A predicative proof of exponentiation of locally compact formal topologies.

Maria Emilia Maietti
Dipartimento di Matematica Pura ed Applicata
Università di Padova
via G. Belzoni n.7, I–35131 Padova, Italy
maietti@math.unipd.it

Abstract

We provide an intuitionistic predicative proof of the well known result proved by M.Hyland that locally compact locales are the exponentiable objects in the category of locales.

More precisely, by adapting Hyland's proofs to the context of formal topology, we prove that locally compact formal covers are the exponentiable objects in the category of inductively generated formal covers. Then, we deduce the analogous result for inductively generated formal topologies.

Contents

1 Introduction

It is well known that the category of topological spaces is not closed and that the exponentiable objects are exactly those spaces whose frame is continuous (see for example the overview [EH02]).

M. Hyland in [Hyl81] proved the analogous result for locales, namely that locally compact locales are the exponentiable objects in the category of locales. Later, I. Sigstam in [Sig95] adapted his proof to show that locally compact locales are exponentiable in the category of (effectively given) formal spaces, but still working within a (classical) impredicative setting. [1] Moreover, Vickers reproduced most of Hyland's results using geometric reasoning (see [Vic04]).

The aim of this paper is to give an intuitionistic predicative proof of the whole characterization in [Hyl81], namely we want to characterize the exponentiable objects both in the category of inductively generated formal covers and of formal topologies. Indeed, the category of inductively generated formal covers [CSSV03] with a corresponding notion of continuous relation [Sam87] constitutes a predicative presentation of the category of locales (see [Joh82]), and also of the category of frames by taking the

[1] Indeed, she proves exponentiation for a locally compact formal space only over formal spaces admitting generators made of directed subsets.

opposite relations (in fact the operation of taking the opposite of the category of frames to get that of locales is here internalized because morphisms are defined in terms of relation). Instead the full subcategory of inductively generated formal topologies provides a predicative presentation of open locales (see [JT84]).

A predicative study of exponentiation in the context of formal topology has been started in [MV04] where it is proved predicatively that unary formal topologies are exponentiable in the category of inductively generated formal topologies. This result was deduced after proving the corresponding result for inductively generated formal covers and the existence of a coreflection of inductively generated formal covers into inductively generated formal topologies thanks to which exponentiation within the subcategory of formal topologies is inherited from the category of inductively generated formal covers.

Here, we show predicatively that locally compact formal covers are the exponentiable objects in the category of inductively generated formal covers by adapting Hyland's proofs in [Hyl81]. In particular, the construction of the exponential topology of a locally compact formal cover over an inductively generated one follows the original idea in [Hyl81] of generating the exponential topology by means of the "way-below" component of a locale morphism, since this construction fits perfectly with the use of the predicative method for the inductive generation of a formal cover in [CSSV03], provided that we can define predicatively the 'way-below" relation between open subsets, which was shown to be possible for inductively generated formal covers in [Cur04]. Hence, inspired by Hyland's construction, based on the predicative justification of the "way-below" relation and thanks to the analysis in [MV04] of the conditions defining a continuous relation, we characterize a continuous relation via its way-below component in terms of conditions that can be put as basic axioms from which to generate the exponential topology candidate as in [Hyl81, Joh84, Sig95]. Then, the proof of exponentiation mostly consists of inductions over the inductive generation of the covers involved.

Afterwards, we prove that any exponentiable inductively generated formal cover is locally compact following very closely the original proof in [Hyl81] that, being almost completely based on categorical properties of exponentiation and on properties of the Sierpinski locale, is already essentially predicative.

Finally, thanks to the coreflection proved in [MV04], we deduce the analogous characterization for inductively generated formal topologies, namely that locally compact formal topologies are the exponentiable objects in the category of inductively generated formal topologies.

2 Preliminaries on formal covers

There are different equivalent presentations of formal topology based on a monoid [Sam87] or on a preorder [CSSV03]. Here, we deal with formal covers and formal topologies based on a preorder, called $\leq$-formal cover and $\leq$-formal topology.

Note that when we speak of "set" we mean a set in Martin-Löf constructive set theory [NPS90].

Here, we recall the definition of $\leq$-formal cover, $\leq$-formal topology and their inductively generated versions.

Definition 2.1 ($\leq$-Formal cover) *A* $\leq$-formal cover *is a structure*

$$\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}})$$

where A is a set, $\leq_{\mathcal{A}}$, is a preorder between elements of A and $\lhd_{\mathcal{A}}$ is an infinitary relation, called cover *relation, between elements and subsets of A satisfying the following conditions:*

$$\textit{(reflexivity)}\quad \frac{a\varepsilon U}{a \lhd U} \qquad\qquad \textit{(transitivity)}\quad \frac{a \lhd U \quad U \lhd V}{a \lhd V}$$

$$\textit{($\leq$-left)}\quad \frac{a \leq b \qquad b \lhd U}{a \lhd U} \qquad\qquad \textit{($\leq$-right)}\quad \frac{a \lhd U \qquad a \lhd V}{a \lhd U \downarrow_{\leq} V}$$

where $U \lhd V \equiv (\forall u\varepsilon U)\ u \lhd V$ *and* $U \downarrow_{\leq} V \equiv \{x \in A \mid \exists u\varepsilon U\ x \leq u \quad \& \quad \exists v\varepsilon V\ x \leq v\ \}$.

Definition 2.2 ($\leq$-Formal topology) *A* $\leq$-formal topology *is a structure*

$$\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}}, \mathsf{Pos}_{\mathcal{A}})$$

where $(A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}})$ *is a formal cover and* $\mathsf{Pos}_{\mathcal{A}}$ *is a predicate over* A*, called* positivity predicate*, satisfying the following conditions:*

$$\textit{(monotonicity)}\ \frac{\mathsf{Pos}_{\mathcal{A}}(a) \qquad a \lhd U}{(\exists u \varepsilon U)\ \mathsf{Pos}_{\mathcal{A}}(u)} \quad \textit{(positivity axiom)}\quad a \lhd a^{+}$$

where $a^{+} \equiv \{x \in A \mid x = a \ \&\ \mathsf{Pos}_{\mathcal{A}}(x)\ \}$.

Definition 2.3 (Inductively generated formal cover) *A $\leq$-formal cover $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}})$ is inductively generated if and only if there exists a localized axiom-set $I(-)$ and $C(-,-)$ (as defined in [CSSV03]) such that $\lhd_{\mathcal{A}}$ is inductively generated by using* reflexivity, $\leq$-left *and*

$$\textit{(infinity)}\ \frac{i \in I(a) \qquad C(a,i) \lhd U}{a \lhd U}$$

We recall from [CSSV03] that in general the relation generated by an axiom-set by means of *reflexivity*, $\leq$*-left* and *infinity* does not necessarily satisfy $\leq$*-right* unless the axiom-set is localized. However, given any axiom-set we can always localize it as shown in [CSSV03].

Definition 2.4 (Ind. gen. formal topology) *A $\leq$-formal topology $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}}, Pos_{\mathcal{A}})$ is inductively generated if and only if the formal cover $(A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}})$ is inductively generated.*

Then, we recall the definition of continuous relation between $\leq$-formal covers:

Definition 2.5 (Continuous relation) *Suppose that $\mathcal{A} = (A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}})$ and $\mathcal{B} = (B, \leq_{\mathcal{B}}, \lhd_{\mathcal{B}})$ are two formal covers. Then a* continuous relation *from $\mathcal{A}$ to $\mathcal{B}$ is a binary proposition $a\, F\, b$, for $a \in A$ and $b \in B$, satisfying the following conditions:*

$$\text{(totality)} \quad A \lhd_{\mathcal{A}} F^{-}(B) \qquad (\leq\text{-convergence}) \quad \frac{aFb \qquad aFd}{a \lhd_{\mathcal{A}} F^{-}(b \downarrow_{\leq} d)}$$

$$\text{(saturation)} \quad \frac{a \lhd_{\mathcal{A}} W \quad (\forall w \varepsilon W)\ wFb}{aFb} \qquad \text{(continuity)} \quad \frac{aFb \qquad b \lhd_{\mathcal{B}} V}{a \lhd_{\mathcal{A}} F^{-}(V)}$$

In the following we will often use this property of continuous relations:

Lemma 2.6 *Let $\mathcal{A}$ and $\mathcal{B}$ be formal covers and F be a continuous relation between them. Then the following condition holds*

$$\text{(weak-continuity)} \quad \frac{a\, F\, b \qquad b \lhd_{\mathcal{B}} d}{a\, F\, d}$$

The definition of continuous relation simplifies when we consider continuous relations between inductively generated formal covers as proved in [MV04].

Lemma 2.7 *Let $\mathcal{A}$ and $\mathcal{B}$ be an inductively generated formal covers, $\mathcal{B}$ and suppose that $I_{\mathcal{A}}(-)$,$C_{\mathcal{A}}(-,-)$ and $I_{\mathcal{B}}(-), C_{\mathcal{B}}(-,-)$ are the axiom-sets for $\mathcal{A}$ and $\mathcal{B}$ respectively. Then, a relation F from A to B is continuous if and only if it satisfies the following conditions:*

$$\text{(totality)} \quad A \lhd_{\mathcal{A}} F^{-}(B) \qquad (\leq\text{-convergence}) \quad \frac{aFb \qquad aFd}{a \lhd_{\mathcal{A}} F^{-}(b \downarrow_{\leq} d)}$$

$$(\leq\text{-saturation}) \quad \frac{a \leq c \quad c\, F\, b}{a\, F\, b} \qquad \text{(axiom-saturation)} \quad \frac{i \in I_{\mathcal{A}}(a) \quad (\forall x \varepsilon C_{\mathcal{A}}(a,i))\ x\, F\, b}{a\, F\, b}$$

$$(\leq\text{-continuity}) \quad \frac{a\, F\, b \quad b \leq d}{a\, F\, d} \qquad \text{(axiom-continuity)} \quad \frac{a\, F\, b \quad j \in I_{\mathcal{B}}(b)}{a \lhd F^{-}(C_{\mathcal{B}}(b,j))}$$

Note that to get a continuous relation it is sufficient to saturate a relation satisfying all the conditions except of *axiom saturation*:

Proposition 2.8 *Let $\mathcal{A}$ and $\mathcal{B}$ be two inductively generated formal covers and suppose that F is a relation satisfying* totality, $\leq$-convergence, $\leq$-saturation, $\leq$-continuity *and* axiom-continuity. *Then*

$$a\, F^{\lhd}\, b \equiv a \lhd_{\mathcal{A}} \{c \in A \mid cFb\}$$

is the minimal continuous relation extending F.

Proof. The proof is completely analogous to the analogous lemma for continuous relations between formal covers (not necessarily inductively generated) in [MV04].

Moreover, it is sufficient to prove *axiom-continuity* of a relation just with respect to an axiom-set whose localization gives the proper axiom-set generating the cover.

Lemma 2.9 *Let $\mathcal{A}$ be an inductively generated formal cover and $\mathcal{B}$ be an inductively generated formal cover having $I'_{\mathcal{B}}(-)$ and $C'_{\mathcal{B}}(-,-)$ as the axiom-set localizing $I_{\mathcal{B}}(-)$ and $C_{\mathcal{B}}(-,-)$ according to [CSSV03]. Let F be a relation from $\mathcal{A}$ to $\mathcal{B}$ satisfying* totality, $\leq$-continuity, $\leq$-saturation, $\leq$-convergence *and*

$$(\text{non-loc-axiom-continuity}) \quad \frac{aFb \quad j \in I_{\mathcal{B}}(b)}{a \lhd_{\mathcal{A}} F^{-}(C_{\mathcal{B}}(b,j))}$$

then F satisfies axiom-continuity *with respect to $I'_{\mathcal{B}}(-)$ and $C'_{\mathcal{B}}(-,-)$. And, hence its saturation $F^{\lhd}$ is a continuous relation from $\mathcal{A}$ to $\mathcal{B}$.*

Proof. We recall that according to [CSSV03]

$$I'_{\mathcal{B}}(b) \equiv \{< b', k > \mid (b \leq_{A^{\vee}} b') \;\&\; k \in I_{\mathcal{B}}(b')\ \}$$

and

$$C'_{\mathcal{B}}(b, < b', k >) \equiv b \downarrow_{\leq} C_{\mathcal{B}}(b', k)$$

Suppose $a\, F\, b$ and $i \in I'_{\mathcal{B}}(b)$ with $i \equiv < b', k >$ for some $b' \in B$ such that $b \leq b'$ and $k \in I_{\mathcal{B}}(b')$. Now, by *$\leq$-continuity* we get $a\, F\, b'$ and hence by *non-loc-axiom-continuity* $a \lhd_{\mathcal{A}} F^{-}(C_{\mathcal{B}}(b', j))$. From the hypothesis $a\, F\, b$ we find $a \lhd_{\mathcal{A}} F^{-}(b)$ and hence by *$\leq$-right* we get $a \lhd_{\mathcal{A}} F^{-}(b) \downarrow_{\leq} F^{-}(C_{\mathcal{B}}(b', j))$. Now, note that we can prove $F^{-}(b) \downarrow_{\leq} F^{-}(C_{\mathcal{B}}(b', j)) \lhd_{\mathcal{A}} F^{-}(b \downarrow_{\leq} C_{\mathcal{B}}(b', k))$ by *$\leq$-saturation* , *$\leq$-convergence*, *transitivity* and *reflexivity*. Therefore, by *transitivity* we conclude $a \lhd_{\mathcal{A}} F^{-}(C'_{\mathcal{B}}(b, i))$ as claimed, that is *axiom-continuity* holds.

Finally, by lemma 2.8 we conclude that $F^{\lhd}$ is a continuous relation.

We denote with $\mathsf{FTop_i}^{-}$ the category of inductively generated $\leq$-formal covers and continuous relations. We recall that the composition of a continuous relation G from $\mathcal{B}$ to $\mathcal{C}$ with a continuous relation F from $\mathcal{A}$ to $\mathcal{B}$ is defined as $(G \circ F)^{\lhd}$ that is

$$a \lhd_{\mathcal{A}} \{x \in A \mid\ \exists y \in B\ x\, F\, y \;\&\; y\, G\, b\}$$

with A the base of $\mathcal{A}$, and two continuous relations are equal if they are equivalent relations. This category provides a constructive presentation of the category of locales [BS01]. Moreover, we denote with $\mathsf{FTop_i}$ the full subcategory of $\mathsf{FTop_i}^{-}$ with inductively generated formal topologies. This category provides a constructive presentation of the category of open locales [Neg02].

In the next we report useful lemmas proved in [MV04]:

Lemma 2.10 *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be formal covers, F be a relation from $\mathcal{A}$ to $\mathcal{B}$ which satisfies* continuity *and G be a relation from $\mathcal{B}$ to $\mathcal{C}$. Then*

$$(G \circ F^{\lhd})^{\lhd} = (G \circ F)^{\lhd} \qquad \textit{and} \qquad (G^{\lhd} \circ F)^{\lhd} = (G \circ F)^{\lhd}$$

In [MV04] the reader may also find the construction of products of formal covers in $\mathsf{FTop_i}^{-}$ and the proof of the following useful lemma:

Lemma 2.11 *Let $\mathcal{A}$ and $\mathcal{B}$ be inductively generated formal covers and let a be an element of A, b be an element of B, U be a subset of A and V be a subset of B. Then the following conditions are valid:*

$$(1)\ \frac{a \lhd_{\mathcal{A}} U}{(a,b) \lhd_{A\times B} U \times \{b\}} \quad (2)\ \frac{b \lhd_{\mathcal{B}} V}{(a,b) \lhd_{A\times B} \{a\} \times V} \quad (3)\ \frac{a \lhd_{\mathcal{A}} U \qquad b \lhd_{\mathcal{B}} V}{(a,b) \lhd_{A\times B} U \times V}$$

Note that projections satisfy the following property:

Lemma 2.12 (projections) *The first and second projections of the product of the inductively generated formal cover $\mathcal{C} \equiv (C, \leq_C, \lhd_C)$ with the inductively generated formal cover $\mathcal{A} \equiv (A, \leq_A, \lhd_A)$ are defined as*

$$\Pi_1 \equiv p_1^{\lhd} \qquad \Pi_2 \equiv p_2^{\lhd}$$

where $(c,a)\, p_1\, c' \equiv c \lhd_A c'$ and $(c,a)\, p_2\, a' \equiv a \lhd_A a'$.

Recalling from [MV04] that we indicate with $\langle F, G\rangle$ the relation from $\mathcal{C}$ to $\mathcal{A} \times \mathcal{B}$ that is the pairing of the relation F from $\mathcal{C}$ to $\mathcal{A}$ with the relation G from $\mathcal{C}$ to $\mathcal{B}$, we prove:

Lemma 2.13 *Let $\mathcal{A}$ and $\mathcal{B}$ be inductively generated formal covers and let F be a relation from $\mathcal{C}$ to $\mathcal{A}$ and G be a relation from $\mathcal{C}$ to $\mathcal{B}$ such that F and G satisfies all the conditions of lemma 2.7 except for* axiom-saturation*. Then*

$$\langle F^{\lhd}, G^{\lhd}\rangle = \langle F, G\rangle^{\lhd}$$

Proof. If $c\,\langle F^{\lhd}, G^{\lhd}\rangle\,(a,b)$ holds then from $c\,F^{\lhd}\,a$ and $c\,G^{\lhd}\,b$ by *$\leq$-right* we get $c \lhd_{\mathcal{C}} F^{-}(a) \downarrow_{\leq} G^{-}(b)$. Then, $F^{-}(a) \downarrow_{\leq} G^{-}(b) \subseteq \langle F, G\rangle^{-}((a,b))$ follows by *$\leq$-saturation* of F and G. Therefore, we conclude $c\,\langle F, G\rangle^{\lhd}\,(a,b)$ by *transitivity* and *reflexivity*.

Conversely, if $c\,\langle F, G\rangle^{\lhd}\,(a,b)$, since by definition $\langle F, G\rangle^{-}((a,b)) \subseteq F^{-}(a)$ and $\langle F, G\rangle^{-}((a,b)) \subseteq G^{-}(b)$, by *transitivity* and *reflexivity* we conclude $c \lhd_{\mathcal{C}} F^{-}(a)$ and $c \lhd_{\mathcal{C}} G^{-}(b)$, that is $c\,\langle F^{\lhd}, G^{\lhd}\rangle\,(a,b)$.

2.1 Join formal covers

Observe that any formal cover $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}})$ is isomorphic to a join-formal cover, that is a formal cover whose base is closed under a binary join operation and has a bottom element. Then, we show that any inductively generated formal cover is isomorphic to an inductively generated join-formal cover whose cover is generated from an axiom-set by means of the rules of *reflexivity*, *$\leq$-left* and *infinity*. The same holds with respect to the notion of bottom-formal cover, that is a formal cover whose base has a bottom element.

Definition 2.14 *We call $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \vee_{\mathcal{A}}, \perp_{\mathcal{A}}, \lhd_{\mathcal{A}})$ a join-formal cover if $(A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}})$ is a formal cover whose base A is closed under the binary join operation $\vee_{\mathcal{A}}$ and has a bottom element $\perp_{\mathcal{A}}$ satisfying*

$$(\perp\text{-left}) \qquad \perp_{\mathcal{A}} \lhd_{\mathcal{A}} U$$

$$(\vee\text{-left}) \qquad \frac{a \lhd_{\mathcal{A}} U \qquad b \lhd_{\mathcal{A}} U}{a \vee_{\mathcal{A}} b \lhd_{\mathcal{A}} U}$$

$$(\vee\text{-right}) \qquad \frac{a \lhd_{\mathcal{A}} b}{a \lhd_{\mathcal{A}} \{b \vee_{\mathcal{A}} c\}} \qquad \frac{a \lhd_{\mathcal{A}} b}{a \lhd_{\mathcal{A}} \{c \vee_{\mathcal{A}} b\}}$$

Proposition 2.15 *Any formal cover $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}})$ is isomorphic to a join-formal $\mathcal{A}^{\vee} \equiv (A^{\vee}, \leq_{\mathcal{A}^{\vee}}, \vee_{\mathcal{A}^{\vee}}, \perp_{\mathcal{A}^{\vee}}, \lhd_{\mathcal{A}^{\vee}})$.*

Proof. The formal topology $\mathcal{A}^{\vee} \equiv (A^{\vee}, \leq_{\mathcal{A}^{\vee}}, \vee_{\mathcal{A}^{\vee}}, \perp_{\mathcal{A}^{\vee}}, \lhd_{\mathcal{A}^{\vee}})$ is defined as follows. $A^{\vee} \equiv \mathit{List}(A)$ which is the set of lists on A, $t \vee_{\mathcal{A}^{\vee}} s \equiv t \cdot s$ where $\cdot$ is the append operation of the list t with s, and $\perp_{\mathcal{A}^{\vee}} \equiv \mathsf{nil}$ which is the empty list. Then, we put for $l, s \in A^{\vee}$

$$l \leq_{\mathcal{A}^{\vee}} s \equiv \forall x\, \varepsilon\, l \quad \exists y\, \varepsilon\, s \quad x \leq_{\mathcal{A}} y$$

where the proposition $x \varepsilon l$ is defined by induction on the construction of l by setting $x \varepsilon \mathsf{nil} \equiv \mathsf{False}$ and $x \varepsilon (a,b) \cdot l \equiv (x = (a,b)) \vee x \varepsilon l$. Finally, we define

$$s \lhd_{\mathcal{A}^\vee} W \equiv \mathcal{U}(s) \lhd_{\mathcal{A}} \bigcup_{w \in W} \mathcal{U}(w)$$

where $\mathcal{U}(s) \equiv \begin{cases} \emptyset & \text{if } s \equiv \mathsf{nil} \\ \mathcal{U}(s') \bigcup \{a\} & \text{if } s \equiv s' \cdot a \text{ for } s' \varepsilon \mathit{List}(A),\ a \in A \end{cases}$

Proposition 2.16 *For any inductively generated formal cover $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}})$ with axiom-set $I_{\mathcal{A}}(-)$ and $C_{\mathcal{A}}(-,-)$, the associated isomorphic join-formal cover $\mathcal{A}^\vee \equiv (A^\vee, \leq_{\mathcal{A}^\vee}, \vee_{\mathcal{A}^\vee}, \perp_{\mathcal{A}^\vee}, \lhd_{\mathcal{A}^\vee})$ is inductively generated with an axiom-set $I^\vee(-)$ and $C^\vee(-,-)$ generating the cover $\lhd_{\mathcal{A}^\vee}$ by means of the rules of* reflexivity*,* $\leq$-left *and* infinity*.*

Proof. Following the method in [CSSV03], the axiom-set $I^\vee(-)$ and $C^\vee(-,-)$ is defined as the localization of the following axiom-set $I^{\vee'}(-)$ and $C^{\vee'}(-,-)$ defined by induction on the formation of terms $t \vee_{\mathcal{A}^\vee} s$ as follows: (note that in the following we simply write $t \vee s$ instead of $t \vee_{\mathcal{A}^\vee} s$ and $a \cdot \mathsf{nil}$ indicates the list with only one element a.)

- We put

  $$I^{\vee'}(\mathsf{nil}) \equiv A^\vee + \{*\}$$

  and

  $$C^{\vee'}(\mathsf{nil}, k) \equiv \begin{cases} \{t\} & \text{if } k = \mathsf{in}_1(t) \text{ for } t \in A^\vee \\ \emptyset & \text{if } k = \mathsf{in}_2(*) \end{cases}$$

- for $a \in A$ we put

  $$I^{\vee'}(a) \equiv I_{\mathcal{A}}(a) + A^\vee + A^\vee$$

  and

  $$C^{\vee'}(a, k) \equiv \begin{cases} \bigcup_{x \in C(a,j)} \{x \cdot \mathsf{nil}\} & \text{if } k = \mathsf{in}_1(j) \text{ for } j \in I_{\mathcal{A}}(a) \\ \{a \vee t\} & \text{if } k = \mathsf{in}_2(t) \text{ for } t \in A^\vee \\ \{t \vee a\} & \text{if } k = \mathsf{in}_3(t) \text{ for } t \in A^\vee \\ \{a\} & \text{if } k = \mathsf{in}_4(*) \end{cases}$$

- (inductive case) for $s, s' \in A^\vee$ we put

  $$I^{\vee'}(s \vee s') \equiv (I^{\vee'}(s) \times I^{\vee'}(s')) + A^\vee + A^\vee + \{*\}$$

  and

  $$C^{\vee'}(s \vee s', k) \equiv \begin{cases} C(s, j_1) \cup C(s', j_2) & \text{if } k = \mathsf{in}_1(\langle j_1, j_2 \rangle) \text{ for } j_1 \in I^{\vee'}(s), j_2 \in I^{\vee'}(s') \\ \{(s \vee s') \vee t\} & \text{if } k = \mathsf{in}_2(t) \text{ for } t \,\varepsilon\, A^\vee \\ \{t \vee (s \vee s')\} & \text{if } k = \mathsf{in}_3(t) \text{ for } t \,\varepsilon\, A^\vee \\ \{s \vee s'\} & \text{if } k = \mathsf{in}_4(*) \end{cases}$$

  where in_i for $i = 1, .., 4$ is the i-nth injection of the finite sum.

Finally, we localize the given axiom-set by obtaining for $s \in A^\vee$

$$I^\vee(s) \equiv \{\langle t, k \rangle \mid (s \leq_{\mathcal{A}^\vee} t) \,\&\, k \in I^{\vee'}(t)\}$$

and

$$C^\vee(s, \langle t, k \rangle) \equiv s \downarrow_{\leq} C^{\vee'}(t, k)$$

With this axiom set, it is immediate to realize the validity of $\vee$-*right* rules. First note that for $s, t \in A^{\vee}$ there exists $k \in I^{\vee'}(s)$ such that $C^{\vee'}(s,k) \equiv \{s \vee t\}$. Now note that by *reflexivity* and by $\leq$-*left* rule we deduce

$$s \downarrow_{\leq} C^{\vee'}(s,k) \lhd_{\mathcal{A}^{\vee}} \{s \vee t\}$$

and hence, by *infinity*, recalling that $C^{\vee}(s, \langle s, k\rangle) \equiv s \downarrow_{\leq} C^{\vee'}(s,k)$, we conclude $s \lhd_{\mathcal{A}^{\vee}} \{s \vee t\}$. Analogously, we can prove the validity of the other $\vee$-*right* rule.

Then, to see the validity of $\vee$-*left rule* we argue as follows. Suppose $s \lhd_{\mathcal{A}^{\vee}} W$ and $s' \lhd_{\mathcal{A}^{\vee}} W$. Then, there exists $j_1 \in I^{\vee'}(s)$, $j_2 \in I^{\vee'}(s')$ such that $C(s, j_1) \equiv \{s\}$ and $C(s', j_1) \equiv \{s'\}$. Hence, we get that $C^{\vee'}(s \vee s', k) \equiv C(s, j_1) \cup C(s', j_2)$ with $k \equiv \mathsf{in}_1(\langle j_1, j_2\rangle)$. Then, by hypothesis and $\leq$-*left* rule we obtain

$$(s \vee s') \downarrow_{\leq} C^{\vee'}(s \vee s', k) \lhd_{\mathcal{A}^{\vee}} W$$

and hence, by *infinity*, recalling that $C^{\vee}(s \vee s', \langle s \vee s', k\rangle) \equiv (s \vee s') \downarrow_{\leq} C^{\vee'}(s \vee s', k)$, we conclude $s \vee s' \lhd_{\mathcal{A}^{\vee}} W$.

If we consider formal covers whose preordered base has only a bottom element we can prove analogous propositions.

Definition 2.17 *We call $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \perp_{\mathcal{A}}, \lhd_{\mathcal{A}})$ a bottom-formal cover if $(A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}})$ is a formal cover whose base A has a bottom element $\perp_{\mathcal{A}}$ satisfying* $\perp$-left.

Lemma 2.18 *Any formal cover $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}})$ is isomorphic to a bottom-formal cover $\mathcal{A}^{\perp} \equiv (A^{\perp}, \leq_{\mathcal{A}^{\perp}}, \perp_{\mathcal{A}^{\perp}}, \lhd_{\mathcal{A}^{\perp}})$.*

Proof. We define

$$A^{\perp} \equiv \Sigma_{l \in List(A)} lh(l) \leq 1$$

that is the set of lists on A with only one element plus the empty list, which stands for $\perp_{\mathcal{A}^{\perp}}$. Then $\leq_{\mathcal{A}^{\perp}}$ and $\lhd_{\mathcal{A}^{\perp}}$ are defined as $\leq_{\mathcal{A}^{\vee}}$ and $\lhd_{\mathcal{A}^{\vee}}$ restricted to the lists with at most one element.

Lemma 2.19 *For any inductively generated formal cover $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}})$ with axiom-set $I_{\mathcal{A}}(-)$, $C_{\mathcal{A}}(-,-)$, the associated isomorphic bottom-formal cover $\mathcal{A}^{\perp} \equiv (A^{\perp}, \leq_{\mathcal{A}^{\perp}}, \perp_{\mathcal{A}^{\perp}}, \lhd_{\mathcal{A}^{\perp}})$ is inductively generated with an axiom-set $I^{\perp}(-)$, $C^{\perp}(-,-)$ generating the cover $\lhd_{\mathcal{A}^{\perp}}$ by means of the rules of* reflexivity, $\leq$-left *and* infinity.

Proof. We define the axiom-set $I^{\perp}(-)$, $C^{\perp}(-,-)$ by using the abbreviation $\widetilde{a} \equiv \langle \mathsf{nil} \cdot a, \mathsf{id}(a)\rangle \in A^{\perp}$ for $a \in A$ as follows: $I^{\perp}(\widetilde{a}) \equiv I_{\mathcal{A}}(a)$ with $C^{\perp}(\widetilde{a}, j) \equiv C_{\mathcal{A}}(a, j)$ for $a \in A$ and $j \in I_{\mathcal{A}}(a)$, and $I^{\perp'}(\perp_{\mathcal{A}^{\perp}}) \equiv \{*\}$ with $C^{\perp'}(\perp_{\mathcal{A}^{\perp}}, *) \equiv \emptyset$.

Note that for bottom-formal covers we can prove:

Lemma 2.20 *Let $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \perp_{\mathcal{A}}, \lhd_{\mathcal{A}})$ and $\mathcal{B} \equiv (B, \leq_{\mathcal{B}}, \perp_{\mathcal{B}}, \lhd_{\mathcal{B}})$ be bottom-formal covers and let F be a continuous relation from $\mathcal{A}$ to $\mathcal{B}$. If $c \lhd_{\mathcal{A}} \perp_{\mathcal{A}}$, or $c \lhd_{\mathcal{A}} \emptyset$, then $c\, F\, b$ follows for every $b \in B$.*

Proof. Since $\forall x \,\varepsilon\, \emptyset \; x\, F\, b$ is valid by ex-falso quodlibet, $\perp_{\mathcal{A}} \; F\, b$ follows by *saturation* from $c \lhd_{\mathcal{A}} \emptyset$, which follows by *transitivity* from $\perp_{\mathcal{A}} \lhd_{\mathcal{A}} \emptyset$ in the case $c \lhd_{\mathcal{A}} \perp_{\mathcal{A}}$.

Moreover, we can prove for continuous relations:

Lemma 2.21 *Let $\mathcal{A}$ be an inductively generated formal cover and $\mathcal{B}$ be an inductively generated formal cover with axiom-set $I_{\mathcal{B}}(-)$ and $C_{\mathcal{B}}(-,-)$. Let $\mathcal{B}^{\perp}$ be the bottom-formal cover defined in lemma 2.19 and let F be a relation from $\mathcal{A}$ to $\mathcal{B}$ satisfying* totality, $\leq$-continuity, axiom-continuity, $\leq$-saturation, $\leq$-convergence. *Then, the relation $F^{\perp}$ from $\mathcal{A}$ to $\mathcal{B}^{\perp}$ defined as*

$$a\, F^{\perp}\, b \equiv \begin{cases} a\, F\, b' & \text{if} \quad b \equiv \tilde{b'} \\ a \lhd_{\mathcal{A}} \emptyset & \text{if} \quad b \equiv \perp_{\mathcal{B}^{\perp}} \end{cases}$$

satisfies totality, $\leq$-continuity, axiom-continuity, $\leq$-saturation, $\leq$-convergence, *and hence $(F^{\perp})^{\lhd}$ is a continuous relation from $\mathcal{A}$ to $\mathcal{B}^{\perp}$.*

Proof. We prove that $F^{\perp}$ satisfies the various conditions.
(*Totality*) It is inherited from that of F.
($\leq$-*continuity*) Suppose $b_1 \leq_{\mathcal{B}^{\perp}} b_2$ and $a\, F^{\perp}\, b_1$. Then, if $b_1 \equiv \tilde{b_1'}$ for $b_1' \in B$, by definition of $\leq_{\mathcal{B}^{\perp}}$ we get $b_2 \equiv \tilde{b_2'}$ for some $b_2' \in B$ and hence we conclude by $\leq$-*continuity* for F, since by definition $b_1' \leq_{\mathcal{B}} b_2'$.

Otherwise, if $b_1 \equiv \perp_{\mathcal{B}^{\perp}}$ and $b_2 \equiv \perp_{\mathcal{B}^{\perp}}$ we conclude immediately. Instead, if $b_1 \equiv \perp_{\mathcal{B}^{\perp}}$ and $b_2 \equiv \tilde{b_2'}$ for some $b_2' \in B$, by definition of $a\, F^{\perp}\, b_2$ we get $a \lhd_{\mathcal{A}} \emptyset$ and hence by lemma 2.20 we conclude $a\, F\, b_2'$, that is $a\, F^{\perp}\, b_2$.
(*axiom-continuity*) Suppose $a\, F^{\perp}\, b$ and $j \in I^{\perp}(b)$. If $b \equiv \tilde{b'}$ for $b' \in B$ and $C^{\perp}(b,j) \equiv C(b',j)$ for $j \in I(b')$, then we conclude by *axiom-continuity* for F. Otherwise, if $b \equiv \perp_{\mathcal{B}^{\perp}}$ then by hypothesis $a \lhd_{\mathcal{A}} \emptyset$ and hence $a \lhd_{\mathcal{A}} {F^{\perp}}^{-}(C^{\perp}(b,j))$ by *transitivity*.
($\leq$-*saturation*) Suppose $a\, F^{\perp}\, b$ and $a' \leq a$. If $b \equiv \tilde{b'}$ for $b' \in B$ then we conclude by $\leq$-*saturation* for F. Otherwise, $b \equiv \perp_{\mathcal{B}^{\perp}}$ and hence by definition $a \lhd_{\mathcal{A}} \emptyset$ and by $\leq$-*left* we get $a' \lhd_{\mathcal{A}} \emptyset$, that is $a'\, F^{\perp}\, b$.
($\leq$-*convergence*) Suppose $a\, F^{\perp}\, b_1$ and $a\, F^{\perp}\, b_2$. If $b_1 \equiv \tilde{b_1'}$ for $b_1' \in B$, and $b_2 \equiv \tilde{b_2'}$ for $b_2' \in B$, then we conclude by $\leq$-*convergence* for F. If $b_1 \equiv \perp_{\mathcal{B}^{\perp}}$ then by definition $a \lhd_{\mathcal{A}} \emptyset$ and hence by *transitivity* we conclude $a \lhd_{\mathcal{A}} {F^{\perp}}^{-}(b_1 \downarrow_{\leq} b_2)$. Analogously, we conclude $a \lhd_{\mathcal{A}} {F^{\perp}}^{-}(b_1 \downarrow_{\leq} b_2)$ if $b_2 \equiv \perp_{\mathcal{B}^{\perp}}$.

Finally, after proving the above conditions we conclude that $(F^{\perp})^{\lhd}$ is a continuous relation by lemma 2.8.

3 Locally compact formal covers

Here, we give a definition of locally compact formal cover equivalent to the original one in [Neg02] and based on the following proposition proved in [Cur04].

Proposition 3.1 *Let $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}})$ be an inductively generated formal cover. We can predicatively define the way-below relation $U \ll V$ between subsets $U, V \subseteq A$, meaning that for all $W \subseteq A$, if $V \lhd W$ then there is a finite (possibly empty) list $w_1, .., w_n$ of elements in W such that $U \lhd \{w_1, .., w_n\}$.*

Then, following [Cur04] we give the definition of locally compact formal cover.

Definition 3.2 *A formal cover $\mathcal{A}$ is locally compact if it is inductively generated and $a \lhd \{b : b \ll a\}$, for all $a \in S$ – where $b \ll a$ stands for the predicative definition mentioned in prop. 3.1 –.*

The notion of locally compact formal cover is preserved by isomorphisms as shown in [Cur04]:

Lemma 3.3 *Let be $\mathcal{A}$ and $\mathcal{A}'$ two formal covers isomorphic via continuous relations. If $\mathcal{A}$ is locally compact, then $\mathcal{A}'$ is locally compact, too.*

Moreover, the following properties about the way-below relation can be proved predicatively:

Lemma 3.4 *Let $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \vee_{\mathcal{A}}, \perp_{\mathcal{A}}, \lhd_{\mathcal{A}})$ be a join-locally compact formal cover. The following statement hold: let U, V, W subsets of A*

1. *If $U \ll V$ then $U \lhd_{\mathcal{A}} V$.*
2. *If $U \ll V$ and $V \lhd_{\mathcal{A}} W$ then $U \ll W$.*
3. *If $U \lhd_{\mathcal{A}} V$ and $V \ll W$ then $U \ll W$.*
4. *(interpolation property) If $a \ll V$ there exists $a' \in A$ such that $a \ll a' \ll V$.*

Remark 3.5 Note that we can express the above *interpolation property* because of the presence of joins in the base. In absence of joins in the base, we need to express the interpolation property by saying that: if $a \ll V$ then there exists a finite list $a_1, .., a_n$, possibly empty, such that $a \ll \{a_1, .., a_n\}$ and $\{a_1, .., a_n\} \ll V$.

Lemma 3.6 *Let $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \vee_{\mathcal{A}}, \perp_{\mathcal{A}}, \lhd_{\mathcal{A}})$ be a join-locally compact formal cover. If $d \ll a$ and $a \lhd_{\mathcal{A}} U$, then there exist a list $u_1 \ll u'_1, .., u_n \ll u'_n$ such that $d \lhd_{\mathcal{A}} \{u_1, .., u_n\}$ and $u'_i \,\varepsilon\, U$ for $i = 1, .., n$, or $d \lhd_{\mathcal{A}} \perp_{\mathcal{A}}$.*

Proof. Since by definition $a \lhd_{\mathcal{A}} wb(a) = \{x \in A \mid x \ll a\}$ for every $a \in A$, then by *transitivity* we deduce $a \lhd_{\mathcal{A}} \bigcup_{u \varepsilon U}\{y \in A \mid y \ll u\}$. Now, for any $d \ll a$ by definition we have that either there exists a list $u_1, .., u_n$ such that $d \lhd_{\mathcal{A}} \{u_1, .., u_n\}$ and $u_i \,\varepsilon \bigcup_{u \varepsilon U}\{y \in A \mid y \ll u\}$ for $i = 1, .., n$, that is there exists $u'_1, .., u'_n$ in U such that $u_i \ll u'_i$ for $i = 1, .., n$, or $d \lhd_{\mathcal{A}} \emptyset$ and hence $d \lhd_{\mathcal{A}} \perp_{\mathcal{A}}$.

Lemma 3.7 *Let $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \vee_{\mathcal{A}}, \perp_{\mathcal{A}}, \lhd_{\mathcal{A}})$ be a join-locally compact formal cover. If $u_1 \ll U, .., u_n \ll U$ then $\{u_1, .., u_n\} \ll U$.*

The following lemma on the product of any inductively generated formal cover with a locally compact join-formal cover will be crucial to exponentiate a locally compact join-formal cover.

Lemma 3.8 *Consider the topological product of an inductively generated formal cover $\mathcal{C} \equiv (C, \leq_{\mathcal{C}}, \lhd_{\mathcal{C}})$ with axiom-set $I_{\mathcal{C}}(-)$, $C_{\mathcal{C}}(-,-)$ and a locally compact join-formal cover $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \vee_{\mathcal{A}}, \perp_{\mathcal{A}}, \lhd_{\mathcal{A}})$ in $\mathsf{FTop_i}^-$. Suppose that $c \in C$, $a \in A$, $(c, a) \lhd_{C \times A} W$ where $W \subseteq C \times A$ satisfies the following properties:*

1. *if $(c', a') \,\varepsilon\, W$, then $(c'', a') \,\varepsilon\, W$ for every c'' such that $c'' \leq c'$;*
2. *$(c, \perp_{\mathcal{A}}) \,\varepsilon\, W$.*

Then, for any $d \ll a$ there exists a subset U of C such that $c \lhd_{\mathcal{C}} U$ and, for every $u \,\varepsilon\, U$ there exists $e_1, e'_1, .., e_n, e'_n \in A$ such that $e_j \ll e'_j$ for $j = 1, .., n$ and $d \lhd_{\mathcal{A}} \{e_1, .., e_n\}$ and $(u, e'_j) \,\varepsilon\, W$ for $j = 1, .., n$.

Proof. Suppose $d \ll a$. We prove the statement by induction on the derivation of $(c, a) \lhd_{C \times A} W$.

- *Reflexity.* If $(c, a) \lhd_{C \times A} W$ follows by *reflexivity* then $(c, a) \,\varepsilon\, W$. Hence, take $U \equiv \{c\}$ and for $u \equiv c$ take $e_1 \equiv d$ and $e'_1 \equiv a$.
- *Infinity.* If $(c, a) \lhd_{C \times A} W$ follows by *infinity* then we have to consider two cases.
  1. If $(c, a) \lhd_{C \times A} W$ follows by *infinity* from $\{c\} \times C_{\mathcal{A}}(a, j) \lhd_{C \times A} W$. Then, we apply lemma 3.6 to $d \ll a$ and $a \lhd_{\mathcal{A}} C_{\mathcal{A}}(a, j)$. If $d \lhd_{\mathcal{A}} \perp_{\mathcal{A}}$ then we put $U \equiv \{c\}$ and $e_1 = e'_1 \equiv \perp_{\mathcal{A}}$, since $\perp_{\mathcal{A}} \ll \perp_{\mathcal{A}}$ and $(c, \perp_{\mathcal{A}}) \in W$. Instead, if we have $h_1 \ll h'_1, .., h_m \ll h'_m$ with $h'_i \,\varepsilon\, C_{\mathcal{A}}(a, j)$ for $i = 1, .., m$ such that $d \lhd_{\mathcal{A}} \{h_1, .., h_m\}$, then by inductive hypothesis for every $h'_i \,\varepsilon\, C_{\mathcal{A}}(a, j)$ with $i = 1, .., m$ there exists a subset $U_{h'_i}$ of C such that $c \lhd_{\mathcal{C}} U_{h'_i}$ and for every $u_{h'_i} \,\varepsilon\, U_{h'_i}$ there exist $e_1^{h'_i} \ll (e^{h'_i})'_1, .., e_n^{h'_i} \ll (e^{h'_i})'_{n_i}$ such that $h_i \lhd_{\mathcal{A}} \{e_1^{h'_i}, .., e_{n_i}^{h'_i}\}$ and $(u, (e^{h'_i})'_j) \,\varepsilon\, W$ for $j = 1, .., n_i$. Therefore, we put $U \equiv U_{h'_1} \downarrow_{\leq} .. \downarrow_{\leq} U_{h'_m}$. Indeed, for every $u \,\varepsilon\, U$ there exist $u_{h'_i} .., u_{h'_m}$ such that $u \leq u_{h'_1} .. u \leq u_{h'_m}$. So, we take all the $e_1^{h'_i} \ll (e^{h'_i})'_1, .. e_{n_i}^{h'_i} \ll (e^{h'_i})'_{n_i}$ associated to each h'_i for $i = 1, .., m$ since $d \lhd_{\mathcal{A}} \cup_{i=1,..,m}\{e_1^{h'_i}, .., e_{n_i}^{h'_i}\}$. Moreover, observe also that for any $j = 1, .., n$ and $i = 1, .., m$ then $(u, (e^{h'_i})'_j) \,\varepsilon\, W$ follows from the hypothesis on W since $u \leq u_{h'_j}$ and $(u_{h'_i}, (e^{h'_i})'_j) \,\varepsilon\, W$.
  2. Suppose that $(c, a) \lhd_{C \times A} W$ follows by using *infinity* from $C_{\mathcal{C}}(c, j) \times \{a\} \lhd_{C \times A} W$. Then, for every $v \,\varepsilon\, C_{\mathcal{C}}(c, j)$, there exists by inductive hypothesis, a subset U_v of C such that for every $u \,\varepsilon\, U_v$ there exists ${e_1}^v, (e^v)'_1, .., {e^v}_n, (e^v)'_n \in A$ such that ${e_j}^v \ll (e^v)'_j$ for $j = 1, .., n$ and $d \lhd_{\mathcal{A}} \{{e_1}^v, .., {e_n}^v\}$ and $(u, (e^v)'_j) \,\varepsilon\, W$ for $j = 1, .., n$. Hence, we take $U \equiv \bigcup_{v \varepsilon C_{\mathcal{C}}(c,j)} U_v$.
- *$\leq$-left.* If $(c, a) \leq (c', a)$ and $(c', a') \lhd_{C \times A} W$, we take U and for each $u \,\varepsilon\, U$ we take $e_j \ll e'_j$ for $j = 1, .., n$ obtained by the inductive hypothesis on $(c', a') \lhd_{C \times A} W$ with respect to d since from $d \ll a$ and $a \leq a'$ we get $a \lhd_{\mathcal{A}} a'$ and hence $d \ll a'$ by lemma 3.4.

4 Exponentiation

In this section we prove that any locally compact bottom-formal cover is exponentiable in the full subcategory $\mathsf{FTop_i}^-_{\perp}$ of $\mathsf{FTop_i}^-$ with inductively generated bottom-formal covers, which is equivalent to $\mathsf{FTop_i}^-$ thanks to lemma 2.16 and the fact that the embedding of $\mathsf{FTop_i}^-_{\perp}$ into $\mathsf{FTop_i}^-$ is full and faithful. To be precise we show how to exponentiate a locally compact join-formal cover. This suffices to prove that any locally compact bottom-formal cover is exponentiable in $\mathsf{FTop_i}^-_{\perp}$ since by lemma 3.3 and lemma 2.15 we know that any locally compact formal cover is isomorphic to a locally compact join-formal cover and if an object is isomorphic to an exponentiable object then it is exponentiable, too. To exponentiate a locally compact join-formal cover $\mathcal{A}$ over any bottom-formal cover $\mathcal{B}$ in $\mathsf{FTop_i}^-_{\perp}$, we need to define the exponential cover of $\mathcal{A}$ over $\mathcal{B}$ in such a way that its formal points are in bijection with the continuous relations from $\mathcal{A}$ to $\mathcal{B}$.

As seen in [MV04] we are able to perform such a correspondence by encoding a continuous relation via lists of pairs of basic open (a, b) for $a \in A$ and $b \in B$ (where A is the set of basic opens for $\mathcal{A}$ and $\mathcal{B}$ the set of basic opens for $\mathcal{B}$) if the conditions defining a continuous relation can be formulated according to one of the form

$$\frac{a\ R\ b \qquad P(a,b,a',b')}{a'\ R\ b'} \qquad \frac{a\ R\ b \qquad Q(a,b,V)}{(\exists y\,\varepsilon\, V)\, a\ R\ y}$$

But, note that *axiom-saturation* condition of a continuous relation from a locally compact formal cover to any other inductively generated formal cover is not of the above forms – and it does not seem to be reducible to one of the above forms –, contrary to continuous relations from a unary topology (as we noticed in [MV04]). However, inspired by the fact that the exponential topology built in [Hyl81], [Joh84], [Sig95] is generated from axioms expressing the properties of the relation $a \ll f^*(b)$ for $a \in A$ and $b \in B$ and f locale morphism, observe that to any continuous relation R, from a locally compact join-formal cover to any inductively generated bottom-formal cover, we can associate in a unique way its way-below component, defined as $a \ll R^-(b)$, whose properties can be expressed by conditions of the above form and hence can be used to generate an exponential cover. Therefore, we give the definition of *wb-relation* in terms of the conditions satisfied by $a \ll R^-(b)$. Then, after proving that wb-relations from a locally compact join-formal cover to any inductively generated bottom-formal cover are in bijection with the corresponding continuous relations, we build the exponential topology according to the semantics given in [MV04] with reference to wb-relations instead of referring to continuous relations.

Definition 4.1 *Let $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \vee_{\mathcal{A}}, \perp_{\mathcal{A}}, \lhd_{\mathcal{A}})$ be a locally compact join-formal cover and $\mathcal{B} \equiv (B, \leq_{\mathcal{B}}, \perp_{\mathcal{B}}, \lhd_{\mathcal{B}})$ be an inductively generated bottom-formal cover with axiom-set $I_{\mathcal{B}}(-)$, $C_{\mathcal{B}}(-,-)$. Then, F^{wb} is a* wb-relation *from $\mathcal{A}$ to $\mathcal{B}$ if it is a relation from A to B and it satisfies the following conditions:*

(wb-bottom continuity)

$$\frac{c\, F^{wb}\ \perp_{\mathcal{B}}}{c \lhd_{\mathcal{A}} \perp_{\mathcal{A}}}$$

(wb-bottom saturation)

$$\frac{b \in B}{\perp_{\mathcal{A}}\ F^{wb}\, b}$$

(wb-totality)

$$\frac{d \ll a}{\exists y_1, .., y_n \in B \quad \exists x_1, .., x_n \in A \quad x_i\, F^{wb}\, y_i \quad i = 1, .., n \quad \& \quad d \lhd_{\mathcal{A}} \{x_1, .., x_n\}}$$

(wb-$\leq$-convergence)

$$\frac{a\, F^{wb}\, b \qquad a\, F^{wb}\, c \qquad d \ll a}{\exists y_1, .., y_n \in (b \downarrow_{\leq} c) \cup \{\perp_{\mathcal{B}}\} \quad \exists x_1, .., x_n \in A \quad x_i\, F^{wb}\, y_i \quad i = 1, .., n \quad \& \quad d \lhd_{\mathcal{A}} \{x_1, .., x_n\}}$$

(wb-finite-saturation)

$$\frac{a \lhd_{\mathcal{A}} \{c_1, .., c_n\} \qquad \forall x\, \varepsilon\, \{c_1, .., c_n\} \qquad x\, F^{wb}\, b}{a\, F^{wb}\, b}$$

(wb-$\leq$-continuity)
$$\frac{a\, F^{wb}\, b \qquad b \leq d}{a\, F^{wb}\, d}$$

(wb-axiom-continuity)
$$\frac{aF^{wb}b \qquad j \in J(b) \qquad d \ll a}{\exists y_1, .., y_n \,\varepsilon\, C(b,j) \cup \{\bot_{\mathcal{A}}\} \quad \exists x_1, .., x_n \in A \quad x_i\, F^{wb}\, y_i \quad i = 1, .., n \quad \& \quad d \lhd_{\mathcal{A}} \{x_1, .., x_n\}}$$

(interpolation)
$$\frac{a\, F^{wb}\, b}{\exists x \in A \quad x\, F^{wb}\, b \quad \& \quad a \ll x}$$

Lemma 4.2 *Let $\mathcal{A}$ be a locally compact join-formal cover and $\mathcal{B}$ any inductively generated bottom-formal cover with axiom-set $I_{\mathcal{B}}(-)$, $C_{\mathcal{B}}(-,-)$. Then, the collection of continuous relations from $\mathcal{A}$ to $\mathcal{B}$ is in bijection with the collection of wb-relations from $\mathcal{A}$ to $\mathcal{B}$. Indeed,*

1. *Given any continuous relation F from $\mathcal{A}$ to $\mathcal{B}$, the relation $a\, F^{wb}\, b \equiv a \ll F^{-}(b)$, called* the wb-component of F, *is a wb-relation from $\mathcal{A}$ to $\mathcal{B}$.*
2. *Given any wb-relation R from $\mathcal{A}$ to $\mathcal{B}$ then $a\, R^{\lhd}\, b \equiv a \lhd_{\mathcal{A}} \{x \in S \mid x\, R\, b\}$ is a continuous relation from $\mathcal{A}$ to $\mathcal{B}$.*
3. *Given any continuous relation F from $\mathcal{A}$ to $\mathcal{B}$ then $F = (F^{wb})^{\lhd}$.*
4. *Given any wb-relation R from $\mathcal{A}$ to $\mathcal{B}$ then $R = (R^{\lhd})^{wb}$.*

Proof.

1. Let F be a continuous relation from $\mathcal{A}$ to $\mathcal{B}$. We prove that F^{wb} satisfies the various conditions:
   - (*wb-bottom continuity*) If $c\, F^{wb}\, \bot_{\mathcal{B}}$, that is $c \ll F^{-}(\bot_{\mathcal{A}})$, by *continuity* of F on $\bot_{\mathcal{A}} \lhd_{\mathcal{A}} \emptyset$ we get $F^{-}(\bot_{\mathcal{A}}) \lhd_{\mathcal{A}} F^{-}(\emptyset)$, that is $F^{-}(\bot_{\mathcal{A}}) \lhd_{\mathcal{A}} \emptyset$, and by lemma 3.4 $c \lhd_{\mathcal{A}} \bot_{\mathcal{A}}$.
   - (*wb-bottom saturation*) $\bot_{\mathcal{A}} \ll F^{-}(b)$, that is $\bot_{\mathcal{A}}\, F^{wb} b$, trivially holds for any $b \in B$.
   - (*wb-totality*) By *totality* of F we get $a \lhd_{\mathcal{A}} F^{-}(B)$. Since $\mathcal{A}$ is locally compact we get $a \lhd_{\mathcal{A}} \bigcup_{z \in F^{-}(B)} \{x \in A \mid x \ll z\}$. Now, from $d \ll a$ by lemma 3.6 we find either $d \lhd_{\mathcal{A}} \bot_{\mathcal{A}}$ from which by lemma 2.20 $\bot_{\mathcal{A}}\ F\ \bot_{\mathcal{B}}$ and hence we conclude $\bot_{\mathcal{A}}\ F^{wb}\ \bot_{\mathcal{B}}$, or there exists $x_1 \ll z_1, .., x_n \ll z_n \in A$ and $y_1, .., y_n \in B$ such that $z_i\, F\, y_i$, and hence $x_i \ll F^{-}(y_i)$, that is $x_i\, F^{wb}\, y_i$ for $i = 1, .., n$ and $d \lhd_{\mathcal{A}} \{x_1, .., x_n\}$.
   - (*wb-$\leq$-convergence*) Suppose $a \ll F^{-}(b)$ and $a \ll F^{-}(c)$. Then, we get $a \lhd_{\mathcal{A}} F^{-}(b)$ and $a \lhd_{\mathcal{A}} F^{-}(c)$ by lemma 3.4. Afterwards, by *$\leq$-right* we obtain $a \lhd_{\mathcal{A}} F^{-}(b) \downarrow_{\leq} F^{-}(c)$ and by *transitivity*, recalling that $F^{-}(b) \downarrow_{\leq} F^{-}(c) \lhd_{\mathcal{A}} F^{-}(b \downarrow_{\leq} c)$ follows by *$\leq$-saturation* and *$\leq$-convergence*, we conclude $a \lhd_{\mathcal{A}} F^{-}(b \downarrow_{\leq} c)$. Then, we get $a \lhd_{\mathcal{A}} \bigcup_{z \in F^{-}(b \downarrow_{\leq} c)} \{x \in A \mid x \ll z\}$ being $\mathcal{A}$ locally compact. Now, from $d \ll a$ by lemma 3.6 either $d \lhd_{\mathcal{A}} \bot_{\mathcal{A}}$ from which by lemma 2.20 $\bot_{\mathcal{A}}\, F\, \bot_{\mathcal{B}}$ and hence we conclude $\bot_{\mathcal{A}}\, F^{wb}\, \bot_{\mathcal{B}}$, or we find $x_1 \ll z_1, .., x_n \ll z_n \in A$ and $y_1, .., y_n \in b \downarrow_{\leq} c$ such that $z_i\, F\, y_i$, and hence $x_i \ll F^{-} y_i$, that is $x_i\, F^{w} b\, y_i$ for $i = 1, .., n$ and $d \lhd_{\mathcal{A}} \{x_1, .., x_n\}$.
   - (*wb-finite saturation*) Suppose $a \lhd_{\mathcal{A}} \{c_1, .., c_n\}$ and that $x \ll F^{-}(b)$ for all $x \,\varepsilon\, \{c_1, .., c_n\}$. Therefore, by lemma 3.7 we get $\{c_1, .., c_n\} \ll F^{-}(b)$ and by lemma 3.4 we conclude $a \ll F^{-} b$, that is $a\, F^{wb}\, b$.
   - (*wb-$\leq$-continuity*) Suppose $a\, F^{wb}\, b$, that is $a \ll F^{-}(b)$, and $b \leq d$ from which we deduce $b \lhd_{\mathcal{B}} d$ by *reflexivity* and *$\leq$-left*. By *continuity* of F we get $F^{-}(b) \lhd_{\mathcal{A}} F^{-}(d)$ and hence by lemma 3.4 we conclude $a \ll F^{-}(d)$, that is $a\, F^{wb}\, d$.
   - (*wb-axiom-continuity*) The proof is analogous to (*wb-$\leq$-convergence*).

- (*wb-axiom-interpolation*) Suppose $a\, F^{wb}\, b$, that is $a \ll F^-(b)$, then by the interpolation property of lemma 3.4 we derive $\exists x\, a \ll x \,\&\, x \ll F^-(b)$.

2. We apply lemma 2.8 to show that $a\, R^{\lhd}\, b$ is a continuous relation, by showing that R satisfies all the condition of a continuous relation except of *axiom saturation*.

   - (*totality*) Being $\mathcal{A}$ locally compact $a \lhd_{\mathcal{A}} wb(a)$ holds for any $a \in A$. Now, for any $d \ll a$ by *wb-totality* we get $y_1, .., y_n \in B$ and $x_1, .., x_n \in A$ such that $x_i\, R\, y_i$ for $i = 1, .., n$ and $d \lhd_{\mathcal{A}} \{x_1, .., x_n\}$, from which we get $d \lhd_{\mathcal{A}} R^-(\{y_1, .., y_n\})$ and since $R^-(\{y_1, .., y_n\}) \subseteq R^-(B)$ by *transitivity* we conclude $d \lhd_{\mathcal{A}} R^-(B)$. Therefore, by *transitivity* on $a \lhd_{\mathcal{A}} wb(a)$ we obtain $a \lhd_{\mathcal{A}} R^-(B)$ as claimed.
   - ($\leq$*-convergence*) Suppose $a\, R\, b$ and $a\, R\, c$ and $d \ll a$. Then, by *wb-*$\leq$*-convergence* we get $y_1, .., y_n \,\varepsilon\, (b \downarrow_{\leq} c) \cup \{\bot_{\mathcal{B}}\}$ and $x_1, .., x_n \in A$ such that $x_i\, R\, y_i$ for $i = 1, .., n$ together with $d \lhd_{\mathcal{A}} \{x_1, .., x_n\}$, from which we get $d \lhd_{\mathcal{A}} R^-(\{y_1, .., y_n\})$ by *transitivity* and *reflexivity*. Since $R^-(\{y_1, .., y_n\}) \subseteq R^-((b \downarrow_{\leq} c) \cup \{\bot_{\mathcal{B}}\})$, we obtain $d \lhd_{\mathcal{A}} R^-((b \downarrow_{\leq} c) \cup \{\bot_{\mathcal{B}}\})$ by *transitivity* and *reflexivity*. Now, note that $R^-((b \downarrow_{\leq} c) \cup \{\bot_{\mathcal{B}}\}) \lhd_{\mathcal{A}} R^-(b \downarrow_{\leq} c)$. Indeed, for any $a' \,\varepsilon\, R^-((b \downarrow_{\leq} c) \cup \{\bot_{\mathcal{B}}\})$ we get either $a'\, R\, y$ for $y \,\varepsilon\, b \downarrow_{\leq} c$, that is $a' \lhd_{\mathcal{A}} R^-(b \downarrow_{\leq} c)$, or $a' R\, \bot_{\mathcal{B}}$ from which by *wb-bottom continuity* we get $a' \lhd_{\mathcal{A}} \bot_{\mathcal{A}}$ and hence $a' \lhd_{\mathcal{A}} R^-(b \downarrow_{\leq} c)$.
     Then, from $R^-((b \downarrow_{\leq} c) \cup \{\bot_{\mathcal{B}}\}) \lhd_{\mathcal{A}} R^-(b \downarrow_{\leq} c)$ and $d \lhd_{\mathcal{A}} R^-((b \downarrow_{\leq} c) \cup \{\bot_{\mathcal{B}}\})$ we conclude $d \lhd_{\mathcal{A}} R^-(b \downarrow_{\leq} c)$. Since this holds for any $d \ll a$, from $a \lhd_{\mathcal{A}} wb(a)$ by *transitivity* we conclude $a \lhd_{\mathcal{A}} R^-(b \downarrow_{\leq} c)$ as claimed.
   - ($\leq$*-saturation*) Suppose $a \leq a'$ and $a'\, R\, b$. Then, by *wb-finite-saturation* since $a \lhd_{\mathcal{A}} a'$, which follows from $a \leq a'$ by *reflexivity* and $\leq$*-left*, we get $a\, R\, b$.
   - ($\leq$*-continuity*) Suppose $a\, R\, b$ and $b \leq b'$, then by *wb-*$\leq$*-continuity* we get $a\, R\, b'$.
   - (*axiom continuity*) The proof is analogous to $\leq$*-convergence*.

3. Given any continuous relation F from $\mathcal{A}$ to $\mathcal{B}$ we prove that $F = (F^{wb})^{\lhd}$. Suppose $a\, F\, b$. Then $a \lhd_{\mathcal{A}} F^-(b)$ and for any $d \ll a$ we get $d \ll F^-(b)$ by lemma 3.4. Therefore, we obtain $d\, F^{wb}\, b$ and from $a \lhd_{\mathcal{A}} wb(a)$ we conclude $a (F^{wb})^{\lhd} b$.

   Conversely, if $a (F^{wb})^{\lhd} b$, that is $a \lhd_{\mathcal{A}} \{x \in A \mid x \ll F^-(b)\}$, then we get $a \lhd_{\mathcal{A}} \{x \in A \mid x \lhd_{\mathcal{A}} F^-(b)\}$ by lemma 3.4 and hence by *transitivity* and *saturation* of F we conclude $a\, F\, b$ as claimed.

4. Given any wb-relation R from $\mathcal{A}$ to $\mathcal{B}$ we prove that $R = (R^{\lhd})^{wb}$. If $a\, R\, b$ then by *interpolation* for R we find a' such that $a \ll a'$ and $a'\, R\, b$ and hence by *reflexivity* $a'\, R^{\lhd}\, b$, from which we deduce $a' \lhd_{\mathcal{A}} (R^{\lhd})^-(b)$ and by lemma 3.4 we conclude $a \ll (R^{\lhd})^-(b)$, that is $a (R^{\lhd})^{wb} b$ as claimed.

   Conversely, if $a (R^{\lhd})^{wb} b$, that is $a \ll (R^{\lhd})^-(b)$, then by lemma 3.6 either we find $a \lhd_{\mathcal{A}} \emptyset$ and hence $a\, R\, b$ by lemma 2.20, or we find $x_1, .., x_n$ such that $a \lhd_{\mathcal{A}} \{x_1, .., x_n\}$ and $x_1\, R\, b$ for $i = 1, .., n$. In this latter case, by *wb-finite-saturation* we also conclude $a\, R\, b$ as claimed.

4.0.1 The exponential formal cover

Given a locally compact join-formal cover $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \vee_{\mathcal{A}}, \bot_{\mathcal{A}}, \lhd_{\mathcal{A}})$ and an inductively generated bottom-formal cover $\mathcal{B} \equiv (B, \leq_{\mathcal{B}}, \bot_{\mathcal{B}}, \lhd_{\mathcal{B}})$ with axiom-set $I_{\mathcal{B}}(-)$, $C_{\mathcal{B}}(-,-)$ the exponential cover candidate $\mathcal{B}^{\mathcal{A}}$ of $\mathcal{A}$ over $\mathcal{B}$ is defined as $(\mathcal{A} \to \mathcal{B})^{\bot}$, that is the closure under a bottom element of a formal cover $\mathcal{A} \to \mathcal{B}$ whose base does not have naturally a bottom element. (Note that passing from a formal cover $\mathcal{A}$ to its bottom form $\mathcal{A}^{\bot}$ does not change the formal points).

The set of basic neighbourhoods of $\mathcal{A} \to \mathcal{B}$ is $List(A \times B)$, that is lists of pairs of elements in the cartesian product $A \times B$. The intended meaning of a list $l \in List(A \times B)$ is to give a partial information on a wb-relation from $\mathcal{A}$ to $\mathcal{B}$.

The order on $List(A \times B)$ is

$$l \leq_{\mathcal{A} \to \mathcal{B}} m \equiv (\forall (a, b) \in A \times B)\, (a, b) \,\varepsilon\, m \to (a, b) \,\varepsilon\, l$$

stating that the list l is more precise, that is, it approximates fewer wb-relations than the list m. This order relation is a refinement of the reverse sub-list relation, which states that m is a sub-list of l, because it does not consider the order among the elements in a list and their repetitions.

We define the cover relation $\lhd_{\mathcal{A}\to\mathcal{B}}$ according to the usual semantics of formal topology (see [Sam03]) as follows: $l \lhd_{\mathcal{A}\to\mathcal{B}} U$ *means that every formal point containing* l *also contains a basic ove that* $F^b ot$ *satisfies the various conditions. of* U. Recalling that formal points are expected to be wb-relations, this means that we have to prove that *for any wb-relation* R, *if* $a\,R\,b$ *holds for each pair* $(a,b)\,\varepsilon\, l$, *then there exists* $m\,\varepsilon\, U$ *such that* $a\,R\,b$ *holds for each pair* $(a,b)\,\varepsilon\, m$.

Since we want to obtain an inductively generated formal cover, we apply the method in [CSSV03] and we define the axiom-set $I_{(\mathcal{A}\to\mathcal{B})^\perp}(-)$ and $C_{(\mathcal{A}\to\mathcal{B})^\perp}(-,-)$ as the localization of another set of axioms $I'_{\mathcal{A}\to\mathcal{B}}(-)$ and $C'_{\mathcal{A}\to\mathcal{B}}(-,-)$ that we describe in the following: we express each axiom in the form $l \lhd_{\mathcal{A}\to\mathcal{B}} U$ by intending that for some index $j \in I'_{\mathcal{A}\to\mathcal{B}}(l)$ then $C'_{\mathcal{A}\to\mathcal{B}}(l,j) \equiv U$. We hope that it will be clear how such a formalization can be actually performed.

The general recipe is that each axiom is in correspondence with a condition defining a wb-relation in definition 4.1.

The set of axioms $I'_{\mathcal{A}\to\mathcal{B}}(-)$ and $C'_{\mathcal{A}\to\mathcal{B}}(-,-)$ include only:

(*wb-bottom continuity*)
$(a,\perp_{\mathcal{B}}) \lhd_{\mathcal{A}\to\mathcal{B}} \{(\perp_{\mathcal{A}},\perp_{\mathcal{B}}) \mid a \lhd_{\mathcal{A}} \perp_{\mathcal{A}}\}$

(*wb-bottom saturation*)
$\mathsf{nil} \lhd_{\mathcal{A}\to\mathcal{B}} (\perp_{\mathcal{A}}, b)$
provided that $b \in B$

(*wb-totality*)
$\mathsf{nil} \lhd_{\mathcal{A}\to\mathcal{B}} \{(x_1,y_1)\cdot ..\cdot (x_n,y_n)\cdot \mathsf{nil} \mid y_1,..,y_n \in B \quad \& \quad d \lhd_{\mathcal{A}} \{x_1,..,x_n\}\}$
provided that $d \ll a$ for some $a \in A$

(*wb-$\leq$-convergence*)
$(a,b)\cdot(a,b')\cdot \mathsf{nil} \lhd_{\mathcal{A}\to\mathcal{B}} \{(x_1,y_1)\cdot ..\cdot (x_n,y_n)\cdot \mathsf{nil} \mid y_i\,\varepsilon\,(b \downarrow_{\leq} b') \cup \{\perp_{\mathcal{B}}\} \text{ for } i=1,..,n \quad \& \quad d \lhd_{\mathcal{A}} \{x_1,..,x_n\}\}$
provided that $d \ll a$

(*wb-finite-saturation*)
$(d_1,b)\cdot ..\cdot (d_n,b)\cdot \mathsf{nil} \lhd_{\mathcal{A}\to\mathcal{B}} (a,b)\cdot \mathsf{nil}$
provided that $a \lhd_{\mathcal{A}} \{d_1,..,d_n\}$ for some $a \in A$

(*wb-$\leq$-continuity*)
$(a,b)\cdot \mathsf{nil} \lhd_{\mathcal{A}\to\mathcal{B}} (a,b')\cdot \mathsf{nil}$
provided that $b \leq b'$

(*wb-axiom-continuity*)
$(a,b)\cdot \mathsf{nil} \lhd_{\mathcal{A}\to\mathcal{B}} \{(x_1,y_1)\cdot ..\cdot (x_n,y_n)\cdot \mathsf{nil} \mid y_i\,\varepsilon\, C(b,j) \cup \{\perp_{\mathcal{B}}\} \text{ for } i=1,..,n \;\&\; d \lhd_{\mathcal{A}} \{x_1,..,x_n\}\}$
provided that $d \ll a$ and $j \in J(b)$

(*interpolation*)
$(a,b)\cdot \mathsf{nil} \lhd_{\mathcal{A}\to\mathcal{B}} \{(x,b)\cdot \mathsf{nil} \mid a \ll x\}$

Then, we localize such an axiom-set $I'_{\mathcal{A}\to\mathcal{B}}(-)$ and $C'_{\mathcal{A}\to\mathcal{B}}(-,-)$ according to [CSSV03] and we obtain the axiom-set $I_{(\mathcal{A}\to\mathcal{B})^\perp}(-)$, $C_{(\mathcal{A}\to\mathcal{B})^\perp}(-,-)$ that generates the formal cover $\mathcal{B}^{\mathcal{A}} \equiv (\mathcal{A}\to\mathcal{B})^\perp$.

Note that if we work with a locally compact bottom-formal cover having a base without joins, then as remarked in 3.5 the interpolation property must be expressed with finite lists and in this case we would need to change the base of the exponential to $List(List(A)\times B)$ making the various proofs more complicated.

Let us recall now useful lemmas on the exponential topology proved in [MV04].

First, for any $m_1, m_2 \in List(A \times B)^{\perp}$ we define $m_1 \cdot m_2 \equiv \begin{cases} \widetilde{l_1 \cdot l_2} & \text{if } m_1 \equiv \widetilde{l_1} \quad m_2 \equiv \widetilde{l_2} \text{ for } l_1, l_2 \in List(A \times B) \\ m_1 & \text{if } m_2 \equiv \perp_{(\mathcal{A} \to \mathcal{B})^{\perp}} \\ m_2 & \text{if } m_1 \equiv \perp_{(\mathcal{A} \to \mathcal{B})^{\perp}} \end{cases}$
and we will write $U_1 \cdot U_2$ to mean the subset $U_1 \cdot U_2 \equiv \{m_1 \cdot m_2 \mid m_1 \,\varepsilon\, U_1 \,\&\, m_2 \,\varepsilon\, U_2\}$.

Lemma 4.3 *Let $l \in List(A \times B)^{\perp}$ and $U_1, U_2 \subseteq (A \to B)^{\perp}$. Then:*

$$(\cdot\text{-right}) \quad \frac{l \lhd_{(\mathcal{A} \to \mathcal{B})^{\perp}} U_1 \qquad l \lhd_{(\mathcal{A} \to \mathcal{B})^{\perp}} U_2}{l \lhd_{(\mathcal{A} \to \mathcal{B})^{\perp}} U_1 \cdot U_2}$$

Lemma 4.4 *Let F be a continuous relation from $\mathcal{C}$ to $\mathcal{A} \to \mathcal{B}$. Then, for any $c \in C$ and any $l_1, l_2 \in List(A \times B)^{\perp}$,*

$$\frac{c\,F\,l_1 \qquad c\,F\,l_2}{c\,F\,l_1 \cdot l_2}$$

As proved in [MV04], the exponential cover axioms do not depend on the particular axiom-set generating the cover of $\mathcal{B}$ since we can show

Lemma 4.5 *Let $\mathcal{A}$ be a locally compact join-formal cover and $\mathcal{B}$ be an inductively generated bottom-formal cover. Then, for any list $l' \in List(A \times B)$ and any $V \subseteq B$, if $(a,b) \,\varepsilon\, l'$ and $b \lhd_{\mathcal{B}} V$ then $\widetilde{l'} \lhd_{(\mathcal{A} \to \mathcal{B})^{\perp}} \{(a,y) \mid y \,\varepsilon\, V\}$.*

4.0.2 Application and abstraction

Now we show that the formal cover $\mathcal{B}^{\mathcal{A}} \equiv (\mathcal{A} \to \mathcal{B})^{\perp}$ is the exponential of the locally compact join-formal cover $\mathcal{A}$ over the inductively generated bottom-formal cover $\mathcal{B}$. From a categorical point of view, we prove that for any locally compact join-formal cover $\mathcal{A}$ the functor $- \times \mathcal{A} : \mathsf{FTop}_{\mathsf{i}\perp}^{-} \Rightarrow \mathsf{FTop}_{\mathsf{i}\perp}^{-}$ has got a right adjoint $(-)^{\mathcal{A}} : \mathsf{FTop}_{\mathsf{i}\perp}^{-} \Rightarrow \mathsf{FTop}_{\mathsf{i}\perp}^{-}$. Equivalently, this amounts to define, for any locally compact join-formal cover $\mathcal{A}$ and any inductively generated bottom-formal cover $\mathcal{B}$, an *application relation* Ap from $(\mathcal{B}^{\mathcal{A}}) \times \mathcal{A}$ to $\mathcal{B}$ such that for any continuous relation F from $\mathcal{C} \times \mathcal{A}$ to $\mathcal{B}$ there exists a continuous relation $\Lambda(F)$ from $\mathcal{C}$ to $\mathcal{B}^{\mathcal{A}}$, called *abstraction* of F, such that, for any continuous relation G from $\mathcal{C}$ to $\mathcal{B}^{\mathcal{A}}$, the following equations are satisfied

$$\mathsf{Ap} * \langle \Lambda(F) * \Pi_1, \Pi_2 \rangle = F$$
$$\Lambda(\mathsf{Ap} * \langle G * \Pi_1, \Pi_2 \rangle) = G$$

For any $l \in List(A \times B)^{\perp}$, $a \in A$, $b \in B$ and $c \in C$, we propose the following definition for the application

$$\mathsf{Ap} \equiv \widetilde{\mathsf{Ap}}^{\lhd} \qquad \text{with} \qquad (l,a)\,\widetilde{\mathsf{Ap}}\,b \equiv l \lhd_{(\mathcal{A} \to \mathcal{B})^{\perp}} (a,b) \cdot \mathsf{nil}$$

where we simply write $(a,b) \cdot \mathsf{nil}$ instead of $\widetilde{(a,b) \cdot \mathsf{nil}}$ (and we do the same from now on).

Then, we propose the following definition for the abstraction

$$\Lambda(F) \equiv (\widetilde{\Lambda(F)}^{\perp})^{\lhd}$$

where

$$c\,\widetilde{\Lambda(F)}\,l \equiv \begin{cases} (\forall (a,b) \,\varepsilon\, l')\ a \ll \{x \in A \mid (c,x)\,F\,b\} & \text{if } l = \widetilde{l'} \text{ for } l' \in List(A \times B) \\ c \lhd_{\mathcal{C}} \perp_{\mathcal{C}} & \text{if } l = \perp_{(\mathcal{A} \to \mathcal{B})^{\perp}} \end{cases}$$

To prove that Ap is a continuous relation, it is sufficient to show that $\widetilde{\mathsf{Ap}}$ satisfies all the conditions in proposition 2.8, being Ap the saturation of $\widetilde{\mathsf{Ap}}$. Before doing this prove the following useful lemma.

Lemma 4.6 *Let $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \vee_{\mathcal{A}}, \perp_{\mathcal{A}}, \lhd_{\mathcal{A}})$ be a locally compact join-formal cover and $\mathcal{B} \equiv (B, \leq_{\mathcal{B}}, \perp_{\mathcal{B}}, \lhd_{\mathcal{B}})$ be an inductively generated bottom-formal cover. Then, for any subset U of B*

$$\widetilde{\mathsf{Ap}}^{-}(U \cup \{\perp_{\mathcal{B}}\}) \lhd_{(\mathcal{A} \to \mathcal{B})^{\perp} \times \mathcal{A}} \widetilde{\mathsf{Ap}}^{-}(U)$$

Proof. Suppose $(l,a) \,\varepsilon\, \widetilde{\mathsf{Ap}}^{-}(U \cup \{\bot_{\mathcal{B}}\})$. Then, either there exists $b \in U$ such that $(l,a)\,\widetilde{\mathsf{Ap}}\, b$, and hence $(l,a) \lhd_{(\mathcal{A}\to\mathcal{B})^{\bot}\times \mathcal{A}} \widetilde{\mathsf{Ap}}^{-}(U)$ by *reflexivity*, or $(l,a)\,\widetilde{\mathsf{Ap}}\, \bot_{\mathcal{B}}$. In this latter case, by definition we get $l \lhd_{(\mathcal{A}\to\mathcal{B})^{\bot}} (a, \bot_{\mathcal{B}})$ and hence by *transitivity* with *wb-bottom continuity* axiom we obtain $l \lhd_{(\mathcal{A}\to\mathcal{B})^{\bot}} \{(\bot_{\mathcal{A}}, \bot_{\mathcal{B}}) \mid a \lhd_{\mathcal{A}} \bot_{\mathcal{A}}\}$. Then, by lemma 2.11 we deduce $(l,a) \lhd_{(\mathcal{A}\to\mathcal{B})^{\bot}\times\mathcal{A}} \{(\bot_{\mathcal{A}}, \bot_{\mathcal{B}}) \mid a \lhd_{\mathcal{A}} \bot_{\mathcal{A}}\} \times \{a\}$. Now, note that $\{(\bot_{\mathcal{A}}, \bot_{\mathcal{B}}) \mid a \lhd_{\mathcal{A}} \bot_{\mathcal{A}}\} \times \{a\} \lhd_{(\mathcal{A}\to\mathcal{B})^{\bot}\times\mathcal{A}} \emptyset$ and hence by *transitivity* we conclude $(l,a) \lhd_{(\mathcal{A}\to\mathcal{B})^{\bot}\times\mathcal{A}} \widetilde{\mathsf{Ap}}^{-}(U)$. Therefore, we have proved $\widetilde{\mathsf{Ap}}^{-}(U \cup \{\bot_{\mathcal{B}}\}) \lhd_{(\mathcal{A}\to\mathcal{B})^{\bot}\times\mathcal{A}} \widetilde{\mathsf{Ap}}^{-}(U)$ as claimed.

Lemma 4.7 *Let $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \vee_{\mathcal{A}}, \bot_{\mathcal{A}}, \lhd_{\mathcal{A}})$ be a locally compact join-formal cover and $\mathcal{B} \equiv (B, \leq_{\mathcal{B}}, \bot_{\mathcal{B}}, \lhd_{\mathcal{B}})$ be an inductively generated bottom-formal cover. Then, $\widetilde{\mathsf{Ap}}$ is a relation from $(\mathcal{A} \to \mathcal{B})^{\bot} \times \mathcal{A}$ to $\mathcal{B}$ satisfying* totality*,* $\leq$-convergence*,* axiom continuity*,* $\leq$-continuity *and* $\leq$-saturation*.*

Proof. We prove the various conditions as follows:

- (*totality*) Let $a \in A$ and $l \in (List(A \times B)^{\bot}$. For any $d \ll a$, by *wb-totality axiom* and *transitivity* on $l \lhd_{(\mathcal{A}\to\mathcal{B})^{\bot}} \mathsf{nil}$ obtained from $l \leq \mathsf{nil}$ by *reflexivity* and *$\leq$-left*, we get

  $$l \lhd_{(\mathcal{A}\to\mathcal{B})^{\bot}} \{(x_1,y_1)\cdot..\cdot(x_n,y_n)\cdot\mathsf{nil} \mid y_1,..,y_n \in B \,\&\, d \lhd_{\mathcal{A}} \{x_1,..,x_n\}\}$$

  and by lemma 2.11

  $$(l,d) \lhd_{(\mathcal{A}\to\mathcal{B})^{\bot}\times\mathcal{B}} \{(x_1,y_1)\cdot..\cdot(x_n,y_n)\cdot\mathsf{nil} \mid y_1,..,y_n \in B \,\&\, d \lhd_{\mathcal{A}} \{x_1,..,x_n\}\} \times \{d\}$$

  and hence for every $(x_1,y_1)\cdot..\cdot(x_n,y_n)\cdot\mathsf{nil}$ in the above subset covering l we get

  $$((x_1,y_1)\cdot..\cdot(x_n,y_n)\cdot\mathsf{nil}, d) \lhd_{(\mathcal{A}\to\mathcal{B})^{\bot}\times\mathcal{A}} \{(x_1,y_1)\cdot..\cdot(x_n,y_n)\cdot\mathsf{nil}\} \times \{x_1,..,x_n\}$$

  Then, for any $i = 1,..,n$ we obtain $((x_1,y_1)\cdot..\cdot(x_n,y_n)\cdot\mathsf{nil}, x_i) \lhd_{\mathcal{A}\to\mathcal{B}^{\bot}} ((x_i,y_i)\cdot\mathsf{nil}, x_i)$ by lemma 2.11 from $(x_1,y_1)\cdot..\cdot(x_n,y_n)\cdot\mathsf{nil} \lhd_{(\mathcal{A}\to\mathcal{B})^{\bot}} (x_i,y_i)\cdot\mathsf{nil}$ which follows by *$\leq$-left* and *reflexivity* from $(x_1,y_1)\cdot..\cdot(x_n,y_n)\cdot\mathsf{nil} \leq (x_i,y_i)\cdot\mathsf{nil}$.

  Now, since $((x_i,y_i)\cdot\mathsf{nil}, x_i)\,\widetilde{\mathsf{Ap}}\, y_i$ holds with $y_i \,\varepsilon\, B$, we get $((x_i,y_i)\cdot\mathsf{nil}, x_i) \lhd_{(\mathcal{A}\to\mathcal{B})^{\bot}\times\mathcal{A}} \widetilde{\mathsf{Ap}}^{-}(B)$, and by *transitivity* we obtain $((x_1,y_1)\cdot..\cdot(x_n,y_n)\cdot\mathsf{nil}, d) \lhd_{(\mathcal{A}\to\mathcal{B})^{\bot}\times\mathcal{A}} \widetilde{\mathsf{Ap}}^{-}(B)$. Since this holds for any list $(x_1,y_1)\cdot..\cdot(x_n,y_n)\cdot\mathsf{nil}$ under the above hypothesis, we conclude $(l,d) \lhd_{(\mathcal{A}\to\mathcal{B})^{\bot}\times\mathcal{A}} \widetilde{\mathsf{Ap}}^{-}(B)$. Since this holds for any $d \ll a$, from $(l,a) \lhd_{(\mathcal{A}\to\mathcal{B})^{\bot}\times\mathcal{B}} \{(l,d) \mid d \ll a\}$, obtained by lemma 2.11 from $a \lhd_{\mathcal{A}} wb(a)$, by *transitivity* we conclude $(l,a) \lhd_{(\mathcal{A}\to\mathcal{B})^{\bot}\times\mathcal{A}} \widetilde{\mathsf{Ap}}^{-}(B)$.

- (*$\leq$-convergence*) Suppose $(l,a)\,\widetilde{\mathsf{Ap}}\, b$ and $(l,a)\,\widetilde{\mathsf{Ap}}\, b'$. The assumptions yields $l \lhd_{(\mathcal{A}\to\mathcal{B})^{\bot}} (a,b)\cdot\mathsf{nil}$ and $l \lhd_{(\mathcal{A}\to\mathcal{B})^{\bot}} (a,b')\cdot\mathsf{nil}$. Then, by lemma 4.3 we get $l \lhd_{(\mathcal{A}\to\mathcal{B})^{\bot}} (a,b)\cdot(a,b')\cdot\mathsf{nil}$ and by *transitivity* and *wb-$\leq$-convergence* axiom we obtain

  $$l \lhd_{(\mathcal{A}\to\mathcal{B})^{\bot}} \{\,(x_1,y_1)\cdot..\cdot(x_n,y_n)\cdot\mathsf{nil} \mid y_i \,\varepsilon\, (b \downarrow_{\leq} b') \cup \{\bot_{\mathcal{B}}\} \text{ for } i = 1,..,n \,\&\, d \lhd_{\mathcal{A}} \{x_1,..,x_n\}\,\}$$

  and by lemma 2.11 we deduce

  $$(l,d) \lhd_{(\mathcal{A}\to\mathcal{B})^{\bot}} \{(x_1,y_1)\cdot..\cdot(x_n,y_n)\cdot\mathsf{nil} \mid y_i \varepsilon (b \downarrow_{\leq} b') \cup \{\bot_{\mathcal{B}}\} \text{ for } i = 1,..,n \,\&\, d \lhd_{\mathcal{A}} \{x_1,..,x_n\}\} \times \{d\}$$

  and hence, for every $(x_1,y_1)\cdot..\cdot(x_n,y_n)\cdot\mathsf{nil}$ in the above subset covering l we obtain

  $$((x_1,y_1)\cdot..\cdot(x_n,y_n)\cdot\mathsf{nil}, d) \lhd_{(\mathcal{A}\to\mathcal{B})^{\bot}\times\mathcal{A}} \{(x_1,y_1)\cdot..\cdot(x_n,y_n)\cdot\mathsf{nil}\} \times \{x_1,..,x_n\}$$

  Then, for any $i = 1,..,n$, we get $((x_1,y_1)\cdot..\cdot(x_n,y_n)\cdot\mathsf{nil}, x_i) \lhd_{(\mathcal{A}\to\mathcal{B})^{\bot}} ((x_i,y_i)\cdot\mathsf{nil}, x_i)$ since from $(x_1,y_1)\cdot..\cdot(x_n,y_n)\cdot\mathsf{nil} \leq (x_i,y_i)\cdot\mathsf{nil}$ by *$\leq$-left* and *reflexivity* $(x_1,y_1)\cdot..\cdot(x_n,y_n)\cdot\mathsf{nil} \lhd_{(\mathcal{A}\to\mathcal{B})^{\bot}} (x_i,y_i)\cdot\mathsf{nil}$ follows. Now, we obtain $((x_i,y_i)\cdot\mathsf{nil}, x_i) \lhd_{(\mathcal{A}\to\mathcal{B})^{\bot}\times\mathcal{A}} \widetilde{\mathsf{Ap}}^{-}((b \downarrow b') \cup \{\bot_{\mathcal{B}}\})$ since $((x_i,y_i)\cdot\mathsf{nil}, x_i)\,\widetilde{\mathsf{Ap}}\, y_i$ holds with $y_i \,\varepsilon\, (b \downarrow_{\leq} b') \cup \bot_{\mathcal{B}}$, and $((x_1,y_1)\cdot..\cdot(x_n,y_n)\cdot\mathsf{nil}, d) \lhd_{(\mathcal{A}\to\mathcal{B})^{\bot}\times\mathcal{A}} \widetilde{\mathsf{Ap}}^{-}((b \downarrow b') \cup \{\bot_{\mathcal{B}}\})$ follows by *transitivity.* Since this holds for any list $(x_1,y_1)\cdot..\cdot(x_n,y_n)\cdot\mathsf{nil}$

under the above hypothesis, we conclude $(l,d) \lhd_{(\mathcal{A}\to\mathcal{B})^\perp\times\mathcal{A}} \widetilde{\mathsf{Ap}}^-((b\downarrow b')\cup\{\bot_\mathcal{B}\})$. Lastly, since this holds for any $d \ll a$, from $(l,a) \lhd_{(\mathcal{A}\to\mathcal{B})^\perp\times\mathcal{B}} \{(l,d) \mid d \ll a\}$, obtained by lemma 2.11 from $a \lhd_\mathcal{A} wb(a)$, we get $(l,a) \lhd_{(\mathcal{A}\to\mathcal{B})^\perp\times\mathcal{A}} \widetilde{\mathsf{Ap}}^-((b\downarrow b')\cup\{\bot_\mathcal{B}\})$ by *transitivity*. Then, by lemma 4.6 we conclude $(l,a) \lhd_{(\mathcal{A}\to\mathcal{B})^\perp\times\mathcal{A}} \widetilde{\mathsf{Ap}}^-(b\downarrow b')$.

- ($\leq$-*continuity*) Suppose $(l,a)\,\widetilde{\mathsf{Ap}}\,b$ and $b \leq b'$. Then, we get $l \lhd_{(\mathcal{A}\to\mathcal{B})^\perp} (a,b)\cdot\mathsf{nil}$ and by *transitivity* on *wb-$\leq$-continuity* axiom we conclude $l \lhd_{(\mathcal{A}\to\mathcal{B})^\perp} (a,b')\cdot\mathsf{nil}$, that is $(l,a)\,\widetilde{\mathsf{Ap}}\,b'$.
- (*axiom continuity*) The proof is analogous to $\leq$-*convergence* thanks to *wb-axiom-continuity* axiom.
- ($\leq$-*saturation*) Suppose $(s,a') \leq (l,a)$ and $(l,a)\,\widetilde{\mathsf{Ap}}\,b$. Then, by definition $s \leq l$ and $a' \leq a$ and $l \lhd_{(\mathcal{A}\to\mathcal{B})^\perp} (a,b)\cdot\mathsf{nil}$, from which by *transitivity* with *wb-$\leq$-saturation* axiom we get $l \lhd_{(\mathcal{A}\to\mathcal{B})^\perp} (a',b)\cdot\mathsf{nil}$. Now by $\leq$-*left* from $s \leq l$ we conclude $s \lhd_{(\mathcal{A}\to\mathcal{B})^\perp} (a',b)\cdot\mathsf{nil}$, that is $(s,a)\,\widetilde{\mathsf{Ap}}\,b$.

Corollary 4.8 *Let $\mathcal{A}$ be a locally compact join-formal cover and $\mathcal{B}$ be an inductively generated bottom-formal cover. Then,* Ap *is a continuous relation from $(\mathcal{A}\to\mathcal{B}^\perp)\times\mathcal{A}$ to $\mathcal{B}$.*

Proof. By lemma 4.7 and lemma 2.8 we conclude that Ap is a continuous relation.

To prove that the *abstraction* $\Lambda(F)$ of a continuous relation F is a continuous relation, too, being $\Lambda(F)$ the saturation of $(\widetilde{\Lambda(F)})^\perp$, by lemma 2.21 and lemma 2.9 it is sufficient to show that $\widetilde{\Lambda(F)}$ satisfies all the conditions in lemma 2.9.

Before showing this, we prove this useful lemma.

Lemma 4.9 *Let $\mathcal{A}$ be a locally compact join-formal cover and $\mathcal{C}$ and $\mathcal{B}$ be inductively generated bottom-formal covers. Suppose that F is any continuous relation from $\mathcal{C}\times\mathcal{A}$ to $\mathcal{B}$. Then $a \ll \{x \mid (c,x)\,F\,b\}$ if and only if $\exists x\ a \ll x\ (c,x)\,F\,b$.*

Proof. Suppose $a \ll \{x \mid (c,x)\,F\,b\}$. By the *interpolation property* of lemma 3.4 we find a' such that $a \ll a'$ and $a' \ll \{x \mid (c,x)\,F\,b\}$. Then, by lemma 3.6 either $a' \lhd_\mathcal{A} \{u_1,..,u_n\}$ such that $(c,u_i)\,F\,b$ for $i = 1,..,n$ from which by *saturation* of F on $(c,a') \lhd_{\mathcal{C}\times\mathcal{A}} \{c\}\times\{u_1,..,u_n\}$ – obtained by lemma 2.11 – we conclude $(c,a')\,F\,b$ as claimed, or $a' \lhd_\mathcal{A} \bot_\mathcal{A}$ and hence $(c,a') \lhd_{\mathcal{C}\times\mathcal{A}} \emptyset$ from which by lemma 2.20 we conclude $(c,a')\,F\,b$.

Conversely, suppose that there exists a' such that $a \ll a'$ and $(c,a')\,F\,b$. Then $a' \lhd_\mathcal{A} \{x \mid (c,x)\,F\,b\}$ and by lemma 3.4 we conclude $a \ll \{x \mid (c,x)\,F\,b\}$.

Lemma 4.10 *Let $\mathcal{A}$ be a locally compact join-formal cover and $\mathcal{C}$ and $\mathcal{B}$ be inductively generated bottom-formal covers. Suppose that F is any continuous relation from $\mathcal{C}\times\mathcal{A}$ to $\mathcal{B}$. Then $\widetilde{\Lambda(F)}$ is a relation from $\mathcal{C}$ to $\mathcal{A}\to\mathcal{B}$ satisfying* totality, $\leq$-convergence, non-loc-axiom-continuity, $\leq$-continuity *and* $\leq$-saturation.

Proof. We prove the various conditions as follows:

- (*totality*) Let $c \in C$. Then, $c \lhd_\mathcal{C} \widetilde{\Lambda(F)}^-(List(A\times B))$ follows by *reflexivity* since $c\,\widetilde{\Lambda(F)}\,\mathsf{nil}$ holds by intuitionistic logic.
- ($\leq$-*convergence*) If $c\,\widetilde{\Lambda(F)}\,l_1$ and $c\,\widetilde{\Lambda(F)}\,l_2$ then by definition we immediately get $c\,\widetilde{\Lambda(F)}\,l_1\cdot l_2$. and hence by *reflexivity* $c \lhd_\mathcal{C} \widetilde{\Lambda(F)}^-(l_1\downarrow l_2)$ since $l_1\cdot l_2 \in l_1\downarrow l_2$.
- (*non-loc-axiom-continuity*) We have to check that if $c\,\widetilde{\Lambda(F)}\,l$ and $j \in J_{\mathcal{A}\to\mathcal{B}}(l)$ then $c \lhd_\mathcal{C} \widetilde{\Lambda(F)}^-(C_{\mathcal{A}\to\mathcal{B}}(l,j))$ holds. The proof depends on the particular shape of the axiom indexed by j.

- (*wb-bottom continuity*) Suppose $c\,\widetilde{\Lambda(F)}\,(a,\bot_{\mathcal{B}})\cdot\mathsf{nil}$, that is $a\ll\{x\in A\mid(c,x)\,F\ \bot_{\mathcal{B}}\}$. Since $F^{-}(\bot_{\mathcal{B}})\ \lhd_{\mathcal{C}\times\mathcal{A}}\ F^{-}(\emptyset)$ by *continuity* of F, then for any $a'\in A$ such that $(c,a')\,F\ \bot_{\mathcal{B}}$ we get $(c,a')\ \lhd_{\mathcal{C}\times\mathcal{A}}\ \emptyset$, from which by *transitivity* $(c,a')\ \lhd_{\mathcal{C}\times\mathcal{A}}\ \{(x,\bot_{\mathcal{A}})\mid x\leq c\}$ and on this we apply lemma 3.8 since its hypothesis are satisfied. Therefore, for any $d\ll a'$ we get a subset U of C such that $c\lhd_{\mathcal{C}}U$ and, for every $u\,\varepsilon\,U$ there exists $e_1,e_1',..,e_n,e_n'\in A$ such that $e_j\ll e_j'$ for $j=1,..,n$ and $d\lhd_{\mathcal{A}}\ \{e_1,..,e_n\}$ and $(u,e_j')\,\varepsilon\,\{(x,\bot_{\mathcal{A}})\mid x\leq c\}$ for $j=1,..,n$. Now, notice that for every $j=1,..,n$ then $e_j'=\bot_{\mathcal{A}}$ and hence $d\lhd_{\mathcal{A}}\emptyset$. Since $a'\lhd_{\mathcal{A}}wb(a')$ and for any $d\ll a'$ we have proved $d\lhd_{\mathcal{A}}\emptyset$, we conclude $a'\lhd_{\mathcal{A}}\emptyset$. Since this holds for any $a'\,\varepsilon\,\{x\,\varepsilon\,A\mid(c,x)\,F\ \bot_{\mathcal{B}}\}$ we get $\{x\in A\mid(c,x)\,F\ \bot_{\mathcal{B}}\}\lhd_{\mathcal{A}}\emptyset$ and by lemma 3.4 we obtain $a\ll\emptyset$ and hence $a\lhd_{\mathcal{A}}\emptyset$. Therefore, from this, since $c\,\widetilde{\Lambda(F)}\,(\bot_{\mathcal{A}},\bot_{\mathcal{B}})$ trivially holds, we conclude $c\lhd_{\mathcal{C}}\widetilde{\Lambda(F)}^{-}(\{(\bot_{\mathcal{A}},\bot_{\mathcal{B}})\mid a\lhd_{\mathcal{A}}\bot_{\mathcal{A}}\})$.
- (*wb-bottom-saturation*) If $c\,\widetilde{\Lambda(F)}\,\mathsf{nil}$ we need to prove $c\,\widetilde{\Lambda(F)}\,(\bot_{\mathcal{A}},b)$, which trivially holds since $\bot_{\mathcal{A}}\ll\{x\in A\mid(c,x)\,F\,b\}$ for any $b\in B$.
- (*wb-totality axiom*) Suppose $c\,\widetilde{\Lambda(F)}\,\mathsf{nil}$ and $d\ll a$. Then by *totality* of F we get $(c,a)\ \lhd_{\mathcal{C}\times\mathcal{A}}\ F^{-}(B)$, that is $a\lhd_{\mathcal{A}}\{(c,x)\mid\exists b\in B\ (c,x)\,F\,b\}$. By lemma 3.6 applied on $d\ll a$ either there exists $x_i\ll x_1',..,x_n\ll x_n'$ and $y_1,..,y_n$ such that $(c,x_j)\,F\,y_j$ for $i=1,..,n$, that is $c\,\widetilde{\Lambda(F)}\,(x_i,y_i)\cdot\mathsf{nil}$ and hence $c\,\widetilde{\Lambda(F)}\,(x_1,y_1)\cdot..\cdot(x_n,y_n)\cdot\mathsf{nil}$ by lemma 4.4, with $d\lhd_{\mathcal{A}}\{x_1,..,x_n\}$, or $d\lhd_{\mathcal{A}}\emptyset$ from which by lemma 2.20 $(c,\bot_{\mathcal{A}})\,F\ \bot_{\mathcal{B}}$ and hence $c\,\widetilde{\Lambda(F)}\,(\bot_{\mathcal{A}},\bot_{\mathcal{B}})\cdot\mathsf{nil}$. In both cases we conclude $c\lhd_{\mathcal{C}}\widetilde{\Lambda(F)}^{-}(\{(x_1,y_1)\cdot..\cdot(x_n,y_n)\cdot\mathsf{nil}\mid y_1,..,y_n\in B\ \&\ d\lhd_{\mathcal{A}}\{x_1,..,x_n\}\})$ by *reflexivity*.
- (*wb-continuity axiom*) Suppose $c\,\widetilde{\Lambda(F)}\,(a,b)\cdot\mathsf{nil}$ and $j\in J(b)$ and $d\ll a$. Then, by lemma 4.9 there exists a' such that $(c,a')\,F\,b$ and $a\ll a'$. Then, $(c,a')\ \lhd_{\mathcal{C}\times\mathcal{A}}\ F^{-}(C(b,j))$ follows by *axiom-continuity* of F from which we get $(c,a')\ \lhd_{\mathcal{C}\times\mathcal{A}}\ F^{-}(C(b,j)\cup\{\bot_{\mathcal{B}}\})$ by *continuity* of F since $C(b,j)\lhd_{\mathcal{B}}C(b,j)\cup\{\bot_{\mathcal{B}}\}$.

  Now, note that $F^{-}(C(b,j)\cup\{\bot_{\mathcal{B}}\})$ satisfies the hypothesis of lemma 3.8: indeed the first condition is satisfied by $\leq$ *saturation*, while the second is satisfied since $(c,\bot_{\mathcal{A}})\lhd_{\mathcal{C}\times\mathcal{A}}\bot_{\mathcal{C}\times\mathcal{A}}$ and by lemma 2.20 we get $(c,\bot_{\mathcal{A}})\,F\ \bot_{\mathcal{B}}$, that is $(c,\bot_{\mathcal{A}})\,\varepsilon\,F^{-}(C(b,j)\cup\{\bot_{\mathcal{B}}\})$. Then, we apply lemma 3.8 on $d\ll a'$, which follows from $d\ll a$, and $(c,a')\ \lhd_{\mathcal{C}\times\mathcal{A}}\ F^{-}(C(b,j)\cup\{\bot_{\mathcal{B}}\})$ by getting a subset V of C such that $c\lhd_{\mathcal{C}}V$ and for every $v\,\varepsilon\,V$ there exists $e_1\ll e_1',..,e_n\ll e_n'\in A$ with $d\lhd_{\mathcal{A}}\{e_1,..,e_n\}$ and $(v,e_i')\,\varepsilon\,F^{-}(C(b,j)\cup\{\bot_{\mathcal{B}}\})$ for $i=1,..,n$. This means that there exist $y_1,..,y_n\,\varepsilon\,C(b,j)\cup\{\bot_{\mathcal{B}}\}$ such that $(v,e_i')\,F\,y_i$ for $i=1,..,n$, and hence $v\,\widetilde{\Lambda(F)}\,(e_i,y_i)\cdot\mathsf{nil}$ by lemma 4.9 for $i=1,..,n$. Now, by lemma 4.4 we get $v\widetilde{\Lambda(F)}(e_1,y_1)\cdot..\cdot(e_n,y_n)\cdot\mathsf{nil}$, from which we conclude

  $$v\,\varepsilon\,\widetilde{\Lambda(F)}^{-}\ \{(x_1,v_1)\cdot..\cdot(x_n,v_n)\mid v_i\,\varepsilon\,C(b,j)\cup\{\bot_{\mathcal{B}}\}\text{ for }i=1,..,n\ \&\ d\lhd_{\mathcal{A}}\{x_1,..,x_n\}\}$$

  Since this holds for every $v\,\varepsilon\,V$ by *transitivity* we conclude

  $$c\lhd_{\mathcal{C}}\widetilde{\Lambda(F)}^{-}\ \{(x_1,v_1)\cdot..\cdot(x_n,v_n)\mid v_i\,\varepsilon\,C(b,j)\cup\{\bot_{\mathcal{B}}\}\text{ for }i=1,..,n\ \&\ d\lhd_{\mathcal{A}}\{x_1,..,x_n\}\}$$

- (*wb-$\leq$-convergence*) The proof is analogous to that for *wb-axiom-continuity* axiom.
- (*wb-finite-saturation*) Suppose $c\,\widetilde{\Lambda(F)}\,(d_1,b)\cdot..\cdot(d_n,b)\cdot\mathsf{nil}$ for $a\lhd_{\mathcal{A}}\{d_1,..,d_n\}$ for some $a\in A$. Then, by definition $d_i\ll\{x\in A\mid(c,x)\,F\,b\}$ for $i=1,..,n$ and hence $\{d_1,..,d_n\}\ll\{x\in A\mid(c,x)\,F\,b\}$ by lemma 3.7. Therefore, by lemma 3.4 we conclude $a\ll\{x\in A\mid(c,x)\,F\,b\}$, that is $c\,\widetilde{\Lambda(F)}\,(a,b)\cdot\mathsf{nil}$ and by *reflexivity* $c\lhd_{\mathcal{C}}\widetilde{\Lambda(F)}^{-}\,(a,b)\cdot\mathsf{nil}$.
- (*wb-$\leq$-continuity*) Suppose $c\,\widetilde{\Lambda(F)}\,(a,b)\cdot\mathsf{nil}$, that is $a\ll\{x\in A\mid(c,x)\,F\,b\}$, and $b\leq b'$. By *$\leq$-continuity* of F we get $\{x\in A\mid(c,x)\,F\,b\}\subseteq\{x\in A\mid(c,x)\,F\,b'\}$ and hence by lemma 3.4 we conclude $a\ll\{x\in A\mid(c,x)\,F\,b'\}$, that is $c\,\widetilde{\Lambda(F)}\,(a,b')\cdot\mathsf{nil}$ as claimed.

- (*interpolation*) Suppose $c\,\widetilde{\Lambda(F)}\,(a,b)\cdot\mathsf{nil}$, that is $a \ll \{x \in A \mid (c,x)\,F\,b\}$. By the interpolation property in lemma 3.4 there exists a' such that $a \ll a'$ and $a' \ll \{x \in A \mid (c,x)\,F\,b\}$, that is $c\,\widetilde{\Lambda(F)}\,(a',b)\cdot\mathsf{nil}$, and hence by *reflexivity* $c \lhd_{\mathcal{C}} \widetilde{\Lambda(F)}^{-}(\{(x,b)\cdot\mathsf{nil} \mid a \ll x\})$ as claimed.

- ($\leq$-*continuity*) If $l \leq s$ and $c\,\widetilde{\Lambda(F)}\,l$ holds then $c\,\widetilde{\Lambda(F)}\,s$ follows since if $(a,b)\,\varepsilon\,l$ then $(a,b)\,\varepsilon\,s$ by definition of $\leq$ on $List(A \times B)$.

- ($\leq$-*saturation*) Suppose $c' \leq c$ and $c\,\widetilde{\Lambda(F)}\,l$. Then, for all $(a,b)\,\varepsilon\,l$ we have $a \ll \{x \in A \mid (c,x)\,F\,b\}$ by definition. Now, note that $\{x \in A \mid (c,x)\,F\,b\} \subseteq \{x \in A \mid (c',x)\,F\,b\}$ since F enjoys $\leq$-*saturation* on each $(c',x) \leq (c,x)$. Hence we obtain $a \ll \{x \in A \mid (c',x)\,F\,b\}$ by lemma 3.4. Therefore, we conclude $c'\,\widetilde{\Lambda(F)}\,l$.

Corollary 4.11 *Let $\mathcal{A}$ be a locally compact join-formal cover and $\mathcal{C}$ and $\mathcal{B}$ be inductively generated bottom-formal covers. Suppose that F is any continuous relation from $\mathcal{C} \times \mathcal{A}$ to $\mathcal{B}$. Then $\Lambda(F)$ is a continuous relation from $\mathcal{C}$ to $(\mathcal{A} \to \mathcal{B})^{\perp}$.*

Proof. By lemma 4.10, lemma 2.9 and lemma 2.21 we conclude that $\Lambda(F)$ is a continuous relation.

Note that $\widetilde{\Lambda(F)}$ enjoys this useful property:

Lemma 4.12 *Let $\mathcal{A}$ be a locally compact join-formal cover and $\mathcal{C}$ and $\mathcal{B}$ be inductively generated bottom-formal covers. Suppose that F is any continuous relation from $\mathcal{C} \times \mathcal{A}$ to $\mathcal{B}$. If $c\,\Lambda(F)\,(a,b)\cdot\mathsf{nil}$ holds, then $(c,a)\,F\,b$ holds, too.*

Proof. By hypothesis we have that $c \lhd_{\mathcal{C}} U$ where $U \equiv \{y \in C \mid y\,(\widetilde{\Lambda(F)})^{\perp}\,(a,b)\cdot\mathsf{nil}\}$. For every u such that $u\,\widetilde{\Lambda(F)}^{\perp}\,(a,b)\cdot\mathsf{nil}$ by definition we get $a \ll \{x \in A \mid (u,x)\,F\,b\}$ and hence $a \lhd_{\mathcal{A}} \{x \in A \mid (u,x)\,F\,b\}$ and by lemma 2.11 also $(u,a) \lhd_{\mathcal{C}\times\mathcal{A}} \{u\} \times \{x \in A \mid (u,x)\,F\,b\}$. Hence, by *saturation* of F we get $(u,a)\,F\,b$. Moreover, by lemma 2.11 we deduce $(c,a) \lhd_{\mathcal{C}\times\mathcal{A}} U \times \{a\}$ and since $(u,a)\,F\,b$ holds for all $u \in U$, we conclude by *saturation* of F that $(c,a)\,F\,b$ holds.

To finish the proof that the formal cover $\mathcal{B}^{\mathcal{A}} \equiv (\mathcal{A} \to \mathcal{B})^{\perp}$ is the exponential cover of $\mathcal{B}$ over $\mathcal{A}$ we need to show the validity of the adjunction equations about abstraction and application.

Before proving the equations we show the following lemma.

Lemma 4.13 *Let $\mathcal{A}$ be a locally compact join-formal cover and $\mathcal{C}$ and $\mathcal{B}$ be inductively generated bottom-formal covers. For any continuous relation G from $\mathcal{C}$ to $\mathcal{B}^{\mathcal{A}}$ we can prove:*

1. $\mathsf{Ap} * \langle G * \Pi_1, \Pi_2\rangle = (\widetilde{\mathsf{Ap}} \cdot \langle G \cdot p_1, p_2\rangle)^{\lhd}$
2. *if* $(c,a)\ \mathsf{Ap} * \langle G * \Pi_1, \Pi_2\rangle\ b$ *then*

$$(c,a) \lhd_{\mathcal{C}\times\mathcal{A}} \{\, (x,y) \in C \times A \mid x\,G\,(y,b)\cdot\mathsf{nil}\,\}$$

Proof. (1) Observe that by lemma 2.12

$$\begin{array}{rcl} \mathsf{Ap} * \langle G * \Pi_1, \Pi_2\rangle & = & \mathsf{Ap} * \langle G * p_1^{\lhd}, p_2^{\lhd}\rangle \\ \text{by definition of } * & = & \mathsf{Ap} * \langle (G \cdot p_1^{\lhd})^{\lhd}, p_2^{\lhd}\rangle \\ \text{by lemma 2.10} & = & \mathsf{Ap} * \langle (G \cdot p_1)^{\lhd}, p_2^{\lhd}\rangle \\ \text{by lemma 2.13} & = & \mathsf{Ap} * (\langle G \cdot p_1, p_2\rangle)^{\lhd} \\ \text{by definition of } * & = & (\widetilde{\mathsf{Ap}}^{\lhd} \cdot (\langle G \cdot p_1, p_2\rangle)^{\lhd})^{\lhd} \\ \text{by lemma 2.10} & = & (\widetilde{\mathsf{Ap}} \cdot \langle G \cdot p_1, p_2\rangle)^{\lhd} \end{array}$$

(2) Suppose $(c,a)\ (\mathsf{Ap} * \langle G * \Pi_1, \Pi_2\rangle)^{\lhd}\,b$. Then, we get $(c,a)\ (\widetilde{\mathsf{Ap}} \cdot \langle G \cdot p_1, p_2\rangle)^{\lhd}\,b$ from point (1). Now, for any $(c',a')\varepsilon\{\,(x,y) \in C \times A \mid (x,y)\,\widetilde{\mathsf{Ap}} \cdot \langle G \cdot p_1, p_2\rangle\,b\,\}$ there exist $l' \in \mathsf{List}(A \times B)$ and $a'' \in A$ such that $(l',a'')\,\widetilde{\mathsf{Ap}}\,b$, that is $l' \lhd_{\mathcal{A}\to\mathcal{B}} (a'',b)\cdot\mathsf{nil}$, and $(c',a')\,\langle G \cdot p_1, p_2\rangle\,(l',a'')$ from which we get $(c',a')\,G \cdot p_1\,l'$ and $(c',a')\,p_2\,a''$, that is $a' \lhd_{\mathcal{A}} a''$. From $(c',a')\,G \cdot p_1\,l'$ by definition there exists $c'' \in C$ such that

$c'' G\, l'$ and $(c', a')\, p_1\, c''$, that is $c' \lhd_{\mathcal{C}} c''$. Now, by *saturation* of G we get $c' G\, l'$. Since $a' \lhd_{\mathcal{A}} a''$ and $l' \lhd_{\mathcal{A}\to\mathcal{B}} (a'', b) \cdot \mathsf{nil}$ by *transitivity* and *wb-finite-saturation axiom* we get $l' \lhd_{\mathcal{A}\to\mathcal{B}} (a', b) \cdot \mathsf{nil}$. Then, by *weak-continuity* of G we conclude $c'\, G\, (a', b) \cdot \mathsf{nil}$, as claimed.

Proposition 4.14 *Let $\mathcal{A}$ be a locally compact join-formal cover and $\mathcal{C}$ and $\mathcal{B}$ be inductively generated bottom-formal topologies. Then*

1. *for every continuous relation F from $\mathcal{C} \times \mathcal{A}$ to $\mathcal{B}$,*

$$\mathsf{Ap} * \langle \Lambda(F) * \Pi_1, \Pi_2 \rangle = F$$

2. *for every continuous relation G from $\mathcal{C}$ to $\mathcal{B}^{\mathcal{A}}$,*

$$\Lambda(\mathsf{Ap} * \langle G * \Pi_1, \Pi_2 \rangle) = G$$

Proof. We prove the two implications of the considered equations one after the other.

- *(1. Right to left)* We have to prove that, for any $c \in C$, $a \in A$ and $b \in B$, if $(c, a)\, F\, b$ then $(c, a)\ \mathsf{Ap} * \langle \Lambda(F) * \Pi_1, \Pi_2 \rangle\ b$. Now, for any $d \ll a$ by lemma 4.9 we get $c\ \Lambda(F)\ (d, b) \cdot \mathsf{nil}$ and hence $(c, d) \langle \Lambda(F) * \Pi_1, \Pi_2 \rangle ((d, b) \cdot \mathsf{nil}, d)$ follows. Then, we conclude $(c, d)\ \mathsf{Ap} * \langle \Lambda(F) * \Pi_1, \Pi_2 \rangle\ b$ since $((d, b) \cdot \mathsf{nil}, d)\ \mathsf{Ap}\ b$ obviously holds. By *saturation* on $(c, a) \lhd_{C \times A} \{c\} \times wb(a)$, obtained by lemma 2.11 from $a \lhd_{\mathcal{A}} wb(a)$ being $\mathcal{A}$ locally compact, we conclude $(c, a)\ \mathsf{Ap} * \langle \Lambda(F) * \Pi_1, \Pi_2 \rangle\ b$.

- *(1. Left to right)* We have to prove that $(c, a)\ \mathsf{Ap} * \langle \Lambda(F) * \Pi_1, \Pi_2 \rangle\ b$ yields $(c, a) F b$. Observe that by lemma 4.13 from $(c, a)\ \mathsf{Ap} * \langle \Lambda(F) * \Pi_1, \Pi_2 \rangle\ b$ we get

$$(c, a) \lhd_{\mathcal{C} \times \mathcal{A}} \{\, (x, y) \in C \times A \mid x\, \Lambda(F)\, (y, b) \cdot \mathsf{nil} \,\}$$

  To this we apply lemma 3.8. Indeed, note that the first condition of the lemma on W is satisfied by $\leq$*-saturation* of $\Lambda(F)$, while the second condition is satisfied since $c\, \Lambda(F)\, (\bot_{\mathcal{A}}, b) \cdot \mathsf{nil}$, that is $\bot_{\mathcal{A}} \ll \{\, x \in A \mid (c, x)\, F\, b \,\}$, trivially holds for any $c \in C$ and $b \in B$.

  Therefore, by lemma 3.8 for any $d \ll a$ we find a subset U such that $c \lhd_{\mathcal{C}} U$ and for every $u \varepsilon U$ there exists $e_1, e'_1, ..e_n, e'_n \in A$ such that $e_j \ll e'_j$ for $j = 1, .., n$ and $d \lhd_{\mathcal{A}} \{e_1, .., e_n\}$ and $(u, e'_j) \varepsilon \{\, (x, y) \in C \times A \mid x\, \Lambda(F)\, (y, b) \cdot \mathsf{nil} \,\}$, that is $u\, \Lambda(F)\, (e'_j, b) \cdot \mathsf{nil}$, for $j = 1, .., n$. Hence, by lemma 4.4 we deduce $u\, \Lambda(F)\, (e'_1, b) \cdot .. \cdot (e'_n, b) \cdot \mathsf{nil}$. Then, by *weak-continuity* of $\Lambda(F)$ on *wb-finite-saturation* axiom considering that $d \lhd_{\mathcal{A}} \{e'_1, .., e'_n\}$, which follows from $d \lhd_{\mathcal{A}} \{e_1, .., e_n\}$ and $e_j \ll e'_j$ for $j = 1, .., n$ by lemma 3.4 and *transitivity*, we get $u\, \Lambda(F)\, (d, b) \cdot \mathsf{nil}$. Since this holds for each $u \,\varepsilon\, U$ we conclude $c\, \Lambda(F)\, (d, b) \cdot \mathsf{nil}$ by *saturation* of $\Lambda(F)$. Then, we obtain $(c, d)\, F\, b$ by lemma 4.12. Finally, by *saturation* of F on $(c, a) \lhd_{C \times A} \{c\} \times wb(a)$, obtained by lemma 2.11 from $a \lhd_{\mathcal{A}} wb(a)$ – being $\mathcal{A}$ locally compact – we conclude $(c, a)\, F\, b$ as claimed.

- *(2. Right to left)* We have to prove that for any $c \in C$ and for any $l \in List(A \times B)^{\bot}$, if $c\, G\, l$ holds then also $c\, \Lambda(\mathsf{Ap} * \langle G * \Pi_1, \Pi_2 \rangle)\, l$ holds. If $l \equiv \bot_{(\mathcal{A}\to\mathcal{B})^{\bot}}$ then since G is continuous we get $c \lhd_{\mathcal{C}} \emptyset$ and hence by lemma 2.20 we conclude $c\ \Lambda(\mathsf{Ap} * \langle G * \Pi_1, \Pi_2 \rangle)\, \bot_{(\mathcal{A}\to\mathcal{B})^{\bot}}$.

  Instead, if $l \equiv \tilde{l'}$ with $l' \in List(A \times B)$, then for every $(a, b) \epsilon l'$ we get $c\ G\ \{(a, b) \cdot \mathsf{nil}\}$ by *weak-continuity* of G on $l \lhd_{(\mathcal{A}\to\mathcal{B})^{\bot}} \{(a, b) \cdot \mathsf{nil}\}$ which follows by $\leq$*-left* and *reflexivity* from $l \leq (a, b) \cdot \mathsf{nil}$. By *continuity* of G on *interpolation* axiom we get $c \lhd_{\mathcal{C}} G^{-}(\{(x, b) \cdot \mathsf{nil} \mid a \ll x\})$. Now, for any $c' \,\varepsilon\, G^{-}(\{(x, b) \cdot \mathsf{nil} \mid a \ll x\})$, that is $c'\, G\, (a', b) \cdot \mathsf{nil}$ for some a' such that $a \ll a'$ we obtain $(c, a')\ \langle G * \Pi_1, \Pi_2 \rangle\ ((a', b) \cdot \mathsf{nil}, a')$ and hence also $(c, a')\ \mathsf{Ap} * \langle G * \Pi_1, \Pi_2 \rangle\ b$ since $((a', b) \cdot \mathsf{nil}, a')\ \mathsf{Ap}\ b$ obviously holds. Therefore, by lemma 4.9 we get $c\ \Lambda(\mathsf{Ap} * \langle G * \Pi_1, \Pi_2 \rangle)\ (a, b) \cdot \mathsf{nil}$ and by successive application of lemma 4.4 on each $(a, b) \epsilon l'$ we conclude $c\ \Lambda(\mathsf{Ap} * \langle G * \Pi_1, \Pi_2 \rangle)\ l$.

- *(2. Left to right)* We have to prove that, for every $c \in C$ and $l \in (A \to B)^{\bot}$, we get $c\ G\ l$ from $c\ \Lambda(\mathsf{Ap} * \langle G * \Pi_1, \Pi_2 \rangle)\ l$. Now, if $l \equiv \bot_{(A\to B)^{\bot}}$ then by *continuity* of $\Lambda(\mathsf{Ap} * \langle G * \Pi_1, \Pi_2 \rangle)$ we get

$c \lhd_{\mathcal{C}} \emptyset$ and by lemma 2.20 we conclude $c\ G\ l$. Otherwise, suppose $l \equiv \tilde{l'}$ for $l' \in A \to B$ and $c\ \Lambda((\widetilde{\mathsf{Ap}}^{\lhd}) * \langle G * \Pi_1, \Pi_2\rangle)\ l'$, that is $c \lhd_{\mathcal{C}} \{\, y \in C \mid y\ \widetilde{\Lambda}((\widetilde{\mathsf{Ap}}^{\lhd}) * \langle G * \Pi_1, \Pi_2\rangle)\ l' \,\}$.

Now, for any c' such that $c'\ \widetilde{\Lambda}((\widetilde{\mathsf{Ap}}^{\lhd}) * \langle G * \Pi_1, \Pi_2\rangle)\ l'$ by definition and lemma 4.9 we get that for every $(a, b)\epsilon l'$, there exists a' such that $a \ll a'$ and $(c', a')\ \mathsf{Ap} * \langle G * \Pi_1, \Pi_2\rangle\ b$. Now, by lemma 4.13 we get

$$(c', a') \lhd_{C\times A} \{(x, y) \in C \times A \mid x\, G\, (y, b) \cdot \mathsf{nil}\,\}$$

Here we proceed analogously to the proof of *(1. Left to right)* since we can apply lemma 3.8. Indeed, the first condition on W of the lemma is satisfied by *$\leq$-saturation* of G, while the second condition is satisfied since $c'\, G\, (\bot_{\mathcal{A}}, b) \cdot \mathsf{nil}$ follows by *weak-continuity* of G on the axiom $\mathsf{nil} \lhd_{(\mathcal{A}\to B)^{\perp}} (\bot_{\mathcal{A}}, b)$ being $c'\, G\, \mathsf{nil}$ valid. Indeed, $c'\, G\, \mathsf{nil}$ is valid because by *totality* of G we get $c' \lhd_{\mathcal{C}} G^{-}(List(A \times B)^{\perp})$ and since $List(A \times B)^{\perp} \lhd_{(\mathcal{A}\to\mathcal{B})^{\perp}} \mathsf{nil}$ follows by *$\leq$-left* and *reflexivity* from the fact that for every $l \in List(A \times B)^{\perp}$ we have $l \leq \mathsf{nil}$, by *continuity* and *transitivity* we get $c' \lhd_{\mathcal{C}} G^{-}(\mathsf{nil})$ and finally by *saturation* of G we conclude $c'\, G\, \mathsf{nil}$.

Then, analogously to the proof of *(1. Left to right)* we obtain $c'\, G\, (a, b) \cdot \mathsf{nil}$ and, since this holds for every $(a, b)\epsilon l'$, by successive applications of lemma 4.4 we conclude $c'\ G\ l$.

Since c' was arbitrary, by *saturation* for G we conclude also $c\ G\ l$.

So, we are arrived at the main theorem of this section.

Theorem 4.15 *Locally compact join-formal covers are exponentiable in the category of inductively generated bottom-formal covers* $\mathsf{FTop_i}^{-}_{\bot}$.

Since $\mathsf{FTop_i}^{-}_{\bot}$ is equivalent to $\mathsf{FTop_i}^{-}$, by lemma 2.15, 2.16, 3.3 we deduce the following corollary:

Corollary 4.16 *Locally compact formal covers are exponentiable in the category of inductively generated formal covers* $\mathsf{FTop_i}^{-}$.

Thanks to the predicative coreflection of inductively generated formal covers into formal topologies in [MV04] we conclude

Corollary 4.17 *Locally compact formal topologies are exponentiable in the category of inductively generated formal topologies.*

5 Any exponentiable formal cover is locally compact

Here, we show that any formal cover that is exponentiable in the category of inductively generated formal covers is locally compact, and we deduce the same result for inductively generated formal topologies. The proof is obtained by following closely the corresponding proof in [Hyl81], which is essentially based on categorical properties related to exponentiation and on properties of the Sierpinski locale and hence it is essentially predicative (see also [Vic04]). The only point where to pay attention is the use of the locale of upward closed subsets of an ideal, which is replaced here by its predicative representation.

We recall that $\top \equiv (\{1\}, \leq_1, \lhd_1)$ indicates the terminal formal cover in $\mathsf{FTop_i}^{-}$and $!_{\mathcal{A}}$ the unique continuous relation from any formal cover $\mathcal{A}$ to $\top$.

To start with the adaptation of Hyland's proof we reproduce for formal covers a well-known property of the Sierpinski locale, namely that the morphisms from any given formal cover $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}})$ to the Sierpinski cover are in bijection with the opens of a formal cover. We recall that the opens of a formal cover are its saturated subsets, that is subsets U such that $U = \{x \in A \mid x \lhd_{\mathcal{A}} U\}$.

Definition 5.1 *Let* $\mathcal{S} \equiv (\{0, 1\}, \leq, \lhd_{\mathcal{S}})$ *be the Sierpinski formal cover where* $0 \leq 1$ *and* $a \lhd_{\mathcal{S}} U \equiv \exists u \varepsilon U\ a \leq u$

Clearly, the Sierpinski formal cover is inductively generated being a unary formal topology [CSSV03]. Then, we define an order among continuous relations towards the Sierpinski formal cover:

Definition 5.2 *Given two continuous relations $R, H : \mathcal{A} \to \mathcal{S}$ we define*

$$R \leq H \equiv \forall a \in A \quad \forall s \in \{0,1\} \quad a\, R\, s \;\to\; a\, H\, s$$

Finally, we prove the expected property of the Sierpinski formal cover:

Proposition 5.3 *The saturated subsets of a formal cover $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}})$ are in bijection with the continuous relations from $\mathcal{A}$ to $\mathcal{S}$, that is for any saturated subset U there exists a continuous relation $R_U : \mathcal{A} \to \mathcal{S}$ and conversely, given a relation $R : \mathcal{A} \to \mathcal{S}$ then $R^{-}(0)$ is a saturated subset of $\mathcal{A}$. Moreover, the bijection preserves the order associated to each collection as follows: if $U \subseteq V$ then $R_U \leq R_V$, and if $R \leq H$ then $R^{-}(0) \subseteq H^{-}(0)$.*

Proof. We just give the definition of R_U as

$$a\, R_U\, s \equiv s =_{\{0,1\}} 0 \to a \varepsilon U$$

Following [Hyl81] we prove

Corollary 5.4 *Let $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \vee_{\mathcal{A}}, \lhd_{\mathcal{A}})$ be an inductively generated join-formal cover which is exponentiable in the category of inductively generated formal covers, $\Pi_{\mathcal{A}} : \top \times \mathcal{A} \to \mathcal{A}$ be the second projection of the cartesian product $\top \times \mathcal{A}$. Moreover, for any saturated subset U let $P_U \equiv \Lambda(R_U * \Pi_{\mathcal{A}})$ be the continuous relation from the terminal formal cover $\top$ to the exponential cover $\mathcal{A} \to \mathcal{S}$ associated to R_U.*

Then, for any saturated subsets U, V we have $U \subseteq V$ if and only if $P_U \leq P_V$.

Proof. Clearly, if $P_U \leq P_V$ then, recalling that from $R_U * \Pi_{\mathcal{A}} = \mathsf{Ap} * \langle \Lambda(R_U * \Pi_{\mathcal{A}}) * \Pi_{\top}, \Pi_{\mathcal{A}} \rangle$ we get $R_U = \mathsf{Ap} * \langle \Lambda(R_U * \Pi_{\mathcal{A}}) \cdot *!_{\mathcal{A}}, \mathsf{id}_{\mathcal{A}} \rangle$, we deduce

$$R_U = \mathsf{Ap} * \langle P_U *!_{\mathcal{A}}, \mathsf{id}_{\mathcal{A}} \rangle \;\leq\; \mathsf{Ap} * \langle P_V *!_{\mathcal{A}}, \mathsf{id}_{\mathcal{A}} \rangle = R_V$$

and from $R_U \leq R_V$ by proposition 5.3 we conclude $U \subseteq V$.

Conversely, suppose $U \subseteq V$. Now, we define a continuous relation $H : \mathcal{S} \times \mathcal{A} \to \mathcal{S}$ as follows

$$(a, s)\, H\, v \equiv v =_{\{0,1\}} 0 \to (a, s) \varepsilon (\{0,1\} \times U) \bigcup (\{0\} \times V)$$

a continuous relation $\mathsf{true} : \top \to \mathcal{S}$ as

$$x\, \mathsf{true}\, y \equiv y =_{\{0,1\}} 0 \to x =_{\top} 1$$

and a continuous relation $\mathsf{false} : \top \to \mathcal{S}$ as

$$x\, \mathsf{false}\, y \equiv y =_{\{0,1\}} 0 \to \neg(x =_{\top} 1)$$

Now, suppose that $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}})$ is an inductively generated formal cover which is exponentiable in the category of inductively generated formal covers. Then, observe that $H * \langle \mathsf{true} *!_{\mathcal{A}}, \mathsf{id}_{\mathcal{A}} \rangle = R_V$ and $H * \langle \mathsf{false} *!_{\mathcal{A}}, \mathsf{id}_{\mathcal{A}} \rangle = R_U$ from which by the uniqueness property of abstraction $P_V = \Lambda(H) * \mathsf{true}$ and $P_U = \Lambda(H) * \mathsf{false}$. Therefore, since $\mathsf{false} \leq \mathsf{true}$ we conclude $P_U \leq P_V$.

Here, we prove a lemma that we will use later. It states how to get a formal cover whose saturated subsets are in correspondence with the upward closed subsets of the directed subset generated by a subset U.

Proposition 5.5 *Let $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \vee_{\mathcal{A}}, \perp_{\mathcal{A}}, \lhd_{\mathcal{A}})$ a join-formal cover, and let U be any subset of A. Then, $\uparrow U \equiv (\overline{U}, \leq_{\uparrow U}, \lhd_{\uparrow U})$ where $\overline{U}$ is the set corresponding to the subset* [2]

$$\{u_1 \vee_{\mathcal{A}} \ldots \vee_{\mathcal{A}} u_n \;\mid u_i \varepsilon U \;\; \textit{for } i = 1, .., n\} \bigcup \{\perp_{\mathcal{A}}\}$$

[2] Given a subset of the set A represented by the propositional function U then $\Sigma_{x \in A} U(x)$ is the set associated.

and then

$$u \leq_{\uparrow U} v \equiv v \lhd_{\mathcal{A}} u$$

$$a \lhd_{\uparrow U} W \equiv \exists w \varepsilon W \;\; w \lhd_{\mathcal{A}} a$$

is an inductively generated formal cover.

Proof. Note that since $a \lhd_{\uparrow U} W$ means $\exists w \varepsilon W \; a \leq_{\uparrow U} w$ then $\uparrow U$ is a unary topology and is inductively generated as proved in [CSSV03].

Then, following [Hyl81] we prove that

Lemma 5.6 *Let $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \lhd_{\mathcal{A}})$ be an inductively generated formal cover which is exponentiable in the category of inductively generated formal covers. Let $\mathcal{F}(a) \equiv \{x \in A \mid x \lhd_{\mathcal{A}} a\}$. Then, for any $a \in A$*

$$a \lhd_{\mathcal{A}} \{x \in A \mid \exists d \; 1 \, P_{\mathcal{F}(a)} \, d \;\&\; (d,x) \, \mathsf{Ap} \, 0\} \quad \& \quad \{x \in A \mid \exists d \; 1 \, P_{\mathcal{F}(a)} \, d \;\&\; (d,x) \, \mathsf{Ap} \, 0\} \lhd_{\mathcal{A}} a$$

Proof. Suppose $a \in A$. Then $a \, R_{\mathcal{F}(a)} \, 0$ follows since $a \varepsilon \mathcal{F}(a)$. Therefore, since by exponentiation $R_{\mathcal{F}(a)} = \mathsf{Ap} * \langle P_{\mathcal{F}(a)} * !_{\mathcal{A}}, \mathsf{id}_{\mathcal{A}} \rangle$ we get that for any a'

$$\begin{array}{c} a' \, R_{\mathcal{F}(a)} \, 0 \\ \text{iff} \\ a' \varepsilon \{x \in A \mid x \, \mathsf{Ap} * \langle P_{\mathcal{F}(a)} * !_{\mathcal{A}}, \mathsf{id}_{\mathcal{A}} \rangle \, 0\} \\ \text{iff} \\ a' \lhd_{\mathcal{A}} \{x \in A \mid \exists d \; 1 \, P_{\mathcal{F}(a)} \, d \;\&\; (d,x) \, \mathsf{Ap} \, 0\} \end{array}$$

Therefore, from $a \, R_{\mathcal{F}(a)} \, 0$ we get $a \lhd_{\mathcal{A}} \{x \in A \mid \exists d \; 1 \, P_{\mathcal{F}(a)} \, d \;\&\; (d,x) \, \mathsf{Ap} \, 0\}$ and conversely from $a' \varepsilon \{x \in A \mid \exists d \; 1 \, P_{\mathcal{F}(a)} \, d \;\&\; (d,x) \, \mathsf{Ap} \, 0\}$ we get $a' \, R_{\mathcal{F}(a)} \, 0$ that is $a' \varepsilon \mathcal{F}(a)$ and hence $a' \lhd_{\mathcal{A}} a$.

Then, we prove that

Lemma 5.7 *Let $\mathcal{A} \equiv (A, \leq_{\mathcal{A}}, \vee_{\mathcal{A}}, \perp_{\mathcal{A}}, \lhd_{\mathcal{A}})$ be an inductively generated join-formal cover which is exponentiable in the category of inductively generated formal covers. Then, for any $a \in A$*

$$\textit{if} \qquad a' \;\varepsilon\; wb(a) \equiv \{x \in A \mid \exists d \quad (d,x) \, \mathsf{Ap} \, 0 \quad \& \quad 1 \, P_{\mathcal{F}(a)} \, d\} \qquad \textit{then} \qquad a' \ll a$$

Proof. Suppose $a' \varepsilon \{x \in A \mid \exists d \; (d,x) \, \mathsf{Ap} \, 0 \;\&\; 1 \, P_{\mathcal{F}(a)} \, d\}$, that is there exists d such that $(d,a') \, \mathsf{Ap} \, 0$ and $1 \, P_{\mathcal{F}(a)} \, d$. Suppose also $a \lhd_{\mathcal{A}} U$. Then, we consider the formal cover $\uparrow U$ in prop. 5.5 and we define the global point $Q : \top \rightarrow \uparrow U$ as:

$$1 \, Q \, z \equiv z =_{\overline{U}} z$$

and for any $h \in \overline{U}$ we define the global point $Q_h : \top \rightarrow \uparrow U$ as:

$$1 \, Q_h \, z \equiv \; z \lhd_{\mathcal{A}} h$$

Finally, we define a relation F from $\uparrow U$ to $\mathcal{S}^{\mathcal{A}}$ as follows: for $u \in \overline{U}$ and $d \in List(A \times \{0,1\})$

$$u \, F \, d \equiv 1 \, P_{\mathcal{F}(u)} \, d$$

Lemma 5.8 *F is a continuous relation from $\uparrow U$ to $\mathcal{S}^{\mathcal{A}}$.*

Proof.

- *(totality)* For any $u \in \overline{U}$, by *totality* of $P_{\mathcal{F}(u)}$ there exists d such that $1 \, P_{\mathcal{F}(u)} \, d$. Hence, $u \, F \, d$ from which $u \lhd_{\top} F^{-}(List(A \times B)^{\perp})$.
- *(continuity)* Suppose $d \lhd_{\mathcal{S}^{\mathcal{A}}} W$ and $u \, F \, d$, that is $1 \, P_{\mathcal{F}(u)} \, d$. Then, by *continuity* of $P_{\mathcal{F}(u)}$ we have $P^{-}_{\mathcal{F}(u)}(d) \lhd_{\{1\}} P^{-}_{\mathcal{F}(u)}(W)$ that is there exists $w \varepsilon W$ such that $1 \, P_{\mathcal{F}(u)} \, w$, and hence $u \varepsilon F^{-}(W)$ from which we conclude $u \lhd_{\top} F^{-}(W)$.

- *($\leq$-convergence)* Analogous to *(continuity)*.
- *(saturation)* Suppose $u \lhd_U W$ and for every $w \varepsilon W$ we have $w\,F\,d$. Then from the hypothesis there exists $w \varepsilon W$ $w \lhd_{\mathcal{A}} u$ and $1\,P_{\mathcal{F}(w)}\,d$. Since $\mathcal{F}(w) \subseteq \mathcal{F}(u)$, by corollary 5.4 we conclude $P_{\mathcal{F}(w)} \leq P_{\mathcal{F}(u)}$ and hence $1\,P_{\mathcal{F}(u)}\,d$, that is $u\,F\,d$.

Of course, for each $h \in \overline{U}$ then $Q_h \leq Q$. Therefore we get $F * Q_h \leq F * Q$ and hence $P_{\mathcal{F}(h)} \leq F * Q$ since we can prove that $F * Q_h = P_{\mathcal{F}(h)}$. Indeed, if $1\,F * Q_h\,z$ then there exists $y \in \overline{U}$ such that $1\,Q_h\,y$, that is $y \lhd_{\mathcal{A}} h$, and $y\,F\,z$, that is $1\,P_{\mathcal{F}(y)}\,z$. Therefore $\mathcal{F}(y) \subseteq \mathcal{F}(h)$ and hence by cor. 5.4 we get $1\,P_{\mathcal{F}(h)}\,z$. Conversely, if $1\,P_{\mathcal{F}(h)}\,z$ then obviously $1\,Q_h\,h$ and $h\,F\,z$ and hence $1\,F * Q_h\,z$. From $P_{\mathcal{F}(h)} \leq F * Q$ and $h \varepsilon R^{-}_{\mathcal{F}(h)}(0)$ by cor. 5.4 we conclude $h \varepsilon \widetilde{F * Q}^{-}(0)$ where $\widetilde{F * Q} \equiv \mathsf{Ap} * \langle (F * Q) * !_{\mathcal{A}}, \mathsf{id}_{\mathcal{A}} \rangle$. Since h was arbitrary we conclude $U \subseteq \widetilde{F * Q}^{-}(0)$ and from the hypothesis $a \lhd_{\mathcal{A}} U$ we get $\mathcal{F}(a) \subseteq \widetilde{F * Q}^{-}(0)$. Therefore, from the hypothesis $1\,P_{\mathcal{F}(a)}\,d$ by corollary 5.4 we deduce $1\,P_{\widetilde{F*Q}^{-}(0)}\,d$, that is $1\,F * Q\,d$, since $P_{\widetilde{F*Q}^{-}(0)} = F * Q$ by uniqueness of abstraction. Hence, there exists $u \varepsilon \overline{U}$ such that $1\,P_{\mathcal{F}(u)}\,d$. By lemma 5.6 applied to u we obtain $a' \lhd_{\mathcal{A}} u$ from $1\,P_{\mathcal{F}(u)}\,d$ since by hypothesis $(d, a')\,\mathsf{Ap}\,0$. Since $u \varepsilon \overline{U}$ we conclude that either there exists a list $u_1, ...u_n$ in U such that $a' \lhd_{\mathcal{A}} \{u_1, .., u_n\}$ or $a' \lhd_{\mathcal{A}} \perp_{\mathcal{A}}$ if $u \equiv \perp_{\mathcal{A}}$.

Now, from lemmas 5.6, 5.7, 2.15, 3.3 we conclude

Theorem 5.9 *Let $\mathcal{A}$ be an inductively generated formal cover that is exponentiable in the category of inductively generated formal covers. Then, it is locally compact.*

Corollary 5.10 *An inductively generated formal topology that is exponentiable in the category of inductively generated formal topologies is locally compact.*

Proof. Since the Sierpinski cover and $\uparrow U$ are formal topology (with $Pos(a)$ always true for every element of the corresponding base), then also an inductively generated formal topology that is exponentiable in the category of inductively generated formal topologies is locally compact.

Acknowledgements. I am very grateful to Giovanni Curi for various discussions on this topic and for pointing out to me the results in his PhD thesis [Cur04] stating that we can represent predicatively the way-below relation on subsets thanks to which morphisms from a locally compact join-formal cover to any inductively generated join-formal cover could be put in bijection with formal points of an exponential topology candidate following [Sig95]. I also thank Giovanni Sambin for general discussions on formal topology and Silvio Valentini for his collaboration on exponentiation in formal topology.

References

[BS01] G. Battilotti and G. Sambin. Pretopologies and a uniform presentation of sup-lattice, quantales and frames. Draft, 2001.

[CSSV03] T. Coquand, G. Sambin, J. Smith, and S. Valentini. Inductively generated formal topologies. *Annals of Pure and Applied Logic*, 124(1-3):71–106, 2003.

[Cur04] G. Curi. Geometry of observations: some contributions to (constructive) point-free topology. Tesi di dottorato, University of Siena, 2004.

[EH02] M. Escardo and R. Heckmann. Topologies on spaces of continuous functions. *Topology proceedings.*, 26:545–564, 2001-2002.

[Hyl81] J.M.E. Hyland. Function spaces in the category of locales. *Lecture Notes in Mathematics*, 871:264–281, 1981.

[Joh82] P.T. Johnstone. Stone spaces. Cambridge University Press, 1982.

[Joh84] P.T. Johnstone. Open locales and exponentiation. *Contemporary Mathematics*, 30:84–116, 1984.

[JT84] A. Joyal and M. Tierney. An extension of the Galois theory of Grothendieck. *Mem. Amer. Math. Soc.*, 51:no.309, 1984.

[MV04] M.E. Maietti and S. Valentini. A structural investigation on formal topology: coreflection of formal covers and exponentiability. *Journal of Symbolic Logic*, 2004. To appear.

[Neg02] S. Negri. Continuous domains as formal spaces. *Mathematical Structures in Computer Science.*, 12:19–52, 2002.

[NPS90] B. Nordström, K. Peterson, and J. Smith. *Programming in Martin Löf's Type Theory.* Clarendon Press, Oxford, 1990.

[Sam87] G. Sambin. Intuitionistic formal spaces - a first communication. *Mathematical logic and its applications*, pages 187–204, 1987.

[Sam03] G. Sambin. Some points in formal topology. *Theoretical Computer Science*, 305(1-3):347–408, 2003.

[Sig95] I. Sigstam. Formal spaces and their effective presentations. *Arch. Math. Logic*, 34:211–246, 1995.

[Vic04] S.J. Vickers. The double powerlocale and exponentiation: A case study in geometric reasoning. *Theory and Applications of Categories*, 12:372–422, 2004.