LOCAL CARTESIAN CLOSURE OF ELEMENTARY QUOTIENT COMPLETIONS

MARIA EMILIA MAIETTI, FABIO PASQUALI AND GIUSEPPE ROSOLINI

Abstract. The elementary quotient completion of an elementary doctrine in the sense of Lawvere was introduced in previous work by the first and third authors. It generalizes the exact completion of a category with finite products and weak pullbacks. In this paper, we characterize when an elementary quotient completion is locally cartesian closed in terms of properties of the elementary doctrine which generates it. It generalizes the characterization of locally cartesian closed exact completions given by the third author with Carboni, in the case that the exact completion is performed on a finite product category with weak pullbacks.

1. Introduction

The study of constructions for completing a category with quotients is a central topic in mathematics and in computer science. Well-known related notions are those of the exact completions of a category with finite limits, and of a regular category, see [Carboni and Celia Magno, 1982, Carboni and Vitale, 1998], which have been widely studied and applied.

In [Maietti and Rosolini, 2013b], the first and third authors generalized the notion of exact completion on a category with weak finite limits to that of an elementary quotient completion of an elementary doctrine [Lawvere, 1970] as a universal construction to provide such a doctrine with quotients. The fundamental contribution of Bill Lawvere in determining structure in logic cannot be overestimated. Hyperdoctrines, introduced in [Lawvere, 1969], have proved to be a crucial tool for the study of logic and its applications, see *e.g.* the survey in [Pitts, 2000] and the references there.

The exact completion of a category $\mathcal{C}$ with finite products and weak pullbacks is a principal instance of such a construction since the doctrine of subobjects of an exact completion is the elementary quotient completion of the weak subobject doctrine of $\mathcal{C}$, see [Lawvere, 1996], see also [Grandis, 2000] for the notion of weak subobject. However, not all elementary quotient completions produce exact categories. Notable examples of

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non-exact categories arising as quotient completions include the category of equilogical spaces of [Scott, 1996, Bauer et al., 2004] and that of assemblies over a partial combinatory algebra, see [Hyland, 1982, van Oosten, 2008]). Other relevant examples are provided by the syntactic category of total setoids in dependent type theory, in the style of [Bishop, 1967], over the Calculus of Inductive Constructions [Coquand and Huet, 1988, Coquand and Paulin, 1990], the theory at the base of the proof-assistant Rocq, as well as the syntactic quotient category used to build the Minimalist Foundation in [Maietti, 2009], described in [Maietti and Rosolini, 2013b]. In these examples, the base of the syntactic doctrine generating the elementary quotient completion has only weak pullbacks.

In the paper, we study sufficient and necessary conditions that a Lawverian elementary doctrine satisfies, to guarantee that the base of its elementary quotient completion is locally cartesian closed. Such conditions amount to require that the elementary doctrine is slicewise weakly cartesian closed, see Definition 3.5.

That characterization generalizes that for the locally cartesian closed exact completions in [Carboni and Rosolini, 2000], as well as its revision in [Emmenegger, 2020]. In particular, when applied to categories with finite products and weak equalizers, the notion of slicewise weakly cartesian closed category provides necessary and sufficient conditions equivalent to those in [Emmenegger, 2020] but formulated only in terms of adjunctions or their weakened form. We also give a negative answer (see Proposition **??**) to the possibility of extending the elementary quotient completion bi-adjunction of [Maietti and Rosolini, 2013b] to a 2-category of elementary doctrines admitting weak comprehension with comprehensive diagonals, but we established a universal property, under suitable hypotheses, in parallel with what is proved in [Carboni and Vitale, 1998] for the exact completion of a category with weak finite limits.

In future work, we will investigate how the characterization presented here can be extended to locally cartesian closed elementary quotient completions of doctrines on a base category with weak finite products, such as the "biased doctrines" considered in [Cioffo, 2022, Cioffo, 2023]. In [Cioffo, 2022], the author characterized locally cartesian closed elementary quotient completions for biased doctrines that are also universal, by adapting notions inspired by [Emmenegger, 2020]. As a subsequent development, we intend to study how to enforce such a characterization in the general case of biased doctrines in such a way that the characterizations in [Carboni and Rosolini, 2000, Emmenegger, 2020] arise as special instances.

The proofs presented in the paper are carried out mainly using the internal logic of the doctrines involved, producing explicit calculations that make it evident that they are entirely constructive and suitable for formalisation in a proof assistant.

2. Basic properties of elementary doctrines

Mainly to introduce the notation that we use in the rest of the paper, we briefly recall the construction of the *elementary quotient completion* of an elementary doctrine from [Maietti and Rosolini, 2013b], together with some properties of a doctrine.

2.1. Elementary doctrines

An indexed poset $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Pos}$ on a category $\mathcal{C}$ with finite products is a ***primary doctrine*** when each fiber is an inf-semilattice, and any reindexing functor $P_f\colon P(B) \longrightarrow P(A)$, for $f\colon A \longrightarrow B$ in $\mathcal{C}$, preserves finite infs, *i.e.* P takes values in the category InfSL of inf-semilattices and finite infs preserving homomorphisms. A primary doctrine is an ***elementary doctrine*** when, for every object A in $\mathcal{C}$, there is an object δ_A in $P(A \times A)$ such that for every arrow $e := \langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_2\rangle\colon X \times A \longrightarrow X \times A \times A$ in $\mathcal{C}$, where pr_i denotes the homonymous projection from the product $X \times A$, the assignment

$$\mathit{\exists}_e(\alpha) := P_{\langle \mathrm{pr}'_1, \mathrm{pr}'_2\rangle}(\alpha) \wedge_{X\times A\times A} P_{\langle \mathrm{pr}'_2, \mathrm{pr}'_3\rangle}(\delta_A)$$

for α in $P(X \times A)$ and pr'_i the homonymous projection from the product $X \times A \times A$ determines a left adjoint to

$$P_e\colon P(X \times A \times A) \longrightarrow P(X \times A).$$

We call $\mathcal{C}$ the ***base*** of the doctrine. One says that α is ***over*** A when α is an element of $P(A)$. The top element over the object A of $\mathcal{C}$ is denoted by $\top_A$, and given α and β over A, their inf is $\alpha \wedge_A \beta$ (we may drop subscripts when these are clear from the context).

Elementary doctrines are just another, conceptually slightly different, presentation of the amnestic Eq-fibrations of [Jacobs, 1999]. An obvious 1-1 correspondence is just a direct extension of the usual indexed family/function correspondence.

For an elementary doctrine, the assignment

$$\mathit{\exists}_{\langle \mathrm{id}_A, \mathrm{id}_A\rangle}(\alpha) := P_{\mathrm{pr}_1}(\alpha) \wedge_{A\times A} \delta_A$$

for α in $P(A)$ determines a left adjoint to the reindexing

$$P_{\langle \mathrm{id}_A, \mathrm{id}_A\rangle}\colon P(A \times A) \longrightarrow P(A).$$

Thus, δ_A is determined uniquely for each object A in $\mathcal{C}$. We will refer to δ_A as the ***fibered equality*** on A.

Elementary doctrines were inspired by Lawvere's notion of hyperdoctrine [Lawvere, 1970], that provide an appropriate mathematical structure to study logical theories independently of their presentation. Indeed, the examples of elementary doctrines that come directly from first-order logic are given by the indexed posets of Lindenbaum–Tarski algebras of well-formed formulas. In detail, given a theory $\mathscr{T}$ in the first order language $\mathscr{L}$, the base category is the category $\mathcal{V}$ whose objects are lists of distinct variables and where an arrow is a list $(t_j)_{j=1}^m\colon (x_i)_{i=1}^n \longrightarrow (y_j)_{j=1}^m$ of terms of $\mathscr{L}$ where each term t_j is in the variables $x_1, \ldots, x_n$. To obtain the primary doctrine $LT_{\mathscr{T}}\colon \mathcal{V}^{\mathrm{op}} \longrightarrow \mathsf{Pos}$, the fiber over the list of variables $(x_i)_{i=1}^n$ in $\mathcal{V}$ is the Lindenbaum–Tarski algebra $LT_{\mathscr{T}}(\vec{x})$ of the equivalence classes of well-formed formulas in the free variables $(x_i)_{i=1}^n$ with respect to provable equivalence in $\mathscr{T}$. Infs in $LT_{\mathscr{T}}(x_i)$ are given by conjunctions. The top element is any true formula. The primary doctrine $LT_{\mathscr{T}}$ is elementary if and only the equality is definable in $\mathscr{T}$. We refer the reader to [Maietti and Rosolini, 2013a] for more details.

Direct examples of elementary doctrines from categories are given by

- the subobject functor $\mathrm{Sub}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ for $\mathcal{C}$ a category with finite limits. The object δ_A is the subobject represented by the diagonal on A;
- the functor of ***variations*** $\Psi_{\mathcal{D}}\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$, introduced in [Lawvere, 1996], when the category $\mathcal{D}$ has binary products and weak pullbacks. Recall that $\Psi_{\mathcal{D}}(A)$ is the poset reflection of the comma category $\mathcal{D}/A$.

Another family of examples is that of evaluations into an inf-semilattice: given an inf-semilattice $\mathcal{H}$, consider the primary doctrine $\mathcal{P}_{\mathcal{H}}\colon \mathcal{S}et^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ which sends a set A to the poset $\mathcal{H}^A$ of $\mathcal{H}$-valued functions, and a function $f\colon A \longrightarrow B$ to pre-composition with it: $\mathcal{P}_{\mathcal{H}}(f) = - \circ f$. It is an elementary doctrine if and only if $\mathcal{H}$ has a bottom element, see [Emmenegger et al., 2020]. In this case δ_A is the function that maps (a, a') to $\top$ when $a = a'$, and to $\bot$ otherwise.

Observe that an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ gives the possibility of "evaluating" how equal two arrows $f, g\colon X \longrightarrow A$ in the category $\mathcal{C}$ are with respect to the "logic" of P by looking at the object $P_{\langle f,g\rangle}(\delta_A)$ in $P(X)$. This object could be $\top_X$ without f and g being actually equal as arrows in $\mathcal{C}$.

We say that diagonals are ***comprehensive*** in the elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ when $\top_X \leq P_{\langle f,g\rangle}(\delta_A)$ always yields that $f = g$.

The elementary doctrine $\mathrm{Sub}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ of subobjects, for any category $\mathcal{C}$ with finite limits, has comprehensive diagonals as well as that of variations $\Psi_{\mathcal{D}}\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ when $\mathcal{D}$ has finite products and weak pullbacks.

Elementary doctrines are the objects of the 2-category $\mathbf{ED}$ where a 1-arrow $P \longrightarrow D$ is a pair made by a functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ that preserves finite products and a natural transformation $f\colon P \longrightarrow D \circ F^{\mathrm{op}}$ that preserves finite meets and fibered equality (we refer the reader to [Maietti and Rosolini, 2013a] for the explicit description). We limit ourselves to note that the category $\mathbf{ED}(LT_{\mathscr{T}}, \mathrm{Sub}_{\mathcal{S}et})$ is equivalent to the category of models of the theory $\mathscr{T}$ and 1-1 homomorphisms, for $\mathscr{T}$ a Horn theory, *i.e.* a theory in the $\wedge$ =-fragment.

We recall from [Maietti and Rosolini, 2013a] that it is possible to force comprehensive diagonals in an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ as follows: Consider the quotient category $\mathcal{X}_P$ where two arrows $f, g\colon A \longrightarrow B$ are equivalent if $\top_A \leq P_{f,g}(\delta_B)$. And note that the action of P on equivalent arrows is the same so that P actually factors through the (opposite of the) quotient functor $Q\colon \mathcal{C} \longrightarrow \mathcal{X}_P$.

2.2. Existential doctrines

Similarly to the way elementary doctrines structure logical theories with equality, existential doctrines provide the doctrinal structure for existential quantification. In the spirit of [Lawvere, 1970] we say that a primary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is ***existential*** when, for very A_1 and A_2 in $\mathcal{C}$, for a(ny) projection $\mathrm{pr}_i\colon A_1 \times A_2 \longrightarrow A_i$, $i = 1, 2$, the functor $P_{\mathrm{pr}_i}\colon P(A_i) \longrightarrow P(A_1 \times A_2)$ has a left adjoint $\exists_{\mathrm{pr}_i}$ and those left adjoints satisfy the following two conditions.

***Beck-Chevalley condition*:** for any pullback diagram

$$\begin{tikzcd} X' \arrow[r, "\mathrm{pr}'"] \arrow[d, "f'"'] & A' \arrow[d, "f"] \\ X \arrow[r, "\mathrm{pr}"] & A \end{tikzcd}$$

with pr a projection (hence also pr′ a projection), for any β in $P(X)$, the canonical inequality $\mathit{\exists}_{\mathrm{pr}'} P_{f'}(\beta) \leq P_f \mathit{\exists}_{\mathrm{pr}}(\beta)$ in $P(A')$ is an equality.

***Frobenius Reciprocity*:** for $\mathrm{pr}\colon X \longrightarrow A$ a projection, α in $P(A)$, β in $P(X)$, the canonical inequality $\mathit{\exists}_{\mathrm{pr}}(P_{\mathrm{pr}}(\alpha) \wedge \beta) \leq \alpha \wedge \mathit{\exists}_{\mathrm{pr}}(\beta)$ in $P(A)$ is actually an equality.

The primary doctrine $LT_{\mathscr{T}}\colon \mathcal{V}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ associated to a first order theory $\mathscr{T}$ is existential: left adjoints along projections are given by existential quantification. The doctrine $\mathrm{Sub}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ of subobjects on a category $\mathcal{C}$ with finite limits is existential if and only if $\mathcal{C}$ is regular (see [Jacobs, 1999] on page 258). On the other hand, when $\mathcal{D}$ has finite products and weak pullbacks, the weak subobject doctrine $\Psi_{\mathcal{D}}\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is existential as left adjoints are given by post-composition. For $\mathcal{H}$ an inf-semilattice, the primary doctrine $\mathcal{P}_{\mathcal{H}}\colon \mathcal{S}et^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is existential if and only if $\mathcal{H}$ is a frame, see [Emmenegger et al., 2020].

2.3. Remark.

As shown in [Lawvere, 1970], for an elementary existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ every reindexing P_f has a left adjoint $\mathit{\exists}_f$. For $f\colon A \longrightarrow B$ the map $\mathit{\exists}_f\colon P(A) \longrightarrow P(B)$ is

$$\mathit{\exists}_f(\alpha) := \mathit{\exists}_{\mathrm{pr}_2}[P_{f\times \mathrm{id}_B}(\delta_B) \wedge P_{\mathrm{pr}_1}(\alpha)]$$

see also [Pitts, 2002]. Moreover, such a left adjoint satisfies the Frobenius Reciprocity. But they need not satisfy the Beck–Chevalley condition, see *e.g.* [Maietti and Trotta, 2023] for a counterexample.

Elementary existential doctrines are the 0-cells of the 2-category **EED**, the 2-full subcategory of **ED** whose 1-cells are those 1-cells $(F, b)\colon P \longrightarrow R$ which also preserves the left adjoints to reindexing along projections. For a theory $\mathscr{T}$ axiomatised in the $\exists\wedge =$-fragment, the category $\mathbf{EED}(LT_{\mathscr{T}}, \mathrm{Sub}_{\mathcal{S}et})$ is equivalent to the category of models of $\mathscr{T}$ with elementary homomorphisms (in the model-theoretic sense of "elementary", *i.e.* those 1-1 homomorphisms which reflect existential quantification).

2.4. Hyperdoctrines

It should already be clear how a primary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ provides the core structure to be adjusted to model other logical operations: so we say that P is

- ***implicational*** if, for every object A in $\mathcal{C}$, every α in $P(A)$, the functor $\alpha \wedge -\colon P(A) \longrightarrow P(A)$ has a right adjoint $\alpha \Rightarrow -\colon P(A) \longrightarrow P(A)$, and reindexing preserves the operation, *i.e.* $P_f(\alpha \Rightarrow \beta) = P_f(\alpha) \Rightarrow P_f(\beta)$;

- ***universal*** if, for A_1 and A_2 in $\mathcal{C}$, for a(ny) projection $\mathrm{pr}_i\colon A_1 \times A_2 \longrightarrow A_i$, $i = 1, 2$, the functor $P_{\mathrm{pr}_i}\colon P(A_i) \longrightarrow P(A_1 \times A_2)$ has a right adjoint $\forall_{\mathrm{pr}_i}$, and these right adjoints satisfy the ***Beck-Chevalley condition***: for any pullback diagram

$$\begin{tikzcd} X' \arrow[r, "\mathrm{pr}'"] \arrow[d, "f'"'] & A' \arrow[d, "f"] \\ X \arrow[r, "\mathrm{pr}"] & A \end{tikzcd}$$

  with pr a projection (hence also pr′ a projection), for any β in $P(X)$, the canonical inequality $P_f \forall_{\mathrm{pr}}(\beta) \leq \forall_{\mathrm{pr}'} P_{f'}(\beta)$ in $P(A')$ is an equality.

A ***hyperdoctrine*** is a primary doctrine which is elementary, existential, implicational, and universal. Note that our notion of hyperdoctrine differs from the original in [Lawvere, 1969] in that in our case the base category $\mathcal{C}$ is not required to be closed and the preordered fibers must be strict orders equipped with left and right adjoints satisfying Beck-Chevalley conditions.

2.5. Remark.

Let P be an elementary doctrine such that every reindexing P_f has a left adjoint $\exists_f$ and every inf-semilattice $P(A)$ has pseudocomplements, that is, $P(A)$, seen as a category, is cartesian closed. Then the left adjoints satisfy the Frobenius Reciprocity (in the general sense of Remark 2.3) if and only if P is implicational. Indeed, the compositions $P_f(\alpha \Rightarrow -)$ and $P_f(\alpha) \Rightarrow P_f(-)$ coincides if and only if the compositions of their left adjoints $\alpha \wedge \exists_f(-)$ and $\exists_f(P_f(\alpha) \wedge P_f(-))$ coincide.

Clearly the primary doctrine $LT_{\mathcal{T}}\colon \mathcal{V}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ associated to a first-order theory $\mathcal{T}$ is a hyperdoctrine: the cartesian closed structure in a fiber is provided by the logical implication; the right adjoint to reindexing along a projection is provided by the universal quantification.

If $\mathcal{C}$ is a Heyting category (also called logos in [Freyd and Scedrov, 1990]), the primary doctrine $\mathrm{Sub}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ of subobjects is a hyperdoctrine.

Assuming that $\mathcal{D}$ has finite limits and each comma category $\mathcal{D}/A$ is weakly cartesian closed, the primary weak subobject doctrine $\Psi_{\mathcal{D}}\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is a hyperdoctrine. We shall see later how to generalize the description to the case when $\mathcal{D}$ is assumed to have only (finite products and) weak pullbacks.

Finally, similar to the existential case, given an inf-semilattice $\mathcal{H}$, the primary doctrine $\mathcal{P}_{\mathcal{H}}\colon \mathcal{S}et^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is implicational and universal if and only if $\mathcal{H}$ is a frame, see [Pitts, 2002].

2.6. Remark.

As for the left adjoint in Remark 2.3, Lawvere showed that in an implicational and universal doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \textsf{Pos}$ every map of the form P_f has a right adjoint $\forall_f$. For $f\colon A \longrightarrow B$ the map $\forall_f\colon P(A) \longrightarrow P(B)$ is $\forall_f(\alpha)\colon= \forall_{\mathrm{pr}_2}[P_{f\times \mathrm{id}_B}(\delta_B) \Rightarrow P_{\mathrm{pr}_1}(\alpha)]$, see also [Pitts, 2002].

2.7. Comprehension

A primary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \textsf{Pos}$ is said to admit ***comprehension*** if for every A in $\mathcal{C}$ and every α in $P(A)$ there is an arrow $\{\!|\alpha|\!\}\colon X \longrightarrow A$ such that $\top_X \le P_{\{\!|\alpha|\!\}}(\alpha)$, and for every arrow $f\colon Y \longrightarrow A$ such that $\top_Y \le P_f(\alpha)$ there is a unique arrow $k\colon Y \longrightarrow X$ such that $\{\!|\alpha|\!\}k = f$, see [Jacobs, 1999, Maietti and Rosolini, 2013b].

In case the mediating arrows k are not required to be unique, one says that P admits ***weak comprehension*** as in [Maietti and Rosolini, 2013b]. Sometimes, we may add the adjective "strong" when P admits comprehension to stress that fact in comparison with the weak condition.

Furthermore, we shall often refer to the arrow $\{\!|\alpha|\!\}$ as the ***comprehending*** arrow of α, or just as a ***comprehensive*** arrow if we drop the reference to the object α in the fiber.

2.8. Remark.

Suppose P is a primary doctrine admitting weak comprehension. To check the uniqueness of the mediating arrow, it is sufficient (and necessary) that $\{\!|\alpha|\!\}\colon X \longrightarrow A$ is monic. So P admits strong comprehension if (and only if) each comprehensive arrow $\{\!|\alpha|\!\}\colon X \longrightarrow A$ is monic.

Note that, for a doctrine P admitting comprehension, if $\alpha \le \beta$ in $P(X)$, then $\top_X \le P_{\{\!|\alpha|\!\}}(\beta)$. When also the converse holds, one says that comprehension is ***full***, *i.e.* for every A and every α, β over A, if $\top_X \le P_{\{\!|\alpha|\!\}}(\beta)$, then $\alpha \le \beta$.

Examples of primary doctrines admitting full comprehension are the subobject doctrines $\mathrm{Sub}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \textsf{Pos}$, for $\mathcal{C}$ with finite limits—a comprehending arrow of a subobject is any representative of the subobject.

On the other hand, when $\mathcal{D}$ has finite products and weak pullbacks, the doctrine $\Psi_{\mathcal{D}}\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \textsf{Pos}$ of variations admits full weak comprehension—any representative of a variation is a comprehending arrow for the variation.

For $\mathcal{H}$ an inf-semilattice, the doctrine $\mathcal{P}_{\mathcal{H}}$ admits strong comprehension. Given α in $\mathcal{P}_{\mathcal{H}}(A)$ the arrow $\{\!|\alpha|\!\}$ is the inclusion $\left\{a \in A \;\middle|\; \alpha(a) = \top\right\} \hookrightarrow A$. In general, the comprehension on $\mathcal{P}_{\mathcal{H}}$ is not full, *e.g.* take $\mathcal{H}$ with at least three elements and $\alpha, \beta\colon A \longrightarrow \mathcal{H}$ which agree only on $\alpha^{-1}\{\top\} = \beta^{-1}\{\top\}$.

In general, a doctrine of the form $LT_{\mathcal{T}}\colon \mathcal{V}^{\mathrm{op}} \longrightarrow \textsf{Pos}$ does not admit comprehension.

Comprehensive diagonals were originally introduced in [Maietti and Rosolini, 2013b] with the name of "comprehensive equalizers" in light of the property that an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ has comprehensive diagonals if and only if diagonals in $\mathcal{C}$ are the comprehending arrows of the corresponding fibered equalities, *i.e.* $\langle \mathrm{id}_A, \mathrm{id}_A\rangle = \{\!|\delta_A|\!\}$, see [Maietti and Rosolini, 2013b, Proposition 4.6]. Furthermore, when P admits full comprehension, comprehensive diagonals assure that its base $\mathcal{C}$ has equalizers.

2.9. Proposition.

Suppose $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{Pos}$ is an elementary doctrine with comprehensive diagonals.

(i) *If P admits weak comprehension, then $\mathcal{C}$ has weak equalizers, hence weak finite limits.*

(ii) *If P admits strong comprehension, then $\mathcal{C}$ has equalizers, hence all finite limits.*

Proof. A weak equalizer of $f, g\colon X \longrightarrow A$ is $\{\!|P_{\langle f,g\rangle}(\delta_A)|\!\}$. ∎

2.10. Proposition.

Let P be an elementary existential doctrine admitting weak comprehension. Then weak comprehension is full if and only if $\exists_{\{\!|\alpha|\!\}}(\top_X) = \alpha$ for every $\{\!|\alpha|\!\}\colon X \longrightarrow A$.

Proof. Assume P admits full weak comprehension. By the adjunction $\exists_{\{\!|\alpha|\!\}} \dashv P_{\{\!|\alpha|\!\}}$, from $\top_X = P_{\{\!|\alpha|\!\}}(\alpha)$ it follows that $\exists_{\{\!|\alpha|\!\}}(\top_X) \leq \alpha$. Fullness applied to the adjunction unit $\top_X \leq P_{\{\!|\alpha|\!\}}(\exists_{\{\!|\alpha|\!\}}(\top_X))$ ensures instead that $\alpha \leq \exists_{\{\!|\alpha|\!\}}(\top_X)$ holds.
Conversely, if $\exists_{\{\!|\alpha|\!\}}(\top_X) = \alpha$, from $\top_X = P_{\{\!|\alpha|\!\}}(\beta)$ it follows $\alpha = \exists_{\{\!|\alpha|\!\}}(\top_X) = \exists_{\{\!|\alpha|\!\}}P_{\{\!|\alpha|\!\}}(\beta) \leq \beta$ using the counit of the adjunction $\exists_{\{\!|\alpha|\!\}} \dashv P_{\{\!|\alpha|\!\}}$. ∎

2.11. Proposition.

Let $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{Pos}$ be an elementary existential doctrine admitting full weak comprehension.

(i) *If P has comprehensive diagonals, and the diagram*

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[d, "k"'] & A \arrow[d, "f"] \\ B \arrow[r, "h"] & C \end{tikzcd}$$

is a weak pullback, then $P_f\exists_h = \exists_g P_k$.

(ii) *If every reindexing P_f has a right adjoint, then P is implicational.*

Proof. (i) See [Maietti et al., 2017, Theorem 2.19].
(ii) See [Maietti and Rosolini, 2013b, Lemma 4.9]. ∎

We end this part on comprehension by recalling the free completion of an elementary doctrine to one admitting comprehension, and the logical notation that one can employ with elementary doctrines.

Given an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$, the free comprehension completion is the doctrine $P_{\mathrm{c}}\colon {\mathcal{C}_{\mathrm{c}}}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ where the base $\mathcal{C}_{\mathrm{c}}$ has objects pairs (A,α) where A is in $\mathcal{C}$ and α is in $P(A)$. An arrow $f\colon(A,\alpha) \longrightarrow (B,\beta)$ is an arrow $f\colon A \longrightarrow B$ in $\mathcal{C}$ such that $\alpha \le P_f(\beta)$. The functor P_{c} maps each (A,α) to $P_{\mathrm{c}}(A,\alpha) = \left\{\phi \in P(A) \;\middle|\; \phi \le \alpha\right\}$ and each $f\colon(A,\alpha) \longrightarrow (B,\beta)$ to the function $P_{\mathrm{c}}(f)\colon P_{\mathrm{c}}(B,\beta) \longrightarrow P_{\mathrm{c}}(A,\alpha)$ determined by the assignment $\psi \mapsto P_f(\psi) \wedge \alpha$. For ϕ in $P_{\mathrm{c}}(A,\alpha)$ it is $\{\!|\phi|\!\} = \mathrm{id}_A\colon (A,\phi) \longrightarrow (A,\alpha)$.

2.12. PROPOSITION.

Suppose $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{Pos}$ is elementary existential admitting full weak comprehension with comprehensive diagonals. If the variational doctrine $\Psi_{\mathcal{C}}$ has a right adjoint for every reindexing $(\Psi_{\mathcal{C}})_f$, then P is a hyperdoctrine.

PROOF. Proposition 2.9 (ii) ensures that $\mathcal{C}$ has weak pullbacks. The universal quantifier of α in $P(A)$ along $f\colon A \longrightarrow B$ is $\exists_{\Pi_f(\{\!|\alpha|\!\})}\top_X$ where $\Pi_f(\{\!|\alpha|\!\})\colon X \longrightarrow A$ is the universal quantifier $\{\!|\alpha|\!\}$ along f in $\Psi_{\mathcal{C}}(B)$, thanks to [Maietti et al., 2019, Proposition 2.3] that crucially employs Proposition 2.10, see also [Maietti et al., 2017, Remark 2.10]. The fact that P is implicational follows from Proposition 2.11 (ii). ∎

As explained in [Jacobs, 1999, Maietti and Rosolini, 2016], one can associate a deductive calculus to elementary doctrines, that of the $\wedge{=}$-fragment over type theory with just a unit type and a binary product type constructor. We will use it with the following notation.

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ be an elementary doctrine. We write

$$a_1\colon A_1, \ldots, a_k\colon A_k \mid \phi_1, \ldots, \phi_n \vdash \psi$$

in place of

$$\phi_1 \wedge \ldots \wedge \phi_n \le \psi \quad \text{ in } P(A_1 \times \ldots \times A_k).$$

where we identify formulas $\phi_1, \ldots \phi_n$ and ψ with their interpretation. We write simply $a_1\colon A_1, \ldots, a_k\colon A_k \vdash \psi$ in case there are no premisses ($n = 0$). The "binary predicate" δ_A in $P(A \times A)$ will be written as $a\colon A, a'\colon A \mid a{=}_A a'$. Also we write $a_1\colon A_1, \ldots, a_k\colon A_k \mid \alpha \dashv\vdash \beta$ to abbreviate $a_1\colon A_1, \ldots, a_k\colon A_k \mid \alpha \vdash \beta$ and $a_1\colon A_1, \ldots, a_k\colon A_k \mid \beta \vdash \alpha$.

If P is also existential and $a\colon A, x\colon X \mid \phi$, *i.e.* ϕ is in $P(A \times X)$, we write $a\colon A \mid \exists_{x:X}\phi$ in place of $\mathit{\exists}_{\mathrm{pr}_1}\phi$ in $P(A)$. And when P is universal, for ϕ in $P(A \times X)$ we write $a\colon A \mid \forall_{x:X}\phi$ in place of $\mathit{\forall}_{\mathrm{pr}_1}\phi$ in $P(A)$. We will denote $\phi \Rightarrow \psi$ the logical implication of two formulas ϕ and ψ and $\phi \wedge \psi$ their conjunction.

We shall employ the logical notation whenever we feel that intuition or readability is improved.

2.13. The elementary quotient completion

To present the construction of the elementary quotient completion for an elementary doctrine, we start by recalling from [Maietti and Rosolini, 2013b, Maietti and Rosolini, 2013a] the notion of P-equivalence relation and P-quotient for one such relation.

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ be an elementary doctrine. A ***P-equivalence relation*** ρ on the object A of $\mathcal{C}$ is an element in $P(A \times A)$ such that

(i) $a\colon A \vdash \rho(a,a)$;

(ii) $a_1\colon A, a_2\colon A \mid \rho(a_1,a_2) \vdash \rho(a_2,a_1)$;

(iii) $a_1\colon A, a_2\colon A, a_3\colon A \mid \rho(a_1,a_2) \wedge \rho(a_2,a_3) \vdash \rho(a_1,a_3)$.

When no confusion may arise, we drop the reference to the doctrine P in the locution P-equivalence relation.

An obvious example of a P-equivalence relation is the fibered equality on an object in $\mathcal{C}$. Also, given an arrow for $f\colon A \longrightarrow B$, the reindexing $P_{f\times f}(\sigma)$ of a P-equivalence relation on B is a P-equivalence relation on A.

It is also clear that in an elementary doctrine of subobjects $\mathrm{Sub}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ over a category $\mathcal{C}$ with finite limits, a $\mathrm{Sub}_{\mathcal{C}}$-equivalence relation is precisely an equivalence relation $\rho\colon R \rightarrowtail A \times A$ in $\mathcal{C}$, see [Barr, 1971].

The relevance of the notion of P-equivalence relation may appear when instantiating the doctrine to the case of variations: consider a category $\mathcal{D}$ with finite products and weak pullbacks, and let $\Psi_{\mathcal{D}}\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ be the elementary weak subobject doctrine. An object ρ of $\Psi_{\mathcal{D}}(A\times A)$ is a $\Psi_{\mathcal{D}}$-equivalence relation on A if and only if it is a pseudo-equivalence relation of $\mathcal{C}$ in the sense of [Carboni and Celia Magno, 1982, Carboni, 1995].

Just to complete the review of the examples of elementary doctrines treated in the previous subsections, in an elementary doctrine $LT_{\mathscr{T}}\colon \mathcal{V}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ built out of a first order theory $\mathscr{T}$, an $LT_{\mathscr{T}}$-equivalence relation on $(x_i)_{i=1}^n$ is a formula $\phi(\ldots,x_i,\ldots,x'_i,\ldots)$ in $2n$-variables such that

- $\vdash_{\mathscr{T}} \phi(\ldots,x_i,\ldots,x_i,\ldots)$;
- $\vdash_{\mathscr{T}} \phi(\ldots,x_i,\ldots,x'_i,\ldots) \Rightarrow \phi(\ldots,x'_i,\ldots,x_i,\ldots)$;
- $\vdash_{\mathscr{T}} \phi(\ldots,x_i,\ldots,x'_i,\ldots) \wedge \phi(\ldots,x'_i,\ldots,x''_i,\ldots) \Rightarrow \phi(\ldots,x_i,\ldots,x''_i,\ldots)$

When $\mathcal{H}$ is an inf-semilattice, a $\mathcal{P}_{\mathcal{H}}$-equivalence relation on A is an $\mathcal{H}$-valued ***ultra-pseudodistance*** on A, *i.e.* a function $\rho\colon A \times A \longrightarrow \mathcal{H}$ such that for all a, a', a'' in A

- $\rho(a,a) = \top$;
- $\rho(a,a') = \rho(a',a'')$;
- $\rho(a,a') \wedge \rho(a',a'') \leq \rho(a,a'')$.

2.14. Definition.

Let P be elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \textbf{Pos}$. An arrow $q\colon A \longrightarrow A/\rho$ is said to be a P-***quotient*** of the P-equivalence relation ρ when $a_1\colon A, a_2\colon A \mid \rho(a_1,a_2) \vdash q(a_1) =_{A/\rho} q(a_2)$ and for every arrow $f\colon A \longrightarrow Y$ such that $a_1\colon A, a_2\colon A \mid \rho(a_1,a_2) \vdash f(a_1) =_Y f(a_2)$ there exists a unique $h\colon A/\rho \longrightarrow Y$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "q"] \arrow[dr, "f"'] & A/\rho \arrow[d, dashed, "h"] \\ & Y \end{tikzcd}$$

commutes in $\mathcal{C}$.

2.15. Definition.

An elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \textbf{Pos}$ on a base with pullbacks is said to have ***stable effective quotients*** when

- each P-equivalence relation has a P-quotient;
- any P-quotient $q\colon A \longrightarrow A/\rho$ is ***effective***, namely in the internal language the judgement $a_1\colon A, a_2\colon A \mid q(a_1) =_{A/\rho} q(a_2) \vdash \rho(a_1,a_2)$ holds;
- any P-quotient is ***stable***, namely in every pullback

$$\begin{tikzcd} P \arrow[r, "q'"] \arrow[d, "f'"'] & B \arrow[d, "f"] \\ A \arrow[r, "q"] & A/\rho \end{tikzcd}$$

  the arrow q' is a quotient.

2.16. Definition.

Given an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \textbf{Pos}$ and a P-equivalence relation ρ on the object A of $\mathcal{C}$, the inf-semilattice of ***descent data*** $\mathscr{D}es_\rho$ is the sub-semilattice of $P(A)$ on those α such that

$$a_1\colon A, a_2\colon A \mid \alpha(a_1) \wedge \rho(a_1,a_2) \vdash \alpha(a_2).$$

For an arrow $f\colon A \longrightarrow B$ in $\mathcal{C}$ the function $P_f\colon P(B) \longrightarrow P(A)$ takes values in $\mathscr{D}es_{P_{f\times f}(\delta_B)}$, the inf-semilattice of descent data on the "kernel pair" of f. We say that f is ***of effective descent*** when $P_f\colon P(B) \longrightarrow \mathscr{D}es_{P_{f\times f}(\delta_B)}$ is an isomorphism.

2.17. Remark.

To justify the terminology introduced above, note that when $\mathcal{C}$ is a category with finite limits, the elementary doctrine $\mathrm{Sub}_{\mathcal{C}}$ has stable quotients if and only if $\mathcal{C}$ is exact. When this is the case, then quotients are of effective descent.

We are ready to review the construction of elementary quotient completion introduced in [Maietti and Rosolini, 2013b]. The construction freely adds stable quotients of effective descent and comprehensive diagonals as the result of composing the completion in [Maietti and Rosolini, 2013a, Theorem 4.5], which freely adds stable effective quotients, and the completion in [Maietti and Rosolini, 2013a, Theorem 5.7] which freely adds comprehensive diagonals. In this way, it determines a left bi-adjoint $\widehat{(-)}\colon \mathbf{ED} \longrightarrow \mathbf{QED}$ to the inclusion of the 2-subcategory $\mathbf{QED}$ of $\mathbf{ED}$ on those elementary doctrines with stable effective quotients and comprehensive diagonals whose 1-cells are 1-cells in $\mathbf{ED}$ preserving quotients. The original universal property in [Maietti and Rosolini, 2013b] was instead shown for elementary doctrines with full comprehension and comprehensive diagonals.

Given an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$, let $\mathcal{Q}_P$ be the category whose objects are pairs (A,ρ) where A is an object in $\mathcal{C}$ and ρ is a P-equivalence relation on A. An arrow in $\mathcal{Q}_P$ is $[f]\colon (A,\rho) \longrightarrow (B,\sigma)$, an equivalence class of arrows $f\colon A \longrightarrow B$ in $\mathcal{C}$ such that

$$a_1\colon A, a_2\colon A \mid \rho(a_1,a_2) \vdash \sigma(f(a_1), f(a_2))$$

and where f and g are equivalent if

$$a\colon A \vdash \sigma(f(a), g(a)).$$

Composition is defined by composing in $\mathcal{C}$ the representative arrows of the equivalence classes. The category $\mathcal{Q}_P$ has finite products: we just point out that a product of objects (A,ρ) and (B,σ) of $\mathcal{Q}_P$ is the diagram

$$(A,\rho) \xleftarrow{[\mathrm{pr}_1]} (A\times B, \rho \boxtimes \sigma) \xrightarrow{[\mathrm{pr}_2]} (B,\sigma)$$

where $(\rho\boxtimes\sigma)(a_1,b_1,a_2,b_2)$ is the P-equivalence relation $\rho(a_1,a_2)\wedge\sigma(b_1,b_2)$.

The ***elementary quotient completion*** of P is the doctrine $\widehat{P}\colon \mathcal{Q}_P^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ where

$$\widehat{P}(A,\rho) = \mathscr{D}es_\rho \qquad \widehat{P}_{[f]} = P_f$$

The elementary structure of $\widehat{P}$ is obtained by choosing $\delta_{(A,\rho)} = \rho$. Moreover, if σ is a $\widehat{P}$-equivalence relation on (A,ρ), then a $\widehat{P}$-quotient for it is $[\mathrm{id}_A]\colon (A,\rho) \longrightarrow (A,\sigma)$. One sees also that quotients in $\widehat{P}$ are stable and of effective descent. We refer the reader to [Maietti and Rosolini, 2013b] for all the details.

There are important remarks about the elementary quotient completion.

2.18. Remarks.

One of the motivating examples for the study of the elementary quotient completion is given by doctrines of the form $\Psi_{\mathcal{D}} \colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ where $\mathcal{D}$ is a category with finite products and weak pullbacks. As proved in [Maietti and Rosolini, 2013b], the doctrine $\widehat{\Psi_{\mathcal{C}}}$ is $\mathrm{Sub}_{\mathcal{C}_{\mathrm{ex/lex}}}$. So, in particular, $\mathcal{Q}_{\Psi_{\mathcal{C}}}$ is equivalent to $\mathcal{C}_{\mathrm{ex/lex}}$.

In [Maietti and Rosolini, 2013a] it is shown that the elementary quotient completion is the extensional collapse of another quotient completion, which is also a co-completion, see [Pasquali, 2015]. For doctrines of the form $LT_{\mathscr{T}} \colon \mathcal{V}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ that co-completion is related to the elimination of imaginaries of the theory $\mathscr{T}$, as analysed in [Emmenegger et al., 2020].

Assuming the Axiom of Choice, *i.e.* epis in $\mathcal{S}et$ split, the base of $\widehat{\mathcal{P}_{\mathcal{H}}} \colon \mathcal{Q}_{\mathcal{P}_{\mathcal{H}}}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is equivalent to $\mathbf{UM}_{\mathcal{H}}$, the category of $\mathcal{H}$-valued ultrametric spaces. Indeed, the functor mapping $f \colon (A, \rho) \longrightarrow (B, \sigma)$ in $\mathbf{UM}_{\mathcal{H}}$ to $[f] \colon (A, \rho) \longrightarrow (B, \sigma)$ in $\mathcal{Q}_{\mathcal{P}_{\mathcal{H}}}$ is full and faithful as $\rho(x, x') = \top$ implies $x = x'$. For essential surjectivity, take (A, ρ) in $\mathcal{Q}_{\mathcal{P}_{\mathcal{H}}}$ and consider the quotient $q \colon A \longrightarrow A/{\sim}$ where $\sim$ is the equivalence relation satisfying $a \sim a'$ if and only if $\rho(a, a') = \top$. The arrow $[q] \colon (A, \rho) \longrightarrow (A/{\sim}, \exists_{q \times q}\rho)$ in $\mathcal{Q}_{\mathcal{P}_{\mathcal{H}}}$ is an isomorphism whose inverse is represented by any section of q.

2.19. Remark.

It is quite evident that the elementary structure plays no role in the construction of $\widehat{P}$, but it is necessary to embed $\mathcal{C}$ into $\mathcal{Q}_P$, see [Pasquali, 2015, Emmenegger et al., 2020] for a detailed analysis of the situation.

The embedding is given by the functor $J \colon \mathcal{C} \longrightarrow \mathcal{Q}_P$ that assigns to each $f \colon X \longrightarrow Y$ the arrow $[f] \colon (X, \delta_X) \longrightarrow (Y, \delta_Y)$. This functor preserves binary products and is full; it is faithful when P has comprehensive diagonals.

2.20. Proposition.

Let $P \colon \mathcal{C}^{op} \longrightarrow \mathbf{Pos}$ be an elementary doctrine. Then

(i) *P is existential if and only if $\widehat{P}$ is existential;*

(ii) *P is implicational if and only if $\widehat{P}$ is implicational;*

(iii) *P is universal if and only if $\widehat{P}$ is universal.*

Proof. The sufficient conditions are proved in [Maietti and Rosolini, 2013b, Propositions 6.1 and 6.7]. Each necessary condition follows immediately since $\widehat{P}$ restricts to P along J_P. ∎

2.21. Proposition.

Let $P\colon \mathcal{C}^{op} \longrightarrow \textsf{Pos}$ be an elementary doctrine with comprehensive diagonals. Consider the square $\boxed{1}$ of arrows in $\mathcal{Q}_P$, and the corresponding square $\boxed{2}$ of representative arrows in $\mathcal{C}$:

$$
\begin{tikzcd}
(S,\theta) \arrow[r, "{[k]}"] \arrow[d, "{[h]}"'] \arrow[dr, phantom, "\boxed{1}"] & (X,\delta_X) \arrow[d, "{[f]}"] \\
(Y,\delta_Y) \arrow[r, "{[g]}"'] & (A,\delta_A)
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[r, "k"] \arrow[d, "h"'] \arrow[dr, phantom, "\boxed{2}"] & X \arrow[d, "f"] \\
Y \arrow[r, "g"'] & A
\end{tikzcd}
$$

(i) *Square $\boxed{1}$ commutes if and only if square $\boxed{2}$ commutes.*

(ii) *If $\boxed{1}$ is a pullback in $\mathcal{Q}_P$, then $\boxed{2}$ is a weak pullback in $\mathcal{C}$.*

(iii) *If $\boxed{2}$ is a weak pullback in $\mathcal{C}$ and $\theta = P_{k\times k}(\delta_X) \wedge P_{h\times h}(\delta_Y)$, then $\boxed{1}$ is a pullback in $\mathcal{Q}_P$.*

Proof. (i) If $\boxed{1}$ commutes, then $\top_S \leq P_{\langle fk,gh\rangle}(\delta_A)$. Besides, the hypothesis on comprehensive diagonals ensures the identity $fk = gh$. The converse is immediate.
(ii) Assume $\boxed{1}$ is a pullback in $\mathcal{Q}_P$, and suppose $a\colon C \longrightarrow X$ and $b\colon C \longrightarrow Y$ are such that $fa = gb$. By (i) the diagram

$$
\begin{tikzcd}
(C,\delta_C) \arrow[r, "{[a]}"] \arrow[d, "{[b]}"'] & (X,\delta_X) \arrow[d, "{[f]}"] \\
(Y,\delta_Y) \arrow[r, "{[g]}"'] & (A,\delta_A)
\end{tikzcd}
$$

commutes. So there is an arrow $[u]\colon (C,\delta_C) \longrightarrow (S,\theta)$ filling in the diagram

$$
\begin{tikzcd}
(C,\delta_C) \arrow[dr, dashed, "{[u]}"'] \arrow[drr, "{[a]}"] \arrow[ddr, "{[b]}"'] & & \\
& (S,\theta) \arrow[r, "{[k]}"'] \arrow[d, "{[h]}"] & (X,\delta_X) \arrow[d, "{[f]}"] \\
& (Y,\delta_Y) \arrow[r, "{[g]}"'] & (A,\delta_A)
\end{tikzcd}
$$

Again, by (i) the diagram

$$\begin{tikzcd} C \arrow[dr, "u"] \arrow[drr, "a"] \arrow[ddr, "b"'] & & \\ & S \arrow[r, "k"'] \arrow[d, "h"] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"'] & A \end{tikzcd}$$

commutes.
(iii) Assume $\boxed{2}$ is a weak pullback in $\mathcal{C}$, and $\theta = P_{k\times k}(\delta_X) \wedge P_{h\times h}(\delta_Y)$, *i.e.*

$$x\colon S, y\colon S \mid \theta(x,y) \dashv\vdash k(x) =_X k(y) \wedge h(x) =_Y h(y).$$

Consider $[a]\colon (C,\rho) \longrightarrow (X,\delta_X)$ and $[b]\colon (C,\rho) \longrightarrow (Y,\delta_Y)$ such that the diagram

$$\begin{tikzcd} (C,\rho) \arrow[r, "{[a]}"] \arrow[d, "{[b]}"'] & (X,\delta_X) \arrow[d, "{[f]}"] \\ (Y,\delta_Y) \arrow[r, "{[g]}"'] & (A,\delta_A) \end{tikzcd}$$

commutes. By (i), one has that $fa = gb$. Since $\boxed{2}$ is a weak pullback, there is $u\colon C \longrightarrow S$ with $ku = a$ and $hu = b$ in $\mathcal{C}$. Moreover, since $[a]$ and $[b]$ are arrows of $\mathcal{Q}_P$, one has that

$$\begin{aligned} x\colon C, y\colon C \mid \rho(x,y) &\vdash a(x) =_X a(y) \wedge b(x) =_Y b(y) \\ &\vdash k(u(x)) =_X k(u(y)) \wedge h(u(x)) =_Y h(u(y)) \\ &\vdash \theta(u(x), u(y)) \end{aligned}$$

which proves that $[u]\colon (C,\rho) \longrightarrow (S,\theta)$ is an arrow in $\mathcal{Q}_P$. Finally, assume $[u']$ is such that $[k][u'] = [a]$ and $[h][u'] = [b]$. Then $\vdash \theta(u(c), u'(c))$, hence $[u] = [u']$. ∎

Recall now that, as first observed in [Maietti and Rosolini, 2013b, Propositions 4.6, Lemma 5.3], an elementary quotient completion of an elementary doctrine admitting weak comprehension has strong comprehension and, hence, pullbacks.

2.22. Proposition.

*Let $P\colon \mathcal{C}^{op} \longrightarrow$ **Pos** be an elementary doctrine admitting weak (full) comprehension. Then $\mathcal{Q}_P$ admits strong (full) comprehension.*

Proof. Suppose α is in $\mathcal{Q}_P(A,\rho)$. So α is in $P(A)$; let $\{\!|\alpha|\!\}\colon X \longrightarrow A$ be a weak comprehending arrow of α with respect to P. Then $[\{\!|\alpha|\!\}]\colon (X, P_{\{\!|\alpha|\!\}\times\{\!|\alpha|\!\}}(\rho)) \longrightarrow (A,\rho)$ is a strong comprehending arrow with respect to $\mathcal{Q}_P$, as one derives from [Maietti and Rosolini, 2013b, Lemma 5.3]. ∎

2.23. Proposition.

Let P be an elementary doctrine on $\mathcal{C}$ with weak comprehension. Then $\mathcal{Q}_P$ has pullbacks.

Proof. This follows from Propositions 2.9 and 2.22. ∎

In the base category of an elementary doctrine admitting full comprehension with comprehensive diagonals, monic arrows coincide with injection arrows as shown in [Maietti and Rosolini, 2013b]. This characterization can be generalized to doctrines admitting just full weak comprehension.

2.24. Proposition.

Let $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{Pos}$ be an elementary doctrine admitting full weak comprehension with comprehensive diagonals. An arrow $m\colon X \longrightarrow A$ is monic if and only if $P_{m\times m}(\delta_A) = \delta_X$.

Proof. If $P_{m\times m}(\delta_A) = \delta_X$, then a representative of the weak comprehension $\{\!|P_{m\times m}(\delta_A)|\!\}$ can be taken to be that of $\{\!|\delta_X|\!\}$ which, thanks to comprehensive diagonals, can be taken to be $\langle \mathrm{id}_X, \mathrm{id}_X\rangle$. Hence, the weak pullback of m along itself, whose projections can be computed as $\mathrm{pr}_1\{\!|P_{m\times m}\delta_A)|\!\}$ and $\mathrm{pr}_2\{\!|P_{m\times m}(\delta_A)|\!\}$ following [Maietti et al., 2017, Remark 2.14], can also be taken to be identities, and hence m is monic.

For the converse, suppose that $m\colon X \longrightarrow A$ is monic, then the commutative square $\Delta_A m = (m\times m)\Delta_X$ is a pullback. Since P admits weak full comprehension, by Proposition 2.11 it holds that $P_{m\times m}(\delta_A) = P_{m\times m}\mathit{\exists}_{\Delta_A}(\top_A) = \mathit{\exists}_{\Delta_X} P_m(\top_A) = \mathit{\exists}_{\Delta_X}(\top_X) = \delta_X$. ∎

As mentioned above, from [Maietti and Rosolini, 2013b] we know that the elementary quotient completion freely adds stable effective quotients (of effective descent type) to elementary doctrines admitting full comprehension with comprehensive diagonals. Let $\mathbf{ECD}$ denote the subcategory of doctrines in $\mathbf{ED}$ admitting full comprehension with comprehensive diagonals, and 1-cells those of $\mathbf{ED}$ preserving them (it was denoted EqD in [Maietti and Rosolini, 2013b]). Let $\mathbf{QCD}$ be the subcategory of doctrines in $\mathbf{ECD}$ which have stable effective quotients of effective descent and 1-cells are those 1-cells of $\mathbf{ECD}$ which preverse them (it was QD in [Maietti and Rosolini, 2013b]).

Furthermore, write $\mathcal{Q}\colon \mathbf{ECD} \longrightarrow \mathbf{QCD}$ the left bi-adjoint to the inclusion of $\mathbf{QCD}$ into $\mathbf{ECD}$ which maps a doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ to $\widehat{P}\colon \mathcal{Q}_P^{\mathrm{op}} \longrightarrow \mathbf{Pos}$. The unit of this adjunction at P is the pair $(\mathsf{H}_P, \eta_P)\colon P \longrightarrow \widehat{P}$ where $\mathsf{H}_P\colon \mathcal{C} \longrightarrow \mathcal{Q}_P$ maps $f\colon A \longrightarrow B$ to $[f]\colon (A,\delta_A) \longrightarrow (B,\delta_B)$ and where $\eta_P(A)\colon P(A) \longrightarrow \mathscr{D}es_{\delta_A}$ is the identity on $P(A)$.

The counit is the pair $(\mathsf{E}_D, \varepsilon_D)\colon \widehat{D} \longrightarrow D$ for Q a doctrine in $\mathbf{QCD}$, where E_D maps an object (A,ρ) in $\mathcal{Q}_D$ to A/ρ, (a choice of) the D-quotient of the D-equivalence relation ρ in the base of Q, and an arrow $[f]\colon (A,\rho) \longrightarrow (B,\sigma)$ to the unique arrow $f'\colon A/\rho \longrightarrow B/\sigma$ induced on the quotients.

One may wonder if the above left bi-adjoint can be extended to the inclusion of $\mathbf{QCD}$ into some category $\mathbf{C}$ having elementary doctrines with full weak comprehension and comprehensive diagonals as objects. As in the case of the exact completion of a category with weak finite limits in [Carboni and Vitale, 1998], we will answer this question in the negative in Proposition 2.28. But we can provide a universal property of a special kind in Proposition 2.29.

2.25. Remark.

One reason for the name EqD in [Maietti and Rosolini, 2013b] instead of **ECD** is that doctrines admitting full comprehension with comprehensive diagonals have equalisers in the base. Along this line, note that the category **Lex** of finite limit categories can be seen as a full reflective subcategory of **ECD** by taking a category $\mathcal{C}$ to $\mathrm{Sub}_{\mathcal{C}}$. The reflector R from **ECD** to **Lex** maps a doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ to its base $\mathcal{C}$ which is a finite limit category by (ii) of Lemma 2.9. In particular, the unit from a doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ to $\mathrm{Sub}_{\mathcal{C}}$ is the pair $(\mathrm{Id}_{\mathcal{C}}, \{\!|-|\!\})$ that is an elementary morphism of doctrines as shown in [Maietti et al., 2017, Theorem 2.15], which trivially preserves comprehending arrows.

2.26. Lemma.

Let $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{Pos}$ be an elementary doctrine on the regular category $\mathcal{C}$, suppose P admits weak comprehension with comprehensive diagonals. If $\mathcal{Q}(\mathsf{H}_P, \eta_P)\colon \widehat{P} \longrightarrow \widehat{\widehat{P}}$ preserves comprehending arrows, then the doctrine P actually admits strong comprehension.

Proof. One can read in Proposition 2.22 that the elementary quotient completion of a doctrine turns weak comprehension into strong comprehension: in details, given (A,ρ) in $\mathcal{Q}_P$ and α in $\widehat{P}(A,\rho)$, let $\{\!|\alpha|\!\}\colon X \longrightarrow A$ be a weak P-comprehending arrow for α in $\mathcal{C}$. Then $[\{\!|\alpha|\!\}]\colon (X, P_{\{\!|\alpha|\!\}\times\{\!|\alpha|\!\}}(\rho)) \longrightarrow (A,\rho)$ is a strong $\widehat{P}$-comprehending arrow for α in $\mathcal{Q}_P$.

The functor component of $\mathcal{Q}(\mathsf{H}_P, \eta_P)$ maps that arrow to

$$[[\{\!|\alpha|\!\}]] : ((X,\delta_X), P_{\{\!|\alpha|\!\}\times\{\!|\alpha|\!\}}(\rho)) \longrightarrow ((A,\delta_A),\rho)$$

where the arrow within the outer pair of brackets is $[\{\!|\alpha|\!\}]\colon (X,\delta_X) \longrightarrow (A,\delta_A)$ in $\mathcal{Q}_P$. On the other hand, the action of $\mathcal{Q}(\mathsf{H}_P, \eta_P)$ on the fiber $\widehat{P}(A,\rho)$ takes α to itself as an object in $\widehat{\widehat{P}}((A,\delta_A),\rho)$. Its comprehending arrow in $\mathcal{Q}_{\widehat{P}}$ is

$$[[\{\!|\alpha|\!\}]] : ((X, P_{\{\!|\alpha|\!\}\times\{\!|\alpha|\!\}}(\delta_A)), P_{\{\!|\alpha|\!\}\times\{\!|\alpha|\!\}}(\rho)) \longrightarrow ((A,\delta_A),\rho)$$

where, this time, the arrow within the outer pair of brackets is the strong comprehending arrow $[\{\!|\alpha|\!\}]\colon (X, P_{\{\!|\alpha|\!\}\times\{\!|\alpha|\!\}}(\delta_A)) \longrightarrow (A,\delta_A)$ in $\mathcal{Q}_P$. The assumption that $\mathcal{Q}(\mathsf{H}_P, \eta_P)\colon \widehat{P} \longrightarrow \widehat{\widehat{P}}$ preserves comprehensive arrows yields that there is an arrow in $\mathcal{Q}_{\widehat{P}}$

$$\begin{tikzcd}
((X, P_{\{\!|\alpha|\!\}\times\{\!|\alpha|\!\}}(\delta_A)), P_{\{\!|\alpha|\!\}\times\{\!|\alpha|\!\}}(\rho)) \arrow[rr] \arrow[dr, "{[[\{\!|\alpha|\!\}]]}"'] & & ((X,\delta_X), P_{\{\!|\alpha|\!\}\times\{\!|\alpha|\!\}}(\rho)) \arrow[dl, "{[[\{\!|\alpha|\!\}]]}"] \\
& ((A,\delta_A),\rho) &
\end{tikzcd}$$

Say $[k]\colon (X, P_{\{\!|\alpha|\!\}\times\{\!|\alpha|\!\}}(\delta_A)) \longrightarrow (X, \delta_X)$ is a representative in $\mathcal{Q}_P$ for that iso in $\mathcal{Q}_{\widehat{P}}$, and

$$x\colon X, y\colon X \mid \{\!|\alpha|\!\}(x) =_A \{\!|\alpha|\!\}(y) \vdash k(x) =_X k(y) \tag{1}$$

The case of interest for the proof is when $\rho = \delta_A$; so also

$$x\colon X \vdash \{\!|\alpha|\!\}(x) =_A \{\!|\alpha|\!\}(k(x)). \tag{2}$$

Factor $\{\!|\alpha|\!\}\colon X \longrightarrow A$ in the regular category $\mathcal{C}$ as $X \xrightarrow{e} Y \xrightarrow{m} A$ where e is a regular epi and m is monic. Since P has comprehensive diagonals, (1) and (2) ensure that k factors through e and there is a commutative triangle

$$\begin{tikzcd}
X \arrow[rr, "k"] \arrow[dr, two heads, "e"] \arrow[ddr, "\{\!|\alpha|\!\}"'] & & X \arrow[ddl, "\{\!|\alpha|\!\}"] \\
& Y \arrow[ur, "t"] \arrow[d, tail, "m"] & \\
& A &
\end{tikzcd}$$

So $m\colon Y \longrightarrow A$ is also a comprehending arrow for α with respect to P. Being monic, it is a strong one for α. ■

2.27. Lemma.

There is a doctrine P admitting weak full comprehension and with comprehensive diagonals such that $\mathcal{Q}(\mathsf{H}_P, \eta_P)\colon \widehat{P} \longrightarrow \widehat{\widehat{P}}$ is not in **QCD**.

Proof. Suppose $\mathcal{C}$ has finite limits and take the doctrines $\Psi_{\mathcal{C}}$ as P. Suppose $\mathcal{Q}(\mathsf{H}_{\Psi_{\mathcal{C}}}, \eta_{\Psi_{\mathcal{C}}})$ is in **QCD**. So $\Psi_{\mathcal{C}}$ is equivalent to $\mathrm{Sub}_{\mathcal{C}}$ by Lemma 2.26. And, since $\Psi_{\mathcal{C}}$ satisfies the rule of choice by [Maietti et al., 2017, Theorem 5.9], also the doctrine $\mathrm{Sub}_{\mathcal{C}}$ satisfies the rule of choice. But this is not always the case, take *e.g.* the subobject doctrine of a non-boolean topos $\mathcal{C}$. Indeed, if $\mathrm{Sub}_{\mathcal{C}}$ satisfied the rule of choice, then it would satisfy the axiom of choice, as shown in [Maietti, 2017], and hence it would have to be boolean by Diaconescu's Theorem, see [Diaconescu, 1975]. ■

2.28. Proposition.

It is impossible to equip the collection of elementary doctrines admitting weak full comprehension with comprehensive diagonals, with a structure of 2-category **C** *in such a way that* **C** *includes* **ECD***, is included in* **ED** *and the family* (H, η) *is the unit of a bi-adjunction between the inclusion of* **QCD** *into* **C** *and the extension of* $\mathcal{Q}$: **ECD** $\longrightarrow$ **QCD** *to* **C**.

Proof. Ifn (H, η) is the unit of such an adjunction, then for every P in **C**, the arrow (H_P, η_P) in **ED** has to be also an arrow in **C**. Therefore $\mathcal{Q}(\mathsf{H}_P, \eta_P)$ is an arrow in **QCD** for every P, which is impossible by Lemma 2.27. ■

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \textsf{\textbf{Pos}}$ and $Q\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \textsf{\textbf{Pos}}$ be elementary doctrines admitting full weak comprehension, and assume that Q has stable effective quotients of effective descent (so Q admits full strong comprehension). Following the analogy with [Carboni and Vitale, 1998], we say that a 1-cell $(F, f)\colon P \longrightarrow Q$ in $\textsf{\textbf{ED}}$ is ***comprehensive-covering*** if for every α in $P(A)$ and every weak P-comprehensive arrow $\{\!|\alpha|\!\}\colon X \longrightarrow A$ in $\mathcal{C}$ there is a Q-comprehending arrow $\{\!|f_A(\alpha)|\!\}\colon Y \longrightarrow FA$ in $\mathcal{C}'$ such that the arrow $q\colon FX \longrightarrow Y$, filling in

$$\begin{tikzcd} FX \arrow[rr, "q"] \arrow[dr, "F(\{\!|\alpha|\!\})"'] & & Y \arrow[dl, "\{\!|f_A(\alpha)|\!\}"] \\ & F(A) & \end{tikzcd}$$

by the universal property of comprehension, is a quotient arrow.

Whilst the elementary quotient completion of elementary doctrines admitting full weak comprehension with comprehensive diagonals can not be characterized as part of a left bi-adjoint, it enjoys a universal property with respect to comprehensive-covering morphisms as follows.

For P a doctrine admitting full weak comprehension with comprehensive diagonals and Q in $\textsf{\textbf{QCD}}$, denote $\textsf{\textbf{clcED}}(P, Q)$ the full subcategory of $\textsf{\textbf{ED}}(P, Q)$ whose objects are the comprehensive-covering $(F, f)\colon P \longrightarrow Q$.

Recall that the elementary quotient completion is a left bi-ajoint $\widehat{(-)}\colon \textsf{\textbf{ED}} \longrightarrow \textsf{\textbf{QED}}$ to the inclusion of the subcategory $\textsf{\textbf{QED}}$ of $\textsf{\textbf{ED}}$ on those elementary doctrines with stable effective quotients and comprehensive diagonals whose 1-cells are 1-cells in $\textsf{\textbf{ED}}$ preserving quotients.

2.29. PROPOSITION.

Suppose $P\colon \mathcal{C}^{op} \longrightarrow \textsf{\textbf{Pos}}$ is a doctrine admitting full weak comprehension with comprehensive diagonals and $Q\colon \mathcal{D}^{op} \longrightarrow \textsf{\textbf{Pos}}$ is in $\textsf{\textbf{QCD}}$. The categorical equivalence $\textsf{\textbf{ED}}(P, Q) \equiv \textsf{\textbf{QED}}(\widehat{P}, Q)$ determined by the elementary quotient completion restricts to an equivalence between the categories $\textsf{\textbf{clcED}}(P, Q)$ and $\textsf{\textbf{QCD}}(\widehat{P}, Q)$.

PROOF. By Proposition 2.22, the doctrine $\widehat{P}$ is in $\textsf{\textbf{QED}}$. Consider next (F, f) in $\textsf{\textbf{clcED}}(P, Q)$ and its corresponding 1-cell in (F', f') in $\textsf{\textbf{QED}}(\widehat{P}, Q)$. In the following, given $\alpha \in \mathscr{D}es_{(A,\rho)} \subseteq P(A)$, we are going to consider two comprehending arrows: one for $\alpha \in \widehat{P}(A, \rho) = \mathscr{D}es_{(A,\rho)}$; one for $\alpha \in \widehat{P}(A, \delta_A) = P(A)$. We shall distinguish the two occurrences of α, keeping the original name for the second, and writing the first as α_ρ. The action of $F'\colon \mathcal{Q}_P \longrightarrow \mathcal{D}$ takes the comprehensive arrow $[\{\!|\alpha_\rho|\!\}]\colon (X, P_{\{\!|\alpha|\!\}\times\{\!|\alpha|\!\}}(\rho)) \longrightarrow (A, \rho)$ in $\mathcal{Q}_P$ to

$$F'([\{\!|\alpha_\rho|\!\}])\colon F(X)/_{f_{X\times X}(P_{\{\!|\alpha|\!\}\times\{\!|\alpha|\!\}}(\rho))} \longrightarrow F(A)/_{f_{A\times A}(\rho)}.$$

as one easily sees recalling that $f\colon P \stackrel{\bullet}{\longrightarrow} QF^{\mathrm{op}}$ so, in particular,

$$f_{X\times X}(P_{\{\!|\alpha|\!\}\times\{\!|\alpha|\!\}}(\rho)) = Q_{F(\{\!|\alpha|\!\})\times F(\{\!|\alpha|\!\})} f_{A\times A}(\rho). \tag{3}$$

Since the arrow $\{\!|\alpha|\!\}\colon X \longrightarrow A$ is a comprehending arrow in $\mathcal{C}$ for $\alpha \in P(A)$, there is a commutative diagram in $\mathcal{D}$

$$\begin{tikzcd}
F(X) \arrow[r, "g"] \arrow[dr, "F(\{\!|\alpha|\!\})"'] & Y \arrow[d, tail, "\{\!|f_A(\alpha)|\!\}"] \\
& F(A)
\end{tikzcd} \tag{4}$$

with the comprehending arrow of $f_A(\alpha)$ a monic in $\mathcal{D}$. Since $F(A)/_{f_{A\times A}(\rho)}$ is a quotient of $F(A)$ and α is a descent data, one can complete the commutative diagram in (4) with a pullback square, obtaining

$$\begin{tikzcd}
& Y \arrow[rr] \arrow[d, tail, "\{\!|f_A(\alpha)|\!\}"] & & Z \arrow[d, tail, "\{\!|f'_{(A,\rho)}(\alpha_\rho)|\!\}"] \\
F(X) \arrow[ur, "g"] \arrow[r, "F(\{\!|\alpha|\!\})"'] \arrow[drrr, bend right] & F(A) \arrow[rr, "F'[\mathrm{id}_A]"'] & & F(A)/_{f_{A\times A}(\rho)} = F'(A,\rho) \\
& & & F(X)/_{f_{X\times X}(P_{\{\!|\alpha|\!\}\times\{\!|\alpha|\!\}}(\rho))} \arrow[u, tail, "F'([\{\!|\alpha_\rho|\!\}])"']
\end{tikzcd}$$

Since Q is a functor, $Q_{F(\{\!|\alpha|\!\})\times F(\{\!|\alpha|\!\})} f_{A\times A}(\rho) = Q_{g\times g} Q_{\{\!|f_A(\alpha)|\!\}\times\{\!|f_A(\alpha)|\!\}} f_{A\times A}(\rho)$, and from (3) it follows that the mono $\{\!|f'_{(A,\rho)}(\alpha_\rho)|\!\}$ is (isomorphic to) the mono $F'([\{\!|\alpha_\rho|\!\}])$ for every object (A,ρ) in $\mathcal{Q}_P$, and every α_ρ in $\widehat{P}(A,\rho)$, if and only if the arrow $q\colon F(X) \longrightarrow Y$ is a quotient for every object A in $\mathcal{C}$, and every α in $P(A)$. ∎

2.30. Remark.

Another way to prove Proposition 2.28 would be to show that comprehensive-covering morphisms *do not compose in general*, as in [Carboni and Vitale, 1998] for the case of the exact completion of a category with weak finite limits. Indeed, every pair of the form $(\mathsf{H}_P, \eta_P)\colon P \longrightarrow \widehat{P}$ is comprehensive-covering, but, with arguments similar to those used in the proof of Lemmas 2.26 and 2.27, one sees that the composition of the two units $(\mathsf{H}_{\widehat{P}}, \eta_{\widehat{P}}) \circ (\mathsf{H}_P, \eta_P)$ is not comprehensive-covering when P is the weak subobject doctrine of a non-boolean topos.

However, this fact does not prevent to show any universal property for the elementary quotient completion of elementary doctrines admitting full weak comprehension with comprehensive diagonals. Indeed, as showed in Proposition 2.29, when restricting to small doctrines the functor $\mathbf{clcED}(P,-)$ on $\mathbf{QCD}$ is bi-represented through the elementary quotient completion, since comprehensive-covering morphisms are closed under post-composition with doctrines in $\mathbf{QCD}$.

3. Preparatory results for the characterization theorem

In this section, we introduce some technical notions that will be used in the proof of the characterization Theorem 4.12 in §4.

3.1. Definition.

Let $\mathcal{C}$ be a category with finite products and let $J\colon \mathcal{D} \hookrightarrow \mathcal{C}$ be an inclusion of a subcategory in it. We say that an object X is ***weakly exponentiable relative to*** $\mathcal{D}$ if the functor

$$X \times (-)\colon \mathcal{D} \longrightarrow \mathcal{C}$$

is a weak left adjoint, in the sense of [Kainen, 1971]: for every object Y in $\mathcal{C}$ there are an object W in $\mathcal{D}$ and an arrow in $\mathcal{C}$

$$X \times J(W) \xrightarrow{\mathrm{ev}} Y$$

such that for every D in $\mathcal{D}$ and every arrow $f\colon X \times J(D) \longrightarrow Y$ in $\mathcal{C}$ there is a commutative diagram

$$\begin{tikzcd} X \times J(D) \arrow[d, "\mathrm{id}_X \times J\left(\widehat{f}\right)"'] \arrow[dr, "f"] & & D \arrow[d, dotted, "\widehat{f}"'] \\ X \times J(W) \arrow[r, "\mathrm{ev}"] & Y & W \end{tikzcd}$$

where $\widehat{f}\colon D \longrightarrow W$ is in $\mathcal{D}$ and the dotted arrow indicates that the condition need not determine it uniquely.

3.2. Remark.

The condition of weak left adjoint in Definition 3.1 provides a family of surjective functions

$$\mathcal{D}(D, W) \longrightarrow \mathcal{C}(X \times J(D), Y)$$

natural in D.

We shall be interested in weak relative exponentiability in slice categories of the form $\mathcal{Q}_P/(A, \delta_A)$. They shall involve specific objects, which will be introduced in Definition 3.4.

3.3. Remark.

Consider a category $\mathcal{C}$ with finite products. An object Y is weakly exponentiable in $\mathcal{C}$ in the usual sense if (and only if) it is weakly exponentiable relative to $\mathcal{C}$. So $\mathcal{C}$ is weakly cartesian closed if and only if every object is weakly exponentiable relative to $\mathcal{C}$.

3.4. Definition.

Let $P\colon \mathcal{C}^{\rm op} \longrightarrow \textsf{Pos}$ be an elementary doctrine, and let $\mathcal{Q}_P$ be its elementary quotient completion. An arrow in $\mathcal{Q}_P$ of the form $[f]\colon (X,\delta_X) \longrightarrow (A,\delta_A)$ is called a ***dependent*** P***-projective***. We write as $\mathcal{D}_A$ the full sucategory of $\mathcal{Q}_P/(A,\delta_A)$ on the dependent P-projectives in it.

In case $\mathcal{C}$ has pullbacks, local cartesian closure suffices to show that the doctrine $\Psi_{\mathcal{C}}$ is universal. In the weak case, we choose a definition that requires it explicitly.

Let $P\colon \mathcal{C}^{\rm op} \longrightarrow \textsf{Pos}$ be an elementary doctrine admitting weak full comprehension with comprehensive diagonals. We know that its base $\mathcal{C}$ has weak pullbacks and $\mathcal{Q}_P$ has pullbacks by Propositions 2.9 and 2.23.

3.5. Definition.

We say that an elementary doctrine $P\colon \mathcal{C}^{\rm op} \longrightarrow \textsf{Pos}$ admitting weak full comprehension with comprehensive diagonals is ***slicewise weakly cartesian closed*** when the following conditions are satisfied:

(i) the doctrine P is implicational and universal;

(ii) for every object A in $\mathcal{C}$, each dependent P-projective is weakly exponentiable in $\mathcal{Q}_P/(A,\delta_A)$ relative to $\mathcal{D}_A$.

3.6. Remark.

It may be useful to expand on condition (ii), taking advantage of the full embedding $J\colon \mathcal{C} \longrightarrow \mathcal{Q}_P$ introduced in Remark 2.19—so, in particular, $JA = (A,\delta_A)$. Given objects $Jf\colon JX \longrightarrow JA$ and $[g]\colon (Y,\rho) \longrightarrow JA$ in the slice category $\mathcal{Q}_P/JA$, there is a diagram of arrows in $\mathcal{C}$

$$\begin{tikzcd}
S \arrow[r, "p_2"'] \arrow[d, "p_1"'] \arrow[rr, bend left, "\mathrm{ev}"] & W \arrow[d, "w"] & Y \arrow[dl, "g"] \\
X \arrow[r, "f"'] & A &
\end{tikzcd}$$

where the inner square is a weak pullback by Proposition 2.21. The arrow $\mathrm{ev}\colon S \longrightarrow A$ is the representative of an arrow $[\mathrm{ev}]\colon Jf \times_{JA} Jw \longrightarrow [g]$ in $\mathcal{Q}_P/JA$ such that, for any arrow $u\colon U \longrightarrow A$ in $\mathcal{C}$ and any arrow $[k]\colon Jf \times_{JA} Ju \longrightarrow [g]$ in $\mathcal{Q}_P/JA$, there exists $\widehat{k}\colon U \longrightarrow W$

in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd}
Jf \times_{JA} Ju \arrow[d, "J\mathrm{id}_X \times_{JA} J\widehat{k}"'] \arrow[dr, "{[k]}"] & \\
Jf \times_{JA} Jw \arrow[r, "{[\mathrm{ev}]}"] & {[g]}
\end{tikzcd}$$

commutes in $\mathcal{Q}_P/JA$.

3.7. Remark.

In [Cioffo, 2023], it is shown that a category $\mathcal{Q}_P/(A, \delta_A)$ is an example of elementary quotient completion of a suitable biased elementary doctrine $P_{/A}$ for which dependent P-projectives $[f]\colon (X, \delta_X) \longrightarrow (A, \delta_A)$ are covering projections. It makes sense thus to introduce the notion of slicewise weakly cartesian closed biased elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ when P is implicational and universal and every $P_{/A}$-projective is weakly exponentiable in $\mathcal{Q}_P/(A, \delta_A)$ relative to $P_{/A}$-projectives, for every object A of $\mathcal{C}$.

3.8. Definition.

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ be an elementary doctrine with comprehensive diagonals whose base $\mathcal{C}$ has weak pullbacks. We say that P is ***slicewise cartesian closed on dependent projectives*** if the following conditions are satisfied.

(i) The doctrine P is implicational and universal;

(ii) For every object A in $\mathcal{C}$, each dependent P-projective is exponentiable in $\mathcal{Q}_P/(A, \delta_A)$.

3.9. Remark.

The notion of extensional exponential is introduced [Emmenegger, 2020] in a category with weak finite limits. Such a notion is equivalent to the universal property in Remark 3.6 in the case of a variational doctrine of a category with finite products and weak equalizers.

Later, in his Ph.D. thesis [Cioffo, 2022], Cipriano Cioffo Jr. extended Emmenegger's notion to biased elementary doctrines. Such a notion is also equivalent to the universal property in Remark 3.6 for doctrines with the properties listed at the beginning of Definition 3.5.

4. Local cartesian closure for an elementary quotient completion

In this section, we generalize to the elementary quotient completion the well-known fact that the exact completion $\mathcal{C}_{\mathrm{ex/lex}}$ of a category $\mathcal{C}$ with finite limits (a.k.a. ex/lex completion) of [Carboni and Celia Magno, 1982] transforms a weak locally cartesian structure on $\mathcal{C}$ into a locally cartesian closed structure on $\mathcal{C}_{\mathrm{ex/lex}}$.

4.1. LEMMA.

Suppose that $P\colon \mathcal{C}^{op} \longrightarrow \mathsf{Pos}$ is elementary, admits full weak comprehension with comprehensive diagonals (hence $\mathcal{C}$ has weak pullbacks by Proposition 2.9). If $\mathcal{Q}_P$ is locally cartesian closed and P is existential, then P is a hyperdoctrine.

PROOF. Note first of all that, since $\mathcal{Q}_P$ is locally cartesian closed, the variational doctrine $\Psi_{\mathcal{Q}_P}$ has right adjoints along any reindexing $(\Psi_{\mathcal{Q}_P})_{[f]}$, hence it is implicational and universal, and admits full weak comprehension. By Proposition 2.20, $\widehat{P}$ is existential since P is existential. Since $\widehat{P}$ admits full strong comprehension, by Proposition 2.12 it follows that $\widehat{P}$ is universal and implicational. Applying again Proposition 2.20 yields that P is universal and implicational, and hence a hyperdoctrine. ∎

4.2. LEMMA.

Suppose that $P\colon \mathcal{C}^{op} \longrightarrow \mathsf{Pos}$ is elementary with comprehensive diagonals, and admits full weak comprehension (hence $\mathcal{C}$ has weak pullbacks by Proposition 2.9). If $\mathcal{Q}_P$ is locally cartesian closed, then for every object A in $\mathcal{C}$, a dependent P-projective over A is weakly exponentiable relative to $\mathcal{D}_A$.

PROOF. Let $[f]\colon (X,\delta_X) \longrightarrow (A,\delta_A)$ be a dependent P-projective and $g\colon (Y,\tau) \longrightarrow (A,\delta_A)$ any object in $\mathcal{Q}_P$. Consider the following diagram in $\mathcal{Q}_P$

$$\begin{tikzcd}
(S',\theta') \arrow[r, "{[q_1]}"] \arrow[d, "{[q_2]}"'] & (W,\delta_W) \arrow[d, "{[\mathrm{id}_W]}"'] & \\
(S,\theta) \arrow[r, "{[p_2]}"'] \arrow[d, "{[p_1]}"'] \arrow[rr, bend left, "{[\mathrm{ev}]}"] & (W,\xi) \arrow[d, "{[g]^{[f]}}"] & (Y,\tau) \arrow[dl, "{[g]}"] \\
(X,\delta_X) \arrow[r, "{[f]}"'] & (A,\delta_A) &
\end{tikzcd}$$

where the two squares are pullbacks and $[\mathrm{ev}]\colon (S,\theta) \longrightarrow (Y,\tau)$ is the universal arrow of the exponential. Fix a representative w of the equivalence class $[g]^{[f]}$. Then $w\colon (W,\delta_W) \longrightarrow (A,\delta)$ together with $[\mathrm{ev} q_2]\colon (S',\theta') \longrightarrow (Y,\tau)$ is clearly a weak exponential of $[f]$ over $[g]$ relative to the full subcategory $\mathcal{D}_A$ on the P-dependent projections.

Indeed, for any arrow $u\colon U \longrightarrow A$ in $\mathcal{C}$ and any arrow $[k]\colon Jf \times_{JA} Ju \longrightarrow [g]$ in $\mathcal{Q}_P/JA$, by cartesian closure of $\mathcal{Q}_P/JA$ there exists $[\widehat{k}]\colon (U,\delta_U) \longrightarrow (W,\xi)$ such that in $\mathcal{Q}_P/JA$

$$\begin{tikzcd}
Jf \times_{JA} Ju \arrow[d, "J\mathrm{id}_X \times_{JA} {[\widehat{k}]}"'] \arrow[dr, "{[k]}"] & \\
Jf \times_{JA} [g]^{[f]} \arrow[r, "{[\mathrm{ev}]}"] & {[g]}
\end{tikzcd}$$

Now, observe that $J(\widehat{k}) = [\widehat{k}]\colon (U,\delta_U) \longrightarrow (W,\delta_W)$ satisfies the required condition of weak exponentiability

$$\begin{tikzcd}
Jf \times_{JA} Ju \arrow[d, "J\mathrm{id}_X \times_{JA} J\widehat{k}"'] \arrow[dr, "{[k]}"] & \\
Jf \times_{JA} Jw \arrow[r, "{[\mathrm{ev} q_2]}"] & {[g]}
\end{tikzcd}$$

as described in Remark 3.6. ∎

4.3. Proposition.

*Suppose $P\colon \mathcal{C}^{op} \longrightarrow$ **Pos** is elementary with comprehensive diagonals and admits full weak comprehension. Suppose also that $\mathcal{Q}_P$ is locally cartesian closed. If P is existential, or P is implicational and universal, then P is a slicewise weakly cartesian closed hyperdoctrine.*

Proof. After Lemma 4.2, one needs only to invoke Lemma 4.1 to get that, in case P is existential, P is also universal and implicational. ∎

We now aim at proving a partial converse to Proposition 4.3, where we shall consider only the case when the elementary doctrine P is universal and implicational because of Proposition 4.3. To that purpose, we produce an equivalent presentation of objects of $\mathcal{C}/A$, giving an algebraic presentation in line with the characterization in [Maietti, 2009, Proposition 4.12].

For the sake of simplicity, we introduce some explicit notations for certain arrows related to constructions in the base $\mathcal{Q}_P$ of the elementary quotient completion.

4.4. Notation.

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow$ **Pos** be an elementary doctrine admitting full weak comprehension. Let $f\colon B \longrightarrow A$ be a representative of an arrow $[f]\colon (B,\sigma) \longrightarrow (A,\rho)$ in $\mathcal{Q}_P$, and write $c_{\rho,f}\colon X \longrightarrow B \times A$ a comprehending arrow for $P_{f\times \mathrm{id}_A}(\rho)$ in $P(B\times A)$. We use the following notation for the compositions in the diagram

$$
\begin{tikzcd}
& B \\
X \arrow[ur, "f^\sigma"] \arrow[r, "c_{\rho,f}"'] \arrow[dr, "f_\rho"'] & B \times A \arrow[u, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"] \\
& A
\end{tikzcd}
$$

Write σ_f for the P-equivalence relation on X determined by the conjunction $P_{f^\sigma \times f^\sigma}(\sigma) \wedge P_{f_\rho \times f_\rho}(\rho) = P_{f^\sigma \times f^\sigma}(\sigma)$ so that in the internal logic $\sigma_f(x, x')$ abbreviates the formula $\sigma(f^\sigma(x), f^\sigma(x'))$.

4.5. Remark.

It is immediate to see from the definition of the relation σ_f that the arrow $f^\sigma : X \longrightarrow B$ determines an arrow $[f^\sigma] : (X, \sigma_f) \longrightarrow (B, \sigma)$ in $\mathcal{Q}_P$ as well as the arrow $f_\rho : X \longrightarrow A$ gives an arrow $[f_\rho] : (X, \sigma_f) \longrightarrow (A, \rho)$.

Also, there is a commutative diagram in $\mathcal{C}$

$$
\begin{tikzcd}
& & B \arrow[r, "f"] & A \\
B \arrow[urr, bend left, "\mathrm{id}_B"] \arrow[rr, bend left=15, "\langle \mathrm{id}_B, f \rangle"] \arrow[r, dotted, "k"'] \arrow[drr, bend right, "f"'] & X \arrow[ur, "f^\sigma"] \arrow[r, "c_{\rho,f}"'] \arrow[dr, "f_\rho"] & B \times A \arrow[u, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"] \arrow[r, "f \times \mathrm{id}_A"] & A \times A \arrow[u, "\mathrm{pr}'_1"'] \arrow[d, "\mathrm{pr}'_2"] \\
& & A \arrow[r, "\mathrm{id}_A"'] & A
\end{tikzcd}
\tag{5}
$$

where the arrow k exists by weak universality of $c_{\rho,f} : X \longrightarrow B \times A$, since ρ is a P-equivalence relation and

$$
P_{\langle \mathrm{id}_B, f \rangle} P_{f \times \mathrm{id}_A}(\rho) = P_{\langle f, f \rangle}(\rho) = \top_B .
$$

In particular, it gives a retraction pair

$$
\begin{tikzcd}
B \arrow[loop left, "\mathrm{id}_B"] \arrow[r, bend right, "k"'] & X . \arrow[l, bend right, "f^\sigma"']
\end{tikzcd}
$$

Moreover, $\top_X = P_{\langle f f^\sigma, f_\rho \rangle}(\rho)$ since $\langle f f^\sigma, f_\rho \rangle = (f \times \mathrm{id}_A) c_{\rho,f}$, *i.e.*

$$
x : X \vdash \rho(f f^\sigma(x), f_\rho(x)) \tag{6}
$$

in the internal logic of the doctrine P.

4.6. Proposition.

In the notations of Remark 4.5, the following diagram

$$
\begin{tikzcd}
(B,\sigma) \arrow[loop left, "{[\mathrm{id}_B]}"] \arrow[rr, bend right, "{[k]}"'] \arrow[dr, "{[f]}"'] & & (X,\sigma_f) \arrow[ll, bend right, "{[f^\sigma]}"'] \arrow[loop right, "{[\mathrm{id}_X]}"] \arrow[dl, "{[f_\rho]}"] \\
& (A,\rho) &
\end{tikzcd}
$$

commutes in $\mathcal{Q}_P$.

Proof. To complete the proof after Remark 4.5, one must show that $[k]\colon (B,\sigma) \longrightarrow (X,\sigma_f)$, and $[k][f^\sigma] = [\mathrm{id}_X]\colon (X,\sigma_f) \longrightarrow (X,\sigma_f)$. They both easily follow from the fact that $f^\sigma k = \mathrm{id}_B$. Indeed, in the internal logic

$$x\colon B, x'\colon B \mid \sigma(x,x') \vdash \sigma(f^\sigma(k(x)), f^\sigma(k(x')))$$

says that $[k]\colon (B,\sigma) \longrightarrow (X,\sigma_f)$ is well defined, while

$$\begin{aligned} x\colon X \vdash\, & \sigma(f^\sigma(x), f^\sigma(x)) \\ \vdash\, & \sigma(f^\sigma(kf^\sigma(x)), f^\sigma(x)) \end{aligned}$$

proves that $[k][f^\sigma] = [\mathrm{id}_X]$. ∎

4.7. Remark.

Proposition 4.6 shows that $[f^\sigma]\colon [f_\rho] \xrightarrow{\sim} [f]$ in the slice category $\mathcal{Q}_P/(A,\rho)$. Note, though, that $[f]$ and $[f_\rho]$ need not factor through each other in $\mathcal{C}$. Indeed, Remark 4.5 shows that f factors through f_ρ in $\mathcal{C}$, but nothing guarantees the other factorisation may occur. Since $\mathcal{Q}_P$ is a category with finite limits, one can see that $[f_\rho]$ is isomorphic to $\Sigma_{[\mathrm{id}_A]}[\mathrm{id}_A]^*([f])$, where $\Sigma_{[\mathrm{id}_A]}$ denotes the left adjoint to the pullback functor $[\mathrm{id}_A]^*\colon \mathcal{Q}_P/(A,\rho) \longrightarrow \mathcal{Q}_P/(A,\delta_A)$ along the map $[\mathrm{id}_A]\colon (A,\delta_A) \longrightarrow (A,\rho)$. This is similar to the homotopical account of partial equivalence relations given in [Frey, 2023] (see diagram (5.2) p. 15).

The following is the fundamental step toward the proof of the main result. It takes advantage of the iso $[f^\sigma]\colon [f_\rho] \xrightarrow{\sim} [f]$ in the slice category $\mathcal{Q}_P/(A,\rho)$ to compute explicitly any product of $[f]$ in the slice category starting from a product in the slice category $\mathcal{C}/A$.

Like before, we employ Notation 4.4.

4.8. LEMMA.

Let $P\colon \mathcal{C}^{op} \longrightarrow \mathsf{Pos}$ be an elementary existential doctrine admitting weak comprehension with comprehensive diagonals. Consider two arrows

$$\begin{tikzcd} & (W,\theta) \arrow[d, "{[q]}"] \\ (B,\sigma) \arrow[r, "{[f]}"] & (A,\rho) \end{tikzcd}$$

in $\mathcal{Q}_P$, and consider the following diagrams

$$\begin{tikzcd} Z \arrow[r, "f'_\rho"] \arrow[d, "q'"'] \arrow[dr, phantom, "\boxed{3}"] & W \arrow[d, "q"] \\ X \arrow[r, "f_\rho"'] & A \end{tikzcd} \qquad \begin{tikzcd} (Z, P_{q'\times q'}(\sigma_f) \wedge P_{f'_\rho \times f'_\rho}(\theta)) \arrow[r, "{[f'_\rho]}"] \arrow[d, "{[q']}"'] \arrow[dr, phantom, "\boxed{4}"] & (W,\theta) \arrow[d, "{[q]}"] \\ (X,\sigma_f) \arrow[r, "{[f_\rho]}"'] & (A,\rho). \end{tikzcd}$$

If $\boxed{3}$ is a weak pullback in $\mathcal{C}$, then $\boxed{4}$ is a pullback in $\mathcal{Q}_P$.

PROOF. We write ζ for the P-equivalence relation $P_{q'\times q'}(\sigma_f) \wedge P_{f'_\rho\times f'_\rho}(\theta)$ on Z. Clearly, if diagram $\boxed{3}$ commutes in $\mathcal{C}$, then so does $\boxed{4}$ in $\mathcal{Q}_P$. Consider a commutative diagram

$$\begin{tikzcd} (C,\gamma) \arrow[drr, "{[\ell]}"] \arrow[ddr, "{[h]}"'] & & \\ & (Z,\zeta) \arrow[r, "{[f'_\rho]}"'] \arrow[d, "{[q']}"] & (W,\theta) \arrow[d, "{[q]}"] \\ & (X,\sigma_f) \arrow[r, "{[f_\rho]}"'] & (A,\rho) \end{tikzcd} \tag{7}$$

in $\mathcal{Q}_P$. So, in the internal logic of P, we have that

(a) $x, x'\colon C \mid \gamma(x,x') \vdash \rho(f_\rho(h(x)), f_\rho(h(x')))$;

(b) $x, x'\colon C \mid \gamma(x,x') \vdash \sigma(f^\sigma(h(x)), f^\sigma(h(x')))$.

Recall that $[f_\rho] = [f f^\sigma]$ by Proposition 4.6. So from (a) we get

$$x\colon C \vdash \rho(f f^\sigma h(x), q\ell(x)).$$

Hence, weak universality of $c_{\rho,f}\colon X \longrightarrow B \times A$ produces a filler j in the following diagram in $\mathcal{C}$

$$\begin{tikzcd} C \arrow[r, dotted, "j"] \arrow[d, "{\langle h,\ell\rangle}"] \arrow[dd, bend right, "h"'] & X \arrow[d, "c_{\rho,f}"] \arrow[dr, "{\langle f f^\sigma, f_\rho\rangle}"] \arrow[dd, bend right, "f^\sigma"'] & \\ X\times W \arrow[r, "f^\sigma\times q"] \arrow[d, "\mathrm{pr}'_1"] & B\times A \arrow[r, "f\times \mathrm{id}_A"] \arrow[d, "\mathrm{pr}_1"] & A\times A \\ X \arrow[r, "f^\sigma"] & B & \end{tikzcd}$$

From $f^\sigma h = f^\sigma j$ and (b), we get $x, x' \colon C \mid \gamma(x,x') \vdash \sigma(f^\sigma(j(x)), f^\sigma(j(x')))$ showing that $[j] \colon (C,\gamma) \longrightarrow (X, \sigma_f)$ is well defined, and also

$$x, x' \colon C \mid \gamma(x,x') \vdash \sigma(f^\sigma(j(x)), f^\sigma(h(x')))$$

showing that [j] and it is equal to $[h] \colon (C,\gamma) \longrightarrow (X,\sigma_f)$. Moreover, also the following diagram commutes in $\mathcal{C}$

$$\begin{tikzcd}
C \arrow[r, "\ell"] \arrow[d, "j"'] & W \arrow[d, "q"] \\
X \arrow[r, "f_\rho"'] & A
\end{tikzcd}$$

Therefore, since $\boxed{3}$ is a weak pullback, there is an arrow

$$\begin{tikzcd}
C \arrow[drr, bend left, "\ell"] \arrow[dr, dotted, "m"'] \arrow[ddr, "j"'] & & \\
& Z \arrow[r, "f'_\rho"] \arrow[d, "q'"] & W \arrow[d, "q"] \\
& X \arrow[r, "f_\rho"'] & A
\end{tikzcd}$$

Thanks to the definition of the P-equivalence relation ζ, it is immediate to prove that that gives a unique arrow filling in the diagram (7). ∎

4.9. Remark.

It is possible to derive a moral from Lemma 4.8. Even though there are only weak pullbacks in $\mathcal{C}$, each object in a slice category of $\mathcal{Q}_P$ may be replaced by an isomorphic copy on which pullbacks can be computed as if weak pullbacks in $\mathcal{C}$ were actual pullbacks.

As obscure as that moral may be, it is going to be employed in the construction of exponentials in each slice category of $\mathcal{Q}_P$.

We approach the main theorem of the section introducing the explicit construction of an exponential in the slice $\mathcal{Q}_P/(A,\rho)$; the following notation presents the first steps of that construction by producing the relevant P-equivalence relation to be used then in the proof of Theorem 4.12.

4.10. Notation.

The following notation that will be used in the proofs of the Lemma 4.11 and Theorem 4.12.

Let $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ be a slicewise weakly cartesian closed doctrine which admits weak comprehension. Let $[f] \colon (B,\sigma) \longrightarrow (A,\rho)$ and $[g] \colon (C,\tau) \longrightarrow (A,\rho)$ be two objects in the slice category $\mathcal{Q}_P/(A,\rho)$. Consider the arrows $f_\rho\colon X \longrightarrow A$ and $g_\rho\colon Y \longrightarrow A$ in $\mathcal{C}$, introduced in Notation 4.4, as well as the corresponding P-equivalence relations σ_f on X and τ_g on Y. Then, in the slice category $\mathcal{Q}_P/(A,\delta_A)$ take the P-dependent projective $[f_\rho]\colon (X,\delta_X) \longrightarrow (A,\delta_A)$ and the arrow $[g_\rho]\colon (Y, P_{g_\rho\times g_\rho}(\delta_A)\wedge\tau_g) \longrightarrow (A,\delta_A)$ obtained by pulling back $[g_\rho]\colon (Y,\tau_g) \longrightarrow (A,\rho)$ along $[\mathrm{id}_A]\colon (A,\delta_A) \longrightarrow (A,\rho)$ as in the following commutative diagram

$$\begin{tikzcd}
(Y, P_{g_\rho\times g_\rho}(\delta_A)\wedge\tau_g) \arrow[r, "{[\mathrm{id}_Y]}"] \arrow[d, "{[g_\rho]}"'] & (Y,\tau_g) \arrow[d, "{[g_\rho]}"] \\
(A,\delta_A) \arrow[r, "{[\mathrm{id}_A]}"'] & (A,\rho).
\end{tikzcd}$$

which is a pullback thanks to Lemma 4.8. Consider a weak exponential $[p]\colon (V,\delta_V) \longrightarrow (A,\delta_A)$ of $[f_\rho]\colon (X,\delta_X) \longrightarrow (A,\delta_A)$ and $[g_\rho]\colon (Y, P_{g_\rho\times g_\rho}(\delta_A)\wedge\tau_g) \longrightarrow (A,\delta_A)$ which gives, in $\mathcal{C}$, the following arrows

$$\begin{tikzcd}
S \arrow[r, "q_2"'] \arrow[d, "q_1"'] \arrow[rr, bend left, "\mathrm{ev}'"] & V \arrow[d, "q"] & Y \arrow[dl, "g_\rho"] \\
X \arrow[r, "f_\rho"'] & A &
\end{tikzcd} \tag{8}$$

where the inner square is a weak pullback. For a variable $v\colon V$, write $\xi(v)$ for the formula

$$\forall_{s,s'\colon S}\,[[(q_2(s) =_V v \wedge q_2(s') =_V v) \wedge \sigma_f(q_1(s), q_1(s'))] \Rightarrow \tau_g(\mathrm{ev}'(s), \mathrm{ev}'(s))]$$

—note that the antecedent of the implication yields that the pair $\langle s, s'\rangle$ is in the P-equivalence relation imposed on the upper left vertex in the diagram $\boxed{4}$ of Lemma 4.8.

Consider the comprehending arrow $\{\!|\xi|\!\}\colon W \longrightarrow V$. Take the weak pullback of $\{\!|\xi|\!\}$ along q_2 and paste it with that in diagram (8) to obtain another weak pullback and the composition $\mathrm{ev} = \mathrm{ev}'u\colon V \longrightarrow Y$, which will eventually be part of the evaluation arrow:

$$\begin{tikzcd}
Z \arrow[r, "p_2"'] \arrow[d, "u"] \arrow[dd, bend right, "p_1"'] \arrow[rrd, bend left, "\mathrm{ev}"] & W \arrow[d, "\{\!|\xi|\!\}"'] \arrow[dd, bend left, "p"] & \\
S \arrow[r, "q_2"'] \arrow[d, "q_1"] \arrow[rr, bend left, "\mathrm{ev}'"] & V \arrow[d, "q"'] & Y \arrow[dl, "g_\rho"] \\
X \arrow[r, "f_\rho"'] & A &
\end{tikzcd} \tag{9}$$

The necessary final piece of data is the appropriate P-equivalence relation on W: consider variables $w, w' \colon W$ and write $\theta(w, w')$ for the formula

$$\begin{aligned}&\rho(p(w), p(w'))\wedge\\ &\qquad\wedge\forall_{z,z':Z}\left[\left[\begin{aligned}&(p_2(z) =_W w \wedge p_2(z') =_W w')\wedge\\ &\qquad\wedge\sigma_f(p_1(z), p_1(z'))\end{aligned}\right] \Rightarrow \tau_g(\mathrm{ev}(z), \mathrm{ev}(z'))\right]\end{aligned} \tag{10}$$

so that θ is in $P(W \times W)$—the same comment as for the formula $\xi(v)$ above, applies here with the pair $\langle z, z'\rangle$.

With the notation above, the proof of the following lemma is intuitively easy.

4.11. Lemma.

The relation θ in notation 4.10 is a P-equivalence relation over W such that

$$z \colon Z, z' \colon Z \mid \theta(p_2(z), p_2(z')) \wedge \sigma_f(p_1(z), p_1(z')) \vdash \tau_g(\mathrm{ev}(z), \mathrm{ev}(z'))$$

Hence, the relation η defined as

$$\eta(z, z') = \theta(p_2(z), p_2(z')) \wedge \sigma_f(p_1(z), p_1(z')) \tag{11}$$

is a P-equivalence relation over Z.

The following theorem is the first characterization of locally cartesian closed elementary quotient completions.

4.12. Theorem.

Suppose P is an elementary existential doctrine admitting full weak comprehension with comprehensive diagonals. The following are equivalent:

(i) *P is slicewise weakly cartesian closed;*

(ii) *P is slicewise cartesian closed on dependent projectives;*

(iii) *$\mathcal{Q}_P$ is locally cartesian closed.*

Proof. (iii) $\Rightarrow$ (i) follows from Lemma 4.3.
(iii) $\Rightarrow$ (ii): Condition (i) of Definition 3.8 follows from Proposition 4.1, while condition (ii) is immediate.
(i)$\Rightarrow$(iii): Suppose that P is slicewise weakly cartesian closed. Let $[f] \colon (B, \sigma) \longrightarrow (A, \rho)$ and $[g] \colon (C, \tau) \longrightarrow (A, \rho)$ be objects in $\mathcal{Q}_P/(A, \rho)$. Here, we align to the notation used after Lemma 4.11. Consider $\mathrm{ev} \colon Z \longrightarrow Y$ defined as in (9) and η as in (11). By definition of η, the arrow ev determines an arrow

$$[\mathrm{ev}] \colon (Z, \eta) \longrightarrow (Y, \tau_g)$$

in $\mathcal{Q}_P/(A,\rho)$ from $[f_\rho p_1]$ to $[g_\rho]$. Moreover, Lemma 4.8 ensures that (Z,η) is the pullback of $[p]$ along $[f_\rho]$.

Thanks to Proposition 4.6, it suffices to show that $[p]$ is the exponential in $\mathcal{Q}_P/(A,\rho)$ of $[g_\rho]$ and $[f_\rho]$ with evaluation $[\mathrm{ev}]\colon [f_\rho]\times_{(A,\rho)}[p]\longrightarrow[g_\rho]$. Consider an arbitrary object $[h]\colon(D,\nu)\longrightarrow(A,\rho)$ in $\mathcal{Q}_P/(A,\rho)$, and let $[m]\colon[f_\rho]\times_{(A,\rho)}[h]\longrightarrow[g_\rho]$. By Lemma 4.8 we can assume

$$\begin{tikzcd}
(Q, P_{d_1\times d_1}(\sigma_f)\wedge P_{p_2\times p_2}(\nu)) \arrow[rr, "{[m]}"] \arrow[dr, "{[f_\rho d_1]}"'] & & (Y,\tau_g) \arrow[dl, "{[g_\rho]}"] \\
& (A,\rho) &
\end{tikzcd}$$

depicting an arrow in $\mathcal{Q}_P/(A,\rho)$ for an appropriate weak pullback in $\mathcal{C}$

$$\begin{tikzcd}
Q \arrow[r, "p_2"] \arrow[d, "d_1"'] & D \arrow[d, "h"] \\
X \arrow[r, "f_\rho"'] & A.
\end{tikzcd}$$

Consider the commutative diagram

$$\begin{tikzcd}
& (Q, P_{d_1\times d_1}(\sigma_f)\wedge P_{p_2\times p_2}(\nu)) \arrow[dr, "{[m]}"] \arrow[ddr, "{[f_\rho d_1]}"' near start] & \\
(Q, P_{d_1\times d_1}(\delta_X)\wedge P_{p_2\times p_2}(\delta_D)) \arrow[ur, "{[\mathrm{id}_Q]}"] \arrow[dr, "{[m]}"] \arrow[ddr, "{[f_\rho d_1]}"'] & & (Y,\tau_g) \arrow[d, "{[g_\rho]}"] \\
& (Y, P_{g_\rho\times g_\rho}(\delta_A)\wedge\tau_g) \arrow[ur, "{[\mathrm{id}_Y]}" near end] \arrow[d, "{[g_\rho]}"] & (A,\rho) \\
& (A,\delta_A) \arrow[ur, "{[\mathrm{id}_A]}"'] &
\end{tikzcd}$$

where the square on the right face is a pullback. Since P is slicewise weakly cartesian closed,

$$[m]\colon(Q, P_{d_1\times d_1}(\delta_X)\wedge P_{p_2\times p_2}(\delta_D))\longrightarrow(Y, P_{g_\rho\times g_\rho}(\delta_A)\wedge\tau_g)$$

determines a commutative triangle

$$\begin{tikzcd}
D \arrow[rr, "\widehat{m}"] \arrow[dr, "h"'] & & V \arrow[dl, "q"] \\
& A &
\end{tikzcd}$$

in $\mathcal{C}$ where V is a weak exponential, and a commutative diagram

$$\begin{tikzcd}
Jf_\rho\times_{(A,\delta_A)}Jh \arrow[d, "J\mathrm{id}_X\times_{(A,\delta_A)}J\widehat{m}"'] \arrow[dr, "{[m]}"] & \\
Jf_\rho\times_{(A,\delta_A)}Jq \arrow[r, "{[\mathrm{ev}']}"] & {[g_\rho]}
\end{tikzcd}$$

in $\mathcal{Q}_P/(A,\delta_A)$ with $[g_\rho]\colon (Y, P_{g_\rho\times g_\rho}(\delta_A)\wedge P_{g^\tau\times g^\tau}(\tau))\longrightarrow (A,\delta_A)$. Therefore,

$$d\colon D\vdash \forall_{s,s'\colon S}[(q_2(s)=_V \widehat{m}(d)\wedge q_2(s')=_V \widehat{m}(d))\wedge \sigma_f(q_1(s),q_1(s'))\Rightarrow \tau_g(\mathrm{ev}'(s),\mathrm{ev}'(s))]$$

holds and, by the weak universal property of comprehension, there is an arrow $\mu\colon X\longrightarrow W$ such that $\widehat{m}=\mu\{\!|\xi|\!\}$. Thus, the arrow μ determines the required arrow $[\mu]\colon (D,\pi)\longrightarrow (W,\theta)$ in $\mathcal{Q}_P/(A,\rho)$. Uniqueness is a direct consequence of the definition of θ.
(ii) $\Rightarrow$ (iii): The proof is similar to that of (i) $\Rightarrow$ (iii). ∎

4.13. Corollary.

Suppose $\mathcal{C}$ is a category with finite products and weak pullbacks, the following are equivalent:

(i) *the variational doctrine $\Psi_{\mathcal{C}}\colon \mathcal{C}^{op}\longrightarrow \textbf{Pos}$ is slicewise weakly cartesian closed;*

(ii) *$\mathcal{C}_{ex/lex}$ is locally cartesian closed.*

Proof. It follows as a direct application of Theorem 4.12 knowing that $\mathcal{C}_{\mathrm{ex/lex}}$ is equivalent to $\mathcal{Q}_{\Psi_{\mathcal{C}}}$ and that $\Psi_{\mathcal{C}}$ is elementary existential, and it admits full weak comprehension and has comprehensive diagonals, if $\mathcal{C}$ has finite products and weak pullbacks. ∎

4.14. Corollary.

Suppose $\mathcal{C}$ is a category with finite limits; the following are equivalent:

(i) *$\Psi_{\mathcal{C}}\colon \mathcal{C}^{op}\longrightarrow \textbf{Pos}$ is slicewise weakly cartesian closed;*

(ii) *$\mathcal{C}$ has weak dependent products, as in* [Birkedal et al., 1998, Definition 3.5]

Proof. It follows from Corollary 4.13 and [Birkedal et al., 1998, Theorem 3.8]. ∎

4.15. Corollary.

Suppose $\mathcal{C}$ is a category with finite products and weak pullbacks, the following are equivalent:

(i) *$\Psi_{\mathcal{C}}\colon \mathcal{C}^{op}\longrightarrow \textbf{Pos}$ is slicewise weakly cartesian closed;*

(ii) *$\mathcal{C}$ has extensional dependent products as in* [Emmenegger, 2020, Definition 3.1].

Proof. It follows from Corollary 4.13 and [Emmenegger, 2020, Theorem 3.6]. ∎

5. Examples

Main applications of Theorem 4.12 include those for the category $\mathcal{Asm}$ of assemblies over a given partial combinatory algebra, see [Hyland, 1982, van Oosten, 2008]. In [Maietti et al., 2019] we showed that $\mathcal{Asm}$ is the base of the elementary quotient completion of the doctrine of strong subobjects on the category of partitioned assemblies. Hence, what is shown here gives an alternative proof that the locally cartesian closure of $\mathcal{Asm}$ is inherited from that of partitioned assemblies.

Other noteworthy examples arise in type theory through the construction of special kinds of "setoid models". Indeed, as noted in [Maietti and Rosolini, 2013b, §7], the models of *total setoids à la Bishop*, constructed either over Coquand-Huet-Paulin's Calculus of Inductive Constructions [Coquand, 1990, Coquand and Paulin, 1990], or over Martin-Löf's type theory [Nordström et al., 1990], or over the intensional level of the Minimalist Foundation [Maietti, 2009], can be represented as the base of the elementary quotient completion of a suitable syntactic doctrine on a base that merely has weak pullbacks.

In particular, the setoid model used to build the Minimalist Foundation in [Maietti, 2009] was one of the motivating examples for introducing the elementary quotient completion in [Maietti and Rosolini, 2013b].

Another application of Theorem 4.12 is for the category of ultrametric spaces. As shown in [Dagnino and Pasquali, 2022], using the axiom of choice in the metatheory, this category is equivalent to the elementary quotient completion of the elementary doctrine $P_{[0,\infty]}\colon \mathcal{Set}^{\mathrm{op}} \longrightarrow \textsf{Pos}$ employing the complete Heyting algebra of the extended positive reals $[0, \infty]$.

6. Conclusions

We generalized Carboni and Rosolini's characterization of locally cartesian closed exact completions of a category with finite products and weak pullbacks in [Carboni and Rosolini, 2000, Emmenegger, 2020].

An independent characterization of locally cartesian closed quotient completions for doctrines with weak finite limits was given in [Cioffo, 2022], with the additional hypothesis that the considered doctrines are universal. In future work, we explore how such a characterization can be extended to include those for exact completions of categories with weak finite limits in [Carboni and Rosolini, 2000, Emmenegger, 2020] as instances.

A further analysis of the categorical structure inherited by the elementary quotient completion is in [Maietti et al., 2021]. As a subsequent work, we aim to carry out such an analysis by including inductive and coinductive constructions such as those investigated in [Moerdijk and Palmgren, 2000, Emmenegger, 2021, van den Berg and De Marchi, 2007].

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Dipartimento di Matematica "Tullio Levi Civita", Università di Padova, via Trieste 63, 35121 Padova, Italy

Dipartimento di Matematica, Dipartimento di Eccellenza 2023-2027, Università di Genova, via Dodecaneso 35, 16146 Genova, Italy

Email: `maietti@math.unipd.it`
`pasquali@dima.unige.it`
`rosolini@unige.it`