# Consistency of the minimalist foundation with Church thesis and Bar Induction

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**Abstract**

We consider a version of the *minimalist foundation* previously introduced to formalize predicative constructive mathematics. This foundation is equipped with *two levels* to meet the usual informal practice of developing mathematics in an extensional set theory (its *extensional level*) with the possibility of formalizing it in an intensional theory enjoying a proofs as programs semantics (its *intensional level*).

For the intensional level we show a realizability interpretation validating Bar Induction and formal Church thesis for type-theoretic functions. This is possible because in our foundation the well-known result by Kleene that Brouwer's principle of Bar Induction is inconsistent with the formal Church thesis for choice sequences can be decomposed as follows: Brouwer's Bar Induction, where choice sequences are functional relations, is inconsistent with the formal Church thesis for type-theoretic functions (from natural numbers to natural numbers) and the axiom of unique choice transforming a functional relation between natural numbers into a type-theoretic function. As a consequence this model disproves the validity of the axiom of unique choice in our foundation.

This model can serve to interpret the whole foundation in a *classical predicative* set theory by keeping the computational interpretation of predicative sets as data types and their type-theoretic functions as programs. Moreover it shows that choice sequences of Cantor space, those of Baire space, and real numbers both as Dedekind cuts or Cauchy sequences, do not form a set in the minimalist foundation.

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## 1 Introduction

**The need of a minimalist foundation.** Various logical systems are available in the literature to formalize constructive mathematics: they range from axiomatic set theories à la Zermelo-Fraenkel, as Aczel's CZF [AR01, Acz78, Acz82, Acz86] or Friedman's IZF [Bee85], to the internal set theory of categorical universes as topoi or pretopoi [MM92, JM95, Mai05b], to type theories as Martin-Löf's type theory [NPS90] or Coquand's Calculus of Inductive Constructions [Coq90, CP90]. No existing constructive foundation has yet superseded the others as the standard one as Zermelo-Fraenkel set theory did for classical mathematics.

Also various machine-aided proof development systems are available to implement mathematics (see, for example, [Wie06]). Most of those for constructive mathematics, as for example Coq [Coq10, BC04] or Nuprl [The95], distinguish themselves for being based on typed systems, respectively the Calculus of (Co)Inductive Constructions and Nuprl's Computational Type Theory [ABC$^+$06], which are also paradigm of (functional) programming languages with the possibility of extracting the computational contents of constructive mathematical proofs. Indeed a peculiar characteristic of what we call "constructive formal systems", contrary to classical ones, is that they enjoy a computational interpretation, which we can call proofs-as-programs semantics, in which we can extract programs witnessing provable existential statements [BC85]. A paradigmatic example is Kleene realizability interpretation [Tv88a] for the intuitionistic version of Peano arithmetics, called Heyting arithmetics.

Another relevant aspect is that the proof assistants based on an intensional type theory, as for example Agda [BDN09] (on Martin-Löf's type theory), or Coq or Matita [ARCT11] (on the Calculus of Inductive Constructions), enjoy a decidable type checking of proofs (and programs).

Starting from the paper [MS05], together with G. Sambin we embarked on the project of developing a minimalist foundation to be considered as a common core among the most relevant constructive foundations. We wanted to design such a foundation as a theory equipped with two levels in order to meet the usual practice of developing mathematics in an extensional set theory, represented by an extensional level, with the practice of formalizing it in a computer-assisted way within an intensional type theory suitable for program extraction. Then, the compatibility of this minimalist foundation with the most relevant constructive and classical extensions, at the right level, would make a proof-assistant based on such a foundation more suitable for formalizing reusable proofs.

**The notion of constructive foundation.** More in detail, in [MS05] we required that a *constructive* foundation should be equipped with *one level*, called *intensional*, given by a proofs-as-programs theory, *another level*, called *extensional*, given by a set theory where to formalize mathematical proofs, with in addition a requirement on how *to link the levels*: the extensional level should be obtained by abstraction from the intensional one according to Sambin's forget-restore principle in [SV98] so to preserve the extraction of programs from proofs. We also formalized the notion of proofs-as-programs theory in a very technical sense in comparison with the intuitive idea of a theory enjoying a semantics where proofs are interpreted as programs. Indeed, in [MS05] we defined a *proofs-as-programs theory* as one *consistent with the formal Church thesis and the axiom of choice*. The reason is that we had in mind that the theory should satisfy an interpretation similar to Kleene realizability one for Heyting arithmetics, where these principles are validated. Then, this definition of proofs-as-programs theory turned to be very useful in discriminating intensional and constructive theories versus extensional and classical ones (see [MS05]).

**The two levels of the minimalist foundation.** In [Mai09] we built an example of our desired minimalist two level constructive foundation. Its two levels are both given by a type theory à la Martin-Löf: the first is an intensional type theory as [NPS90], called mTT, with propositions defined primitively to avoid the validity of choice principles and the latter is an extensional one, called emTT, with proof-irrelevance of propositions and quotients (similar to those implemented in Nuprl [The95]). The extensional level is then interpreted in the intensional one via a quotient model based on total setoids à la Bishop [Bis67, Hof97, BCP03, Pal05]. The quotient model is an instance of an abstract quotient completion described in [MR13] and shows how extensional concepts are obtained by just abstracting from equalities of intensional ones. In [Mai09] we also emphasized that it is enough to present the extensional level as a fragment of the internal language of a quotient completion of the intensional one in order to meet the abstract link in [MS05] between the two levels of a constructive foundation.

**A realizability interpretation validating Bar Induction and Church thesis for type-theoretic functions.** As advocated in [MS05], a main novelty of our foundation in [Mai09], which is also a major difference with respect to Martin-Löf's type theory, is that it should not validate the axiom of unique choice turning a functional relation into a type-theoretic function, even restricted to natural numbers. Formally, this distinction is possible because, at both levels of our foundation, we discharged the isomorphism "propositions-as-sets" of Martin-Löf's type theory and we built propositions via primitive constructors distinct from those for sets, as it happens in the Calculus of Constructions [Coq90] of which mTT is a predicative version.

Therefore in our foundation, contrary to most extensional constructive theories in the literature, such as Aczel's CZF or the internal theory of a topos (for example in [Mai05b]), we have two distinct notions of function: the usual notion of functional relation and that of type-theoretic function.

The benefit of this is the possibility of revisiting the well known result by Kleene [KV65, Tv88a] that Brouwer's principle of Bar Induction, or better the Fan theorem derived from it, is inconsistent with the formal Church thesis for choice sequences [Tv88a, Rat05, Dum00]. Indeed, at the extensional level of our foundation this result gives that Brouwer's Bar Induction ($BI_{fr}$) where choice sequences are functional relations is inconsistent with the formal Church thesis ($\mathrm{CT}_{\mathrm{tt}}$) for type-theoretic functions from natural numbers to natural numbers *in the presence of the axiom of unique choice* on natural numbers ($\mathrm{AC!}_{\mathrm{N,N}}$) turning a functional relation into a type-theoretic function.

Therefore, in the absence of unique choice, it makes sense to investigate consistency of our foundation with $\mathrm{BI}_{\mathfrak{fr}}$ and $\mathrm{CT}_{\mathrm{tt}}$.

The importance of finding a consistent extension where $\mathrm{BI}_{\mathfrak{fr}}$ and $\mathrm{CT}_{\mathrm{tt}}$ are valid is that of providing a setting, with a denotation for lawlike computable sequences, apt to develop constructive analysis, where $\mathrm{BI}_{\mathfrak{fr}}$, or better the Fan theorem, has already shown to be very useful (see for example [Bri08, BR87]) given its topological meaning. Indeed, topologically, the traditional formulation of Bar Induction on choice sequences on natural numbers, defined as functional relations, is equivalent to spatiality of Baire point-free topology (see [FG82, Sam87, GS07]) since choice sequences on natural numbers amount to be the formal points of Baire topology. In essence Bar Induction says that we can reason on the Baire space of choice sequences in a point-free inductive way (and in our formulation $\mathrm{BI}_{\mathfrak{fr}}$ we extend this to spaces of choice sequences on any set). Furthermore, the topological reading of Kleene's result says that we ca not do this if choice sequences are assumed to be computable. Therefore, in the presence of $\mathrm{BI}_{\mathfrak{fr}}$ we need to keep the concept of choice sequence as a not computable one (see also [Sam08]). Then, if identify *lawlike sequences* with type-theoretic functions and we are interesting in keeping their computational meaning, we end up in a theory where both $\mathrm{BI}_{\mathfrak{fr}}$ and $\mathrm{CT}_{\mathrm{tt}}$ are present.

Here we show a realizability interpretation validating $\mathrm{BI}_{\mathfrak{fr}}$ $+\mathrm{CT}_{\mathrm{tt}}$ for a slightly modified version of our two-level foundation in [Mai09], where we restricted the collection constructors to a minimum to represent the power collection of a set. Its intensional level is called $\mathrm{mTT}_0$ and its extensional one $\mathrm{emTT}_0$.

We interpret $\mathrm{emTT}_0$ validating $\mathrm{BI}_{\mathfrak{fr}}$ $+\mathrm{CT}_{\mathrm{tt}}$ by lifting an interpretation of the intensional level $\mathrm{mTT}_0$ that validates the $\mathrm{mTT}_0$-translations $\mathrm{BI}^{i}_{\mathfrak{fr}} + \mathrm{CT}^{i}_{\mathrm{tt}}$ of the corresponding $\mathrm{emTT}_0$-formulations $\mathrm{BI}_{\mathfrak{fr}}$ $+\mathrm{CT}_{\mathrm{tt}}$. Both interpretations of $\mathrm{mTT}_0$ and of $\mathrm{emTT}_0$, called *proof-irrelevant realizability interpretations*, for short *pf-realizability*, can be placed in an ambient theory which can be either the classical set theory ZFC or Aczel's CZF extended with $\mathrm{BI}_{\mathfrak{fr}}$. We give such a name to these interpretations because propositions are interpreted in a proof-irrelevant way, namely as propositions of the ambient theory, while sets and type-theoretic terms are interpreted as in Kleene realizability (see [Tv88b]).

More in detail, in order to validate $\mathrm{CT}^{i}_{\mathrm{tt}}$ the pf-realizability interprets $\mathrm{mTT}_0$-sets as subsets of natural numbers and their families of elements as suitable computable functions, like in the realizability interpretation à la Kleene built in [Tv88b] for a version of Martin-Löf's type theory. Then, it interprets $\mathrm{mTT}_0$-propositions as subsets of the zero singleton in ZFC (and as subsets/subclasses in CZF). Finally, to validate $\mathrm{BI}^{i}_{\mathfrak{fr}}$ the pf-realizability interpret $\mathrm{mTT}_0$-collections as sets in ZFC (and as classes in CZF) and their families of elements as functions with no computational contents. An advantage of the pf-realizability interpretation is that of making well visible the separation, advocated in [Sam08], between computable concepts as the set of lawlike sequences on natural numbers and those not computable concepts as the collection of formal points of the Baire topology or Cantor topology. Actually, the pf-realizability interpretation in ZFC shows that choice sequences of Baire space and also those of Cantor space do not form a set in $\mathrm{emTT}_0$ but just proper collections. The reason is that such collections are interpreted as not countable sets while all $\mathrm{mTT}_0$-sets are interpreted as ZFC-subsets of natural numbers and hence as countable ZFC-sets. For the same reason, also real numbers, both as Dedekind cuts or as Cauchy sequences, do not form a set in $\mathrm{emTT}_0$ because according to the proof-irrelevant realizability interpretation in ZFC they are interpreted as ZFC-reals which are not countable.

The proof-irrelevant realizability model for $\mathrm{mTT}_0$ can be lifted to interpret the extensional level $\mathrm{emTT}_0$ and provides a natural way to interpret *predicatively* our minimalist foundation in the classical set theory ZFC, or in Feferman's predicative classical/constructive theories [Fef79], by keeping the interpretation of type-theoretic functions as computable ones.

This model cannot validate theories as CZF because the addition of the principle of excluded middle makes it become the whole theory ZF [Acz78], which is not predicative. Actually the interpretation of our foundation described here can be thought of the *intended semantics* of our minimalist foundation both in Aczel's CZF and in classical set theory. In the future we hope to extend this interpretation to model the intensional level of our original foundation in [Mai09].

It is worth recalling that theories including relevant features of a proofs-as-programs semantics, as the identification of existential quantifications with strong indexed sums in Martin-Löf's type theory, do not enjoy an intuitive interpretation in classical set theory preserving logical connectives and quantifiers as our minimalist foundation. Therefore a proof-assistant based on our foundation would be more suitable to develop reusable proofs in a modular way both in its constructive and classical extensions (implementations of systems based on our foundation are under development in the proof assistant Matita [AMC⁺11]).

## 2 The two-level theory: $\mathrm{mTT}_0$ and $\mathrm{emTT}_0$

As described in the introduction, in [Mai09] we built a two-level foundation meeting the requirements in [MS05].

Here we consider a slightly modified version of this foundation where we restrict collection constructors to a minimum to formalize Bar Induction. Its intensional level is called $\mathrm{mTT}_0$ and its extensional one $\mathrm{emTT}_0$.

$\mathrm{mTT}_0$ is a fragment of mTT and, as mTT, it has the following features: it is represented by an intensional type theory as [NPS90] (written by using the same higher-order syntax, see also [Gui09]), with collections distinct from sets to represent the power collection of a set in a predicative way; propositions are defined in a primitive way to avoid the validity of choice principles; we distinguish small propositions as those propositions closed only under quantification over sets to define subsets of a set; we identify any proposition with the collection of its proofs, as well as any small proposition with the set of its proofs to implement useful operations on subsets advocated in [SV98, Sam14]; we replace usual equality rules in [NPS90] with *substitution rules* given explicitly, in order to avoid the presence of the $\xi$-rule for $\lambda$-terms.

We recall from [Mai09] that the mentioned change of equality rules enable us to show consistency of $\mathrm{mTT}_0$ with the axiom of choice and formal Church thesis, as advocated in [MS05] via a realizability interpretation à la Kleene (see [Mar75]). Luckily, *this change of equality rules does not affect the interpretation of the extensional level* $\mathrm{emTT}_0$ where the $\xi$-rule is present and it is equivalent to extensionality of type-theoretic functions.

Also the extensional level $\mathrm{emTT}_0$ shares with the extensional level emTT in [Mai09] the fact that it is an extensional type theory as [Mar84] which is closed under effective quotient sets (similar to those in [The95]). Moreover, its propositions, defined primitively as in $\mathrm{mTT}_0$, are proof-irrelevant, namely they are equipped with at most a unique canonical proof-term.

The only difference between $\mathrm{mTT}_0$ and mTT, as well as between $\mathrm{emTT}_0$ and emTT, is that strong indexed sums of collection families are restricted to strong indexed sums of propositional functions only. In other terms the rules F-Σ), I-Σ), E-Σ), C-Σ) of strong indexed sums in mTT and emTT in [Mai09] are simply replaced by the following ones

**Strong Indexed Sum of a propositional function**

F-ip) $\dfrac{C(x)\ \ prop\ [x \in B]}{\Sigma_{x\in B} C(x)\ col}$ I-ip) $\dfrac{b \in B \quad d \in C(b) \quad C(x)\ \ prop\ [x \in B]}{\langle b, d\rangle \in \Sigma_{x\in B} C(x)}$

E-ip) $\dfrac{\begin{array}{l} M(z)\ col\ [z \in \Sigma_{x\in B} C(x)] \\ d \in \Sigma_{x\in B} C(x) \quad m(x,y) \in M(\langle x, y\rangle)\ [x \in B, y \in C(x)] \end{array}}{El_\Sigma(d, m) \in M(d)}$

C-ip) $\dfrac{\begin{array}{l} M(z)\ col\ [z \in \Sigma_{x\in B} C(x)] \\ b \in B \quad c \in C(b) \quad m(x,y) \in M(\langle x, y\rangle)\ [x \in B, y \in C(x)] \end{array}}{El_\Sigma(\,\langle b, c\rangle, m\,) = m(b, c) \in M(\langle b, c\rangle)}$

In $\mathrm{emTT}_0$ we add also the equality rules of such strong indexed sums corresponding to eq-Σ), I-eq Σ), E-eq Σ) in emTT.

Moreover, in $\mathrm{emTT}_0$ we have also the collection of subsets of the singleton as in emTT (in [Mai09] we forgot to add to emTT the rule sm-eq) saying that the propositional equality of subsets is small):

**Power collection of the singleton**

F-P) $\mathcal{P}(1)\ col$ I-P) $\dfrac{B\ prop_s}{[B] \in \mathcal{P}(1)}$ eq-P) $\dfrac{\mathsf{true} \in B \leftrightarrow C}{[B] = [C] \in \mathcal{P}(1)}$ eff-P) $\dfrac{[B] = [C] \in \mathcal{P}(1)}{\mathsf{true} \in B \leftrightarrow C}$

sm-eq) $\dfrac{U \in \mathcal{P}(1) \qquad V \in \mathcal{P}(1)}{\mathsf{Eq}(\mathcal{P}(1),\, U,\, V\,)\ prop_s}$ $\eta$-P) $\dfrac{U \in \mathcal{P}(1)}{U = [\,\mathsf{Eq}(\mathcal{P}(1),\, U,\, [\mathsf{tt}]\,)\,]}$

where $\mathsf{tt} \equiv \bot \to \bot$ represents the truth constant.

Then, we have also function collections from a set toward $\mathcal{P}(1)$ to represent the power collection of a set:

**Function collection to $\mathcal{P}(1)$**

F-Fc) $\dfrac{B\ set}{B \to \mathcal{P}(1)\ col}$ I-Fc) $\dfrac{c(x) \in \mathcal{P}(1)\ [x \in B] \qquad B\ set}{\lambda x^B.c(x) \in B \to \mathcal{P}(1)}$

E-Fc) $\dfrac{b \in B \quad f \in B \to \mathcal{P}(1)}{\mathsf{Ap}(f,b) \in \mathcal{P}(1)}$ $\beta$C-Fc) $\dfrac{b \in B \quad c(x) \in \mathcal{P}(1)\ [x \in B] \qquad B\ set}{\mathsf{Ap}(\lambda x^B.c(x), b) = c(b) \in \mathcal{P}(1)}$

$\eta$C-Fc) $\dfrac{f \in B \to \mathcal{P}(1)}{\lambda x^B.\mathsf{Ap}(f,x) = f \in B \to \mathcal{P}(1)}(x\ not\ free\ in\ f)$

The above restriction of strong indexed sums is enough to interpret these function collections in $\mathrm{mTT}_0$ as in [Mai09]. Hence we define an interpretation of the extensional level $\mathrm{emTT}_0$ into $\mathrm{mTT}_0$ as that in [Mai09] as follows. Note that we use the word *type* as a meta-variable varying on *collection*, *set*, *proposition*, *small proposition*:

**Def. 2.1** We call

$$(-)^i : \mathrm{emTT}_0 \to \mathrm{mTT}_0$$

the restriction of the interpretation of emTT-dependent types and terms into mTT-extensional dependent types and terms in [Mai09], where, in particular, dependent sets are interpreted as total dependent setoids à la Bishop [Bis67, Pal05] [1].

This is well defined because $\mathrm{emTT}_0$-propositions are interpreted as $\mathrm{mTT}_0$-propositions from which we get that strong indexed sums of propositional functions in $\mathrm{emTT}_0$ are defined via strong indexed sums of propositional functions in $\mathrm{mTT}_0$ as follows (recall that $\sigma^{\overline{x}}_{\overline{x'}}$ are isomorphisms needed to interpret substitution):

**Strong Indexed Sum** :
$(\,\Sigma_{y\in B}C(y)\,)^I\ col\ [\Gamma^I] \equiv \Sigma_{y\in B^I}C^I(y)\ col\ [\Gamma^I]$
and $z =_{\Sigma_{y\in B}C(y)^I} z' \ \equiv\ \exists_{d\in \pi_1(z)=_{B^I}\pi_1(z')}\ \sigma^{\pi_1(z')}_{\pi_1(z)}(\pi_2(z)) =_{C^I(\pi_1(z'))} \pi_2(z')$ for $z, z' \in (\,\Sigma_{y\in B}C(y)\,)^I$.
with terms constructors interpreted exactly as in [Mai09].

Moreover, the function collection towards $\mathcal{P}(1)$ is interpreted by using only strong indexed sums of propositional functions as follows (recall from [Mai09] that the notation $\widetilde{c^I}$ stands for the composition of the interpretation $c^I$ with suitable canonical isomorphisms):

**Function collection toward $\mathcal{P}(1)$** :
$(\,B \to \mathcal{P}(1)\ col\ [\Gamma]\,)^I \equiv \Sigma_{h\in B^I \to \mathsf{props}}\ \ \forall_{y_1\in B^I}\ \forall_{y_2\in B^I}\ (\,y_1 =_{B^I} y_2\ \to\ (\,\mathsf{Ap}(h,y_1) \leftrightarrow \mathsf{Ap}(h,y_2)\,)\,)$
with equality $z =_{\mathcal{P}} z' \equiv \forall_{y\in B^I}\ \ (\,\mathsf{Ap}(\,\pi_1(z)\,,y) \leftrightarrow\ \mathsf{Ap}(\,\pi_1(z')\,,y)\ \,)$ for $z, z' \in (\,B \to \mathcal{P}(1)\,)^I$

[1] Note that we can not turn such an interpretation $(-)^i$ into one that interprets $\mathrm{emTT}_0$ in a categorical model of quotients built over $\mathrm{mTT}_0$ as done over mTT in [Mai09], because there we interpreted contexts via generic strong indexed sums not available in $\mathrm{mTT}_0$. Here we can only turn $(-)^i$ into an interpretation of $\mathrm{emTT}_0$ in a syntactic indexed category built out of $\mathrm{mTT}_0$, for example as in [Hof97], where $\mathrm{emTT}_0$-contexts are interpreted as $\mathrm{mTT}_0$-extensional contexts exactly as done by $(-)^i$.

$( \lambda y^{B}.c )^{I} \equiv \langle \lambda y^{\widetilde{B}}.c^{\widetilde{I}} , p \rangle$ where $p \in \forall_{y_1 \in B^I} \ \forall_{y_2 \in B^I} \ ( y_1 =_{B^I} y_2 \rightarrow ( c^{\widetilde{I}}(y_1) \leftrightarrow c^{\widetilde{I}}(y_2) ) )$
$( \mathsf{Ap}(f, b) )^{I} \equiv \mathsf{Ap}( \pi_1(f^{\widetilde{I}}) , b^{\widetilde{I}})$
$\sigma_{\overline{x}}^{\overline{x'}}(w) \equiv \langle \lambda y'^{B^I(\overline{x'})}. \sigma^{\overline{x'},y'}_{\overline{x},\sigma_{\overline{x'}}^{\overline{x}}(y')}( \mathsf{Ap}( \pi_1(w) , \sigma_{\overline{x'}}^{\overline{x}}(y') ) ), p \rangle$ for $\overline{x}, \overline{x'} \in \Gamma^I$ and $w \in ( B \rightarrow \mathcal{P}(1) )^I(\overline{x})$ where $p$ is a proof-term witnessing the preservation of equalities obtained from $\pi_2(w)$.

### 2.1 Comparison of our minimalist foundation with Aczel's CZF

Recalling from [MS05] that theories satisfying extensionality of functions are not proofs-as-programs ones (i.e. consistent with formal Church thesis and axiom of choice), clearly CZF, being one of those, is not a proofs-as-programs theory, and, of course, is not apt to be the intensional level of a constructive foundation. However it enjoys an interpretation in Martin-Löf's intensional type theory [Acz78, Acz82, Acz86, RT06] (with disjunction and suitable existence properties). One could then view CZF as the extensional level of Martin-Löf's intensional type theory. But, according to our technical notion of constructive foundation in [Mai09], the extensional level should be seen as the internal theory of a quotient completion of the intensional one and we do not know how to accomplish this for CZF.

In any case, it makes sense to compare our extensional level emTT with CZF. We recall from [Mai09] that our emTT is certainly interpretable in Aczel's CZF [AR01] by interpreting sets as CZF sets, collections as classes, propositions as subclasses of the singleton and small propositions as subsets of the singleton (in order to make the rules **prop-into-col** and **prop$_s$-into-set** valid) and typed terms as functions. In particular, the power collection $\mathcal{P}(A)$ of a set $A$ is interpreted as the corresponding power collection of subsets.

This interpretation however looses the computable meaning of typed terms. What we will show in the next is a realizability interpretation where typed terms are seen as computable.

## 3 Formulation of $\mathbf{CT}_{\mathrm{tt}}$, $\mathbf{AC!}_{\mathbf{N},\mathbf{N}}$ and $\mathbf{BI}_{\mathbf{fr}}$

The goal of our work is to build a model of the extensional level $\mathrm{emTT}_0$ of our minimalist foundation extended with Bar Induction where choice sequences are functional relations, for short $\mathrm{BI}_{\mathrm{fr}}$, and the formal Church thesis for type-theoretic functions, for short $\mathrm{CT}_{\mathrm{tt}}$, in a set theory $\mathcal{S}$ as ZFC or CZF+$\mathrm{BI}_{\mathrm{fr}}$.

Given that our extensional level $\mathrm{emTT}_0$ can be interpreted in $\mathrm{mTT}_0$ via quotients, to fulfil our purpose it is enough to provide a model of the intensional level $\mathrm{mTT}_0$ which validates the translations $\mathrm{BI}^i_{\mathrm{fr}}$ and $\mathrm{CT}^i_{\mathrm{tt}}$ of the $\mathrm{emTT}_0$-formulations $\mathrm{BI}_{\mathrm{fr}}$ and $\mathrm{CT}_{\mathrm{tt}}$. Then an interpretation for $\mathrm{emTT}_0$ in the set theory $\mathcal{S}$ will be defined by closing under suitable quotients that of $\mathrm{mTT}_0$.

Here we start by presenting the formulation of $\mathrm{CT}_{\mathrm{tt}}$ and of unique choice on natural numbers $\mathrm{AC!}_{\mathrm{N},\mathrm{N}}$, where N denotes the set of natural numbers in $\mathrm{emTT}_0$ (and $\mathrm{mTT}_0$). Then we pass to formulate $\mathrm{BI}_{\mathrm{fr}}$ in topological terms as pioneered in [FG82] but in the context of formal topology [Sam87, GS07], namely of point-free topology developed in a predicative way (for a survey and related notation see [Sam03]). We then review its connection with the traditional formulation of Bar Induction and Fan theorem [Dum00, Tv88a, Rat05] (see [Sam14] for a survey and further developments about this). Then we use Kleene's result [Tv88a] about inconsistency of Fan theorem with Church thesis for choice sequences to deduce that Bar Induction is inconsistent with $\mathrm{CT}_{\mathrm{tt}}$ and unique choice. Finally we formulate such principles at the intensional level.

### 3.1 Formulation of $\mathbf{CT}_{\mathrm{tt}}$, $\mathbf{AC!}_{\mathbf{N},\mathbf{N}}$ and $\mathbf{BI}_{\mathbf{fr}}$ at the extensional level

One of the benefits of our foundation $\mathrm{emTT}_0$ is that it allows to have two notions of function: one is that of *functional relation* and the other is that of *type-theoretic function* (or *operation* as called in [Sam14]). In $\mathrm{emTT}_0$ a functional relation from a set $A$ to a set $B$ is identified with a small proposition $a\,R\,b$ *prop$_s$* $[a \in A, b \in B]$ satisfying [2]

[2] As usual $\exists ! y \in A\ R(x,y) \equiv \exists y \in B\ R(x,y) \ \wedge\ \forall y_1, y_2 \in B\ (\ R(x,y_1) \ \wedge\ R(x,y_1) \rightarrow \mathsf{Eq}(B, y_1, y_2)\ )$.

$$\forall x \in A\ \exists! y \in B \quad R(x,y)$$

Instead in $\mathrm{emTT}_0$ a type-theoretic function from a set $A$ to a set $B$ is identified with an element of the set $A \to B$

$$f \in A \to B$$

We recall that canonical terms of the set $A \to B$ are of the form $\lambda x.t(x)$ obtained by $\lambda$-abstraction from terms of the form $t(x) \in B\ [x \in A]$.

Note here that while the collection of functional relations from a set $A$ to a set $B$ does not need to form a set, instead by definition the type-theoretic functions from a set $A$ to a set $B$ form a set!
Given that type-theoretic functions are meant to be computable as in Feferman's theories [Fef79], it makes sense to formulate the formal Church thesis only for them, namely to say that all type theoretic functions from natural numbers to natural numbers are internally recursive:

$$(\mathrm{CT_{tt}}) \qquad \forall f \in \mathrm{N} \to \mathrm{N} \quad \exists e \in \mathrm{N} \quad (\, \forall x \in \mathrm{N}\ \exists y \in \mathrm{N}\ T(e,x,y) \wedge U(y) =_{\mathrm{N}} f(x)\,)$$

where $T(e,x,y)$ is the Kleene predicate expressing that $y$ is the computation executed by the program numbered $e$ on the input $x$ and $U(y)$ is output of the computation $y$.
Note that the notion of functional relation is more general than that of type-theoretic function from $A$ to $B$, unless we can derive the *set-theoretic axiom of unique choice on natural numbers*, formulated in $\mathrm{emTT}_0$ as follows: for any small proposition $a\,R\,b\ prop_s\ [a \in A, b \in B]$

$$(\ \mathrm{AC!}_{\mathrm{N,N}}\ ) \qquad \forall x \in \mathrm{N}\ \exists! y \in \mathrm{N}\ R(x,y) \quad \longrightarrow \quad \exists f \in \mathrm{N} \to \mathrm{N}\ \forall x \in \mathrm{N}\ R(x, f(x))$$

The absence of unique choice was exactly one of the key features desired for our minimalist foundation as explained in [MS05]. The main motivation was to be able to identify the notion of *choice sequence* with that of functional relation, as done in the context of axiomatic set theory, and that of *lawlike sequence* with that of type-theoretic function (see [Sam08] for the relevance of this distinction).

To express this we identify the tree with nodes labelled by lists of elements in a set $A$ with $A^* \equiv List(A)$ itself:

**Def. 3.1 (choice sequence)** Given a set $A$, a *choice sequence* on the tree $A^*$ is a functional relation from N to $A$ defined by a small proposition $\alpha(x,y)\ prop_s\ [x \in \mathrm{N}, y \in A]$ in $\mathrm{emTT}_0$.
We write $\alpha \varepsilon \mathrm{CH}(A)$ to mean that $\alpha$ is a choice sequence.

**Def. 3.2 (lawlike sequence)** Given a set $A$, a *lawlike sequence* on the tree $A^*$ is a type-theoretic function $f \in \mathrm{N} \to A$ from natural numbers to $A$ in $\mathrm{emTT}_0$.

**Remark 3.3** Note that in [Tv88a] the notion of choice sequence is identified with that of type-theoretic function $f : \mathrm{N} \to \mathrm{N}$ while in [Rat05] with that of functional relation.

Now we formulate the principle of Bar Induction in topological terms by employing an inductively generated formal topology put on the tree $A^*$ (see [Sam14]). In the next in $\mathrm{emTT}_0$ we use the notion of subset with its $\varepsilon$-relation as in [Mai09]: a subset $V$ of $A^*$ is a term $V \in \mathcal{P}(A^*)$, with $\mathcal{P}(A^*) \equiv A^* \to \mathcal{P}(1)$ and $l \varepsilon V \equiv \mathsf{Eq}(\,\mathcal{P}(1),\, V(l),\, [\mathsf{tt}]\,)$ for a list $l \in A^*$.

**Def. 3.4** *The tree formal topology over A* is the formal topology $(A^*, \lhd_{A^N})$ where $\lhd_{A^N}$ is inductively generated by the following rules (see [CSSV03])

$$\mathrm{rfl}\ \frac{l \varepsilon V}{l \lhd_{A^N} V} \qquad \leq\ \frac{s \sqsubseteq^{op} l \qquad l \lhd_{A^N} V}{s \lhd_{A^N} V} \qquad \mathrm{tr}\ \frac{\forall x \in A \ \ \mathsf{cons}(l,x) \lhd_{A^N} V}{l \lhd_{A^N} V}$$

where $s \sqsubseteq^{op} l \equiv \exists_{t \in A^*}\, s =_{A^*} [l,t]$, i.e. $l$ is an initial segment of $s$.

The above tree topology is called *Cantor formal topology* when $A$ is the boolean set $\{0,1\}$ and we indicate its cover with $\lhd_C \equiv \lhd_{\{0,1\}^N}$. Moreover, it is called *Baire formal topology* when $A$ is the set of natural numbers N and we indicate its cover with $\lhd_B \equiv \lhd_{N^N}$.

We just recall that a subset $V$ of $A^*$ is called a formal open of the formal topology $\lhd_{A^N}$ if $V = \lhd_{A^N}(V) \in \mathcal{P}(A^*)$ holds, where in turn $\lhd_{A^N}(V) \equiv \{\ l \in A^* \ \mid\ l \lhd_{A^N} V\ \}$.

Then the frame associated to the formal topology $\lhd_{A^N}$ is represented by the formal opens of $\lhd_{A^N}$, with inclusion as order. The intersection of two opens $V, W$ in the frame is given by $\lhd_{A^N}(V \downarrow W)$ where $V \downarrow W \equiv \{\ s \in A^* \ \mid\ \exists l \varepsilon V\ s \sqsubseteq^{op} l\ \wedge\ \exists w \varepsilon W\ s \sqsubseteq^{op} w\ \}$. The arbitrary supremum of a family of opens $V_{i \in I}$ in the frame is given by $\lhd_{A^N}(\bigcup_{i \in I} V_i)$ where $\bigcup_{i \in I} V_i \equiv \{\ s \in A^* \ \mid\ \exists i \in I\ s \varepsilon V_i\ \}$.

In [Val07] it is shown how to build the above tree formal topologies in an extension of Martin-Löf's type theory with the help of the axiom of choice (which is a theorem there!). In the quotient model built over Martin-Löf's type theory in the same way as the quotient model over mTT in [Mai09], this axiom of choice survives only when it is applied to copies of sets of the intensional level (and we can call it *intensional axiom of choice*) and hence it does not entail classical logic as the full extensional one (see [ML06, Car04]). Given that we do not have such a choice principle in $\text{emTT}_0$ we *simply postulate the existence of tree formal topologies* as added axioms to $\text{emTT}_0$. We indicate this extension with $\text{emTT}_0 + \lhd_{A^N}$.

Before proceeding we define a useful notation introduced in [Sam03]:

**Def. 3.5 ($\between$-relation )** Given a set $A$ and subsets $V, W$ of $A^*$ we define

$$V \between W \ \equiv\ \exists l \in A^*\ (\ l \varepsilon V\ \wedge\ l \varepsilon W\ )$$

We then recall the notion of a formal point for the tree formal topologies of the form $\lhd_{A^N}$:

**Def. 3.6 (formal point of $\lhd_{A^N}$)** A subset $\alpha$ of $A^*$ for a given set $A$ is a *formal point*, written $\alpha \varepsilon Pt(\lhd_{A^N})$, if it satisfies the following conditions:

$$\begin{array}{l} \exists_{l \in A^*}\ l \varepsilon \alpha \\ \forall_{l_1, l_2 \in A^*}\ (\ l_1 \varepsilon \alpha \wedge l_2 \varepsilon \alpha\ \rightarrow\ \exists_{s \in A^*}\ (\ s \varepsilon \alpha \wedge s \sqsubseteq^{op} l_1 \wedge s \sqsubseteq^{op} l_2\ )\ ) \\ \forall_{s \in A^*}\ (\ s \varepsilon \alpha \rightarrow (\ \forall_{l \in A^*}\ s \sqsubseteq^{op} l\ \rightarrow\ l \varepsilon \alpha\ )\ ) \\ \forall_{l \varepsilon A^*}\ (\ l \varepsilon \alpha\ \rightarrow\ \exists_{a \in A}\ \mathsf{cons}(l, a) \varepsilon \alpha\ ) \end{array}$$

In the next, we use the following abbreviations to quantify over formal points: for any formula $\phi(\alpha)$
$\forall\, \alpha \varepsilon Pt(\lhd_{A^N})\ \ \phi(\alpha) \equiv \forall\, \alpha \in A^* \rightarrow \mathcal{P}(1)\ (\ \alpha \varepsilon Pt(\lhd_{A^N})\ \rightarrow\ \phi(\alpha)\ )$
$\exists\, \alpha \varepsilon Pt(\lhd_{A^N})\ \ \phi(\alpha) \equiv \exists\, \alpha \in A^* \rightarrow \mathcal{P}(1)\ (\ \alpha \varepsilon Pt(\lhd_{A^N})\ \wedge\ \phi(\alpha)\ )$.

Now note that choice sequences on the tree $A^*$ are exactly the formal points of the tree formal topology over $A$:

**Proposition 3.7** *The collection of formal points $Pt(\lhd_{A^N})$ of the tree formal topology over a set A are in bijection with the choice sequences on the tree $A^*$.*

**Proof.** Given a formal point $\alpha$, we can define a functional relation as follows:

$$\alpha_{fr}(n, a)\ \equiv\ \exists\ l \varepsilon \alpha\ \ \mathsf{Eq}(A, l_{n+1}, a)$$

where $l_n$ is the n-th component of $l$.

Conversely, given a functional relation $\alpha(x, y)\ prop_s\ [x \in \mathrm{N}, y \in A]$ the following subset

$$\alpha_{pt}\ \equiv\ \{\ l \in A^*\ \mid\ \forall n \in \mathrm{N}\ (\ 1 \leq n \leq \mathsf{lh}(l)\ \rightarrow\ \alpha(n, l_{n+1})\ )\}$$

where $\mathsf{lh}(l)$ is the length of $l$, turns out to be a formal point.

An alternative proof follows after noting, as observed in [Sig95], that any tree formal topology is the exponential formal topology of the discrete formal topology of natural numbers on itself (see [Mai05a] for a constructive and predicative construction of exponentiation). Therefore its formal points are in bijection with functional relations, being these all continuous. This explains why we label the cover $\lhd_{A^N}$ of the tree formal topology over $A$ with $A^N$.

Then, we are ready to formulate Bar Induction as *spatiality of the tree formal topology on a given set A* similarly to [FG82]:

**Def. 3.8 (Bar Induction in topological form)** In $\mathrm{emTT}_0 + \lhd_{A^N}$ the principle of Bar Induction in topological form is the following statement: for any given set $A$ in $\mathrm{emTT}_0$

$$(\ \mathrm{BI}_{\mathrm{fr}}(\mathrm{A})\ ) \qquad \forall l \in A^* \ \forall V \in \mathcal{P}(A^*) \ \ (\ \forall \alpha \varepsilon Pt(\lhd_{A^N}) \ \ (\ l\varepsilon\alpha \ \rightarrow \ \alpha \between V \ ) \rightarrow l \lhd_{A^N} V \ )$$

This formulation of $\mathrm{BI}_{\mathrm{fr}}(\mathrm{A})$ essentially means that the topology put on the formal points of the tree $A^*$, that are its choice sequences, coincides with the point-free one and hence we can reason on it by induction on finite sequences, being the point-free one inductively generated (see [Sam08, Sam14]).
We give specific names to Bar Induction on the Baire formal topology and on Cantor formal topology:

**Def. 3.9 (Bar Induction on Baire and Cantor formal topologies)** We call $\mathrm{BI}_{\mathrm{fr}}(\mathrm{N})$ the above formulation of $\mathrm{BI}_{\mathrm{fr}}(\mathrm{A})$ on Baire formal topology, namely when $A \equiv \mathrm{N}$.

We call $\mathrm{BI}_{\mathrm{fr}}(\{0,1\})$ the above formulation of $\mathrm{BI}_{\mathrm{fr}}(\mathrm{A})$ on Cantor formal topology, namely when $A \equiv \{0,1\}$.

Note that spatiality of Cantor formal topology allows to derive compactness of Cantor space [FG82].
In the rest of the paper we just say $\mathrm{BI}_{\mathrm{fr}}$ to mean $\mathrm{BI}_{\mathrm{fr}}(\mathrm{A})$ for any given set $A$ and $\mathrm{emTT}_0 + \mathrm{BI}_{\mathrm{fr}}$ to mean $\mathrm{emTT}_0 + \lhd_{A^N} + \mathrm{BI}_{\mathrm{fr}}(\mathrm{A})$ for any given set $A$.

## 3.2 Connection of $\mathrm{BI}_{\mathrm{fr}}$ with traditional formulations

Here, we review the connection of our topological formulation of Bar Induction with more traditional formulations of it and with Fan theorem. We want to make this clear in order to derive an inconsistency of $\mathrm{CT}_{\mathrm{tt}}$ with $\mathrm{AC!}_{\mathrm{N,N}}$ and $\mathrm{BI}_{\mathrm{fr}}$ in $\mathrm{emTT}_0$ from Kleene's proof in [Dum00, Tv88a, Rat05] about inconsistency of the Fan theorem with Church thesis for functional relations.

We start with defining the notion of bar of a list:

**Def. 3.10 (bar of a list)** Given a set $A$, a *bar of a list* $l$ on the tree $A^*$ is a subset $V$ of $A^*$ satisfying

$$\forall \alpha \varepsilon CH(A) \ \ (\ \alpha \between \{\, l \,\} \ \ \rightarrow \ \ \alpha \between V\ )$$

We then say that

- $V$ is *monotone* if $\forall l \in A^* \ (\ l\varepsilon V \ \rightarrow \ \forall a \in A \ \mathsf{cons}(l,a)\varepsilon V\ )$ holds.
- $V$ is *inductive* if $\forall l \in A^* \ (\ \forall a \in A \ \mathsf{cons}(l,a)\varepsilon V \ \rightarrow \ l\varepsilon V\ )$ holds.

Now we are ready to give the traditional formulation of Monotone Bar Induction as in [Dum00, Tv88a, Rat05] for $A \equiv \mathrm{N}$ and here extended also when $A \equiv \{0,1\}$:

**Def. 3.11 (traditional Bar Induction)** For $A \equiv \mathrm{N}$ or $A \equiv \{0,1\}$ the principle of *Bar induction* $\mathrm{BI}^{tr}_A$ says that every inductive subset $Q$ of $A^*$ containing a monotone bar $V$ of the empty list contains the empty list:

$$\begin{array}{lll}(\mathrm{BI}^{tr}_A) & \forall V, Q \in \mathcal{P}(A^*) & (\ \forall \alpha \varepsilon \mathrm{CH}(A) \ \ \alpha \between V \\ & & \wedge \ \ \forall l \in A^* \ (\ \ l\varepsilon V \ \rightarrow \ \forall a \in A \ \mathsf{cons}(l,a)\varepsilon V\ ) \\ & & \wedge \ \ \forall l \in A^* \ (\ \forall a \in A \ \mathsf{cons}(l,a)\varepsilon Q \ \rightarrow \ l\varepsilon Q\ ) \\ & & \wedge \ \ \forall l \in A^* \ (\ l\varepsilon V \ \rightarrow \ l\varepsilon Q\ )\ ) \\ & & \qquad\qquad \rightarrow \ \mathsf{nil}\varepsilon Q \end{array}$$

holds, where $\mathsf{nil}$ is the empty list.

In order to see the connection between the traditional formulation of Bar Induction and our topological form it is convenient to note that monotone inductive subsets of lists over a set $A$ are in bijection with formal opens of the tree formal topology over a set $A$:

**Lemma 3.12** *A subset $V$ of $A^*$ is monotone and inductive if and only if $V$ is a formal open in the tree formal topology over $A$, i.e. we can derive a proof of*

$$\mathsf{Eq}(\ \mathcal{P}(A^*)\,,\ V\ ,\ \lhd_{A^N}(V)\ )$$

**Proof.** Given a monotone inductive subset, we can prove by induction that $\lhd_{A^N}(V) \subseteq V$ and the other inclusion is obvious. The converse is trivial.

This lemma suggests a reformulation of $\mathrm{BI}_A^{tr}$ in terms of monotone inductive bars:

**Def. 3.13** The principle of *monotone bar induction* $\mathrm{MBI}_A^{\mathsf{nil}}$ says that, for a given set $A$, every monotone inductive bar of the empty list in $A^*$ contains the empty list (and hence by monotonicity every list):

$$(\mathrm{MBI}_A^{\mathsf{nil}})\qquad \begin{array}{l}\text{for all } \textit{monotone inductive} \text{ subset } V \text{ of } A^* \\ \qquad\qquad \forall \alpha \varepsilon \mathrm{CH}(A)\ \ \alpha \between V \ \ \rightarrow\ \mathsf{nil}\varepsilon V\end{array}$$

We also give the following more general definition of monotone bar induction:

**Def. 3.14** The general principle of *monotone bar induction* $\mathrm{MBI}_A$ says that, for a given set $A$, every monotone inductive bar of a list $l$ in $A^*$ contains the list $l$:

$$(\mathrm{MBI}_A)\qquad \begin{array}{l}\text{for all } \textit{monotone inductive} \text{ subset } V \text{ of } A^* \\ \qquad\qquad \forall l \in A^* \quad (\ \forall \alpha \varepsilon \mathrm{CH}(A)\ \ (\ \alpha \between \{\,l\,\} \rightarrow \alpha \between V\ )\ \rightarrow\ l\varepsilon V\ )\end{array}$$

Now we show that all the above formulations of Bar Induction are equivalent when applied to Baire and Cantor formal topologies:

**Theorem 3.15** *In* $\mathrm{emTT}_0 + \lhd_{A^N}$*, for* $A \equiv N$ *or* $A \equiv \{\,0,1\,\}$ *the following are equivalent:*

1. $\mathrm{BI}_{\mathrm{fr}}(A)$
2. $\mathrm{MBI}_A$
3. $\mathrm{MBI}_A^{\mathsf{nil}}$
4. $\mathrm{BI}_A^{tr}$

**Proof.** $1 \leftrightarrow 2$ Clearly $\mathrm{BI}_{\mathrm{fr}}(A)$ is equivalent to $\mathrm{MBI}_A$ because of lemma 3.12.
$2 \leftrightarrow 3$ To prove that $\mathrm{MBI}_A^{\mathsf{nil}}$ entails $\mathrm{MBI}_A$ note that given a monotone inductive bar $V$ for $l$ then

$$W \equiv V \bigcup \{ s \in \mathrm{N}^* \ \mid\ l \neq s\ ^3\ \wedge\ \mathsf{Eq}(\mathrm{N}, \mathsf{lh}(s)\,,\,\mathsf{lh}(l))\ \}$$

is a bar of the empty list. Therefore, from lemma 3.12 we get that $\lhd_{A^N}(W)$ is a monotonic inductive bar of the empty list and from $\mathrm{MBI}_A^{\mathsf{nil}}$ we obtain that $\mathsf{nil}\varepsilon\lhd_{A^N}(W)$ and by monotonicity also that $l\varepsilon\lhd_{A^N}(W)$, i.e. $l \lhd_{A^N} W$. Now, by intersecting the open $\lhd_{A^N}(W)$ with the open generated from $\{\,l\,\}$ we get the open $\lhd_{A^N}(\{\,l\,\}\downarrow W)$ with $l \lhd_{A^N} \{\,l\,\}\downarrow W$. Now observe that $\{\,l\,\}\downarrow W \subseteq V$ being $V$ monotone. Hence, by transitivity of $\lhd_{A^N}$ as a formal cover we conclude $l \lhd_{A^N} V$. Being $V$ a monotone inductive bar, by lemma 3.12 we conclude $l\varepsilon V$.
$3 \leftrightarrow 4$ $\mathrm{MBI}_A^{\mathsf{nil}}$ entails $\mathrm{BI}_A^{tr}$, as shown in [Sam14], if we consider the minimum inductive subset containing a given monotone bar V. This is monotone, it is contained in $Q$ and coincides with $\lhd_{A^N}(V)$. By $\mathrm{MBI}_A^{\mathsf{nil}}$ we get that $\mathsf{nil}\varepsilon \lhd_{A^N}(V)$ and hence we conclude $\mathsf{nil}\varepsilon Q$.
Conversely, $\mathrm{BI}_A^{tr}$ implies $\mathrm{MBI}_A^{\mathsf{nil}}$ trivially by taking $Q \equiv V$ for a given monotone inductive bar $V$.

A consequence of Bar Induction on the tree $\mathrm{N}^*$, namely of $\mathrm{BI}_{\mathrm{N}}^{tr}$, is the well known Fan theorem regarding choice sequences from N to the boolean set (see [Dum00, Tv88a, Rat05]):

[3] The equality on $\mathrm{N}^*$ or $\{\,0,1\,\}^*$ is decidable being decidable that on N and on $\{\,0,1\,\}$.

**Def. 3.16 (traditional Fan theorem)** The traditional formulation of Fan theorem, called here $\mathrm{FT}^{\mathsf{nil}}$, says that every bar $V$ of the empty list in $\{0,1\}^*$ is uniform, namely there exists a subset of $V$, which is still a bar of the empty list, with lists bounded by a fixed natural number:

$$(\mathrm{FT}^{\mathsf{nil}})\ \forall V \in \mathcal{P}(\{0,1\})\ \ (\ \forall \alpha \varepsilon \mathrm{CH}(\{0,1\})\ \ \alpha \between V\ \ \rightarrow\ \ \ \exists n \in \mathrm{N}\ \forall \alpha \varepsilon \mathrm{CH}(\{0,1\})\ \alpha \between V_n\ )$$

where $V_n \equiv \{\ v \in \{0,1\}^*\ \mid\ v \varepsilon V\ \wedge\ \mathsf{lh}(v) \leq n\ \}$.

We can extend the formulation of the Fan theorem to bars of a generic list:

**Def. 3.17 (Fan theorem with bars of a generic list)** The more general formulation of Fan theorem, called here FT, says that every bar $V$ of a list $l$ in $\{0,1\}^*$ is uniform, namely there exists a subset of $V$, which is still a bar of the list $l$, with lists bounded by a fixed natural number:

$$(\mathrm{FT})\quad \begin{array}{l} \forall l \in A^*\ \forall V \in \mathcal{P}(\{0,1\}) \\ \qquad (\ \forall \alpha \varepsilon \mathrm{CH}(\{0,1\})\ (\ \alpha \between \{\ l\ \} \rightarrow \alpha \between V\ )\ \rightarrow\ \ \exists n \in \mathrm{N}\ \forall \alpha \varepsilon \mathrm{CH}(\{0,1\})\ \ (\ \alpha \between \{\ l\ \} \rightarrow \alpha \between V_n\ )\ ) \end{array}$$

But with this formulation of Fan theorem on generic lists we do not get a stronger statement than the traditional one and more importantly this is also equivalent to Bar Induction on the Cantor formal topology $\mathrm{BI}_{\mathfrak{fr}}(\{0,1\})$:

**Theorem 3.18** *In* $\mathrm{emTT}_0 + \lhd_{A^N}$ *the following are equivalent*

1. $\mathrm{BI}_{\mathfrak{fr}}(\{0,1\})$
2. FT
3. $\mathrm{FT}^{\mathsf{nil}}$

**Proof.** $1 \leftrightarrow 2$ The proof is given in [GS07] and it can be easily carried out in $\mathrm{emTT}_0$ being based on induction over the generation of Cantor formal topology.
$2 \leftrightarrow 3$ To prove that $\mathrm{FT}^{\mathsf{nil}}$ entail FT note that given a bar $V$ for $l$ then

$$W \equiv V \bigcup \{s \in \{0,1\}^*\ \mid\ l \neq s\ \wedge\ \mathsf{Eq}(\mathrm{N}, \mathsf{lh}(s), \mathsf{lh}(l))\ \}$$

is a bar of the empty list. Hence, by $\mathrm{FT}^{\mathsf{nil}}$ there exists a natural numbers $n$ such that $W_n$ is a bar of the empty list, and hence $V_n$ is a bar of $l$.

We can also show that Bar Induction on the Cantor formal topology, or equivalently the Fan theorem, is a consequence of Bar Induction on the Baire formal topology:

**Proposition 3.19** $\mathrm{BI}_{\mathfrak{fr}}(\mathrm{N})$ *entails* FT *in* $\mathrm{emTT}_0$.

**Proof.** Thanks to theorem 3.18 we just show that $\mathrm{BI}_{\mathfrak{fr}}(\mathrm{N})$ entails $\mathrm{BI}_{\mathfrak{fr}}(\{0,1\})$. As suggested to us by T. Streicher this follows from the fact that Cantor formal topology is a retract of Baire formal topology, i.e. that there exist morphisms $\mathcal{E} : \lhd_C \rightarrow \lhd_B$ , $\mathcal{R} : \lhd_B \rightarrow \lhd_C$ such that $\mathcal{R} \cdot \mathcal{E} = \mathsf{id}_{\lhd_C}$ in the category of inductively generated formal topologies (the definition of such a category can be found in [Mai05a] and in loc. cit.). In particular, the existence of $\mathcal{R}$ is in turn based on a retraction $\sigma : \mathrm{N}^* \rightarrow \{0,1\}^*$ of the embedding $i : \{0,1\}^* \rightarrow \mathrm{N}^*$ of boolean lists into lists of natural numbers, where $\sigma \equiv List(\widetilde{\sigma})$ is the lifting of the operation $\widetilde{\sigma}(x) \in \{0,1\}$ $[x \in \mathrm{N}]$ defined as follows:

$$\widetilde{\sigma}(x) \equiv \begin{cases} 0 & \text{if } x \equiv 0 \\ 1 & \text{otherwise} \end{cases}$$

Indeed, we can define $\mathcal{E} : \lhd_C \rightarrow \lhd_B$ and $\mathcal{R} : \lhd_B \rightarrow \lhd_C$ as follows: given $s \in \mathrm{N}^*$ and $l \in \{ 0,1 \}^*$

$$l\, \mathcal{E}\, s \equiv l \lhd_C \{ x \in \{ 0,1 \}^* \mid \mathsf{Eq}(\mathrm{N}^*, x, s) \} \qquad\qquad s\, \mathcal{R}\, l \equiv \sigma(s) \lhd_C l$$

where for easiness we just consider a list $l \in \{ 0,1 \}^*$ also as a list in $\mathrm{N}^*$. (We just recall that, for any formal open $V$ of $\lhd_B$, then $\mathcal{E}^-(V)$ is a formal open of $\lhd_C$ and this gives rise to a frame morphism from the Baire frame to the Cantor one. Similarly $\mathcal{R}^-$ gives rise to a frame morphism from the Cantor frame to the Baire one.)
Then, one derives $\mathrm{BI}_{\mathrm{fr}}(\{ 0,1 \})$ from $\mathrm{BI}_{\mathrm{fr}}(\mathrm{N})$ by using $\mathcal{E}$ and $\mathcal{R}$. The essence is that any bar $V$ of a list $l$ in the Cantor formal topology yields to a bar $\mathcal{R}^-(V) \equiv \{ s \in \mathrm{N}^* \mid \exists v \varepsilon V\, s\, \mathcal{R}\, v \}$ for the list $l$ in the Baire formal topology. Then by $\mathrm{BI}_{\mathrm{fr}}(\mathrm{N})$ we get $l \lhd_B \mathcal{R}^-(V)$ and hence also that $\mathcal{E}^-(l) \lhd_C \mathcal{E}^-(\mathcal{R}^-(V))$. From this and from $l \varepsilon \mathcal{E}^-(l)$ and $\mathcal{R} \cdot \mathcal{E} = \mathsf{id}_{\lhd_C}$ we conclude $l \lhd_C V$. Hence $\mathrm{BI}_{\mathrm{fr}}(\{ 0,1 \})$ holds, as claimed.

### 3.3 Inconsistency of $\mathbf{AC!_{N,N}}$+ $\mathbf{CT_{tt}}$+ $\mathbf{BI_{fr}}$

Here we reread in our foundation Kleene's well known result that the Fan theorem is inconsistent with the formal Church thesis saying that all choice sequences are recursive [KV65]. We found at least two ways in which choice sequences are defined in the literature. Some authors, like [Tv88a], identify choice sequences with type-theoretic functions and hence their intended formal Church thesis to get Kleene's result coincides with our $\mathrm{CT_{tt}}$. Since in the presence of unique choice our notion of choice sequence coincides with that in [Tv88a], Kleene's result in our setting amounts to say that the Fan theorem together with $\mathrm{AC!_{N,N}}$ is inconsistent with $\mathrm{CT_{tt}}$.

Others authors identify choice sequences with functional relations as in [Rat05] and their intended formal Church thesis to get Kleene's result is then a consequence of combining our Church thesis $\mathrm{CT_{tt}}$ for type-theoretic functions with the axiom of unique choice. Also in this case Kleene's result amounts to inconsistency of the Fan theorem together with $\mathrm{CT_{tt}} + \mathrm{AC!_{N,N}}$.

Therefore we deduce for our foundation:

**Proposition 3.20** *There is no model of* $\mathrm{emTT_0} + \mathrm{FT} + \mathrm{CT_{tt}} + \mathrm{AC!_{N,N}}$.

**Proof.** Given that Heyting arithmetics of finite types can be seen as a fragment of $\mathrm{emTT_0}$, we can mimick Kleene's proof in [Tv88a] because by the presence of $\mathrm{AC!_{N,N}}$ our choice sequences become identified with type-theoretic functions, namely lawlike sequences, as in [Tv88a].

Thanks to propositions 3.18, 3.19 we then conclude:

**Corollary 3.21** $\mathrm{emTT_0} + \mathrm{BI_{fr}(N)} + \mathrm{CT_{tt}} + \mathrm{AC!_{N,N}}$ *is inconsistent.*
*Hence,* $\mathrm{emTT_0} + \mathrm{BI_{fr}} + \mathrm{CT_{tt}} + \mathrm{AC!_{N,N}}$ *is inconsistent, too, where we recall that* $\mathrm{BI_{fr}}$ *means* $\mathrm{BI_{fr}}(A)$ *for all set* $A$.

These inconsistency statements provide a rereading of Kleene's result as follows: in the presence of $\mathrm{BI_{fr}}$, *we can not identify all choice sequences*, defined as functional relations between natural numbers, *with lawlike ones*, defined as terms of type $\mathrm{N} \rightarrow \mathrm{N}$, *if these are also internally recursive* (as stated in our formal Church thesis). Topologically, this implies that we can not reason in a point-free inductive way in the Baire space, or in the Cantor space, if choice sequences are assumed to be computable.

Then, it comes natural to ask whether without unique choice $\mathrm{AC!_{N,N}}$ our $\mathrm{emTT_0}$ turns out to be consistent with $\mathrm{BI_{fr}}$ and $\mathrm{CT_{tt}}$. This is what we are going to show in the next. As a byproduct we will conclude that unique choice on natural numbers does not generally holds in $\mathrm{emTT_0}$.

### 3.4 Formulation of $\mathbf{CT_{tt}}$, $\mathbf{AC!_{N,N}}$ and $\mathbf{BI_{fr}}$ at the intensional level

Now, we describe the interpretation of $\mathrm{CT_{tt}}$ and $\mathrm{AC!_{N,N}}$ and $\mathrm{BI_{fr}}$ at the intensional level $\mathrm{mTT_0}$. The translations of $\mathrm{CT_{tt}}$ and $\mathrm{AC!_{N,N}}$ are essentially the identity while that of $\mathrm{BI_{fr}}$ is not because we need to represent the power collection of subsets as a suitable quotient (see [Mai09]).

We start by giving the definition of tree formal topology on a *setoid* or *extensional set* $(A, =_A)$ in $\mathrm{mTT}_0$, namely on an $\mathrm{mTT}_0$-set $A$ equipped with an equivalence relation

$$x =_A y \; props_s \; [x \in A, y \in A]$$

as in [Mai09]. The following formulation is obtained by translating in $\mathrm{mTT}_0$ the notion of tree formal topology of $\mathrm{emTT}_0$ by using the interpretation $(-)^i$ in definition 2.1:

**Def. 3.22 (tree formal topology in $\mathrm{mTT}_0$)** A tree formal topology in $\mathrm{mTT}_0$ on a *setoid* $(A, =_A)$ in $\mathrm{mTT}_0$ consists of a proposition

$$l \lhd^i_{A^N} V \; \mathsf{prop_s} \; [l \in A^*, V \in A^* \to \mathsf{prop_s}]$$

with a proof of the proposition

$$l_1 \lhd^i_{A^N} V \leftrightarrow l_2 \lhd^i_{A^N} W \; \mathsf{prop_s} \quad [l_1, l_2 \in A^*, \; V, W \in A^* \to \mathsf{prop_s}, \; u \in l_1 =_{A^*} l_2, \; z \in \forall_{x \in A^*} V(x) \leftrightarrow W(x)]$$

where the equality on lists $=_{A^*}$ is defined from $=_A$ as in [Mai09], and $\lhd^i_{A^N}$ is inductively generated (see [CSSV03]) from the rules rfl, $\leq$, tr in def. 3.4 written as axioms in the implicative form with corresponding proof-terms.

Before giving the interpretation of $\mathrm{BI_{fr}}$, $\mathrm{CT_{tt}}$ and $\mathrm{AC!_{N,N}}$ in $\mathrm{mTT}_0$, we need to extend the interpretation $(-)^i$ of $\mathrm{emTT}_0$ into $\mathrm{mTT}_0$ to include the existence of tree formal topologies in both theories:

$$(-)^i : \mathrm{emTT}_0 \; + \lhd_{A^N} \; \to \mathrm{mTT}_0 \; + \lhd^i_{A^{iN}}$$

by simply interpreting each $\lhd_{A^N}$ as $\lhd^i_{A^{iN}}$ supposing $(A^i, =_{A^i})$ the setoid interpretation of the $\mathrm{emTT}_0$-set $A$. Hence, we are ready to prove:

**Lemma 3.23** *According to the interpretation* $(-)^i : \mathrm{emTT}_0 \; + \lhd_{A^N} \to \mathrm{mTT}_0 \; + \lhd^i_{A^N}$ *in definition 2.1, supposing* $(A^i, =_{A^i})$ *the setoid interpretation in* $\mathrm{mTT}_0$ *of the* $\mathrm{emTT}_0$*-set* $A$*, then*
*- The translation of* $\mathrm{BI_{fr}}(\mathrm{A})$ *for an* $\mathrm{emTT}_0$*-set* $A$ *in* $\mathrm{mTT}_0 + \lhd^i_{A^{iN}}$ *is the following:*

$$(\mathrm{BI_{fr}}(A^i)) \qquad \forall\, l \in List(A^i) \;\; \forall\, V \in List(A^i) \to \mathsf{prop_s} \;\; (\; \forall \alpha \in List(A^i) \to \mathsf{prop_s} \;\; (\; \alpha \varepsilon Pt(\lhd^i_{A^{iN}}) \;\wedge\; \alpha(l) \;\to\; \alpha \between V \;) \to l \lhd^i_{A^{iN}} V \;)$$

*where* $\alpha \between V$ *and* $\alpha \varepsilon Pt(\lhd^i_{A^{iN}})$ *are defined as in* $\mathrm{emTT}_0$ *in definition 3.8.*
*- The translation of* $\mathrm{CT_{tt}}$ *and* $\mathrm{AC!_{N,N}}$*, called* $\mathrm{CT}^i_{\mathrm{tt}}$ *and* $\mathrm{AC!}^i_{\mathrm{N,N}}$ *are essentially the same as* $\mathrm{CT_{tt}}$ *and* $\mathrm{AC!_{N,N}}$ [4].

Thanks to the interpretation of $\mathrm{emTT}_0$ into $\mathrm{mTT}_0$ and to prop. 3.21 we get:

**Corollary 3.24**
$\mathrm{mTT}_0 + \mathrm{BI}^i_{\mathrm{fr}}(\mathrm{N}) + \mathrm{CT}^i_{\mathrm{tt}} + \mathrm{AC!}^i_{\mathrm{N,N}}$ *is inconsistent, and hence* $\mathrm{mTT}_0 + \mathrm{BI}^i_{\mathrm{fr}} + \mathrm{CT}^i_{\mathrm{tt}} + \mathrm{AC!}^i_{\mathrm{N,N}}$ *(where* $\mathrm{BI}^i_{\mathrm{fr}}$ *means* $\mathrm{BI}^i_{\mathrm{fr}}(A)$ *for any* $\mathrm{mTT}_0$*-set* $A$ *with an equivalence relation* $=_A$*), is inconsistent, too.*

**Proof.** Thanks to the interpretation $(-)^i$ in def. 2.1, a proof that falsum is true in the extension $\mathrm{emTT}_0 + \mathrm{BI_{fr}}(\mathrm{N}) + \mathrm{CT_{tt}} + \mathrm{AC!_{N,N}}$ converts to the construction of a proof-term for falsum in $\mathrm{mTT}_0 + \mathrm{BI}^i_{\mathrm{fr}}(\mathrm{N}) + \mathrm{CT}^i_{\mathrm{tt}} + \mathrm{AC!}^i_{\mathrm{N,N}}$.

[4] Note that, according to the interpretation $(-)^i$ based on that in [Mai09], the $\mathrm{emTT}_0$-set of natural numbers is interpreted as the $\mathrm{mTT}_0$-set of natural numbers N equipped with the propositional equality of N. Hence, the support of the interpretation of the $\mathrm{emTT}_0$-set $\mathrm{N} \to \mathrm{N}$ turns out to be the $\mathrm{mTT}_0$-set $\mathrm{N} \to \mathrm{N}$ itself because all $\mathrm{mTT}_0$-functions between natural numbers preserve the propositional equality on N.

**Corollary 3.25** *If* $\mathrm{mTT}_0 + \mathrm{BI}^i_{\mathrm{fr}} + \mathrm{CT}^i_{\mathrm{tt}}$ *is consistent, then*

- $\mathrm{mTT}_0$ *does not validate* $\mathrm{AC!}^i_{\mathrm{N,N}}$*;*
- $\mathrm{emTT}_0 + \mathrm{BI}_{\mathrm{fr}} + \mathrm{CT}_{\mathrm{tt}}$ *is consistent;*
- $\mathrm{emTT}_0$ *does not validate unique choice on natural numbers* $\mathrm{AC!}_{\mathrm{N,N}}$.

# 4 The intermediate level $\mathrm{mTT}_0^{eq}$

In building a realizability interpretation for $\mathrm{mTT}_0$ validating $\mathrm{BI}^i_{\mathrm{fr}}$ and $\mathrm{CT}^i_{\mathrm{tt}}$ we encountered some technical obstacles when interpreting the strong indexed sum elimination constructor on proper collections. We are able to solve such difficulties if we adopt projections as strong indexed sum elimination constructors. But, in an intensional type theory as $\mathrm{mTT}_0$, adopting projections as strong indexed sum elimination constructors does not seem to be equivalent to adopting the current elimination constructor $El_\Sigma(d,m)$. This is instead so if we replace the intensional propositional equality $\mathsf{Id}(A,a,b)$ with the extensional propositional equality $\mathsf{Eq}(A,a,b)$ as in [Mar84]. Therefore, we give our realizability interpretation for an extension of $\mathrm{mTT}_0$, called $\mathrm{mTT}_0^{eq}$, where *the propositional equality* $\mathsf{Id}(A,a,b)$ *is replaced by the stronger extensional one* $\mathsf{Eq}(A,a,b)$ whose rules are the following

**Extensional Propositional Equality**

F-Eq) $\dfrac{C\ col\quad c\in C\quad d\in C}{\mathsf{Eq}(C,c,d)\ prop}$ I-Eq) $\dfrac{c\in C}{\mathsf{eq}\in\mathsf{Eq}(C,c,c)}$

E-Eq) $\dfrac{p\in\mathsf{Eq}(C,c,d)}{c=d\in C}$ C-Eq) $\dfrac{p\in\mathsf{Eq}(C,c,d)}{p=\mathsf{eq}\in\mathsf{Eq}(C,c,d)}$

and we adopt projections as indexed sum elimination constructors both on collections and on sets together with $\beta$ and $\eta$-conversion rules as follows:

**Strong Indexed Sum elimination and conversion rules**

$\mathrm{E}_1$-$\Sigma$) $\dfrac{d\in\Sigma_{x\in B}C(x)}{\pi_1(d)\in B}$ $\mathrm{E}_2$-$\Sigma$) $\dfrac{d\in\Sigma_{x\in B}C(x)}{\pi_2(d)\in C(\pi_1(d))}$

$\mathrm{C}_1$-$\Sigma$) $\dfrac{b\in B\quad c\in C(b)}{\pi_1(\langle b,c\rangle)=b\in B}$ $\mathrm{C}_2$-$\Sigma$) $\dfrac{b\in B\quad c\in C(b)}{\pi_2(\langle b,c\rangle)=c\in C(b)}$

$\eta$-$\Sigma$) $\dfrac{d\in\Sigma_{x\in B}C(x)}{\langle\pi_1(d),\pi_2(d)\rangle=d\in\Sigma_{x\in B}C(x)}$

Luckily, the realizability interpretation we intend to build for $\mathrm{mTT}_0$ validates the rules of $\mathrm{mTT}_0^{eq}$. Indeed, this interpretation is based on Kleene's realizability interpretation for a version of Martin-Löf's type theory in [Tv88b], which was already known to validate the rules of $\mathsf{Eq}(A,a,b)$ in the absence of $\xi$-rule for $\lambda$-terms and in the presence of substitution rules in place of usual equality rules in [NPS90] (see [Mar75]).

Hence, we can interpret $\mathrm{mTT}_0$ into $\mathrm{mTT}_0^{eq}$, both extended with $\mathrm{BI}^i_{\mathrm{fr}} + \mathrm{CT}^i_{\mathrm{tt}}$ as follows:

**Proposition 4.1** *We can interpret* $\mathrm{mTT}_0 + \mathrm{BI}^i_{\mathrm{fr}} + \mathrm{CT}^i_{\mathrm{tt}}$ *into* $\mathrm{mTT}_0^{eq} + \mathrm{BI}^i_{\mathrm{fr}} + \mathrm{CT}^i_{\mathrm{tt}}$ *as the identity on all constructors except for those of the propositional equality* $\mathsf{Id}$ *which are interpreted as those of the extensional one* $\mathsf{Eq}$*, and except for the strong indexed sum elimination constructor which is interpreted via projections.*

**Proof.** We interpret the indexed sum elimination constructor of $\mathrm{mTT}_0$ in $\mathrm{mTT}_0^{eq}$ as follows: given $d\in\Sigma_{x\in B}C(x)\ [\Gamma]$ and $m(x,y)\in M(\langle x,y\rangle)\ [\Gamma,x\in B,y\in C(x)]$ then

$$El_\Sigma(d,m)\ \equiv\ m(\pi_1(d),\pi_2(d))$$

that is of type $M(\langle\pi_1(d), \pi_2(d)\rangle)$ by definition. But by the substitution rules and the rule conv) [5] (see the rules of mTT in [Mai09]) and the above $\eta$-$\Sigma$) of $\text{mTT}_0^{eq}$ we conclude that it is of type $M(d)$ as well, as required.
Concerning the propositional equality: the constructor $\mathsf{id}_A(a)$ of $\text{mTT}_0$ is interpreted as $\mathsf{eq}$ of $\text{mTT}_0^{eq}$ and the elimination constructor $El_{\mathsf{Id}}(p, (x)c(x))$ as $c(a)$, given that its type $C(a, a, \mathsf{eq})$ happens to be equal to $C(a, b, p)$ by the rules subT) and conv) in [Mai09] since from $p \in \mathsf{Eq}(A, a, b)$ we get $a = b \in A$ and also $p = \mathsf{eq} \in \mathsf{Eq}(A, a, b)$ by the rules of $\mathsf{Eq}$.

Note that indexed sum projections can be defined in $\text{mTT}_0$ from the original indexed elimination constructor $El_\Sigma$ as follows:

$$\pi_1(z) \equiv El_\Sigma(z, (x, y).x) \qquad \pi_2(z) \equiv El_\Sigma(z, (x, y).y)$$

By the original conversion rule of $\Sigma$, they clearly satisfy $\text{C}_1$-$\Sigma$) and $\text{C}_2$-$\Sigma$) conversions but $\eta$-$\Sigma$) does not seem to follow.

# 5 The proof-irrelevant realizability interpretation of the intensional level

Here we describe an interpretation of $\text{mTT}_0^{eq}$ extended with Bar Induction $\text{BI}^i_{\text{fr}}$ and the formal Church thesis $\text{CT}^i_{\text{tt}}$ in a set theory $\mathcal{S}$ which can be the classical set theory ZFC or Aczel's CZF extended with Bar Induction $\text{BI}_{\text{fr}}$, for short CZF+$\text{BI}_{\text{fr}}$. We call such an interpretation *proof-irrelevant realizability interpretation*, for short *pf-realizability interpretation* of $\text{mTT}_0^{eq}$. Thanks to proposition 4.1 this gives an interpretation also for $\text{mTT}_0$ in $\mathcal{S}$.

The underlying idea of our pf-realizability interpretation is to interpret $\text{mTT}_0^{eq}$-*sets and their elements* in an *effective way*, namely $\text{mTT}_0^{eq}$-sets as subsets of natural numbers and their families of elements as suitable computable functions, like in the realizability interpretation à la Kleene built in [Tv88b] for a version of Martin-Löf's type theory. Then, we interpret $\text{mTT}_0^{eq}$-*propositions* in a *proof-irrelevant way* as subsets of the zero singleton in ZFC, and as subclasses in CZF (only small propositions are interpreted there as subsets). Finally, to validate $\text{BI}^i_{\text{fr}}$ we interpret $\text{mTT}_0^{eq}$-*collections* as sets in ZFC (and as classes in CZF) and their families of elements as functions *with no computational contents*.

Then, in order to validate the formal Church thesis we add a *computational requirement* saying that *a function interpreting a family of set elements* has the property *to be computed by a family of programs* depending on the interpretation of the minimum context part containing all its proper collection assumptions. Indeed, the idea is to interpret *the dependency of a set element on a set assumption* as a *computable functional dependence*, computed by a program represented by a Gödel number. Instead we interpret *the dependency on a proper collection assumption* as a *functional dependence* with no *computational contents*. Finally, note that, given that propositions are interpreted in a proof-irrelevant way as subsets (or subclasses) of the zero singleton, their proofs are interpreted as zero constant functions which are trivially computable.

Now we explain some problems in building such an interpretation separating computable and not computable entities. Recalling that contexts of $\text{mTT}_0^{eq}$ are telescopic, a proper collection assumption could be in the middle of a context after some set assumptions. In this case the dependency on such set assumptions is treated simply as a functional dependence *by loosing its computational contents*. For example, the term

$$x + y \in \text{N}\ [x \in \text{N}, y \in \text{N}]$$

will be interpreted as the sum function computed by a program with the interpretation of both assumptions as inputs. But if we consider its weakening with a proper collection assumption, for example

$$x + y \in \text{N}\ [x \in \text{N}, V \in \mathsf{prop_s}, y \in \text{N}]$$

[5]We just recall that this rule says that from $a \in A$ and $A = B$ *type* we get $a \in B$.

the resulting term turns out to be interpreted as a *family of programs with one input depending on the interpretation of* $V$ *and in turn also of* $x$ *as* non-computable *assumptions.* It is only after a substitution of $V$ with some closed term that we get access to the sum function computed by a program with two inputs.

Given that we follow Kleene's interpretation of set constructors as in [Tv88b] for a version of Martin-Löf's type theory, this problem of restoring missing computable codes appears when interpreting the strong indexed sum elimination constructor of mTT in [Mai09] from a proper strong indexed sum collection toward a set. Luckily we can solve this problem for $\mathrm{mTT}_0^{eq}$ just because there we restricted its strong indexed sum collections to be only strong indexed sums of propositional functions. Indeed, in $\mathrm{mTT}_0^{eq}$ we can only eliminate from a proper collection toward a proper collection, which does not cause any problem of interpretation. Or we can eliminate from a proper collection toward a proposition, whose elements, even after substitutions, are always interpreted as the constant zero function. Hence we can assign to them the constant zero program in a *canonical* way: for example, after interpreting

$$t(z_1, z_2) \in \Sigma_{x \in \mathsf{props}}\ \phi(x)\ [z_1 \in \Sigma_{x \in \mathsf{props}}\ \phi(x), z_2 \in \mathrm{N}]$$

as a suitable function in $\mathcal{S}$, its second projection

$$\pi_2(t(z_1, z_2)) \in \phi(\pi_1(z_1))\ [z_1 \in \Sigma_{x \in \mathsf{props}}\ \phi(x), z_2 \in \mathrm{N}]$$

turns out to be interpreted as its second projection in $\mathcal{S}$. Moreover, for each $w_1 \varepsilon (\Sigma_{x \in \mathsf{props}}\ \phi(x))^{\mathcal{I}}$ the constant zero program code $\lfloor z_2 \mapsto 0 \rfloor$ computes the function

$$\pi_2(t(z_1, z_2))^{\mathcal{I}}(w_1, -) : w_2 \mapsto \pi_2(t(z_1, z_2))^{\mathcal{I}}(w_1, w_2)$$

on the computable input $w_2 \varepsilon \mathrm{N}^{\mathcal{I}}$.

In the more general case of strong indexed sums of a collection family indexed on proper collections or sets we are not able to assign canonical codes that can be restored when we eliminate towards sets. In order to interpret them we need to build a more complex realizability interpretation where the apparently forgotten codes of set inputs, on which proper collection assumptions depend, are *all stored* in order to use them after substitution. This more complex interpretation is left to future work.

The interpretation of propositions as subsets/subclasses of the zero singleton is crucial to validate $\mathrm{BI}^i_{\mathrm{fr}}$(its interpretation will correspond to $\mathrm{BI}_{\mathrm{fr}}$ which is a theorem both in ZFC and, of course, in CZF+$\mathrm{BI}_{\mathrm{fr}}$). Recall also that we can not interpret propositions according to Kleene's realizability interpretation [Tv88a] because this interpretation validates the axiom of choice, and, hence, the axiom of unique choice $\mathrm{AC!}^i_{\mathrm{N,N}}$ which is inconsistent with $\mathrm{BI}^i_{\mathrm{fr}}$ and $\mathrm{CT}^i_{\mathrm{tt}}$. Moreover, interpreting propositions as subsets/subclasses of the zero singleton allows us to validate also the rules **prop-into-col** and **prop$_s$-into-set** in [Mai09].

Now we start to properly define the proof-irrelevant realizability interpretation of $\mathrm{mTT}_0^{eq}$ in a set theory $\mathcal{S}$ as ZFC or CZF+$\mathrm{BI}_{\mathrm{fr}}$. To this purpose we first fix some abbreviations regarding computable functions we are going to use. We denote with $\mathcal{N}$ the set of natural numbers in $\mathcal{S}$. Then $\{n\} : \mathcal{N} \to \mathcal{N}$ stands for the computable partial function with Gödel number $n$. Moreover, we will simply write

$$\{n\}(y)\varepsilon B \;\equiv\; \exists_{w\varepsilon\mathcal{N}}\, T(n, y, w) \,\wedge\, U(w)\varepsilon B$$

for a natural number $n$ and for a subset $B$ of $\mathcal{N}$. Conversely, given a partial computable function $z \mapsto f(z) : \mathcal{N} \to \mathcal{N}$, then $\lfloor z \mapsto f(z) \rfloor$ denotes a natural number such that

$$\{\, \lfloor z \mapsto f(z) \rfloor\, \}(x) \;=\; f(x)$$

Moreover, the isomorphism of the set of natural numbers $\mathcal{N}$ with its binary product $\mathcal{N} \times \mathcal{N}$ is denoted by the following functions in $\mathcal{S}$:

$$\langle \mathsf{pr}_1, \mathsf{pr}_2 \rangle : \mathcal{N} \to \mathcal{N} \times \mathcal{N} \qquad \text{and} \qquad \mathsf{pair} : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$$

Finally, after recalling that the set of natural numbers $\mathcal{N}$ is isomorphic to the set of lists on itself, we denote its list structure as in type theory: the empty list is $\mathsf{nil}$, which is 0 in $\mathcal{N}$, $\mathsf{cons}_{List(\mathcal{N})}(-,-)$ is the list constructor appending an element to a list and $Rec_{List(\mathcal{N})}(-,-,-)$ is the constructor defining a term by recursion on lists.

In order to validate $\mathrm{BI}^{i}_{\mathrm{fr}}$ we need to distinguish sets from proper collections. To this purpose we will use the following decidable function saying when a type is a *proper collection* (for short `is-pc`) that is neither a set nor a proposition for any $\mathrm{mTT}^{eq}_{0}$-expression $A$ for which we know that in $\mathrm{mTT}^{eq}_{0}$ $A\ type\ [\Gamma]$ is derivable [6]:

$$\texttt{is-pc}(A) \equiv \begin{cases} 0 & \text{if } A\ set \text{ or } A\ prop \\ 1 & \text{otherwise} \end{cases}$$

In $\mathrm{mTT}^{eq}_{0}$ we can decompose any context $\Gamma$ in two parts: one, called $\Gamma_p$, is the minimal context part containing all proper collection assumptions, and hence the remaining part, called $\Gamma_t$, is made of set or proposition assumptions only:

**Lemma 5.1** *Any context* $\Gamma$ *can be decomposed into* $\Gamma_p, \Gamma_t$ *where* $\Gamma_p \equiv \emptyset$ *or* $\Gamma_p \equiv x_1 \in A_1, \ldots, x_n \in A_n$ *with* $\texttt{is-pc}\ (A_n) = 1$*, and* $\Gamma_t \equiv \emptyset$ *or* $\Gamma_t \equiv y_1 \in B_1, \cdots \in B_m$ *made only of set or proposition assumptions, namely* $\texttt{is-pc}\ (B_i) = 0$ *for* $i = 1, \ldots, m$.

**Def. 5.2 (pf-realizability interpretation of $\mathrm{mTT}^{eq}_{0}$)** Here we define the interpretation

$$(-)^{\mathcal{I}} : \mathrm{mTT}^{eq}_{0} \longrightarrow \mathcal{S}$$

where $\mathcal{S}$ can be ZFC classical set theory or CZF+ $\mathrm{BI}_{\mathrm{fr}}$.
A context $\Gamma$ is interpreted by induction by means of disjoint unions as follows:

$$(\emptyset\ cont)^{\mathcal{I}} \equiv \{0\} \qquad (\Gamma, x \in A\ cont)^{\mathcal{I}} \equiv \biguplus_{z\varepsilon\Gamma^{\mathcal{I}}} A^{\mathcal{I}}(z)$$

A type judgement is interpreted as a family of sets in ZFC, or a family of sets/classes in CZF

$$(\,B\ type\ [\Gamma]\,)^{\mathcal{I}} \equiv (\,B^{\mathcal{I}}(z)\,)_{z\varepsilon\Gamma^{\mathcal{I}}}$$

In particular, dependent sets turn out to be interpreted as families of subsets of natural numbers:

$$(\,B\ set\ [\Gamma]\,)^{\mathcal{I}} \equiv (\,B^{\mathcal{I}}(z)\,)_{z\varepsilon\Gamma^{\mathcal{I}}} \ \text{ such that } \ B^{\mathcal{I}}(z) \subseteq \mathcal{N}$$

and dependent propositions as subsets of the zero singleton $\{0\}$ in ZFC, or subclasses of the zero singleton in CZF that become subsets when they interpret small propositions:

$$(\,B\ prop\ [\Gamma]\,)^{\mathcal{I}} \equiv (\,B^{\mathcal{I}}(z)\,)_{z\varepsilon\Gamma^{\mathcal{I}}} \ \text{ such that } \ B^{\mathcal{I}}(z) \subseteq \{\, n\varepsilon\mathcal{N} \mid n = 0\,\}$$

A type equality judgement is interpreted as the extensional equality between families

$$(\,B = C\ type\ [\Gamma]\,)^{\mathcal{I}} \equiv \forall_{z\varepsilon\Gamma^{\mathcal{I}}} \quad B^{\mathcal{I}}(z) = C^{\mathcal{I}}(z)$$

A term judgement is interpreted as a function

$$(\,b \in B\ [\Gamma]\,)^{\mathcal{I}} \equiv z\varepsilon\Gamma^{\mathcal{I}} \mapsto b^{\mathcal{I}}(z) \in B^{\mathcal{I}}(z)$$

[6] More formally $\texttt{is-pc}(A)$ is defined by induction on type constructions as follows:
$\texttt{is-pc}(A) \equiv 0$ for $A \equiv N_0, N_1, List(C), B + C, \bot, B \wedge C, B \vee C, B \to C, \exists_{x\in B} C(x), \forall_{x\in B} C(x), \mathsf{Eq}(A, a, b)$
$\texttt{is-pc}(A) \equiv 0$ for $A \equiv \Sigma_{x\in B} C(x)$ iff $\texttt{is-pc}(B) = 0$ and $\texttt{is-pc}(C(x)) = 0$
$\texttt{is-pc}(p) \equiv 0$ for $p \in \mathsf{prop_s}$
$\texttt{is-pc}(A) \equiv 1$ for $A \equiv \mathsf{prop_s}, B \to \mathsf{prop_s}$.

and, if $\texttt{is-pc}(B) = 0$, namely if $B$ is a dependent set or a proposition, this must be equipped with a suitable family of program codes $b^\sharp(z_p)\varepsilon\mathcal{N}$ depending on $\Gamma_p^{\mathcal{I}}$, namely on the interpretation of the minimal context part including all proper collection assumptions, and computing the function $z_t \mapsto b^{\mathcal{I}}(z_p, z_t)$ [7]: i.e. we assume that there exists

$$z_p\varepsilon\Gamma_p^{\mathcal{I}} \mapsto b^\sharp(z_p)\varepsilon\mathcal{N} \qquad \text{s. t.} \qquad \forall_{z_t}\varepsilon\Gamma_t^{\mathcal{I}}(z_p) \;\; b^{\mathcal{I}}(z_p, z_t) = \{b^\sharp(z_p)\}(z_t)$$

where, in the case $\Gamma_t \equiv \emptyset$, i.e. there are no set inputs available, we simply consider the program code coinciding with the output.
In the next we will simply abbreviate $b^\sharp(z_p) \equiv \lfloor z_t \mapsto b^{\mathcal{I}}(z_p, z_t)\rfloor$ when applicable.
A term equality judgement is interpreted as the extensional equality of functions:

$$(\, b = c \in B \; [\Gamma]\,)^{\mathcal{I}} \equiv \forall_z\varepsilon\Gamma^{\mathcal{I}} \; b^{\mathcal{I}}(z) = c^{\mathcal{I}}(z)$$

Now, we give the interpretation of $\mathrm{mTT}_0^{eq}$-constructors. Actually this will be a partial interpretation of the so-called "raw syntax" in [Mai05b], namely of the syntax forming types and typed terms of $\mathrm{mTT}_0^{eq}$, because term equalities are involved in the formation of types and typed terms.

Note that for simplicity, we interpret strong indexed sums of propositional functions indexed on a set in a computational way being $\mathrm{mTT}_0^{eq}$-propositions interpreted as subsets of the zero singleton.

In the following, we denote the power of a set $X$ in $\mathcal{S}$ as $\mathcal{P}(X)$.
Assumption of variables is interpreted as follows:

$$(\, x \in A \; [\Gamma, x \in A, \Delta]\,)^{\mathcal{I}} \equiv \begin{cases} z_p \mapsto \lfloor z_t \mapsto \pi_{n+1}(z_p, z_t)\rfloor & \text{if } \texttt{is-pc}(A) = 0 \\ z \mapsto \pi_{n+1}(z) & \text{if } \texttt{is-pc}(A) = 1 \end{cases}$$

Collection and set constructors are interpreted as follows:

$$(\, \Sigma_{x\in B} D(x) \; col \; [\Gamma]\,)^{\mathcal{I}} \equiv \begin{cases} (\{\, n\varepsilon\mathcal{N} \mid \mathsf{pr}_1(n)\varepsilon B^{\mathcal{I}}(z) \wedge \mathsf{pr}_2(n)\varepsilon C^{\mathcal{I}}(z, \mathsf{pr}_1(n))\,\})_{z\varepsilon\Gamma^{\mathcal{I}}} & \text{if } \texttt{is-pc}\;(B) = 0 \\ (\biguplus_{x\varepsilon B^{\mathcal{I}}(z)} C^{\mathcal{I}}(z, x)\,)_{z\varepsilon\Gamma^{\mathcal{I}}} & \text{if } \texttt{is-pc}\;(B) = 1 \end{cases}$$

$$(\, \langle b, d\rangle \in \Sigma_{x\in B} D(x) \; [\Gamma]\,)^{\mathcal{I}} \equiv \begin{cases} z_p \mapsto \lfloor z_t \mapsto \mathsf{pair}(\, b^{\mathcal{I}}(z_p, z_t)\,,\, c^{\mathcal{I}}(z_p, z_t)\,)\rfloor & \text{if } \texttt{is-pc}\;(B) = 0 \\ z \mapsto (b^{\mathcal{I}}(z), d^{\mathcal{I}}(z)) & \text{if } \texttt{is-pc}\;(B) = 1 \end{cases}$$

$$(\, \pi_1(d) \in B \; [\Gamma]\,)^{\mathcal{I}} \equiv \begin{cases} z_p \mapsto \lfloor z_t \mapsto \mathsf{pr}_1(\, d^{\mathcal{I}}(z_p, z_t)\,)\rfloor & \text{if } \texttt{is-pc}\;(B) = 0 \\ z \mapsto \pi_1(d^{\mathcal{I}}(z)) & \text{if } \texttt{is-pc}\;(B) = 1 \end{cases}$$

$$(\, \pi_2(d) \in C(d) \; [\Gamma]\,)^{\mathcal{I}} \equiv \begin{cases} z_p \mapsto \lfloor z_t \mapsto \mathsf{pr}_2(\, d^{\mathcal{I}}(z_p, z_t)\,)\rfloor & \text{if } \texttt{is-pc}\;(\Sigma_{x\in B} C(x)) = 0 \\ z_p \mapsto \lfloor z_t \mapsto 0\rfloor & \text{if } \texttt{is-pc}\;(\Sigma_{x\in B} C(x)) = 1 \end{cases}$$

$(\, \mathsf{prop_s} \; col \; [\Gamma]\,)^{\mathcal{I}} \equiv (\, \mathcal{P}(\{0\})\,)_{z\varepsilon\Gamma^{\mathcal{I}}}$
$(\, B \to \mathsf{prop_s} \; col \; [\Gamma]\,)^{\mathcal{I}} \equiv (\, B^{\mathcal{I}}(z) \to \mathcal{P}(\{0\})\,)_{z\varepsilon\Gamma^{\mathcal{I}}}$
$(\, \lambda x^B.C \in B \to \mathsf{prop_s} \; [\Gamma]\,)^{\mathcal{I}} \equiv z \mapsto (\, x \mapsto C^{I}(z, x)\,)$
$(\, \mathsf{Ap}(f, b) \in \mathsf{prop_s} \; [\Gamma]\,)^{\mathcal{I}} \equiv z \mapsto f^{\mathcal{I}}(z)\,(\, b^{\mathcal{I}}(z)\,)$
$(\, N_0 \; set \; [\Gamma]\,)^{\mathcal{I}} \equiv (\,\emptyset\,)_{z\varepsilon\Gamma^{\mathcal{I}}}$
$(\, \mathsf{emp_o}\,)^{\mathcal{I}} \equiv z_p \mapsto \lfloor z_t \mapsto 0\rfloor$
$(\, N_1 \; set \; [\Gamma]\,)^{\mathcal{I}} \equiv (\,\{\, 0\,\}\,)_{z\varepsilon\Gamma^{\mathcal{I}}}$
$(\, * \in N_1 \; [\Gamma]\,)^{\mathcal{I}} \equiv z_p \mapsto \lfloor z_t \mapsto 0\rfloor$

$$(\, El_{\mathsf{N_1}}(t, c)\,) \in M(t) \; [\Gamma]\,)^{\mathcal{I}} \equiv \begin{cases} z_p \mapsto \lfloor z_t \mapsto c^{\mathcal{I}}(z_p, z_t)\rfloor & \text{if } \texttt{is-pc}\;(M(t)) = 0 \\ z \mapsto c^{\mathcal{I}}(z) & \text{if } \texttt{is-pc}\;(M(t)) = 1 \end{cases}$$

$(\, List(C) \; set \; [\Gamma]\,)^{\mathcal{I}} \equiv (\,\{\, n\varepsilon\mathcal{N} \mid \forall_{j\varepsilon\mathcal{N}} \; (\, 1 \le j \le \mathsf{lh}(n) \wedge \mathsf{p_j}(n)\varepsilon C^{\mathcal{I}}(z)\,)\,\}\,)_{z\varepsilon\Gamma^{\mathcal{I}}}$
where $\mathsf{lh}(n)$ is the length of the list encoded by $n$ and $\mathsf{p_j}(n)$ its ith-projection.
$(\, \epsilon \in List(C)\,[\Gamma]\,)^{\mathcal{I}} \equiv z_p \mapsto \lfloor z_t \mapsto 0\rfloor$

[7] From now on, when writing the dependency of a function on a disjoint union, we simply write $b^{\mathcal{I}}(z_p, z_t)$ instead of $b^{\mathcal{I}}(\,(z_p, z_t)\,)$, where $(z_p, z_t)$ represents the pairing of $z_p$ with components of $z_t$ to become an element of $\Gamma^{\mathcal{I}}$ made of nested disjoint unions. The same we do for families depending on a disjoint union, i.e. we write $C^{\mathcal{I}}(w, z)$ for $w\varepsilon A^{\mathcal{I}}, z\varepsilon B^{\mathcal{I}}(x)$ instead of $C^{\mathcal{I}}(\,(w, z)\,)$.

$(\,\mathsf{cons}(s,c)\in List(C)\ [\Gamma]\,)^{\mathcal{I}}\ \equiv\ z_p\mapsto\lfloor z_t\mapsto \mathsf{cons}_{List(\mathcal{N})}(\,s^{\mathcal{I}}(z_p,z_t)\,,\,c^{\mathcal{I}}(z_p,z_t)\,)\rfloor$

$$(\,El_{List}(a,l,s)\in L(s)\ [\Gamma]\,)^{\mathcal{I}}\ \equiv\ \begin{cases} z_p\mapsto\lfloor z_t\mapsto \\ \qquad Rec_{List(\mathcal{N})}(\,a^{\mathcal{I}}(z_p,z_t)\,,\,(x,y,w).\,l^{\mathcal{I}}(z_p,z_t,x,y,w)\,,\,s^{\mathcal{I}}(z_p,z_t)\,)\rfloor \\ \qquad\quad \text{if } \texttt{is-pc}\ (L(s))=0 \\ z\mapsto Rec_{List(\mathcal{N})}(\,a^{\mathcal{I}}(z),\,(x,y,w).\,l^{\mathcal{I}}(z,x,y,w)\,,\,s^{\mathcal{I}}(z)\,) \\ \qquad\quad \text{if } \texttt{is-pc}\ (L(s))=1 \end{cases}$$

$(\,B+C\ set\ [\Gamma]\,)^{\mathcal{I}}\ \equiv\ (\,\{\,n\varepsilon\mathcal{N}\ \mid\ (\,n=(0,y)\ \wedge\ y\varepsilon B^{\mathcal{I}}(z)\,)\ \vee\ (\,n=(1,y)\ \wedge\ y\varepsilon C^{\mathcal{I}}(z)\,)\,\}\,)_{z\varepsilon\Gamma^{\mathcal{I}}}$
$(\,\mathsf{inl}(b)\in B+C\ [\Gamma]\,)^{\mathcal{I}}\ \equiv\ z_p\mapsto\lfloor z_t\mapsto \mathsf{pair}(0,b^{\mathcal{I}}(z_p,z_t)\,)\rfloor$
$(\,\mathsf{inl}(c)\in B+C\ [\Gamma]\,)^{\mathcal{I}}\ \equiv\ z_p\mapsto\lfloor z_t\mapsto \mathsf{pair}(1,c^{\mathcal{I}}(z_p,z_t)\,)\rfloor$

$$(\,El_{+}(d,a_B,a_{\{0,1\}^N})\in A(w)\ [\Gamma]\,)^{\mathcal{I}}\ \equiv\ \begin{cases} z_p\mapsto\lfloor z_t\mapsto \begin{cases} a^{\mathcal{I}}_B(z_p,z_t,y) & \text{if } d^{\mathcal{I}}(z_p,z_t)=\mathsf{pair}(0,y) \\ a^{\mathcal{I}}_C(z_p,z_t,y) & \text{if } d^{\mathcal{I}}(z_p,z_t)=\mathsf{pair}(1,y)\end{cases}\rfloor \\ \qquad\qquad \text{if } \texttt{is-pc}\ (A(w))=0 \\ z\mapsto \begin{cases} a^{\mathcal{I}}_B(z,y) & \text{if } d^{\mathcal{I}}(z)=\mathsf{pair}(0,y) \\ a^{\mathcal{I}}_C(z,y) & \text{if } d^{\mathcal{I}}(z)=\mathsf{pair}(1,y)\end{cases} \\ \qquad\qquad \text{if } \texttt{is-pc}\ (A(w))=1 \end{cases}$$

$(\,\Pi_{x\in B}C(x)\ set\ [\Gamma]\,)^{\mathcal{I}}\ \equiv\ (\,\{\,n\varepsilon\mathcal{N}\ \mid\ \forall_{y\varepsilon\mathcal{N}}\ y\varepsilon B^{\mathcal{I}}(z)\to\{n\}(y)\varepsilon C^{\mathcal{I}}(z,y)\ \}\,)_{z\varepsilon\Gamma^{\mathcal{I}}}$
$(\,\lambda x^B.c\in\Pi_{x\in B}C(x)\ [\Gamma]\,)^{\mathcal{I}}\ \equiv\ z_p\mapsto\ \lfloor z_t\mapsto S^1_m(c^{\sharp}(z_p)\,,\,z_t\,)\rfloor$
with $\{\,S^1_m(\,c^{\sharp}(z_p)\,,\,z_t\,)\}(x)=\{\,c^{\sharp}(z_p)\,\}(z_t,x)$ for all $z_t,x$ by s-m-n theorem with $m$ length of $\Gamma_t$
$(\,\mathsf{Ap}(f,b)\in C(b)\ [\Gamma]\,)^{\mathcal{I}}\ \equiv\ z_p\mapsto\lfloor z_t\mapsto\{\,f^{\mathcal{I}}(\,z_p,z_t\,)\,\}(\,b^{\mathcal{I}}(\,z_p,z_t\,)\,)\rfloor.$
Now, we give the interpretation of propositions:
$(\,\bot\ prop\ [\Gamma]\,)^{\mathcal{I}}\ \equiv\ (\,\emptyset\,)_{z\varepsilon\Gamma^{\mathcal{I}}}$
$(\,\mathsf{r_o}(a)\,)^{\mathcal{I}}\ \equiv\ z_p\mapsto\lfloor z_t\mapsto 0\rfloor$
$(\,\exists_{x\in B}C(x)\ prop\ [\Gamma]\,)^{\mathcal{I}}\ \equiv\ (\,\{\,n\varepsilon\mathcal{N}\ \mid\ n=0\ \wedge\ \exists_{x\varepsilon\mathcal{N}}(\,x\varepsilon B^{\mathcal{I}}(z)\ \wedge\ 0\varepsilon C^{\mathcal{I}}(z,x)\,)\,\}\,)_{z\varepsilon\Gamma^{\mathcal{I}}}$
$(\,\langle b,_{\exists}c\rangle\in\exists_{x\in B}C(x)\,)^{\mathcal{I}}\ \equiv\ z_p\mapsto\lfloor z_t\mapsto 0\rfloor$
$(\,El_{\Sigma}(d,m)\in M(d)\ [\Gamma]\,)^{\mathcal{I}}\ \equiv\ z_p\mapsto\lfloor z_t\mapsto 0\rfloor$
$(\,B\vee C\ prop\ \Gamma]\,)^{\mathcal{I}}\ \equiv\ (\,\{\,n\varepsilon\mathcal{N}\ \mid\ (\,n=0\ \wedge\ 0\varepsilon B^{\mathcal{I}}(z)\ \vee\ 0\varepsilon C^{\mathcal{I}}(z)\,)\,\}\,)_{z\varepsilon\Gamma^{\mathcal{I}}}$
$\mathcal{I}(\,\mathsf{inl}_{\vee}(b)\in B\vee C\ [\Gamma]\,)^{\mathcal{I}}\ \equiv\ z_p\mapsto\lfloor z_t\mapsto 0\rfloor$
$(\,\mathsf{inr}_{\vee}(c)\in B\vee C\ [\Gamma]\,)^{\mathcal{I}}\ \equiv\ z_p\mapsto\lfloor z_t\mapsto 0\rfloor$
$(\,El_{\vee}(d,a_B,a_C)\in A\ [\Gamma]\,)^{\mathcal{I}}\ \equiv\ z_p\mapsto\lfloor z_t\mapsto 0\rfloor$
$(\,B\wedge C\ prop\ [\Gamma]\,)^{\mathcal{I}}\ \equiv\ (\,\{\,n\varepsilon\mathcal{N}\ \mid\ n=0\ \wedge\ 0\varepsilon B^{\mathcal{I}}(z)\ \wedge\ 0\varepsilon C^{\mathcal{I}}(z)\ \}\,)_{z\varepsilon\Gamma^{\mathcal{I}}}$
$(\,\langle b,_{\wedge}c\rangle\in B\wedge C\ [\Gamma]\,)^{\mathcal{I}}\ \equiv\ z_p\mapsto\lfloor z_t\mapsto 0\rfloor$
$(\,\pi^B_1(d)\in B\ [\Gamma]\,)^{\mathcal{I}}\ \equiv\ z_p\mapsto\lfloor z_t\mapsto 0\rfloor$
$(\,\pi^C_2(d)\in C\ [\Gamma]\,)^{\mathcal{I}}\ \equiv\ z_p\mapsto\lfloor z_t\mapsto 0\rfloor^{\mathcal{I}}$
$(\,B\to C\ prop\ [\Gamma]\,)^{\mathcal{I}}\ \equiv\ (\,\{\,n\varepsilon\mathcal{N}\ \mid\ n=0\ \wedge\ 0\varepsilon B^{\mathcal{I}}(z)\ \to\ 0\varepsilon C^{\mathcal{I}}(z)\ \}\,)_{z\varepsilon\Gamma^{\mathcal{I}}}$
$(\forall_{x\in B}C(x)\ prop\ [\Gamma]\,)^{\mathcal{I}}\ \equiv\ (\,\{\,n\varepsilon\mathcal{N}\ \mid\ n=0\ \wedge\ \forall_{y\varepsilon\mathcal{N}}\ (\,y\varepsilon B^{\mathcal{I}}(z)\to 0\varepsilon C^{\mathcal{I}}(z,y)\,)\,\}\,)_{z\varepsilon\Gamma^{\mathcal{I}}}$
$(\,\lambda x^B.c\in\Pi_{x\in B}C(x)\ [\Gamma]\,)^{\mathcal{I}}\ \equiv\ z_p\mapsto\lfloor z_t\mapsto 0\rfloor$
$(\,\mathsf{Ap}(f,b)\in C(b)\ [\Gamma]\,)^{\mathcal{I}}\ \equiv\ z_p\mapsto\lfloor z_t\mapsto 0\rfloor$
$(\,\mathsf{Eq}(A,a,b)\ prop\ [\Gamma]\,)^{\mathcal{I}}\ \equiv\ (\,\{\,n\varepsilon\mathcal{N}\ \mid\ n=0\ \wedge\ a^{\mathcal{I}}(z)=b^{\mathcal{I}}(z)\ \}\,)_{z\varepsilon\Gamma^{\mathcal{I}}}$
$\mathcal{I}(\,\mathsf{id_A}(a)\in\mathsf{Id}(A,a,a)\ [\Gamma]\,)^{\mathcal{I}}\ \equiv\ z_p\mapsto\lfloor z_t\mapsto 0\rfloor$
$(\,El_{\mathsf{Id}}(p,(x)c(x))\in C(a,b)\ [\Gamma]\,)^{\mathcal{I}}\ \equiv\ z_p\mapsto\lfloor z_t\mapsto 0\rfloor$

In order to show the validity theorem, we need to show how weakening and substitutions are interpreted.

**Lemma 5.3** *For any judgements $B\ type\ [\Gamma]$ and $b\in B\ [\Gamma]$ derived in* $\mathrm{mTT}^{eq}$ *and interpreted as*

$$(\,B\ type\ [\Gamma]\,)^{\mathcal{I}}\ \equiv\ (\,B^{\mathcal{I}}(z)\,)_{z\varepsilon\Gamma^{\mathcal{I}}}\qquad \text{and}\qquad (\,b\in B\ [\Gamma]\,)^{\mathcal{I}}\ \equiv\ z\varepsilon\Gamma^{\mathcal{I}}\ \mapsto\ b^{\mathcal{I}}(z)\varepsilon B^{\mathcal{I}}(z)$$

weakening *is interpreted as follows:*

$$(\,B\ type\ [\Gamma,\Delta]\,)^{\mathcal{I}}\ =\ (\,B^{\mathcal{I}}(z)\,)_{(z,w)\varepsilon(\,\Gamma,\Delta\,)^{\mathcal{I}}}$$
$$\mathcal{I}(\,b\in B\ [\Gamma,\Delta]\,)\ =\ (z,w)\varepsilon(\,\Gamma,\Delta\,)^{\mathcal{I}}\ \mapsto\ b^{\mathcal{I}}(z)$$

**Proof.** By induction on the derivation of the judgements.

Now we show how substitution is interpreted via composition of functions by using the following abbreviations: given a context $\Gamma \equiv \Sigma, x_n \in A_n, \Delta$ with $\Delta \equiv x_{n+1} \in A_{n+1}, ..., x_k \in A_k$ then for every $a \in A_n\ [\Sigma]$ and for any type $B\ type\ [\Gamma]$ we simply write the type $B$ after substitution of $x_n$ with $a$ in the form $B[x_n/a]\ type\ [\Sigma, \Delta_a]$ instead of the more correct form $B[x_n/a_n][x_i/x'_i]_{i=n+1,...,k}\ type\ [\Sigma, \Delta_a]$ where $\Delta_a \equiv x'_{n+1} \in A'_{n+1}, ..., x'_k \in A'_k$ and $A'_j \equiv A_j\ [x_n/a_n][x_i/x'_i]_{i=n+1,...,j-1}$ for $j = n+1, ..., k$. Similar abbreviations are used also for terms.

**Lemma 5.4** *For any judgements $B\ type\ [\Gamma]$ and $b \in B\ [\Gamma]$ derived in $\mathrm{mTT}_0^{eq}$ and interpreted as*

$$(B\ type\ [\Gamma])^{\mathcal{I}} \equiv (\, B^{\mathcal{I}}(z_1, \ldots, z_n)\,)_{z \varepsilon \Gamma^{\mathcal{I}}} \qquad and \qquad (\, b \in B\ [\Gamma]\,)^{\mathcal{I}} \equiv z\varepsilon\Gamma^{\mathcal{I}} \mapsto b^{\mathcal{I}}(z)\varepsilon B^{\mathcal{I}}(z)$$

substitution *is interpreted as follows:*
*supposed $\Gamma \equiv \Sigma, x_n \in A_n, \Delta$ with $\Delta \equiv x_{n+1} \in A_{n+1}, ..., x_k \in A_k$ then for every $a \in A_n\ [\Sigma]$ interpreted as*

$$(\, a \in A_n\ [\Sigma]\,)^{\mathcal{I}} \equiv w\varepsilon\Sigma^{\mathcal{I}} \mapsto a^{\mathcal{I}}(z)\varepsilon {A_n}^{\mathcal{I}}(w_1, \ldots, w_{n-1})$$

*with $w = (w_1, \ldots, w_{n-1})$, if the interpretations $(B[x_n/a]\ type\ [\Sigma, \Delta_a]\,)^{\mathcal{I}}$ and $(\, b[x_n/a] \in B[x_n/a]\ [\Sigma, \Delta_a]\,)^{\mathcal{I}}$ are well defined, then*

$$(B[x_n/a]\ type\ [\Sigma, \Delta_a]\,)^{\mathcal{I}} = (\, B^{\mathcal{I}}(w,\, a^{\mathcal{I}}(w),\, w')\,)_{(w,w')\varepsilon(\,\Sigma,\Delta_a\,)^{\mathcal{I}}}$$
$$(\, b[x_n/a] \in B[x_n/a]\ [\Sigma, \Delta_a]\,)^{\mathcal{I}} = (w, w')\varepsilon(\,\Sigma, \Delta_a\,)^{\mathcal{I}} \mapsto b^{\mathcal{I}}(w,\, a^{\mathcal{I}}(w),\, w'\,)\varepsilon B^{\mathcal{I}}(w,\, a^{\mathcal{I}}(w),\, w')$$

**Proof.** By induction on the derivation of judgements.

**Theorem 5.5 (pf-realizability validity)** *The calculus $\mathrm{mTT}_0^{eq}$ is validated by the pf-realizability interpretation of definition 5.2 for $\mathcal{S} =$ ZF or $\mathcal{S} =$ CZF, namely:*
*If $A\ type\ [\Gamma]$ is derivable in $\mathrm{mTT}_0^{eq}$ then $(\, A\ type\ [\Gamma]\,)^{\mathcal{I}}$ is well defined.*
*If $a \in A\ [\Gamma]$ is derivable in $\mathrm{mTT}_0^{eq}$ then $(\, a \in A\ [\Gamma]\,)^{\mathcal{I}}$ is well defined.*
*Supposed $A\ type\ [\Gamma]$ and $B\ type\ [\Gamma]$ derivable in $\mathrm{mTT}_0^{eq}$, if $A = B\ type\ [\Gamma]$ is derivable in $\mathrm{mTT}_0^{eq}$, then $(A = B\ type\ [\Gamma]\,)^{\mathcal{I}}$ is valid.*
*Supposed $a \in A\ [\Gamma]$ and $b \in A\ [\Gamma]$ derivable in $\mathrm{mTT}_0^{eq}$, if $a = b \in A\ [\Gamma]$ is derivable in $\mathrm{mTT}_0^{eq}$, then $(\, a = b \in A\ [\Gamma]\,)^{\mathcal{I}}$ is valid.*

*Moreover, the interpretation $(-)^{\mathcal{I}}$ also validates $CT^i_{tt}$.*

*If $\mathcal{S}$ is the classical set theory ZFC or Aczel's CZF + $BI_{fr}$, then the interpretation $(-)^{\mathcal{I}}$ validates $BI^i_{fr}$ when interpreting a generic $\triangleleft^i_{A^N}$, for any $\mathrm{emTT}_0$-set $A$ (interpreted in $\mathrm{mTT}_0^{eq}$ as the setoid $(A^i, =_{A^i})$), as the corresponding $\mathcal{S}$-tree formal topology over $(A^i)^{\mathcal{I}}$ quotiented under the interpretation of its equality $(=_{A^i})^{\mathcal{I}}$.*
*If $\mathcal{S}$ is the classical set theory ZF, then the interpretation $(-)^{\mathcal{I}}$ validates the principle of excluded middle EM, written $P \vee \neg P$ for a proposition $P$.*

**Proof.** By induction on the derivation of judgements. Note that the interpretation of the second projection $\mathrm{E}_2$-ip) is well defined given that a valid proposition is interpreted as the zero singleton. Indeed, if for each $z \in \Gamma^{\mathcal{I}}$ we have that $\pi_2(d^{\mathcal{I}}(z))\varepsilon C^{\mathcal{I}}(z, \pi_1(d^{\mathcal{I}}(z)))$, this means that for $z \in \Gamma^{\mathcal{I}}$ then $C^{\mathcal{I}}(z, \pi_1(d^{\mathcal{I}}(z)))$ is inhabited. Since it is a subset of $\{0\}$ we deduce $C^{\mathcal{I}}(z, \pi_1(d^{\mathcal{I}}(z))) = \{0\}$ and hence $\pi_2(d^{\mathcal{I}}(z)) = 0$. Therefore, for every $z_p\varepsilon\Gamma_p^{\mathcal{I}}$ and $z_t\varepsilon\Gamma_t^{\mathcal{I}}(z_p)$ we conclude $\pi_2(d^{\mathcal{I}}(z_p, z_t)) = \{\lfloor z_t \mapsto 0 \rfloor\}(z_t)$ as wanted. Finally $\eta$-conversion, beside $\beta$-one is valid because $\langle \mathsf{pr}_1, \mathsf{pr}_2 \rangle$ is an isomorphism with $\mathsf{pair}$.
The rules of Extensional Propositional Equality are valid: in particular the rule E-Eq) of $\mathrm{mTT}_0^{eq}$ is valid because from the assumption that $p^{\mathcal{I}}(z)\varepsilon \mathsf{Eq}(A, c, d)^{\mathcal{I}}$ for any $z\varepsilon\Gamma^{\mathcal{I}}$ we conclude $c^{\mathcal{I}}(z) = d^{\mathcal{I}}(z)$ for any $z\varepsilon\Gamma^{\mathcal{I}}$.
The interpretation of lambda abstraction in the rule I-Π) is well defined by s-m-n theorem.
EM is valid for $\mathcal{S}$=ZFC, since propositions are interpreted as their boolean value.

$\mathrm{BI}^i_{\mathrm{fr}}$ is valid for $\mathcal{S}$=ZFC because it turns out to be interpreted as spatiality of a generic tree formal topology which is a ZFC theorem (actually ZF together with the axiom of dependent choices would be enough to validate $\mathrm{BI}^i_{\mathrm{fr}}$).
$\mathrm{CT}^i_{\mathrm{tt}}$ is valid because type-theoretic functions are interpreted as computable functions with a chosen program code.

Note that, to prove the above theorem, it is crucial to have substitution rules in terms derivable in the calculus.

**Corollary 5.6** *The proof-irrelevant realizability interpretation in definition 5.2 can be adapted to interpret the calculus* $\mathrm{mTT}_0$ *extended with* $BI^i_{fr}$ *and* $CT^i_{tt}$.

**Proof.** We interpret the syntax of $\mathrm{mTT}_0$ according to proposition 4.1 in $\mathrm{mTT}_0^{eq}$ which is then interpreted according to the pf-realizability interpretation in definition 5.2. We conclude that it is well defined by proposition 4.1 and theorem 5.5.

# 6 The proof-irrelevant realizability interpretation of the extensional level

Here we show how to lift the proof-irrelevant realizability interpretation $(-)^{\mathcal{I}}$ in definition 5.2 to interpret $\mathrm{emTT}_0$ in the set theory $\mathcal{S}$, where $\mathcal{S}$=ZFC or $\mathcal{S}$=CZF+$\mathrm{BI}_{\mathrm{fr}}$. The pf-realizability interpretation of $\mathrm{emTT}_0$ is defined by first interpreting the $\mathrm{emTT}_0$-syntax via the interpretation of definition 2.1 according to which $\mathrm{emTT}_0$-types are interpreted as extensional dependent types in $\mathrm{mTT}_0$. Then extensional dependent types in $\mathrm{mTT}_0$ are interpreted in $\mathcal{S}$ as suitable families of quotients of the corresponding pf-realizability interpretation of the $\mathrm{mTT}_0$-types representing their supports.

**Def. 6.1** The interpretation of type judgements of $\mathrm{emTT}_0$

$$(-)^q : \mathrm{emTT}_0 \ \rightarrow \ \mathcal{S}$$

is defined as follows by using the interpretations $(-)^{\mathcal{I}}$ in definition 5.2 and $(-)^i$ in definition 2.1:
A context $\Gamma$ is interpreted by induction by means of disjoint unions as follows:

$$(\emptyset \ cont)^q \equiv \{0\} \qquad (\Gamma, x \in A \ cont)^q \equiv \biguplus_{z\varepsilon\Gamma^q} A^q(z)$$

An dependent type judgement is interpreted as a family of quotients

$$(\ B \ type \ [\Gamma]\ )^q \equiv (\ \mathcal{Q}(\ B^i(x)^{\mathcal{I}}/(=_{B^i})^{\mathcal{I}}\ )\ )_{z\varepsilon\Gamma^q}$$

where with the notation $\mathcal{Q}(\ B^i(x)^{\mathcal{I}}/(=_{B^i})^{\mathcal{I}}\ )$ we mean the lifting of the family $(\ B^i(x)^{\mathcal{I}}/(=_{B^i})^{\mathcal{I}}\ )_{x\varepsilon(\Gamma^i)^{\mathcal{I}}}$ on the disjoint union of quotients $\Gamma^q$.

In particular, if $\Gamma$ is the empty context, then $(\ B \ type \ [\,]\ )^q \equiv (B^i)^{\mathcal{I}}/(=_{B^i})^{\mathcal{I}}$ is the quotient of $(B^i)^{\mathcal{I}}$ over the equivalence relation $(=_{B^i})^{\mathcal{I}}$.
A type equality judgement is interpreted as the extensional equality between set/class families

$$(\ B = C \ type \ [\Gamma]\ )^q \equiv \forall_{z\varepsilon\Gamma^q} \quad B^q(z) = C^q(z)$$

A dependent term judgement is interpreted as a function

$$(\ b \in B \ [\Gamma]\ )^q \equiv z\varepsilon\Gamma^q \ \mapsto \mathcal{Q}([(b^i)^{\mathcal{I}}(w)])\varepsilon B^q(z)$$

where $\mathcal{Q}([(b^i)^{\mathcal{I}}(w)])$ is the unique function induced from $[(b^i)^{\mathcal{I}}(w)]$ for $w\varepsilon(\Gamma^i)^{\mathcal{I}}$ on the disjoint union of quotients $\Gamma^q$.

A term equality judgement is interpreted as the extensional equality of functions:

$$( b = c \in B \; [\Gamma] \,)^q \;\equiv\; \forall_z \varepsilon \Gamma^q \quad b^q(z) = c^q(z)$$

Note that the interpretation is obviously well defined when $\mathcal{S} = ZFC$. When $\mathcal{S} = CZF + \mathrm{BI}_{\mathrm{fr}}$, it is also well defined after observing that the equivalence relations used by the interpretation $(-)^i$ on collections in $\mathrm{mTT}_0$ are built up on set-theoretic equivalence relations possibly with the equality of subsets.

**Theorem 6.2 (extensional pf-realizability validity)** *The calculus* $\mathrm{emTT}_0$ *is validated by the pf-realizability interpretation* $(-)^q$ *of definition 6.1 for* $\mathcal{S} = ZF$ *or* $\mathcal{S} = CZF$.

*Moreover, the interpretation* $(-)^q$ *also validates* $CT_{\mathrm{tt}}$.

*If* $\mathcal{S}$ *is the classical set theory* ZFC *or* CZF+$\mathrm{BI}_{\mathrm{fr}}$, *then the interpretation* $(-)^q$ *validates* $BI_{fr}$ *provided that* $\lhd_{A^N}$, *for any* $\mathrm{emTT}_0$*-set* $A$, *is interpreted as the corresponding* $\mathcal{S}$*-tree formal topology over* $A^q$.
*If* $\mathcal{S}$ *is the classical set theory* ZF, *then the interpretation* $(-)^q$ *validates the principle of excluded middle* EM, *written* $P \vee \neg P$ *for any proposition* $P$.

Thanks to this proof-irrelevant realizability interpretation we can show that choice sequences both of Cantor space and of Baire space, as well as real numbers, both as Dedekind cuts and as Cauchy sequences à la Bishop, do not form a set in $\mathrm{emTT}_0$.

To this purpose we recall from [NS99] that Dedekind cuts can be equivalently expresses as formal points of the topology of real numbers, whose definition can be expressed in $\mathrm{emTT}_0$ as follows:

**Def. 6.3 (Formal topology of real numbers)** *The* formal topology of real numbers $(\mathbb{Q} \times \mathbb{Q}, \lhd_{\mathcal{R}})$ *is an inductively generated formal topology defined as follows. The base is* $\mathbb{Q} \times \mathbb{Q}$ *and the basic neighbourhoods are pairs of rational numbers,* $\langle p, q\rangle$ *with* $p, q \in \mathbb{Q}$. *A preorder on* $\mathbb{Q} \times \mathbb{Q}$ *is defined as follows*

$$\langle p, q\rangle \leq \langle p', q'\rangle \;\equiv\; p' \leq p \leq q \leq q'$$

*for* $p, q, p', q'$ *in* $\mathbb{Q}$. *The cover is defined inductively by the following rules (which are a formulation in our context of Joyal axioms, cf. [Joh82], pp. 123-124):*

$$\frac{q \leq p}{\langle p, q\rangle \lhd_{\mathcal{R}} U} \qquad \frac{\langle p, q\rangle \in U}{\langle p, q\rangle \lhd_{\mathcal{R}} U} \qquad \frac{p' \leq p < q \leq q' \qquad \langle p', q'\rangle \lhd_{\mathcal{R}} U}{\langle p, q\rangle \lhd_{\mathcal{R}} U}$$

$$\frac{p \leq r < s \leq q \qquad \langle p, s\rangle \lhd_{\mathcal{R}} U \quad \langle r, q\rangle \lhd_{\mathcal{R}} U}{\langle p, q\rangle \lhd_{\mathcal{R}} U} \qquad \textsf{wc}\ \frac{wc(\langle p, q\rangle) \lhd_{\mathcal{R}} U}{\langle p, q\rangle \lhd_{\mathcal{R}} U}$$

*where in the last axiom we have used the abbreviation*

$$wc(\langle p, q\rangle) \equiv \{ \, \langle p', q'\rangle \in \mathbb{Q} \times \mathbb{Q} \;\; | \;\; p < p' < q' < q \}$$

*(wc stands for 'well-covered').*

As shown in [NS99], also in $\mathrm{emTT}_0$ we can prove that formal points of the formal topology $\lhd_{\mathcal{R}}$ are in bijection with Dedekind cuts:

**Proposition 6.4** *In* $\mathrm{emTT}+\lhd_{\mathcal{R}}$, *formal points of the inductively generated formal topology* $\mathcal{R}$ *are in bijection with the collection of Dedekind cuts on the rationals.*

Given a formal point $\alpha \in Pt(\mathcal{R})$ we can build the following Dedekind cut:

$$L_\alpha \;\equiv\; \{ \, p \in \mathbb{Q} \;\; | \;\; \langle p, q\rangle \,\epsilon\, \alpha \, \} \qquad\qquad U_\alpha \;\equiv\; \{ \, q \in \mathbb{Q} \;\; | \;\; \langle p, q\rangle \,\epsilon\, \alpha \, \}$$ [8]

Conversely, given a Dedekind cut $(L, U)$ we can define the following formal point

$$\alpha_{(L,U)} \;\equiv\; \{ \; \langle p, q\rangle \in \mathbb{Q} \times \mathbb{Q} \;\; | \;\; p \,\epsilon\, L \;\&\; q \,\epsilon\, U \; \}$$

[8]Note that the base of our topology $\mathcal{R}$ does not contain $+\infty, -\infty$ as that in [NS99].

In [NS99] it is proved that formal points of $\lhd_{\mathcal{R}}$, or Dedekind cuts, are also in bijective correspondence with Cauchy sequences à la Bishop [Bis67] of which we recall the definition:

**Def. 6.5 (Cauchy sequence à la Bishop)** *A function $R(x,y)$ $prop_s$ $[x \in N, y \in \mathbb{Q}]$, indicated with the usual notation $(x_n)_{x\in \mathrm{N}}$, is a* Cauchy sequence *in* emTT *if we can prove for any $n, m \in N^+$*

$$| x_n - x_m | \leq 1/n + 1/m^9$$

*where $N^+$ denotes the set of positive natural numbers $\Sigma_{x\in N}\ x \leq 1$.*

However, in $\mathrm{emTT}_0$ only Cauchy sequences à la Bishop can be shown to be formal points of $\lhd_{\mathcal{R}}$ as in [NS99], but not the converse because formal points are shown to give rise to Cauchy sequences by using a choice principle that in $\mathrm{emTT}_0$ is not available.

**Corollary 6.6 (choice sequences as proper collections)** *In the calculus* $\mathrm{emTT}_0 + \lhd_{A^N} + \lhd_{\mathcal{R}}$ *choice sequences of Cantor space $Pt(\lhd_{\{0,1\}^N})$, those of Baire space $Pt(\lhd_{N^N})$, and real numbers both as Dedekind cuts or Cauchy sequences à la Bishop do not form a set but proper collections.*

**Proof.** In ZFC according to the proof-irrelevant realizability interpretation the mentioned collections are interpreted as quotients of non-countable sets while all $\mathrm{emTT}_0$-sets are interpreted as ZFC-quotients of subsets of natural numbers and hence as countable ZFC-sets. Indeed, choice sequences of Baire (Cantor) space are interpreted as ZFC-choice sequences of Baire (Cantor) space, as well as, real numbers both as Dedekind cuts or Cauchy sequences are interpreted as ZFC-real numbers, provided that $\lhd_{\mathcal{R}}$ gets interpreted as the ZFC-formal topology of real numbers.

This result makes more evident the non-validity of unique choice in $\mathrm{emTT}_0$, because lawlike sequences of Cantor or Baire spaces form a set, while generic choice sequences do not. Also for real numbers, Cauchy sequences à la Bishop defined by type-theoretic functions $f \in N^+ \to \mathbb{Q}$, which we can call *lawlike Cauchy sequences à la Bishop*, form a set, but generic Cauchy sequences do not.

**Remark 6.7 (Predicativity of the proof-irrelevant realizability interpretation.)** If we add the principle of excluded middle EM to $\mathrm{emTT}_0$ we do not get that the power-collection of small propositions becomes a set as it happens when it is added to CZF [Acz78]. Indeed the extension CZF+EM is equal to the classical set theory ZF. In particular, when EM is added to CZF, all the subsets of a set get identified with their characteristic boolean functions which form a set by exponentiation. Even in $\mathrm{emTT}_0$ + EM a subset defined by a propositional function $\phi(x)$ $prop_s$ $[x \in A]$ corresponds to a functional relation in the boolean set $\{0,1\}$ given by $R(x,y) \equiv (\, y = 1 \wedge \phi(x)\,) \vee (\, y = 0 \wedge \neg\phi(x)\,)$ for $x \in A$, in a bijective way. But the collection of such functional relations do not necessarily form a set. Only those subsets enjoying a type-theoretic characteristics functions in the boolean set $\{0,1\}$ form a set. The lack of the validity of the axiom of unique choice prevents from identifying the collection of boolean functional relations on $A \equiv \mathrm{N}$ with the corresponding set of type-theoretic characteristics functions. This is more evident in the pf-realizability interpretation because the type theoretic functions from N to the boolean sets are identified with the recursives ones.

**Remark 6.8 (Non validity of unique choice)** Note that, a direct proof that $\mathrm{AC!}_{\mathrm{N,N}}$ is not valid in $\mathrm{mTT}_0^{eq}$ and hence in $\mathrm{mTT}_0$ and $\mathrm{emTT}_0$, can be obtained from th. 5.5 and interpretation $(-)^i$ in section 2 with arguments like those used here where we replace $\mathrm{BI}_{\mathrm{fr}}$ (and $\mathrm{BI}^i_{\mathrm{fr}}$) with the principle of excluded middle EM. This is because $\mathrm{emTT}_0 + \mathrm{CT}_{\mathrm{tt}} + \mathrm{AC!}_{\mathrm{N,N}}$ is inconsistent with EM, thanks to a proof similar to that in prop.0.1 in [MS05] provided that one starts with the formula

$$\forall x \in \mathrm{N}\ \exists! y \in \mathrm{N}\ (\,(\, y = 1 \wedge P(x)\,) \ \vee\ (\, y = 0 \wedge \neg P(x)\,)\,)$$

[9] This is formally written as $\forall p \in \mathbb{Q}\ \forall q \in \mathbb{Q}\ (\ R(n,p)\ \&\ R(m,q)\ \to |\ q - p\ | \leq 1/n + 1/m\ )$ where the definition of module is the usual one.

**Remark 6.9 (Compatibility with Markov principle)** Note from th. 6.2 when $\mathcal{S}$ is ZFC we deduce that $\mathrm{emTT}_0$ is consistent with $\mathrm{BI}_{\mathrm{fr}}$ and $\mathrm{CT}_{\mathrm{tt}}$ and Markov principle

$$\forall x \in \mathrm{N}\ (\, P(x)\ \vee\ \neg P(x)\,)\ \ \rightarrow\ \ (\,\neg\neg\exists y \in \mathrm{N}\ P(x)\ \rightarrow\ \exists y \in \mathrm{N}\ P(x)\,)$$

given that the pf-realizability interpretation for $\mathrm{mTT}^{eq}$ validates the law of excluded middle and hence also Markov principle.

This setting may provide a way to reconciliate Brouwer's intuitionism with Markov's mathematics, as soon as one drops the axiom of unique choice.

# 7 Conclusions: three extensions of our foundation

Thanks to the rereading of Kleene's result in our foundation $\mathrm{emTT}_0$ as stated in cor. 3.21, $\mathrm{emTT}_0$ + $\mathrm{BI}_{\mathrm{fr}}$ + $\mathrm{CT}_{\mathrm{tt}}$ + $\mathrm{AC!}_{\mathrm{N,N}}$ is inconsistent. However any combination of two such principles seems to be consistent with $\mathrm{emTT}_0$, and it gives a specific behavior of choice sequences defined as functional relations between natural numbers.
Here, we list the various cases briefly:

1. $\mathrm{emTT}_0$ **+ $\mathbf{BI}_{\mathbf{fr}}$ + $\mathbf{CT}_{\mathrm{tt}}$** : in this theory type-theoretic functions between natural numbers are recursive thanks to $\mathrm{CT}_{\mathrm{tt}}$, but can not be identified with choice sequences on natural numbers given that $\mathrm{AC!}_{\mathrm{N,N}}$ can not hold; a *model* for this theory is described in this paper and provides an interpretation *in* ZFC or *in* CZF+$\mathrm{BI}_{\mathrm{fr}}$ which preserves the computable meaning of typed terms.

2. $\mathrm{emTT}_0$ **+ $\mathbf{BI}_{\mathbf{fr}}$ + $\mathbf{AC!}_{\mathbf{N,N}}$**: in this theory choice sequences on natural numbers correspond to type-theoretic functions, but such type-theoretic functions can not be internally recursive given that $\mathrm{CT}_{\mathrm{tt}}$ can not hold; a *model* for this theory is the intuitive interpretation of sets and collections (with their elements) as the corresponding sets in ZFC, or sets/collections in CZF, and propositions as subsets of the singleton set in ZFC, and subcollections/subsets in CZF; the model provides a direct interpretation *in* ZFC or *in* CZF+$\mathrm{BI}_{\mathrm{fr}}$, but this does not preserve the computable meaning of typed terms.

3. $\mathrm{emTT}_0$ **+ $\mathbf{CT}_{\mathrm{tt}}$ + $\mathbf{AC!}_{\mathbf{N,N}}$**: in this theory choice sequences on natural numbers are identified with type-theoretic functions, which are also internally recursive but $\mathrm{BI}_{\mathrm{fr}}$, as well as excluded middle, is not valid; a *model* for this theory is current work in progress and it will be defined as the quotient completion of the model for $\mathrm{mTT}_0$ extending the original Kleene's realizability interpretation [Tv88a] (it is straightforward to see that it validates the set-theoretic part of $\mathrm{mTT}_0$) and serving as the *proofs-as-programs semantics* of $\mathrm{mTT}_0$ for program extraction.

In the future we intend to investigate how to extend the realizability interpretation shown here for the whole original intensional theory mTT in [Mai09]. A model for just showing a classical consistency of the theory with $\mathrm{BI}^i_{\mathrm{fr}}$+ $\mathrm{CT}^i_{\mathrm{tt}}$ could be derived from that for the Calculus of Constructions [Coq90] in [Str92], which disproves the axiom of unique choice and validates the principle of excluded middle. Then, it would be worthwhile to investigate whether this model can provide an intuitive predicative interpretation validating $\mathrm{BI}^i_{\mathrm{fr}}$+ $\mathrm{CT}^i_{\mathrm{tt}}$ as the one shown here when based on constructive foundations as CZF.

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# 8 Appendix: The intensional level $\mathrm{mTT}_0$

As mTT in [Mai09], the inference rules of $\mathrm{mTT}_0$ involve judgements written in the style of Martin-Löf's type theory [Mar84, NPS90] that may be of the form:

$$A\ type\ [\Gamma] \quad A = B\ type\ [\Gamma] \quad a \in A\ [\Gamma] \quad a = b \in A\ [\Gamma]$$

where types include collections, sets, propositions and small propositions, namely

$$type \in \{col, set, prop, prop_s\}$$

For easinesss, the piece of context common to all judgements involved in a rule is omitted and typed variables appearing in a context are meant to be added to the implicit context as the last one.
Note that to write the elimination constructors of our types we adopt the higher-order syntax in [NPS90] (see also [Gui09]). According to this syntax the open term $a_B(x) \in A\ [x \in B]$ yields to $(x \in B)\, a_B(x)$ of higher type $(x \in B)\, A$. Then, by $\eta$-conversion among higher types, it follows that $(x \in B)\, a_B(x)$ is equal to $a_B$. Hence, we often simply write the short expression $a_B$ to recall the open term where it comes from..
We also have a form of judgement to build contexts:

$$\Gamma\ cont$$

whose rules are the following

$$\emptyset\ cont \qquad\qquad \text{F-c}\frac{A\ type\ [\Gamma]}{\Gamma, x \in A\ cont}\ (x \in A \notin \Gamma)$$

Then, the first rule to build elements of type is the assumption of variables:

$$\text{var)}\ \frac{\Gamma, x \in A, \Delta \quad cont}{x \in A\ [\Gamma, x \in A, \Delta]}$$

Among types there are the following embeddings: sets are collections and propositions are collections

$$\textbf{set-into-col)}\ \frac{A\ set}{A\ col} \qquad\qquad \textbf{prop-into-col)}\ \frac{A\ prop}{A\ col}$$

**Strong Indexed Sum of a propositional function**

$$\text{F-ip)}\ \frac{C(x)\ \ prop\ [x \in B]}{\Sigma_{x\in B} C(x)\ col} \qquad \text{I-ip)}\ \frac{b \in B \quad d \in C(b) \quad C(x)\ \ prop\ [x \in B]}{\langle b, d\rangle \in \Sigma_{x\in B} C(x)}$$

$$\text{E-ip)}\ \frac{\begin{array}{l} M(z)\ col\ [z \in \Sigma_{x\in B} C(x)] \\ d \in \Sigma_{x\in B} C(x) \quad m(x,y) \in M(\langle x, y\rangle)\ [x \in B, y \in C(x)] \end{array}}{El_\Sigma(d, m) \in M(d)}$$

$$\text{C-ip)}\ \frac{\begin{array}{l} M(z)\ col\ [z \in \Sigma_{x\in B} C(x)] \\ b \in B \quad c \in C(b) \quad m(x,y) \in M(\langle x, y\rangle)\ [x \in B, y \in C(x)] \end{array}}{El_\Sigma(\, \langle b, c\rangle, m\,) = m(b, c) \in M(\langle b, c\rangle)}$$

Sets are generated as follows:

**Empty set**

F-Em) $\mathsf{N}_0\ set$ E-Em) $\dfrac{a \in \mathsf{N}_0 \quad A(x)\ col\ [x \in \mathsf{N}_0]}{\mathsf{emp_o}(a) \in A(a)}$

**Singleton**

S) $\mathsf{N}_1\ set$ I-S) $\star \in \mathsf{N}_1$ E-S) $\dfrac{t \in \mathsf{N}_1 \quad M(z)\ col\ [z \in \mathsf{N}_1] \quad c \in M(\star)}{El_{\mathsf{N}_1}(t, c) \in M(t)}$ C-S) $\dfrac{M(z)\ col\ [z \in \mathsf{N}_1] \quad c \in M(\star)}{El_{\mathsf{N}_1}(\star, c) = c \in M(\star)}$

**Strong Indexed Sum set**

F-Σ) $\dfrac{C(x)\ set\ \ [x \in B] \qquad B\ set}{\Sigma_{x\in B}C(x)\ set}$ I-Σ) $\dfrac{b \in B \quad c \in C(b) \qquad C(x)\ set\ \ [x \in B] \qquad B\ set}{\langle b, c\rangle \in \Sigma_{x\in B}C(x)}$

E-Σ) $\dfrac{\begin{array}{l} M(z)\ col\ [z \in \Sigma_{x\in B}C(x)] \\ d \in \Sigma_{x\in B}C(x) \quad m(x,y) \in M(\langle x, y\rangle)\ [x \in B, y \in C(x)]\end{array}}{El_{\Sigma}(d, m) \in M(d)}$

C-Σ) $\dfrac{\begin{array}{l} M(z)\ col\ [z \in \Sigma_{x\in B}C(x)] \\ b \in B \quad c \in C(b) \quad m(x,y) \in M(\langle x, y\rangle)\ [x \in B, y \in C(x)]\end{array}}{El_{\Sigma}(\langle b, c\rangle, m) = m(b, c) \in M(\langle b, c\rangle)}$

**List set**

F-list) $\dfrac{C\ set}{List(C)\ set}$ I$_1$-list) $\dfrac{List(C)\ set}{\epsilon \in List(C)}$ I$_2$-list) $\dfrac{s \in List(C) \quad c \in C}{\mathsf{cons}(s, c) \in List(C)}$

E-list) $\dfrac{\begin{array}{l} L(z)\ col\ [z \in List(C)] \quad s \in List(C) \qquad a \in L(\epsilon) \\ l(x,y,z) \in L(\mathsf{cons}(x,y))\ [x \in List(C), y \in C, z \in L(x)]\end{array}}{El_{List}(s, a, l) \in L(s)}$

C$_1$-list) $\dfrac{\begin{array}{l} L(z)\ col\ [z \in List(C)] \qquad a \in L(\epsilon) \\ l(x,y,z) \in L(\mathsf{cons}(x,y))\ [x \in List(C), y \in C, z \in L(x)]\end{array}}{El_{List}(\epsilon, a, l) = a \in L(\epsilon)}$

C$_2$-list) $\dfrac{\begin{array}{l} L(z)\ col\ [z \in List(C)] \quad s \in List(C) \quad c \in C \quad a \in L(\epsilon) \\ l(x,y,z) \in L(\mathsf{cons}(x,y))\ [x \in List(C), y \in C, z \in L(x)]\end{array}}{El_{List}(\mathsf{cons}(s, c), a, l) = l(s, c, El_{List}(s, a, l)) \in L(\mathsf{cons}(s, c))}$

**Disjoint Sum set**

F-+) $\dfrac{B\ set \quad C\ set}{B + C\ set}$ I$_1$-+) $\dfrac{b \in B \qquad B\ set \qquad C\ set}{\mathsf{inl}(b) \in B + C}$ I$_2$-+) $\dfrac{c \in C \qquad B\ set \qquad C\ set}{\mathsf{inr}(c) \in B + C}$

E-+) $\dfrac{\begin{array}{l} A(z)\ col\ [z \in B + C] \\ w \in B + C \quad a_B(x) \in A(\mathsf{inl}(x))\ [x \in B] \quad a_C(y) \in A(\mathsf{inr}(y))\ [y \in C]\end{array}}{El_{+}(w, a_B, a_C) \in A(w)}$

C$_1$-+) $\dfrac{\begin{array}{l} A(z)\ col\ [z \in B + C] \\ b \in B \quad a_B(x) \in A(\mathsf{inl}(x))\ [x \in B] \quad a_C(y) \in A(\mathsf{inr}(y))\ [y \in C]\end{array}}{El_{+}(\mathsf{inl}(b), a_B, a_C) = a_B(b) \in A(\mathsf{inl}(c))}$

C$_2$-+) $\dfrac{\begin{array}{l} A(z)\ col\ [z \in B + C] \\ c \in C \quad a_B(x) \in A(\mathsf{inl}(x))\ [x \in B] \quad a_C(y) \in A(\mathsf{inr}(y))\ [y \in C]\end{array}}{El_{+}(\mathsf{inr}(c), a_B, a_C) = a_C(c) \in A(\mathsf{inr}(c))}$

**Dependent Product set**

F-Π) $\dfrac{C(x)\ set\ [x \in B] \qquad B\ set}{\Pi_{x\in B} C(x)\ set}$ I-Π) $\dfrac{c(x) \in C(x)\ [x \in B] \qquad C(x)\ set\ [x \in B] \qquad B\ set}{\lambda x^B.c(x) \in \Pi_{x\in B} C(x)}$

E-Π) $\dfrac{b \in B \quad f \in \Pi_{x\in B} C(x)}{\mathsf{Ap}(f, b) \in C(b)}$

βC-Π) $\dfrac{b \in B \quad c(x) \in C(x)\ [x \in B] \qquad C(x)\ set\ [x \in B] \qquad B\ set}{\mathsf{Ap}(\lambda x^B.c(x), b) = c(b) \in C(b)}$

Propositions are generated as follows:

**Falsum**

F-Fs) $\perp prop$ E-Fs) $\dfrac{a \in \perp \quad A\ prop}{\mathsf{r_o}(a) \in A}$

**Disjunction**

F-∨) $\dfrac{B\ prop \quad C\ prop}{B \vee C\ prop}$ $\mathrm{I}_1$-∨) $\dfrac{b \in B \qquad B\ prop \qquad C\ prop}{\mathsf{inl}_\vee(b) \in B \vee C}$ $\mathrm{I}_2$-∨) $\dfrac{c \in C \qquad B\ prop \qquad C\ prop}{\mathsf{inr}_\vee(c) \in B \vee C}$

E-∨) $\dfrac{\begin{array}{l} A\ prop \\ w \in B \vee C \quad a_B(x) \in A\ [x \in B] \quad a_C(y) \in A\ [y \in C] \end{array}}{El_\vee(w, a_B, a_C) \in A}$

$\mathrm{C}_1$-∨) $\dfrac{\begin{array}{l} A\ prop \qquad B\ prop \qquad C\ prop \\ b \in B \quad a_B(x) \in A\ [x \in B] \quad a_C(y) \in A\ [y \in C] \end{array}}{El_\vee(\mathsf{inl}_\vee(b), a_B, a_C) = a_B(b) \in A}$

$\mathrm{C}_2$-∨) $\dfrac{\begin{array}{l} A\ prop \qquad B\ prop \qquad C\ prop \\ c \in C \quad a_B(x) \in A\ [x \in B] \quad a_C(y) \in A\ [y \in C] \end{array}}{El_\vee(\mathsf{inr}_\vee(c), a_B, a_C) = a_C(c) \in A}$

**Conjunction**

F-∧) $\dfrac{B\ prop \qquad C\ prop}{B \wedge C\ prop}$ I-∧) $\dfrac{b \in B \quad c \in C \qquad B\ prop \qquad C\ prop}{\langle b,_\wedge c\rangle \in B \wedge C}$

$\mathrm{E}_1$-∧) $\dfrac{d \in B \wedge C}{\pi_1^B(d) \in B}$ $\mathrm{E}_2$-∧) $\dfrac{d \in B \wedge C}{\pi_2^C(d) \in C}$

$\beta_1$ C-∧) $\dfrac{b \in B \quad c \in C \qquad B\ prop \qquad C\ prop}{\pi_1^B(\langle b,_\wedge c\rangle) = b \in B}$ $\beta_2$ C-∧) $\dfrac{b \in B \quad c \in C \qquad B\ prop \qquad C\ prop}{\pi_2^C(\langle b,_\wedge c\rangle) = c \in C}$

**Implication**

F-→) $\dfrac{B\ prop \qquad C\ prop}{B \to C\ prop}$

I-→) $\dfrac{c(x) \in C\ [x \in B] \qquad B\ prop \qquad C\ prop}{\lambda_\to x^B.c(x) \in B \to C}$ E-→) $\dfrac{b \in B \quad f \in B \to C}{\mathsf{Ap}_\to(f, b) \in C}$

βC-→) $\dfrac{b \in B \quad c(x) \in C\ [x \in B] \qquad B\ prop \qquad C\ prop}{\mathsf{Ap}_\to(\lambda_\to x^B.c(x), b) = c(b) \in C}$

**Existential quantification**

F-∃) $\dfrac{C(x)\ prop\ \ [x \in B]}{\exists_{x\in B} C(x)\ prop}$ I-∃) $\dfrac{b \in B \quad c \in C(b) \qquad C(x)\ prop\ [x \in B]}{\langle b,_{\exists} c\rangle \in \exists_{x\in B} C(x)}$

E-∃) $\dfrac{\begin{array}{l} M\ prop \\ d \in \exists_{x\in B} C(x) \quad m(x,y) \in M\ [x \in B, y \in C(x)] \end{array}}{El_{\exists}(d,m) \in M}$

C-∃) $\dfrac{\begin{array}{l} M\ prop \qquad C(x)\ prop\ [x \in B] \\ b \in B \quad c \in C(b) \quad m(x,y) \in M\ [x \in B, y \in C(x)] \end{array}}{El_{\exists}(\langle b,_{\exists} c\rangle, m) = m(b,c) \in M}$

**Universal quantification**

F-∀) $\dfrac{C(x)\ prop\ [x \in B]}{\forall_{x\in B} C(x)\ prop}$ I-∀) $\dfrac{c(x) \in C(x)\ [x \in B] \qquad C(x)\ prop\ [x \in B]}{\lambda_{\forall} x^{B}.c(x) \in \forall_{x\in B} C(x)}$

E-∀) $\dfrac{b \in B \quad f \in \forall_{x\in B} C(x)}{\mathsf{Ap}_{\forall}(f,b) \in C(b)}$ βC-∀) $\dfrac{b \in B \quad c(x) \in C(x)\ [x \in B] \qquad C(x)\ prop\ [x \in B]}{\mathsf{Ap}_{\forall}(\lambda_{\forall} x^{B}.c(x), b) = c(b) \in C(b)}$

**Propositional Equality**

F-Id) $\dfrac{A\ col \quad a \in A \quad b \in A}{\mathsf{Id}(A,a,b)\ prop}$ I-Id) $\dfrac{a \in A}{\mathsf{id_A}(a) \in \mathsf{Id}(A,a,a)}$

E-Id) $\dfrac{\begin{array}{l} C(x,y)\ prop\ [x : A, y \in A] \\ a \in A \quad b \in A \quad p \in \mathsf{Id}(A,a,b) \quad c(x) \in C(x,x)\ [x \in A] \end{array}}{El_{\mathsf{Id}}(p,(x)c(x)) \in C(a,b)}$

C-Id) $\dfrac{\begin{array}{l} C(x,y)\ prop\ [x : A, y \in A] \\ a \in A \quad c(x) \in C(x,x)\ [x \in A] \end{array}}{El_{\mathsf{Id}}(\mathsf{id}_A(a),(x)c(x)) = c(a) \in C(a,a)}$

Then, small propositions are generated as follows:

$\bot\ prop_s$ $\dfrac{B\ prop_s \quad C\ prop_s}{B \vee C\ prop_s}$ $\dfrac{B\ prop_s \qquad C\ prop_s}{B \to C\ prop_s}$ $\dfrac{B\ prop_s \qquad C\ prop_s}{B \wedge C\ prop_s}$

$\dfrac{C(x)\ prop_s\ \ [x \in B] \qquad B\ set}{\exists_{x\in B} C(x)\ prop_s}$ $\dfrac{C(x)\ prop_s\ [x \in B] \qquad B\ set}{\forall_{x\in B} C(x)\ prop_s}$ $\dfrac{A\ set \quad a \in A \quad b \in A}{\mathsf{Id}(A,a,b)\ prop_s}$

And we add rules saying that a small proposition is a proposition and that a small proposition is a set:

**prop$_s$-into-prop)** $\dfrac{A\ prop_s}{A\ prop}$ **prop$_s$-into-set)** $\dfrac{A\ prop_s}{A\ set}$

Then, we also have the collection of small propositions and function collections from a set toward it:

**Collection of small propositions**

F-Pr) $\mathsf{props_s}\ col$ I-Pr) $\dfrac{B\ prop_s}{B \in \mathsf{props_s}}$ E-Pr) $\dfrac{B \in \mathsf{props_s}}{B\ prop_s}$

**Function collection to $\mathsf{props_s}$**

F-Fun) $\dfrac{B\ set}{B \to \mathsf{props_s}\ col}$ I-Fun) $\dfrac{c(x) \in \mathsf{props_s}\ [x \in B] \qquad B\ set}{\lambda x^B.c(x) \in B \to \mathsf{props_s}}$

E-Fun) $\dfrac{b \in B \quad f \in B \to \mathsf{props_s}}{\mathsf{Ap}(f,b) \in \mathsf{props_s}}$ $\beta$C-Fun) $\dfrac{b \in B \quad c(x) \in \mathsf{props_s}\ [x \in B] \qquad B\ set}{\mathsf{Ap}(\lambda x^B.c(x), b) = c(b) \in \mathsf{props_s}}$

Equality rules include those saying that type equality is an equivalence relation and substitution of equal terms in a type:

ref) $\dfrac{A\ type}{A = A\ type}$ sym) $\dfrac{A = B\ type}{B = A\ type}$ tra) $\dfrac{A = B\ type \quad B = C\ type}{A = C\ type}$

subT) $\dfrac{\begin{array}{c} C(x_1, \ldots, x_n)\ type\ [\, x_1 \in A_1,\ \ldots,\ x_n \in A_n(x_1, \ldots, x_{n-1})\,] \\ a_1 = b_1 \in A_1\ \ldots\ a_n = b_n \in A_n(a_1, \ldots, a_{n-1}) \end{array}}{C(a_1, \ldots, a_n) = C(b_1, \ldots, b_n)\ type}$

conv) $\dfrac{a \in A \qquad A = B\ type}{a \in B}$ conv-eq) $\dfrac{a = b \in A \qquad A = B\ type}{a = b \in B}$

# 9 Appendix: The intermediate typed calculus $\mathrm{mTT}_0^{eq}$

The typed calculus $\mathrm{mTT}^{eq}$ is an extension of mTT where the propositional equality $\mathsf{Id}$ is replaced by the extensional equality $\mathsf{Eq}$ defined as follows:

**Extensional Propositional Equality**

I-Eq) $\dfrac{C\ \ col \quad c \in C \quad d \in C}{\mathsf{Eq}(C, c, d)\ prop}$ I-Eq) $\dfrac{c \in C}{\mathsf{eq}(c) \in \mathsf{Eq}(C, c, c)}$

E-Eq) $\dfrac{p \in \mathsf{Eq}(C, c, d)}{c = d \in C}$ C-Eq) $\dfrac{p \in \mathsf{Eq}(C, c, d)}{p = \mathsf{eq_C}(c) \in \mathsf{Eq}(C, c, d)}$

Then the rules for indexed sums on collections and sets are the following:

**Strong Indexed Sum of a propositional function**

F-ip) $\dfrac{C(x)\ \ prop\ [x \in B]}{\Sigma_{x \in B} C(x)\ col}$ I-ip) $\dfrac{b \in B \quad c \in C(b) \qquad C(x)\ prop\ [x \in B]}{\langle b, c \rangle \in \Sigma_{x \in B} C(x)}$

$\mathrm{E}_1$-ip) $\dfrac{d \in \Sigma_{x \in B} C(x)}{\pi_1(d) \in B}$ $\mathrm{E}_2$-ip) $\dfrac{d \in \Sigma_{x \in B} C(x)}{\pi_2(d) \in C(\pi_1(d))}$

$\mathrm{C}_1$-ip) $\dfrac{b \in B \quad c \in C(b) \qquad C(x)\ prop\ \ [x \in B]}{\pi_1(\, \langle b, c \rangle\,) = b \in B}$ $\mathrm{C}_2$-ip) $\dfrac{b \in B \quad c \in C(b) \qquad C(x)\ prop\ \ [x \in B]}{\pi_2(\, \langle b, c \rangle\,) = c \in C(b)}$

$\eta$-ip) $\dfrac{d \in \Sigma_{x \in B} C(x)}{\langle \pi_1(\,d\,)\,,\, \pi_2(\,d\,)\,\rangle = d \in \Sigma_{x \in B} C(x)}$

**Strong Indexed Sum set**

F-Σ) $\dfrac{C(x)\ \ set\ [x \in B]}{\Sigma_{x\in B}C(x)\ set}$ I-Σ) $\dfrac{b \in B \quad c \in C(b) \qquad B\ set \qquad C(x)\ set\ [x \in B]}{\langle b, c\rangle \in \Sigma_{x\in B}C(x)}$

E$_1$-Σ) $\dfrac{d \in \Sigma_{x\in B}C(x)}{\pi_1(d) \in B}$ E$_2$-Σ) $\dfrac{d \in \Sigma_{x\in B}C(x)}{\pi_2(d) \in C(\pi_1(d))}$

C$_1$-Σ) $\dfrac{b \in B \quad c \in C(b) \qquad B\ set \qquad C(x)\ set\ [x \in B]}{\pi_1(\ \langle b, c\rangle\ ) = b \in B}$

C$_2$-Σ) $\dfrac{b \in B \quad c \in C(b) \qquad B\ set \qquad C(x)\ set\ [x \in B]}{\pi_2(\ \langle b, c\rangle\ ) = c \in C(b)}$

η-Σ) $\dfrac{d \in \Sigma_{x\in B}C(x)}{\langle \pi_1(\,d\,)\,,\ \pi_2(\,d\,)\,\rangle = d \in \Sigma_{x\in B}C(x)}$

## 10 Appendix: The extensional level emTT$_0$

As emTT to build types and terms of emTT$_0$ we use the same kinds of judgements used in mTT$_0$. Contexts are generated by the same context rules of mTT$_0$.
Also here, the only change we do on emTT$_0$ with respect to emTT is to allow only strong indexed sums of propositional functions as generic collection constructors:

**Strong Indexed Sum of a propositional function**

F-ip) $\dfrac{C(x)\ \ prop\ [x \in B]}{\Sigma_{x\in B}C(x)\ col}$ I-ip) $\dfrac{b \in B \quad c \in C(b) \quad C(x)\ \ prop\ [x \in B]}{\langle b, c\rangle \in \Sigma_{x\in B}C(x)}$

E-*ip*) $\dfrac{\begin{array}{l} M(z)\ col\ [z \in \Sigma_{x\in B}C(x)] \\ d \in \Sigma_{x\in B}C(x) \quad m(x,y) \in M(\langle x, y\rangle)\ [x \in B, y \in C(x)] \end{array}}{El_\Sigma(d, m) \in M(d)}$

C-*ip*) $\dfrac{\begin{array}{l} M(z)\ col\ [z \in \Sigma_{x\in B}C(x)] \qquad C(x)\ prop\ [x \in B] \\ b \in B \quad c \in C(b) \quad m(x,y) \in M(\langle x, y\rangle)\ [x \in B, y \in C(x)] \end{array}}{El_\Sigma(\ \langle b, c\rangle, m\ ) = m(b, c) \in M(\langle b, c\rangle)}$

Sets are generated as follows:

**Empty set**

F-Em) $\mathsf{N}_0\ set$ E-Em) $\dfrac{a \in \mathsf{N}_0 \quad A(x)\ \ col\ [x \in \mathsf{N}_0]}{\mathsf{emp_o}(a) \in A(a)}$

**Singleton set**

S) $\mathsf{N}_1\ set$ I-S) $\star \in \mathsf{N}_1$ E-S) $\dfrac{t \in \mathsf{N}_1 \quad M(z)\ col\ [z \in \mathsf{N}_1] \quad c \in M(\star)}{El_{\mathsf{N}_1}(t, c) \in M(t)}$ C-S) $\dfrac{M(z)\ col\ [z \in \mathsf{N}_1] \quad c \in M(\star)}{El_{\mathsf{N}_1}(\star, c) = c \in M(\star)}$

**Strong Indexed Sum set**

F-Σ) $\dfrac{C(x)\ set\ [x \in B] \qquad B\ set}{\Sigma_{x\in B}C(x)\ set}$ I-Σ) $\dfrac{b \in B \quad c \in C(b) \qquad C(x)\ set\ [x \in B] \qquad B\ set}{\langle b, c\rangle \in \Sigma_{x\in B}C(x)}$

E-Σ) $\dfrac{\begin{array}{l} M(z)\ type\ [z \in \Sigma_{x\in B}C(x)] \\ d \in \Sigma_{x\in B}C(x) \quad m(x,y) \in M(\langle x,y\rangle)\ [x \in B, y \in C(x)] \end{array}}{El_\Sigma(d,m) \in M(d)}$

C-Σ) $\dfrac{\begin{array}{l} M(z)\ type\ [z \in \Sigma_{x\in B}C(x)] \\ b \in B \quad c \in C(b) \quad m(x,y) \in M(\langle x,y\rangle)\ [x \in B, y \in C(x)] \end{array}}{El_\Sigma(\langle b,c\rangle, m) = m(b,c) \in M(\langle b,c\rangle)}$

**List set**

F-list) $\dfrac{C\ set}{List(C)\ set}$ $I_1$-list) $\dfrac{List(C)\ set}{\epsilon \in List(C)}$ $I_2$-list) $\dfrac{s \in List(C) \quad c \in C}{\mathsf{cons}(s,c) \in List(C)}$

E-list) $\dfrac{\begin{array}{c} L(z)\ col\ [z \in List(C)] \quad s \in List(C) \qquad a \in L(\epsilon) \\ l(x,y,z) \in L(\mathsf{cons}(x,y))\ [x \in List(C), y \in C, z \in L(x)] \end{array}}{El_{List}(s,a,l) \in L(s)}$

$C_1$-list) $\dfrac{\begin{array}{c} L(z)\ col\ [z \in List(C)] \qquad a \in L(\epsilon) \\ l(x,y,z) \in L(\mathsf{cons}(x,y))\ [x \in List(C), y \in C, z \in L(x)] \end{array}}{El_{List}(\epsilon,a,l) = a \in L(\epsilon)}$

$C_2$-list) $\dfrac{\begin{array}{c} L(z)\ col\ [z \in List(C)] \quad s \in List(C) \quad c \in C \quad a \in L(\epsilon) \\ l(x,y,z) \in L(\mathsf{cons}(x,y))\ [x \in List(C), y \in C, z \in L(x)] \end{array}}{El_{List}(\mathsf{cons}(s,c),a,l) = l(s,c,El_{List}(s,a,l)) \in L(\mathsf{cons}(s,c))}$

**Disjoint Sum set**

F-+ ) $\dfrac{B\ set \quad C\ set}{B + C\ set}$ $I_1$-+) $\dfrac{b \in B \qquad B\ set \qquad C\ set}{\mathsf{inl}(b) \in B + C}$ $I_2$-+) $\dfrac{c \in C \qquad B\ set \qquad C\ set}{\mathsf{inr}(c) \in B + C}$

E-+) $\dfrac{\begin{array}{l} A(z)\ col\ [z \in B + C] \\ w \in B + C \quad a_B(x) \in A(\mathsf{inl}(x))\ [x \in B] \quad a_C(y) \in A(\mathsf{inr}(y))\ [y \in C] \end{array}}{El_+(w, a_B, a_C) \in A(w)}$

$C_1$-+) $\dfrac{\begin{array}{l} A(z)\ col\ [z \in B + C] \\ b \in B \quad a_B(x) \in A(\mathsf{inl}(x))\ [x \in B] \quad a_C(y) \in A(\mathsf{inr}(y))\ [y \in C] \end{array}}{El_+(\mathsf{inl}(b), a_B, a_C) = a_B(b) \in A(\mathsf{inl}(c))}$

$C_2$-+) $\dfrac{\begin{array}{l} A(z)\ col\ [z \in B + C] \\ c \in C \quad a_B(x) \in A(\mathsf{inl}(x))\ [x \in B] \quad a_C(y) \in A(\mathsf{inr}(y))\ [y \in C] \end{array}}{El_+(\mathsf{inr}(c), a_B, a_C) = a_C(c) \in A(\mathsf{inr}(c))}$

**Dependent Product set**

$$\text{F-}\Pi)\ \frac{C(x)\ set\ [x \in B] \qquad B\ set}{\Pi_{x\in B}C(x)\ set} \qquad \text{I-}\Pi)\ \frac{c(x) \in C(x)\ [x \in B] \qquad C(x)\ set\ [x \in B] \qquad B\ set}{\lambda x^B.c(x) \in \Pi_{x\in B}C(x)}$$

$$\text{E-}\Pi)\ \frac{b \in B \quad f \in \Pi_{x\in B}C(x)}{\mathsf{Ap}(f,b) \in C(b)}$$

$$\beta\text{C-}\Pi)\ \frac{b \in B \quad c(x) \in C(x)\ [x \in B] \qquad C(x)\ set\ [x \in B] \qquad B\ set}{\mathsf{Ap}(\lambda x^B.c(x), b) = c(b) \in C(b)}$$

$$\eta\text{C-}\Pi\ \ \frac{f \in \Pi_{x\in B}C(x)}{\lambda x^B.\mathsf{Ap}(f,x) = f \in \Pi_{x\in B}C(x)}(x\ not\ free\ in\ f)$$

**Quotient set**

$$\text{Q})\ \frac{\begin{array}{ll} A\ set & R(x,y)\ \in \mathsf{props_s}\ [x \in A, y \in A] \\ \mathsf{Equiv}(R) & \begin{array}{l} \mathsf{true} \in R(x,x)\ [x \in A] \\ \mathsf{true} \in R(y,x)\ [x \in A,\ y \in A,\ u \in R(x,y)] \\ \mathsf{true} \in R(x,z)\ [x \in A,\ y \in A,\ z \in A, \\ \qquad\qquad u \in R(x,y),\ v \in R(y,z)] \end{array} \end{array}}{A/R\ set}$$

$$\text{I-Q})\ \frac{a \in A \qquad A/R\ set}{[a] \in A/R} \qquad \text{eq-}Q)\ \frac{a \in A \quad b \in A \quad \mathsf{true} \in R(a,b) \qquad A/R\ set}{[a] = [b] \in A/R}$$

$$\text{E-Q})\ \frac{\begin{array}{l} L(z)\ col\ [z \in A/R] \\ p \in A/R \quad l(x) \in L([x])\ [x \in A] \quad l(x) = l(y) \in L([x])\ [x \in A, y \in A, d \in R(x,y)] \end{array}}{El_Q(p,l) \in L(p)}$$

$$\text{C-Q})\ \frac{\begin{array}{l} L(z)\ col\ [z \in A/R] \\ a \in A \quad l(x) \in L([x])\ [x \in A] \quad l(x) = l(y) \in L([x])\ [x \in A, y \in A, d \in R(x,y)] \end{array}}{El_Q(l,[a]) = l(a) \in L([a])}$$

**Effectiveness**

$$\text{eff})\ \frac{a \in A \quad b \in A \quad [a] = [b] \in A/R \qquad A/R\ set}{\mathsf{true} \in R(a,b)}$$

$\text{emTT}_0$ propositions are mono, namely they are inhabited by at most a canonical proof-term:

$$\textbf{prop-mono})\ \frac{A\ prop \qquad p \in A \quad q \in A}{p = q \in A} \qquad \textbf{prop-true})\ \frac{A\ prop \qquad p \in A}{\mathsf{true} \in A}$$

Propositions are generated as follows:

**Falsum**

$$\text{F-Fs})\ \ \bot\ prop \qquad \text{E-Fs})\ \frac{\mathsf{true} \in \bot \quad A\ prop}{\mathsf{true} \in A}$$

**Extensional Propositional Equality**

$$\text{F-Eq})\ \frac{C\ \ col \quad c \in C \quad d \in C}{\mathsf{Eq}(C,c,d)\ prop} \qquad \text{I-Eq})\ \frac{c \in C}{\mathsf{true} \in \mathsf{Eq}(C,c,c)}$$

$$\text{E-Eq})\ \frac{\mathsf{true} \in \mathsf{Eq}(C,c,d)}{c = d \in C} \qquad \text{C-Eq})\ \frac{p \in \mathsf{Eq}(C,c,d)}{p = \mathsf{eq_C}(c) \in \mathsf{Eq}(C,c,d)}$$

**Implication**

$$\text{F-Im)}\ \frac{B\ prop \quad C\ prop}{B \to C\ prop} \qquad \text{I-Im)}\ \frac{\mathsf{true} \in C\ [x \in B] \qquad B\ prop \qquad C\ prop}{\mathsf{true} \in B \to C}$$

$$\text{E-Im)}\ \frac{\mathsf{true} \in B \quad \mathsf{true} \in B \to C}{\mathsf{true} \in C}$$

$$\beta\text{C-}\!\to\ \frac{B\ prop \qquad b \in B \quad c \in C\ [x \in B]}{\mathsf{Ap}_{\to}(\lambda_{\to} x^{B}.c, b) = c(b) \in C}$$

$$\eta\text{C-}\!\to\ \frac{f \in B \to C}{\lambda_{\to} x^{B}.\mathsf{Ap}_{\to}(f, x) = f}(x\ not\ free\ in\ f)$$

**Conjunction**

$$\text{F-}\wedge)\ \frac{B\ prop \quad C\ prop}{B \wedge C\ prop} \qquad \text{I-}\wedge)\ \frac{\mathsf{true} \in B \quad \mathsf{true} \in C \qquad B\ prop \qquad C\ prop}{\mathsf{true} \in B \wedge C}$$

$$\text{E}_1\text{-}\wedge)\ \frac{\mathsf{true} \in B \wedge C}{\mathsf{true} \in B} \qquad \text{E}_2\text{-}\wedge)\ \frac{\mathsf{true} \in B \wedge C}{\mathsf{true} \in C}$$

**Disjunction**

$$\text{F-}\vee)\ \frac{B\ prop \quad C\ prop}{B \vee C\ prop} \qquad \text{I}_1\text{-}\vee)\ \frac{\mathsf{true} \in B \qquad B\ prop \qquad C\ prop}{\mathsf{true} \in B \vee C} \qquad \text{I}_2\text{-}\vee)\ \frac{\mathsf{true} \in C \qquad B\ prop \qquad C\ prop}{\mathsf{true} \in B \vee C}$$

$$\text{E-}\vee)\ \frac{A\ prop \qquad \mathsf{true} \in B \vee C \quad \mathsf{true} \in A\ [x \in B] \quad \mathsf{true} \in A\ [y \in C]}{\mathsf{true} \in A}$$

$$\text{C}_1\text{-}\vee)\ \frac{\begin{array}{l} A\ prop \\ b \in B \quad a_B(x) \in A\ [x \in B] \quad a_C(y) \in A\ [y \in C] \end{array}}{El_{\vee}(\mathsf{inl}_{\vee}(b), a_B, a_C) = a_B(b) \in A}$$

$$\text{C}_2\text{-}\vee)\ \frac{\begin{array}{l} A\ prop \\ c \in C \quad a_B(x) \in A\ [x \in B] \quad a_C(y) \in A\ [y \in C] \end{array}}{El_{\vee}(\mathsf{inr}_{\vee}(c), a_B, a_C) = a_C(c) \in A}$$

$$\eta\text{-}\vee\ \frac{t \in A\ [z \in C + D]}{El_{\vee}(z, (x)t(\mathsf{inl}_{\vee}(x)), (y)t(\mathsf{inr}_{\vee}(x))) = t(z) \in A}$$

**Existential quantification**

$$\text{F-}\exists)\ \frac{C(x)\ prop\ \ [x \in B]}{\exists_{x \in B} C(x)\ prop} \qquad \text{I-}\exists)\ \frac{b \in B \quad \mathsf{true} \in C(b) \qquad C(x)\ prop\ \ [x \in B]}{\mathsf{true} \in \exists_{x \in B} C(x)}$$

$$\text{E-}\exists)\ \frac{M\ prop \qquad \mathsf{true} \in \exists_{x \in B} C(x) \quad \mathsf{true} \in M\ [x \in B, y \in C(x)]}{\mathsf{true} \in M}$$

$$\text{C-}\exists)\ \frac{\begin{array}{l} M\ prop \\ b \in B \quad c \in C(b) \quad \mathsf{true} \in M\ [x \in B, y \in C(x)] \end{array}}{El_{\exists}(\langle b,_{\exists} c\rangle, m) = m(b, c) \in M}$$

**Universal quantification**

F-∀) $\dfrac{C(x)\ prop\ [x \in B]}{\forall_{x\in B} C(x)\ prop}$ I-∀) $\dfrac{\mathsf{true} \in C(x)\ [x \in B] \qquad C(x)\ prop\ \ [x \in B]}{\mathsf{true} \in \forall_{x\in B} C(x)}$

E-∀) $\dfrac{b \in B \quad \mathsf{true} \in \forall_{x\in B} C(x)}{\mathsf{true} \in C(b)}$ βC-∀) $\dfrac{b \in B \quad c(x) \in C(x)\ [x \in B]}{\mathsf{Ap}_\forall(\lambda_\forall x^B.c(x), b) = c(b) \in C(b)}$

ηC-∀) $\dfrac{f \in \forall_{x\in B} C(x)}{\lambda_\forall x^B.\mathsf{Ap}_\forall(f, x) = f \in \forall_{x\in B} C(x)}$

As in $\mathrm{mTT}_0$, small propositions are generated as follows:

$$\bot\ prop_s \qquad \frac{B\ prop_s \quad C\ prop_s}{B \vee C\ prop_s} \qquad \frac{B\ prop_s \qquad C\ prop_s}{B \to C\ prop_s} \qquad \frac{B\ prop_s \qquad C\ prop_s}{B \wedge C\ prop_s}$$

$$\frac{C(x)\ prop_s\ [x \in B] \qquad B\ set}{\exists_{x\in B} C(x) \in prop_s} \qquad \frac{C(x)\ prop_s\ [x \in B] \qquad B\ set}{\forall_{x\in B} C(x)\ prop_s} \qquad \frac{A\ set \quad a \in A \quad b \in A}{\mathsf{Eq}(A, a, b)\ prop_s}$$

Contrary to $\mathrm{mTT}_0$, in $\mathrm{emTT}_0$ we do not have the intensional collection of small propositions but the quotient of the collection of small propositions under equiprovability representing the power collection of the singleton:

**Power collection of the singleton**

F-P) $\mathcal{P}(1)\ col$ I-P) $\dfrac{B\ prop_s}{[B] \in \mathcal{P}(1)}$ eq-P) $\dfrac{\mathsf{true} \in B \leftrightarrow C}{[B] = [C] \in \mathcal{P}(1)}$ eff-P) $\dfrac{[B] = [C] \in \mathcal{P}(1)}{\mathsf{true} \in B \leftrightarrow C}$

$\dfrac{U \in \mathcal{P}(1) \qquad V \in \mathcal{P}(1)}{\mathsf{Eq}(\mathcal{P}(1), U, V\,)\ prop_s}$ η-P) $\dfrac{U \in \mathcal{P}(1)}{U = [\,\mathsf{Eq}(\mathcal{P}(1), U, [\mathsf{tt}]\,)\,]}$

where $\mathsf{tt} \equiv \bot \to \bot$ represents the truth constant.

Then, we have also function collections from a set toward $\mathcal{P}(1)$:

**Function collection to $\mathcal{P}(1)$**

F-Fc) $\dfrac{B\ set}{B \to \mathcal{P}(1)\ col}$ I-Fc) $\dfrac{c(x) \in \mathcal{P}(1)\ [x \in B] \qquad B\ set}{\lambda x^B.c(x) \in B \to \mathcal{P}(1)}$

E-Fc) $\dfrac{b \in B \quad f \in B \to \mathcal{P}(1)}{\mathsf{Ap}(f, b) \in \mathcal{P}(1)}$ βC-Fc) $\dfrac{b \in B \quad c(x) \in \mathcal{P}(1)\ [x \in B] \qquad B\ set}{\mathsf{Ap}(\lambda x^B.c(x), b) = c(b) \in \mathcal{P}(1)}$

ηC-Fc) $\dfrac{f \in B \to \mathcal{P}(1)}{\lambda x^B.\mathsf{Ap}(f, x) = f \in B \to \mathcal{P}(1)}(x\ not\ free\ in\ f)$

Then, as in $\mathrm{mTT}_0$ we add the embedding rules of sets into collections **set-into-col**, of propositions into collections **prop-into-col**, of small propositions into sets **prop$_s$-into-set** and of small propositions into propositions **prop$_s$-into-prop**.
Moreover, we also add the equality rules ref), sym), tra) both for types and for terms saying that type and term equalities are equivalence relations, and the rules conv), conv-eq).
Contrary to $\mathrm{mTT}_0$, we add all the equality rules about collections and sets saying that their constructors preserve type equality as follows:

**Strong Indexed Sum Collection-eq**

$$\text{eq-}ip)\ \frac{C(x) = D(x)\ prop\ \ [x \in B] \qquad B = E\ col}{\Sigma_{x \in B} C(x) = \Sigma_{x \in E} D(x)\ col}$$

**Function collection-eq**

$$\text{eq-Fc)}\ \frac{B = E\ set}{B \to \mathcal{P}(1) = E \to \mathcal{P}(1)\ col}$$

**Lists-eq**

$$\text{eq-list)}\ \frac{C = D\ set}{List(C) = List(D)\ set}$$

**Strong Indexed Sum set-eq**

$$\text{eq-}\Sigma)\ \frac{C(x) = D(x)\ set\ \ [x \in B] \qquad B = E\ set}{\Sigma_{x \in B} C(x) = \Sigma_{x \in E} D(x)\ set}$$

**Disjoint Sum-eq**

$$\text{eq-+)}\ \frac{B = D\ set \quad C = E\ set}{B + C = D + E\ set}$$

**Dependent Product-eq**

$$\text{eq-}\Pi)\ \frac{C(x) = D(x)\ set\ [x \in B] \qquad B = E\ set}{\Pi_{x \in B} C(x) = \Pi_{x \in E} D(x)\ set}$$

**Quotient set-eq**

$$\text{eq-Q)}\ \frac{A = B\ set \qquad R(x,y) = S(x,y)\ \ prop_s\ [x \in A, y \in A] \qquad \mathsf{Equiv}(R) \qquad \mathsf{Equiv}(S)}{A/R = B/S\ \ set}$$

Then, $\mathrm{emTT}_0$ includes the following equality rules about propositions:

**Disjunction-eq**

$$\text{eq-}\vee)\ \frac{B = D\ prop \quad C = E\ prop}{B \vee C = D \vee E\ prop}$$

**Implication-eq**

$$\text{eq-}\to)\ \frac{B = D\ prop \qquad C = E\ prop}{B \to C = D \to E\ prop}$$

**Conjunction-eq**

$$\text{eq-}\wedge)\ \frac{B = D\ prop \qquad C = E\ prop}{B \wedge C = D \wedge E\ prop}$$

**Propositional equality-eq**

$$\text{eq-Eq)}\ \frac{A = E\ col \quad a = e \in A \quad b = c \in A}{\mathsf{Eq}(A, a, b) = \mathsf{Eq}(E, e, c)\ prop}$$

**Existential quantification-eq**

$$\text{eq-}\exists)\ \frac{C(x) = D(x)\ prop\ \ [x \in B] \qquad B = E\ col}{\exists_{x \in B} C(x) = \exists_{x \in E} D(x)\ prop}$$

**Universal quantification-eq**

$$\text{eq-}\forall)\ \frac{C(x) = D(x)\ prop\ [x \in B] \qquad B = E\ col}{\forall_{x \in B} C(x) = \forall_{x \in E} D(x)\ prop}$$

Analogously, we add eq-$\vee$), eq-$\to$), eq-$\wedge$), eq-Eq), eq-$\exists$), eq-$\forall$) restricted to small propositions. Moreover, equality of propositions is that of collections, that of small propositions coincides with that of $\mathsf{prop_s}$ and is that of propositions and that of sets:

$$\textbf{prop-into-col eq)}\ \frac{A = B\ prop}{A = B\ col} \qquad \textbf{prop}_s\textbf{-eq1)}\ \frac{A = B\ prop_s}{A = B \in \mathsf{prop_s}} \qquad \textbf{prop}_s\textbf{-eq2)}\ \frac{A = B \in \mathsf{prop_s}}{A = B\ prop_s}$$

$$\textbf{prop}_s\textbf{-into-prop eq)}\ \frac{A = B\ prop_s}{A = B\ prop} \qquad \textbf{prop}_s\textbf{-into-set eq)}\ \frac{A = B\ prop_s}{A = B\ set}$$

Equality of sets is that of collections:

$$\textbf{set-into-col eq)}\ \frac{A = B\ set}{A = B\ col}$$

Contrary to $\mathrm{mTT}_0$, also for terms we add equality rules saying that all the constructors preserve equality as in [NPS90]:

I-eq Σ) $$\frac{b = b' \in B \quad c = c' \in C(b) \qquad C(x)\ set\ [x \in B] \qquad B\ set}{\langle b, c\rangle = \langle b', c'\rangle \in \Sigma_{x\in B}C(x)}$$

E-eq Σ) $$\frac{\begin{array}{l} M(z)\ col\ [z \in \Sigma_{x\in B}C(x)] \\ d = d' \in \Sigma_{x\in B}C(x) \quad m(x,y) = m'(x,y) \in M(\langle x, y\rangle)\ [x \in B, y \in C(x)] \end{array}}{El_{\Sigma}(d, m) = El_{\Sigma}(d', m') \in M(d)}$$

E-eq Em) $$\frac{a = a' \in \mathsf{N}_0 \quad A(x)\ col\ [x \in \mathsf{N}_0]}{\mathsf{emp_o}(a) = \mathsf{emp_o}(a') \in A(a)}$$ E-eq S) $$\frac{t = t' \in \mathsf{N}_1 \quad M(z)\ col\ [z \in \mathsf{N}_1] \quad c = c' \in M(\star)}{El_{\mathsf{N}_1}(t, c) = El_{\mathsf{N}_1}(t', c') \in M(t)}$$

$\mathrm{I}_2$-eq list) $$\frac{s = s' \in List(C) \quad c = c' \in C}{\mathsf{cons}(s, c) = \mathsf{cons}(s', c') \in List(C)}$$

E-eq list) $$\frac{\begin{array}{l} L(z)\ col\ [z \in List(C)] \quad s = s' \in List(C) \quad a = a' \in L(\epsilon) \\ l(x,y,z) = l'(x,y,z) \in L(\mathsf{cons}(x,y))\ [x \in List(C), y \in C, z \in L(x)] \end{array}}{El_{List}(s, a, l) = El_{List}(s', a', l') \in L(s)}$$

I-eq Q) $$\frac{a = a' \in A \qquad A/R\ set}{[a] = [a'] \in A/R}$$

E-eq Q) $$\frac{\begin{array}{l} L(z)\ col\ [z \in A/R] \\ p = p' \in A/R \quad l(x) = l'(x) \in L([x])\ [x \in A] \quad l(x) = l(y) \in L([x])\ [x \in A, y \in A, d \in R(x,y)] \end{array}}{El_{Q}(p, l) = El_{Q}(p', l') \in L(p)}$$

$\mathrm{I}_1$-eq +) $$\frac{b = b' \in B \qquad B\ set \qquad C\ set}{\mathsf{inr}(b) = \mathsf{inr}(b') \in B + C}$$ $\mathrm{I}_2$-eq +) $$\frac{c = c' \in C \qquad B\ set \qquad C\ set}{\mathsf{inl}(c) = \mathsf{inl}(c') \in B + C}$$

E-eq +) $$\frac{\begin{array}{l} A(z)\ col\ [z \in B + C] \\ d = d' \in B + C \quad a_B(x) = a'_B(x) \in A(\mathsf{inl}(x))\ [x \in B] \quad a_C(y) = a'_C(y) \in A(\mathsf{inr}(y))\ [y \in C] \end{array}}{El_{+}(d, a_B, a_C) = El_{+}(d', a'_B, a'_C) \in A(w)}$$

I-eq Π) $$\frac{c(x) = c'(x) \in C(x)\ [x \in B] \qquad C(x)\ set\ [x \in B] \quad B\ set}{\lambda x^B.c(x) = \lambda x^B.c'(x) \in \Pi_{x\in B}C(x)}$$ E-eq Π) $$\frac{b = b' \in B \quad f = f' \in \Pi_{x\in B}C(x)}{\mathsf{Ap}(f, b) = \mathsf{Ap}(f', b') \in C(b)}$$

I-eq Fc) $$\frac{c(x) = c'(x) \in \mathcal{P}(1)\ [x \in B] \quad B\ set}{\lambda x^B.c(x) = \lambda x^B.c'(x) \in B \to \mathcal{P}(1)}$$ E-eq Fc) $$\frac{b = b' \in B \quad f = f' \in B \to \mathcal{P}(1)}{\mathsf{Ap}(f, b) = \mathsf{Ap}(f', b') \in \mathcal{P}(1)}$$

Analogously, we define I-eq ip), E-eq ip) for indexed sum collections of propositional functions as I-eq Σ) and E-eq Σ).
Note that I-eq Π) is the so-called $\xi$-rule in [Mar75].