathcal{C}$ of α as an object of $P(A)$ and consider the object $(X, P_{\{\!|\alpha|\!\} \times \{\!|\alpha|\!\}}(\rho))$ in $\mathcal{R}_P$. Clearly $\{\!|\alpha|\!\}$ determines a morphism $(X, P_{\{\!|\alpha|\!\} \times \{\!|\alpha|\!\}}(\rho)) \to (A, \rho)$ in $\mathcal{R}_P$; we intend to show that that morphism is a comprehension of α as an object in $(P)_{\mathrm{q}}(A, \rho)$. The following is a trivial computation in $\mathcal{D}es_{P_{\{\!|\alpha|\!\} \times \{\!|\alpha|\!\}}(\rho)} \subseteq P(X)$:

$$\top_X = P_{\{\!|\alpha|\!\}}(\alpha) = (P)_{\mathrm{q}\,\{\!|\alpha|\!\}}(\alpha).$$

Suppose now that $f \colon (Z, \sigma) \to (A, \rho)$ is such that $\top_Z = (P)_{\mathrm{q}\,f}(\alpha)$. Since $\{\!|\alpha|\!\}$ is a comprehension in $\mathcal{C}$, there is a unique morphism $g \colon Z \to X$ such that $f = \{\!|\alpha|\!\} \circ g$. To conclude, it is enough to show that g determines a morphism $(Z, \sigma) \to (X, P_{\{\!|\alpha|\!\} \times \{\!|\alpha|\!\}}(\rho))$ in $\mathcal{R}_P$, but

$$\sigma \leq_{Z \times Z} P_{f \times f}(\rho) = P_{g \times g}(P_{\{\!|\alpha|\!\} \times \{\!|\alpha|\!\}}(\rho)).$$

As for the second part of the statement, let α be in $P(A)$ and let $\{\!|\alpha|\!\} \colon X \to A$ be a comprehension of α in $\mathcal{C}$. Suppose, first, that $\delta_X = P_{\{\!|\alpha|\!\} \times \{\!|\alpha|\!\}}(\delta_A)$, and consider a morphism $f \colon (Z, \sigma) \to (A, \delta_A)$ such that $((P)_{\mathrm{q}})_f(\alpha) = \top_Z$. By definition of $(P)_{\mathrm{q}}$, there is a unique morphism $g \colon Z \to X$ such $f = \{\!|\alpha|\!\} \circ g$ in $\mathcal{C}$. Thus

$$\sigma \leq_{Z \times Z} P_{f \times f}(\delta_A) = P_{g \times g} P_{\{\!|\alpha|\!\} \times \{\!|\alpha|\!\}}(\delta_A) = P_{g \times g}(\delta_X).$$

Conversely, suppose $\{\!|\alpha|\!\}\colon (X,\delta_X) \to (A,\delta_A)$ in $\mathcal{R}_P$ is a (necessarily monic) comprehension of α in $(P)_{\mathrm{q}}$. Consider $\{\!|\alpha|\!\}\colon (X, P_{\{\!|\alpha|\!\}\times\{\!|\alpha|\!\}}(\delta_A)) \to (A,\delta_A)$. Since $((P)_{\mathrm{q}})_{\{\!|\alpha|\!\}}(\alpha) = P_{\{\!|\alpha|\!\}}(\alpha) = \top_X$, the morphism must factor through $\{\!|\alpha|\!\}\colon (X,\delta_X) \to (A,\delta_A)$, necessarily with the identity morphism. Hence the conclusion follows. $\square$

4.7 Remark. When P has full comprehensions and every diagonal morphism $\langle \mathrm{id}_A, \mathrm{id}_A\rangle\colon A \to A \times A$ is a comprehension, the condition $\delta_X = P_{\{\!|\alpha|\!\}\times\{\!|\alpha|\!\}}(\delta_A)$ is ensured for all A and α.

Observe that the fibration of vertical morphisms on the category $\mathcal{G}r_P$ of points [Jacobs, 1999] universally adds comprehensions to a given fibration producing an indexed poset in case the given fibration is such. In our case of interest, for a doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$, the indexed poset consists of the base category $\mathcal{G}r_P$ where

an object is a pair (A,α) where A is in $\mathcal{C}$ and α is in $P(A)$;

a morphism $f\colon (A,\alpha) \to (B,\beta)$ is a morphism $f\colon A \to B$ in $\mathcal{C}$ such that $\alpha \le P_f(\beta)$.

The category $\mathcal{G}r_P$ has products and there is a natural embedding $I\colon \mathcal{C} \to \mathcal{G}r_P$ which maps A to $(A, \top_A)$. The indexed functor extends to $(P)_{\mathrm{c}}\colon {\mathcal{G}r_P}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ along I by setting $(P)_{\mathrm{c}}(A,\alpha) := \{\gamma \in P(A) \mid \gamma \le \alpha\}$. Moreover, the comprehensions in $(P)_{\mathrm{c}}$ are full. As an immediate corollary, we have the following.

4.8 Theorem. *There is a left biadjoint to the forgetful 2-functor from the full 2-category of* **QED** *on elementary doctrines with comprehensions and stable descent quotients into the 2-category* **ED** *of elementary doctrines.*

Proof. The left biadjoint sends an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ to the elementary doctrine $((P)_{\mathrm{c}})_{\mathrm{q}}\colon {\mathcal{R}_{(P)_{\mathrm{c}}}}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$. $\square$

5 Extensional equality

In [Maietti and Rosolini, 2013a], "extensional" models of constructive theories, presented as doctrines $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$, were obtained by forcing the equality of morphisms $f, g\colon A \to B$ in the base category $\mathcal{C}$ to correspond to the "provable" equality $\top_A = P_{\langle f,g\rangle}(\delta_B)$ in the fibre $P(A)$. We recall from [Jacobs, 1999] the basic property that supports a stronger notion of equality for the case of an elementary doctrine.

5.1 Proposition. *Let $P\colon \mathcal{C}^{op} \longrightarrow \mathcal{I}nf\mathcal{SL}$ be an elementary doctrine and let A be an object in $\mathcal{C}$. The diagonal $\langle \mathrm{id}_A, \mathrm{id}_A\rangle\colon A \to A \times A$ is a comprehension if and only if it is the comprehension of δ_A.*

5.2 DEFINITION. Given an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ we say that it has ***comprehensive diagonals*** if every diagonal morphism $\langle \mathrm{id}_A, \mathrm{id}_A\rangle\colon A \to A \times A$ is a comprehension.

5.3 REMARK. In case $\mathcal{C}$ has equalizers, one finds that P has comprehensive diagonals in the sense of [Maietti and Rosolini, 2013a].

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ be an elementary doctrine for the rest of the section. Consider the category $\mathcal{X}_P$, the "extensional collapse" of P:

the objects of $\mathcal{X}_P$ are the objects of $\mathcal{C}$;

a morphism $[f]: A \to B$ is an equivalence class of morphisms $f\colon A \to B$ in $\mathcal{C}$ such that $\delta_A \leq_{A\times A} P_{f\times f}(\delta_B)$ in $P(A\times A)$ with respect to the equivalence which relates f and f' when $\delta_A \leq_{A\times A} P_{f\times f'}(\delta_B)$.

Composition is given by that of $\mathcal{C}$ on representatives, and identities are represented by identities of $\mathcal{C}$.

5.4 LEMMA. *The quotient functor $\mathcal{C} \longrightarrow \mathcal{X}_P$ preserves finite products.*

Proof. Given a product diagram $A \xleftarrow{\mathrm{pr}_1} A\times B \xrightarrow{\mathrm{pr}_2} B$ in $\mathcal{C}$, the diagram

$$A \xleftarrow{[\mathrm{pr}_1]} A\times B \xrightarrow{[\mathrm{pr}_2]} B$$

in $\mathcal{X}_P$ is clearly a weak product. To check that it is strong, suppose that $f, g\colon X \to A\times B$ are such that $\delta_X \leq_{X\times X} P_{(\mathrm{pr}_1 f)\times(\mathrm{pr}_1 g)}(\delta_A)$ and $\delta_X \leq_{X\times X} P_{(\mathrm{pr}_2 f)\times(\mathrm{pr}_2 g)}(\delta_B)$. Recall from 2.3 that

$$\delta_{A\times B} = \delta_A \boxtimes \delta_B = P_{\langle \mathrm{pr}_1, \mathrm{pr}_3\rangle}(\delta_A) \wedge P_{\langle \mathrm{pr}_2, \mathrm{pr}_4\rangle}(\delta_B)$$

where $\mathrm{pr}_i, i = 1, 2, 3, 4$, are the projections from $A\times B\times A\times B$. So

$$P_{f\times g}(\delta_{A\times B}) = P_{(\mathrm{pr}_1 f)\times(\mathrm{pr}_1 g)}(\delta_A) \wedge P_{(\mathrm{pr}_2 f)\times(\mathrm{pr}_2 g)}(\delta_B).$$

Hence the hypothesis on f and g ensures that $\delta_X \leq_{X\times X} P_{f\times g}(\delta_{A\times B})$ which yields the conclusion. □

The indexed inf-semilattice $(P)_{\mathrm{x}}\colon {\mathcal{X}_P}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ on $\mathcal{X}_P$ will be given essentially by P itself; the following lemma is instrumental to give the assignment on morphisms using the action of P on morphisms.

5.5 LEMMA. *With the notation used above, let $f, g\colon A \to B$ be morphisms in $\mathcal{C}$ and β an object in $P(B)$. If $\delta_A \leq_{A\times A} P_{f\times g}(\delta_B)$, then $P_f(\beta) = P_g(\beta)$.*

Proof. Suppose that $\delta_A \leq_{A\times A} P_{f\times g}(\delta_B)$. Write $\mathrm{pr}_1, \mathrm{pr}_2\colon A\times A \to A$ for the two projections and, similarly, $\mathrm{pr}'_1, \mathrm{pr}'_2\colon B\times B \to B$. By 2.2(b) one has $P_{\mathrm{pr}'_1}(\beta) \wedge \delta_B \leq_{B\times B} P_{\mathrm{pr}'_2}(\beta)$. Thus

$$P_{f\times g}(P_{\mathrm{pr}'_1}(\beta)) \wedge P_{f\times g}(\delta_B) \leq_{A\times A} P_{f\times g}(P_{\mathrm{pr}'_2}(\beta)).$$

From the hypothesis it follows that

$$P_{f\circ\mathrm{pr}_1}(\beta)\wedge\delta_A \le_{A\times A} P_{g\circ\mathrm{pr}_2}(\beta).$$

Taking $P_{\langle\mathrm{id}_A,\mathrm{id}_A\rangle}$ of both sides,

$$P_f(\beta)=P_f(\beta)\wedge\top_A = P_{\langle\mathrm{id}_A,\mathrm{id}_A\rangle}(P_{f\circ\mathrm{pr}_1}(\beta))\wedge P_{\langle\mathrm{id}_A,\mathrm{id}_A\rangle}(\delta_A)\le P_{\langle\mathrm{id}_A,\mathrm{id}_A\rangle}(P_{g\circ\mathrm{pr}_2}(\beta))=P_g(\beta).$$

The other direction follows by symmetry. □

In other words, the elementary doctrine $P\colon\mathcal{C}^{\mathrm{op}}\longrightarrow\mathcal{I}nf\mathcal{SL}$ factors through the quotient functor $K\colon\mathcal{C}^{\mathrm{op}}\longrightarrow\mathcal{X}_P$. That induces a 1-morphism $(K,k)\colon P\to(P)_{\mathrm{x}}$ in **ED**, where k_A is the identity for A in $\mathcal{C}$.

Consider the full 2-subcategory **CED** of **ED** whose objects are elementary doctrines $P\colon\mathcal{C}^{\mathrm{op}}\longrightarrow\mathcal{I}nf\mathcal{SL}$ with comprehensive diagonals.

The following result is now obvious.

5.6 Lemma. *With the notation used above, $(P)_{\mathrm{x}}\colon{\mathcal{X}_P}^{\mathrm{op}}\longrightarrow\mathcal{I}nf\mathcal{SL}$ is an elementary doctrine with comprehensive diagonals.*

Also the following is easy.

5.7 Theorem. *For every elementary doctrine $P\colon\mathcal{C}^{\mathrm{op}}\longrightarrow\mathcal{I}nf\mathcal{SL}$, pre-composition with the 1-morphism*

$$\begin{tikzcd}
\mathcal{C}^{op} \arrow[dd, "K"'] \arrow[drr, "P"] \arrow[dd, phantom, shift left=6ex, "k\,\downarrow\cdot"] & & \\
 & & \mathcal{I}nf\mathcal{SL} \\
{\mathcal{X}_P}^{op} \arrow[urr, "(P)_{\mathrm{x}}"'] & &
\end{tikzcd}$$

in **ED** *induces an essential equivalence of categories*

$$-\circ(K,k)\colon\mathbf{CED}((P)_{\mathrm{x}},Z)\equiv\mathbf{ED}(P,Z)\tag{3}$$

for every Z in **CED**.

We can now mention the explicit connection between the two universal constructions we have considered. For that it is useful to prove the following two lemmas.

Note that for any elementary doctrine $Q\colon\mathcal{C}^{\mathrm{op}}\longrightarrow\mathcal{I}nf\mathcal{SL}$ with effective quotients, the doctrine $(Q)_{\mathrm{x}}$ has only a weak form of quotients. But when $Q=(P)_{\mathrm{q}}$ of an elementary doctrine P, then we can prove:

5.8 Lemma. *Let $P\colon\mathcal{C}^{\mathrm{op}}\longrightarrow\mathcal{I}nf\mathcal{SL}$ be an elementary doctrine. The morphism $(K,k)\colon(P)_{\mathrm{q}}\to((P)_{\mathrm{q}})_{\mathrm{x}}$ preserves quotients and therefore $((P)_{\mathrm{q}})_{\mathrm{x}}$ has effective descent quotients of $((P)_{\mathrm{q}})_{\mathrm{x}}$-equivalence relations.*

Proof. It is easy to check that K preserves quotients of $(P)_{\mathrm{q}}$-equivalence relations. Since the k-components of $(K,k)\colon P\to(P)_{\mathrm{x}}$ are identity functions, a $((P)_{\mathrm{q}})_{\mathrm{x}}$-equivalence relation τ on A is also a $(P)_{\mathrm{q}}$-equivalence relation in $P(A\times A)$. □

Moreover, note that for any elementary doctrine $Q\colon\mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ with comprehensions, the doctrine $(Q)_{\mathrm{x}}$ has only weak comprehensions. But when $Q = (P)_{\mathrm{c}}$ of an elementary doctrine P, then we can prove:

5.9 LEMMA. *Let $P\colon\mathcal{C}^{op} \longrightarrow \mathcal{I}nf\mathcal{SL}$ be an elementary doctrine, then $((P)_c)_x$ has full comprehensions and comprehensive diagonals.*

The results of this section, together with 4.5, produce an extension of the quotient completion of [Maietti and Rosolini, 2013a].

5.10 THEOREM. *There is a left biadjoint to the forgetful 2-functor from the full 2-category of* **QED** *on elementary doctrines with comprehensions, stable descent quotients and comprehensive diagonals into the 2-category* **ED** *of elementary doctrines.*

Proof. The left biadjoint sends an elementary doctrine $P\colon\mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ to the elementary quotient completion $((((P)_{\mathrm{c}})_{\mathrm{x}})_{\mathrm{q}})_{\mathrm{x}}\colon {\mathcal{X}_{(((P)_{\mathrm{c}})_{\mathrm{x}})_{\mathrm{q}}}}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$. □

5.11 COROLLARY. *For $P\colon\mathcal{C}^{op} \longrightarrow \mathcal{I}nf\mathcal{SL}$ an elementary doctrine, the elementary quotient completion $\overline{P}\colon {\mathcal{Q}_P}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ in* [Maietti and Rosolini, 2013a] *coincides with the doctrine $((P)_{\mathrm{q}})_x\colon {\mathcal{X}_{(P)_{\mathrm{q}}}}^{op} \longrightarrow \mathcal{I}nf\mathcal{SL}$.*

5.12 REMARK. Because of the logical setup in [Maietti and Rosolini, 2013a], only a particular case of 5.10 was proved, namely the left biadjoint was restricted to the full sub-2-category of **ED** of elementary doctrines with full comprehensions and comprehensive diagonals, see 5.3. On those doctrines $P\colon\mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$, the action of the left biadjoint was simply $((P)_{\mathrm{q}})_{\mathrm{x}}\colon {\mathcal{X}_{(P)_{\mathrm{q}}}}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$.

6 Comparing some universal contructions

The elementary quotient completion resembles very closely that of exact completion. In fact, one has the following results.

6.1 THEOREM. *Given a category $\mathcal{S}$ with finite products and weak pullbacks, let $\Psi\colon\mathcal{S}^{op} \longrightarrow \mathcal{I}nf\mathcal{SL}$ be the elementary doctrine of weak subobjects. Then the doctrine $((\Psi)_{\mathrm{q}})_x\colon {\mathcal{X}_{(\Psi)_{\mathrm{q}}}}^{op} \longrightarrow \mathcal{I}nf\mathcal{SL}$, is equivalent to the doctrine $S\colon{\mathcal{S}_{\mathrm{ex}}}^{op} \longrightarrow \mathcal{I}nf\mathcal{SL}$ of subobjects on the exact completion $\mathcal{S}_{\mathrm{ex}}$ of $\mathcal{S}$.*

Proof. It follows from 4.3 and the characterization of the embedding of $\mathcal{S}$ into $\mathcal{S}_{\mathrm{ex}}$ in [Carboni and Vitale, 1998]. □

Though an elementary quotient completion with full comprehension is regular, see [Maietti and Rosolini, 2013a], the regular completion is an instance of a completion of a doctrine which is radically different from the elementary quotient completion in 5.10.

6.2 Remark. For an elementary dotrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$, a ***weak comprehension of*** α is a morphism $\{\!|\alpha|\!\}\colon X \to A$ in $\mathcal{C}$ such that $\top_X \leq P_{\{\!|\alpha|\!\}}(\alpha)$ and, for every morphism $g\colon Y \to A$ such that $\top_Y \leq P_g(\alpha)$ there is a (not necessarily unique) $h\colon Y \to X$ such that $g = \{\!|\alpha|\!\} \circ h$, see [Maietti and Rosolini, 2013a].

For an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ with weak comprehensions, it is possible to add (strong) comprehensions to its extensional collapse as formal retracts of weak comprehensions: consider the category $\mathcal{D}_P$ determined by the following data

objects of $\mathcal{D}_P$ are triples (A, α, c) such that A is an object in $\mathcal{C}$, α is an object in $P(A)$, and $c\colon X \to A$ is a weak comprehension α;

a morphism $[f]\colon (A, \alpha, c) \to (B, \beta, d)$ is an equivalence class of morphisms $f\colon X \to Y$ in $\mathcal{C}$ such that $P_{c\times c}(\delta_A) \leq P_{f\times f}(P_{d\times d}(\delta_B))$ with respect to the relation $f \sim f'$ determined by $P_{c\times c}(\delta_A) \leq P_{f\times f'}(P_{d\times d}(\delta_B))$;

composition of $[f]\colon (A, \alpha, c) \to (B, \beta, d)$ and $[g]\colon (B, \beta, d) \to (C, \gamma, e)$ is $[g \circ f]$.

There is a full functor $K\colon \mathcal{C} \to \mathcal{D}_P$ defined on objects A as $K(A) := (A, \top_A, \mathrm{id}_A)$—it factors through $\mathcal{X}_P$. It preserves products and there is an extension $(P)_{\mathrm{r}}\colon {\mathcal{D}_P}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ of $P\colon C^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ defined on objects as $(P)_{\mathrm{r}}(A, \alpha, c) := \mathcal{D}es_{(P_{c\times c}(\delta_A))}$. The doctrine $(P)_{\mathrm{r}}\colon {\mathcal{D}_P}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ is elementary with comprehensions and K preserves all existing comprehensions.

Given a category $\mathcal{S}$ with finite products and weak pullbacks, let $\Psi\colon \mathcal{S}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ be the elementary doctrine of weak subobjects. Then the doctrine $(\Psi)_{\mathrm{r}}\colon {\mathcal{D}_\Psi}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ is equivalent to the doctrine $S\colon {\mathcal{S}_{\mathrm{reg}}}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ of subobjects on the regular completion $\mathcal{S}_{\mathrm{reg}}$ of $\mathcal{S}$.

The proof is similar to that of 6.1 since, in the regular completion $\mathcal{S}_{\mathrm{reg}}$ of $\mathcal{S}$, every object is covered by a regular projective and a subobject of a regular projective, see [Carboni and Vitale, 1998].

Since the construction given in 6.1 factors through that in 6.2 via the exact completion of a regular category, see [Freyd and Scedrov, 1991], and the exact completion of a weakly finitely complete category may appear very similar to the category $\mathcal{X}_{((P)_{\mathrm{q}})_{\mathrm{x}}}$, it is appropriate to mention an example of an elementary quotient completion which is not exact.

For that, consider the indexed poset on the monoid of partial recursive functions $F\colon \mathcal{N}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ whose value on the single object of $\mathcal{N}$ is the powerset of the natural numbers and, for any φ partial recursive function, $F_\varphi := \varphi^{-1}$, the inverse image of a subset along the partial function. It is clearly an elementary doctrine, and the doctrine $((F)_{\mathrm{c}})_{\mathrm{x}}\colon {\mathcal{X}_{(F)_{\mathrm{c}}}}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ is equivalent to the subobject doctrine $S\colon \mathcal{PR}^{\mathrm{op}} \longrightarrow \mathcal{I}nf\mathcal{SL}$ on the category $\mathcal{PR}$ of subsets of natural numbers and (restrictions of) partial recursive functions between them, see [Carboni, 1995] for properties of that category, in particular its exact completion (as a weakly finitely complete category) is the category $\mathcal{D}$ of discrete objects of the effective topos.

Now, if one considers the elementary doctrine $((S)_\mathrm{q})_\mathrm{x}\colon {\mathcal{X}_{(S)_\mathrm{q}}}^{\mathrm{op}} \longrightarrow \mathcal{I}\mathit{nfSL}$, the category $\mathcal{X}_{(S)_\mathrm{q}}$ is equivalent to the category $\mathcal{PER}$ of partial equivalence relations on the natural numbers, and the indexed poset $((S)_\mathrm{q})_\mathrm{x}$ is equivalent to that of subobjects on that category. The category $\mathcal{PER}$ is not exact because there are equivalence relations which are not kernel equivalences. In fact, the exact completion $\mathcal{PER}_{\mathrm{ex/reg}}$ of $\mathcal{PER}$ as a regular category is the category $\mathcal{D}$ of discrete objects.

Similar examples can be produced using topological categories such as those in the papers [Birkedal et al., 1998, Carboni and Rosolini, 2000]. Other examples of elementary quotient completions that are not exact are given in the paper [Maietti and Rosolini, 2013a]: one is applied to the doctrine of the Calculus of Constructions [Coquand, 1990, Streicher, 1992] and the other to the doctrine of the intensional level of the minimalist foundation in [Maietti, 2009].

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