Triposes, exact completions, and Hilbert's ϵ-operator

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Abstract

Triposes were introduced as presentations of toposes by J.M.E. Hyland, P.T. Johnstone and A.M. Pitts. They introduced a construction that, from a tripos $P\colon \mathcal{C}^{\text{op}} \longrightarrow \textsf{Pos}$, produces an elementary topos $\mathcal{T}_P$ in such a way that the fibration of the subobjects of the topos $\mathcal{T}_P$ is freely obtained from P. One can also construct the "smallest" elementary doctrine made of subobjects which fully extends P, more precisely the free full comprehensive doctrine with comprehensive diagonals $P_{\text{cx}}\colon \mathcal{P}\!rd_P{}^{\text{op}} \longrightarrow \textsf{Pos}$ on P. The base category has finite limits and embeds into the topos $\mathcal{T}_P$ via a functor $K\colon \mathcal{P}\!rd_P \longrightarrow \mathcal{T}_P$ determined by the universal property of P_{cx} and which preserves finite limits. Hence it extends to an exact functor $K^{\text{ex}}\colon (\mathcal{P}\!rd_P)_{\text{ex/lex}} \longrightarrow \mathcal{T}_P$ from the exact completion of $\mathcal{P}\!rd_P$.

We characterize the triposes P for which the functor K^{ex} is an equivalence as those P equipped with a so-called ϵ-operator. We also show that the tripos-to-topos construction need not preserve ϵ-operators by producing counterexamples from localic triposes constructed from well-ordered sets.

A characterization of the tripos-to-topos construction as a completion to an exact category is instrumental for the results in the paper and we derived it as a consequence of a more general characterization of an exact completion related to Lawvere's hyperdoctrines.

1 Introduction

The topic of completing a given structure with quotients to get a richer one has been widely employed in logic in order to obtain relative consistency results, and its categorical aspects have been studied extensively. The calculus of Partial Equivalence Relations has many applications in the semantics of programming languages. In Type Theory, models of abstract quotients, known as setoid models, are very useful to formalize mathematical proofs. In category theory

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one finds various notions of completing a category to an exact category initiated by P.J. Freyd's exact completion of a regular category and they include also the exact completion of a category with certain weak finite limits, *e.g.* see [FS91, Car95, CV98].

In recent work [MR15], two of the authors generalized these exact completions by relativizing the basic data to a doctrine equipped with just the structure sufficient to present the notion of an equivalence relation. In particular, they determined the exact completion of an elementary existential doctrine P with (weak) full comprehensions and comprehensive diagonals, see *loc.cit.*. The exact completion of a regular category $\mathcal{R}$ coincides with the exact completion on the existential doctrine of the subobjects of $\mathcal{R}$. The exact completion of a category with finite limits $\mathcal{C}$ is the exact completion of the doctrine of the weak subobjects on $\mathcal{C}$.

But there is also another way of completing an elementary existential doctrine P to an exact category which consists essentially in the tripos-to-topos construction of J.M.E. Hyland, P.T. Johnstone and A.M. Pitts, see [HJP80] and which made apparent the abstract construction behind Higg's complete Heyting valued toposes and toposes obtained from Kleene's realizability like the effective topos, see [Hyl82]. In [MR15, Pas15b] it was shown that the tripos-to-topos construction $\mathcal{T}_P$ of a given tripos $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \textsf{Heyt}$ can be obtained as the exact completion of the doctrine $P_{\mathrm{cx}}\colon \mathcal{Prd}_P{}^{\mathrm{op}} \longrightarrow \textsf{Heyt}$ obtained by freely completing the original tripos with full comprehensions and comprehensive diagonals. In particular, the base category $\mathcal{Prd}_P$ has finite limits and the functor $K\colon \mathcal{Prd}_P \longrightarrow \mathcal{T}_P$, obtained by the universal property, is an embedding into the topos $\mathcal{T}_P$.

In this paper we address the question of characterizing those triposes P for which the exact estension $K^{\mathrm{ex}}\colon (\mathcal{Prd}_P)_{\mathrm{ex/lex}} \longrightarrow \mathcal{T}_P$ of K is an equivalence. We show that this happens if and only if each object in the base of the tripos P is equipped with the logical constructors called ϵ-operator, see [HB01a, HB01b].

This characterization follows from the following facts.

- the starting tripos P is equipped with ϵ-operators if and only if the free full comprehensive doctrine P_{cx} with comprehensive diagonals satisfies the Rule of Choice;
- the doctrine with full comprehensions and comprehensive diagonals P_{cx} of P satisfies the Rule of Choice if and only if a certain "comprehension functor" from the doctrine P_{cx} to the doctrine $\Psi_{\mathcal{Prd}_P}$ of the weak subobjects of $\mathcal{Prd}_P$ is part of an equivalence.

These two facts together with the decomposition results of exact completions in terms of the free full comprehensive completion doctrine P_{cx} with comprehensive diagonals in [MR15] allow us to conclude that, given a tripos P, the exact functor $K^{\mathrm{ex}}\colon (\mathcal{Prd}_P)_{\mathrm{ex/lex}} \longrightarrow \mathcal{T}_P$, extending $K\colon \mathcal{Prd}_P \longrightarrow \mathcal{T}_P$ to the exact completion, is an equivalence if and only if P is equipped with ϵ-operators.

Examples of toposes coming from a tripos equipped with ϵ-operators include toposes of complete Heyting valued sets whose algebra of values is (the opposite of) a well-order. Most notably these toposes are not necessarily boolean even if they satisfy a weak law of excluded middle, see [Bel93a]. This allows to conclude that the tripos-to-topos construction does not preserve ϵ-operators because from [Bel93b] we know that toposes with ϵ-operators satisfy the axiom of choice and hence, by Diaconescu's theorem, are necessarily boolean.

2 Doctrines of weak subobjects

The notion of elementary doctrine is a variation of the notion of hyperdoctrine introduced in a series of seminal papers by F.W. Lawvere to synthetize the structural properties of logical systems, see [Law69a, Law69b, Law70], and also [LR03] for a unified survey.

Lawvere's crucial intuition was to consider logical languages and theories as fibrations to study their 2-categorical properties, *e.g.* connectives, quantifiers and equality are determined by structural adjunctions. That approach proved extremely fruitful, see [MR77, Car82, LS86, Jac99, Tay99, vO08] and references therein.

Taking advantage of the category-theoretical presentation of logic by doctrines, we review from [MR13b, MR15] a general notion of elementary doctrine appropriate to analyse the notion of quotient of an equivalence relation. Let **InfSL** be the locally ordered 2-category of inf-semilattice, *i.e.* posets with finite infima, and functions between them which preserves finite infima, with the pointwise order between those.

2.1 Definition. Let $\mathcal{C}$ be a category with a terminal object T and with a binary product

$$C_1 \xleftarrow{\mathrm{pr}_1} C_1 \times C_2 \xrightarrow{\mathrm{pr}_2} C_2$$

for every pair of objects C_1 and C_2 in $\mathcal{C}$. An ***elementary doctrine on*** $\mathcal{C}$ is an indexed inf-semilattice $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ such that, for every object A in $\mathcal{C}$, there is an object δ_A in $P(A \times A)$ such that

(i) the assignment

$$\mathit{\exists}_{\langle \mathrm{id}_A, \mathrm{id}_A \rangle}(\alpha) := P_{\mathrm{pr}_1}(\alpha) \wedge_{A \times A} \delta_A$$

for α in $P(A)$ determines a left adjoint to

$$P_{\langle \mathrm{id}_A, \mathrm{id}_A \rangle}\colon P(A \times A) \longrightarrow P(A);\,^{1}$$

(ii) for every arrow e of the form $\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_2 \rangle\colon X \times A \longrightarrow X \times A \times A$ in $\mathcal{C}$, the assignment

$$\mathit{\exists}_e(\alpha) := P_{\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle}(\alpha) \wedge_{X \times A \times A} P_{\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle}(\delta_A)$$

for α in $P(X \times A)$ determines a left adjoint to

$$P_e\colon P(X \times A \times A) \longrightarrow P(X \times A).$$

[1] Here and in the following we write P_f for the value of the indexing functor P on an arrow f.

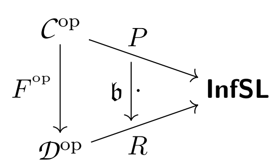
2.2 Remark. (a) Condition (i) determines δ_A uniquely for each object A in $\mathcal{C}$. The object δ_A will be referred to as the ***fibered equality on*** A.
(b) Since $\langle \mathrm{pr}_2, \mathrm{pr}_1\rangle \circ \langle \mathrm{id}_A, \mathrm{id}_A\rangle = \langle \mathrm{id}_A, \mathrm{id}_A\rangle$, from (a) it follows that we can use the second projection in the definition of the left adjoint in (i) in this way

$$\exists_{\langle \mathrm{id}_A, \mathrm{id}_A\rangle}(\alpha) = P_{\mathrm{pr}_2}(\alpha) \wedge_{A\times A} \delta_A$$

for every α in $P(A)$, by uniqueness of left adjoints.
(c) It follows from the fact that $\mathcal{C}$ has a terminal object that condition (ii) entails condition (i).[2]
(d) One has that $\top_A \leq_A P_{\langle \mathrm{id}_A, \mathrm{id}_A\rangle}(\delta_A)$ and $\delta_A \leq_{A\times A} P_{f\times f}(\delta_B)$ when $f\colon A \longrightarrow B$.

To express precisely the relationships between the examples one must consider the 2-category **ED** of elementary doctrines:

the 1-arrows are pairs $(F, \mathfrak{b})$ where $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a functor and $\mathfrak{b}\colon P \stackrel{\cdot}{\longrightarrow} R\circ F^{\mathrm{op}}$ is a natural transformation as in the diagram



where the functor F preserves products and, for every object A in $\mathcal{C}$, the homomorphism $\mathfrak{b}_A\colon P(A) \longrightarrow R(F(A))$ of inf-semilattices is such that $\mathfrak{b}_{A\times A}(\delta_A) = R_{\langle F(\mathrm{pr}_1), F(\mathrm{pr}_2)\rangle}(\delta_{F(A)})$—hence it commutes with all the left adjoints $\exists_f$;

the 2-arrows are natural transformations $\theta\colon F \stackrel{\cdot}{\longrightarrow} G$ such that $\mathfrak{b}_A(\alpha) \leq_{F(A)} R_{\theta_A}(\mathfrak{c}_A(\alpha))$ for every A in $\mathcal{C}$ and α in $P(A)$.

2.3 Examples. The first three examples below are discussed in [Law69a, Law70].
(a) The standard categorical examples of indexed posets are the fibrations of subobjects. For a category $\mathcal{C}$ with finite limits, the functor $\mathrm{Sub}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow$ **InfSL** assigns to an object A in $\mathcal{C}$ the poset $\mathrm{Sub}_{\mathcal{C}}(A)$ of the subobjects of A in $\mathcal{C}$ and, for an arrow $f\colon B \longrightarrow A$, the homomorphism $\mathrm{Sub}_{\mathcal{C}}(f)\colon \mathrm{Sub}_{\mathcal{C}}(A) \longrightarrow \mathrm{Sub}_{\mathcal{C}}(B)$ is given by pulling a subobject back along f. The fibered equalities are the diagonal arrows.
(b) Another example is provided by any category $\mathcal{D}$ with finite products and weak pullbacks: the doctrine is given by the functor of weak subobjects $\Psi_{\mathcal{D}}\colon \mathcal{D}^{\mathrm{op}} \longrightarrow$ **InfSL** where $\Psi_{\mathcal{D}}(A)$ is the poset reflection of the slice category $\mathcal{D}/A$ and, for an arrow $f\colon B \longrightarrow A$, the homomorphism $(\Psi_{\mathcal{D}})_f\colon \Psi_{\mathcal{D}}(A) \longrightarrow \Psi_{\mathcal{D}}(B)$ is given by a(ny) weak pullback of an arrow $g\colon X \longrightarrow A$ with f. This doctrine is studied in [Gra00] where weak subobjects are called ***variations*** and subobjects become ***monic variations***.

[2] Nonetheless we preferred to state condition (i) explicitly in the definition.

The previous two examples are equivalent in case the categories are the same $\mathcal{C} = \mathcal{D}$ if and only if every arrow in $\mathcal{C}$ can be factored as a retraction followed by a mono — for instance, for $\mathcal{C} = \mathcal{S}et$ the category of sets and functions, thanks to the Axiom of Choice.

(c) An example directly from first order logic is the Lindenbaum-Tarski algebras of well-formed formulas of a theory $\mathscr{T}$ with equality. The base category is the category $\mathcal{V}$ of lists of distinct variables and term substitutions, and the elementary doctrine $LT\colon \mathcal{V}^{\mathrm{op}} \longrightarrow \mathsf{InfSL}$ on $\mathcal{V}$ is given on a list of typed variables $\vec{x}$ by taking $LT(\vec{x})$ as the Lindenbaum-Tarski algebra of well-formed formulas with free variables in $\vec{x}$, see [MR13a] for more details.

A set-theoretic model for a first order theory $\mathscr{T}$ with equality determines an 1-arrow from $LT\colon \mathcal{V}^{\mathrm{op}} \longrightarrow \mathsf{InfSL}$ to $\mathrm{Sub}_{\mathcal{S}et}\colon \mathcal{S}et^{\mathrm{op}} \longrightarrow \mathsf{InfSL}$ in ED. And a homomorphism between two set-theoretic models of $\mathscr{T}$ determines a 2-arrow.

(d) Let $\mathcal{S}t$ be a full subcategory of the category $\mathcal{S}et$ of sets and functions, closed under finite products—for instance, $\mathcal{S}t$ can be chosen as the category $\mathcal{S}et_*$ on the non-empty sets, or as the category $\mathcal{F}in\mathcal{S}et$ on the finite sets, or more generally as the category $\mathcal{S}et_{<\lambda}$ on the sets of cardinality less than λ, for λ a limit ordinal, or even $\mathcal{S}et_{<\lambda *}$ on non-empty sets of cardinality less than λ.

Let $\mathcal{B}$ be a poset with a bottom element $\bot$, least upper bounds $\bigvee^I\colon \mathcal{B}^I \longrightarrow \mathcal{B}$ for every indexing set I in $\mathcal{S}t$, and greatest lower bounds of finite subsets which distribute over $\bigvee$. Consider the indexed inf-semilattice $\mathcal{B}^{(-)}\colon \mathcal{S}t^{\mathrm{op}} \longrightarrow \mathsf{InfSL}$ on the category $\mathcal{S}t$. It maps a set I to the power inf-semilattice $\mathcal{B}^I$ and a function $f\colon I \longrightarrow J$ to the homomorphism

$$\mathcal{B}^J \xrightarrow{-\circ f} \mathcal{B}^I$$

given by pre-composition with f. For I in $\mathcal{S}t$, let

$$\delta_I(i_1, i_2) := \begin{cases} \top & \text{if } i_1 = i_2 \\ \bot & \text{otherwise} \end{cases}$$

It is straightforward to see that $\mathcal{B}^{(-)}$ is an elementary doctrine.

The doctrines which are relevant for the present paper are of a special kind.

2.4 Definition. An elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSL}$ is ***existential*** when, for A_1 and A_2 in $\mathcal{C}$, for a(ny) projection $\mathrm{pr}_i\colon A_1 \times A_2 \longrightarrow A_i$, $i = 1, 2$, the functor $P_{\mathrm{pr}_i}\colon P(A_i) \longrightarrow P(A_1 \times A_2)$ has a left adjoint $\exists_{\mathrm{pr}_i}$—we shall call such a left adjoint ***existential***—, and those left adjoints satisfy the

***Beck-Chevalley Condition*:** for any pullback diagram

$$\begin{array}{ccc} X' & \xrightarrow{\mathrm{pr}'} & A' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{\mathrm{pr}} & A \end{array}$$

with pr a projection (hence also pr′ a projection), for any β in $P(X)$, the natural inequality $\exists_{\mathrm{pr}'} P_{f'}(\beta) \leq P_f \exists_{\mathrm{pr}}(\beta)$ in $P(A')$ is an identity;

***Frobenius Reciprocity*:** for $\mathrm{pr}\colon X \longrightarrow A$ a projection, α in $P(A)$, β in $P(X)$, the natural inequality $\mathit{\exists}_{\mathrm{pr}}(P_{\mathrm{pr}}(\alpha) \wedge_X \beta) \leq \alpha \wedge_A \mathit{\exists}_{\mathrm{pr}}(\beta)$ in $P(A)$ is an identity.

2.5 Examples. Among the examples in 2.3, the doctrine in (a) is existential if and only if $\mathcal{C}$ has images. The doctrines in the other examples are existential. For the doctrine in (b) the existential left adjoints are given by post-composition. For the doctrine in (c) the existential left adjoints are constructed with existential quantifier. For the doctrine in (d) the existential left adjoint is computed by $\bigvee$, *e.g.*

$$\mathit{\exists}_{\mathrm{pr}_2}(\alpha)(j) = \bigvee_{i\in I} \alpha(i,j)$$

for $\alpha \in \mathcal{B}^{I\times J}$ and $\mathrm{pr}_2\colon I\times J \longrightarrow J$.

We should remark that the original analysis of existential elementary doctrine in [Law70] was much finer than the one we offer here, yet like in *loc.cit.* a general functor P_f may fail to have a left adjoint. We need only the following result, see [Law70, Jac99].

2.6 Proposition. *If $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow$ **InfSL** is an elementary existential doctrine, then for every $f\colon A \longrightarrow B$ in $\mathcal{C}$ the functor $P_f\colon P(B) \longrightarrow P(A)$ has a left adjoint which is defined for $f\colon A \longrightarrow B$ in $\mathcal{C}$ and α in $P(A)$ as follows*

$$\mathit{\exists}_f(\alpha) := \mathit{\exists}_{\mathrm{pr}_1}(P_{\mathrm{pr}_2}(\alpha) \wedge_{B\times A} P_{\langle \mathrm{pr}_1, f\,\mathrm{pr}_2\rangle}(\delta_B))$$

Moreover, these left adjoints satisfy the Frobenius Reciprocity, i.e. $\mathit{\exists}_f(\alpha \wedge P_f(\beta)) = \beta \wedge \mathit{\exists}_f(\alpha)$ holds for every β in $P(B)$ and α in $P(A)$.

We write **EED** for 2-full subcategoy of **ED** on the existential elementary doctrines with those 1-arrows that commute with the existential left adjoints.

The careful reader will have noticed that 2.6 does not mention any sort of Beck-Chevalley Condition. The crux of the matter is that the fibres of P have very little to do with the constructions in the base category $\mathcal{C}$, in particular pullbacks or equalizers. The technical tool for such a connection are comprehensions.

2.7 Definition. Given an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow$ **InfSL**, and an object α in $P(A)$, a ***weak comprehension*** of α is an arrow $\{\!|\alpha|\!\}\colon X \longrightarrow A$ in $\mathcal{C}$ such that $\top_X \leq_X P_{\{\!|\alpha|\!\}}(\alpha)$, and, for every $f\colon Z \longrightarrow A$ such that $\top_Z \leq_Z P_f(\alpha)$, there is an arrow $f'\colon Z \longrightarrow X$ such that $f = \{\!|\alpha|\!\} \circ f'$.
We say that an elementary doctrine P ***has weak comprehensions*** if every α has a weak comprehension, and that P ***has full weak comprehensions*** if, moreover, $\alpha \leq \beta$ in $P(A)$ whenever $\{\!|\alpha|\!\}$ factors through $\{\!|\beta|\!\}$.

For a given α in $P(A)$, the arrow $\{\!|\alpha|\!\}\colon X \longrightarrow A$ is monic if and only if, for every f, the representation f' is unique. In such a situation, usually one drops the adjective "weak" from "weak comprehension", possibly emphasizing the result with the adjective "strong". We shall align with the standard use and speak of (***strong***) ***comprehension*** for a monic weak comprehension.

2.8 Remark. Note that a weak comprehension, as any weak universal arrow, is *not* determined up to iso. Two weak comprehensions $k\colon X \longrightarrow A$ and $h\colon Y \longrightarrow A$ of the same object α of $P(A)$ are connected by arrows $f\colon X \longrightarrow Y$ and $g\colon y \longrightarrow X$ which need not be inverse of each other, but they do make the following triangles commute

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & X \\ & {\scriptstyle k}\searrow & \downarrow{\scriptstyle h} & \swarrow{\scriptstyle k} & \\ & & A. & & \end{array}$$

Note that that is all is needed to ensure that fullness does not depend on the choice of a particular weak comprehension.

2.9 Remark. Recall from [Law70] that the notion of (strong) comprehension connects an abstract elementary doctrine with that of the subobjects of the base when this has finite limits—see also [Jac99] where a more abstract, elegant view of comprehensions as right adjoint is considered.

Note also that, for $\alpha, \beta \in P(A)$ with weak comprehension, one has that $\{\!|\alpha \wedge_A \beta|\!\}$ is a weak pullback of $\{\!|\alpha|\!\}$ and $\{\!|\beta|\!\}$. So, assuming $\mathcal{C}$ has weak equalizers, the assignment $\{\!|-|\!\}\colon P(A) \longrightarrow \Psi_{\mathcal{D}}(A)$ is a natural homomorphism from P to $\Psi_{\mathcal{D}}$. But it may fail to be a 1-arrow in **ED** because it need not preserve fibered equalities, see 2.10 and 2.15 though.

2.10 Remark. Suppose that, in the elementary existential doctrine $P\colon \mathcal{D}^{\mathrm{op}} \longrightarrow$ **InfSL**, the category $\mathcal{D}$ has all pullbacks. Suppose also that all left adjoints to the action of P on arrows in $\mathcal{D}$ satisfy the Beck-Chevalley Condition for all pullbacks, *i.e.* given any pullback diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & A' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{g} & A \end{array}$$

in $\mathcal{D}$, for any β in $P(X)$, the natural inequality $\exists_{g'} P_{f'}(\beta) \leq P_f \exists_g(\beta)$ in $P(A')$ is an identity. The function

$$\begin{array}{rcl} \Psi_{\mathcal{D}}(A) & \xrightarrow{\exists_{-} \top_A} & P(A) \\ {[f\colon A \to B]} & \longmapsto & \exists_f \top_A \end{array}$$

extends to a homomorphism in **InfSL** which is left adjoint to $\{\!|-|\!\}\colon P(A) \longrightarrow \Psi_{\mathcal{D}}(A)$. Clearly it provides a 1-arrow in **ED**

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{op}} & \xrightarrow{\Psi_{\mathcal{D}}} & \\ {\scriptstyle \mathrm{Id}^{\mathrm{op}}_{\mathcal{D}}}\downarrow & \exists_{-}\top \downarrow \cdot & \mathbf{InfSL} \\ \mathcal{D}^{\mathrm{op}} & \xrightarrow{P} & \end{array}$$

from $\Psi_{\mathcal{D}}$ to P.

A special case of comprehensions are the diagonal arrows and the following definition considers just that possibility.

2.11 Definition. An elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ ***has comprehensive diagonals*** if every diagonal arrow $\langle \mathrm{id}_A, \mathrm{id}_A\rangle\colon A \longrightarrow A \times A$ is the (necessarily strong) comprehension of δ_A.

2.12 Proposition. *Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an elementary doctrine. The following are equivalent:*

(i) *P has comprehensive diagonals.*

(ii) *For any two arrows $f, g\colon A \longrightarrow B$ in $\mathcal{C}$, it is*

$$f = g \qquad \textit{iff} \qquad \top_A \leq_A P_{\langle f,g\rangle}(\delta_B).$$

Proof. Notice that $f = g$ if and only if $\langle f, g\rangle\colon A \longrightarrow B \times B$ factors through the diagonal. □

Thanks to proposition 2.12, there is a 2-reflection of elementary doctrines from **ED** into the full 2-subcategory **CED** of elementary doctrines with comprehensive diagonals once one notices that the condition

$$\top_A \leq_A P_{\langle f,g\rangle}(\delta_B)$$

ensures that $P_f = P_g$. So the reflection takes an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ to the elementary doctrine $P_{\mathrm{x}}\colon \mathcal{X}_P^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, induced by P on the quotient category $\mathcal{X}_P$ of $\mathcal{C}$ with respect to the equivalence relation where $f \sim g$ when

$$\top_A \leq_A P_{\langle f,g\rangle}(\delta_B).$$

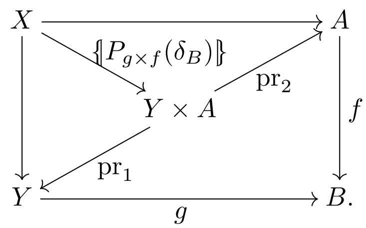
We may refer to the doctrine P_{x} as the ***extensional reflection of*** P, see [MR13a] for the details.

It is easy to see that the extensional reflection of an elementary existential doctrine is existential since the further structure does not involve the base category. Also recall from [MR13b] that, when an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ has full comprehensions and comprehensive diagonals, then the base category $\mathcal{C}$ has equalizers, hence all finite limits.

2.13 Proposition. *Let P be an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ with comprehensive diagonals. If P has weak comprehensions then $\mathcal{C}$ has weak equalizers, and if P has comprehensions then $\mathcal{C}$ has equalizers.*

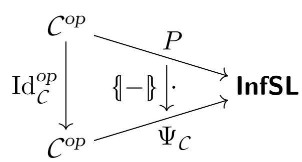
Proof. A weak equalizer of $A \underset{g}{\overset{f}{\rightrightarrows}} B$ is computed as $\{\!|P_{\langle f,g\rangle}(\delta_B)|\!\}\colon E \longrightarrow A$. And this becomes an equalizer as soon as it is monic, see 2.7. □

2.14 Remark. In an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ with comprehensive diagonals and full comprehensions, the pullback of f along g in $\mathcal{C}$ can be computed as



As a follow-up to 2.9, the presence of comprehensive diagonals in an elementay doctrine makes comprehension a 1-arrow in the 2-category **ED**.

2.15 Theorem. *Suppose* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *is an elementary doctrine with weak comprehensions and comprehensive diagonals. The assignment of weak comprehensions extends to a 1-arrow*



from P *to the doctrine of the weak subobjects in* **ED**. *Moreover, if the weak comprehensions are full, then the functors (aka order-preserving functions)* $P(A) \longrightarrow \Psi_{\mathcal{C}}(A)$ *are full.*

Proof. First observe that thanks to proposition 2.13 the base category $\mathcal{C}$ has weak equalizers. Also, by 2.9 $\{\!|-|\!\}\colon P(A) \longrightarrow \Psi_{\mathcal{C}}(A)$ preserves finite meets. Finally note that the natural transformation $\{\!|-|\!\}$ preserves the fibered equality because diagonals are comprehensive. □

So 2.15 provides a representation of an elementary doctrine with weak full comprehension and comprehensive diagonals $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ as a subdoctrine of the doctrine of the weak subobjects $\Psi_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ on $\mathcal{C}$.

Since this can also be strengthened to yield a representation of an elementary doctrine with full (strong) comprehension and comprehensive diagonals $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ as a subdoctrine of the doctrine of the subobjects $\mathrm{Sub}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, we introduce the following definitions, inspired by 2.3(b).

2.16 Definition. Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an elementary doctrine. We say that P is a ***variational doctrine*** if it has weak full comprehensions and comprehensive diagonals. And we say that P is an ***m-variational doctrine*** if it has full comprehensions and comprehensive diagonals.

Recall from [Jac99] that the Grothendieck category $\mathcal{G}_P$ of points of the indexed category $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ provides the free addition of comprehensions. In the posetal case of interest, the category $\mathcal{G}_P$ has objects which are pairs (A, α)

where A is in $\mathcal{C}$ and α is in $P(A)$. An arrow $f\colon (A,\alpha) \longrightarrow (B,\beta)$ in $\mathcal{G}_P$ is an arrow $f\colon A \longrightarrow B$ in $\mathcal{C}$ such that $\alpha \leq P_f(\beta)$. The indexed poset $P_{\mathrm{c}}\colon {\mathcal{G}_P}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ takes an object (A,α) of $\mathcal{G}_P$ to

$$P_{\mathrm{c}}(A,\alpha) := \{\gamma \in P(A) \mid \gamma \leq \alpha\}$$

and an arrow $f\colon (A,\alpha) \longrightarrow (B,\beta)$ to

$$(P_{\mathrm{c}})_f(\phi) = P_f(\phi) \wedge \alpha.$$

If P is an elementary doctrine, P_{c} is an elementary doctrine with comprehensions, and it is the free one on P. The comprehensions in P_{c} are actually full, see [MR13b, Pas15b] for the details in the posetal case.

Let $\mathbf{SD}$ be the 2-full 2-subcategory of $\mathbf{ED}$ on the m-variational doctrines whose 1-arrows preserve comprehensions.

2.17 Theorem. *The association to an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ of the doctrine $P_{cx}\colon {\mathcal{X}_{P_{\mathrm{c}}}}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ determines a left bi-adjoint to the inclusion of $\mathbf{SD}$ into $\mathbf{ED}$. If the doctrine P is existential, then P_{c} and P_{x} are also existential.*

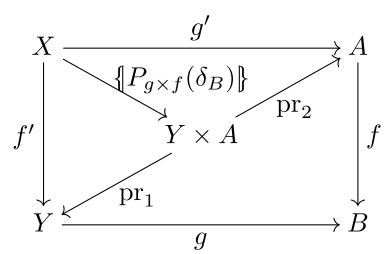
Proof. See [MR13b] for a proof of the first statement, and [MR15] for the second part. □

Inspired by the construction of the category of predicates in Joyal's arithmetic universes, see [Mai10], we shall refer to the category $\mathcal{X}_{P_{\mathrm{c}}}$ as the ***category of predicates*** of the elementary doctrine P and write it as $\mathcal{Prd}_P$, because it is the base of the m-variational doctrine generated by P. Recall from proposition 2.13 that $\mathcal{Prd}_P$ has finite products.

2.18 Example. Consider the functor $D_{\mathcal{S}}$ that maps each object of a Skolem category $\mathcal{S}$ to the poset of its decidable predicates, see [Mai10]. The category $\mathcal{Prd}_{D_{\mathcal{S}}}$ is the second stage of the construction of Joyal's arithmetic universes in *loc.cit.*.

2.19 Proposition. *Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an existential variational doctrine. The left adjoint functors $\exists_f$ satisfy the Beck-Chevalley condition with respect to weak pullbacks.*

Proof. Let $f\colon A \longrightarrow B$ and $g\colon Y \longrightarrow B$ be arrows in $\mathcal{C}$. Consider first the weak pullback of f and g obtained as follows



as in 2.14. Let $\eta := \{\!|P_{g\times f}(\delta_B)|\!\}$. So, by the hypothesis of full weak comprehensions, $\mathit{\exists}_\eta(\top_X) = P_{g\times f}(\delta_B)$. Also, by 2.6, applying Frobenius Reciprocity one has that

$$\mathit{\exists}_\eta P_\eta(P_{\mathrm{pr}_2}(\alpha)) = \mathit{\exists}_\eta(\top_X) \wedge P_{\mathrm{pr}_2}(\alpha).$$

Hence, for ${\mathrm{pr}_1}'$ and ${\mathrm{pr}_2}'$ the projections from $B \times A$,

$$\begin{aligned} P_g \mathit{\exists}_f(\alpha) &= P_g(\mathit{\exists}_{{\mathrm{pr}_1}'}(P_{{\mathrm{pr}_2}'}(\alpha) \wedge P_{\langle {\mathrm{pr}_1}', f{\mathrm{pr}_2}'\rangle}(\delta_B))) \\ &= \mathit{\exists}_{\mathrm{pr}_1}(P_{g\times \mathrm{id}_A}(P_{{\mathrm{pr}_2}'}(\alpha) \wedge P_{\langle {\mathrm{pr}_1}', f{\mathrm{pr}_2}'\rangle}(\delta_B))) \\ &= \mathit{\exists}_{\mathrm{pr}_1}(P_{\mathrm{pr}_2}(\alpha) \wedge P_{f\times g}(\delta_B)) \\ &= \mathit{\exists}_{\mathrm{pr}_1}(P_{\mathrm{pr}_2}(\alpha) \wedge \mathit{\exists}_\eta(\top_X)) \\ &= \mathit{\exists}_{\mathrm{pr}_1}\mathit{\exists}_\eta(P_\eta P_{\mathrm{pr}_2}(\alpha)) = \mathit{\exists}_{f'}(P_{g'}(\alpha)) \end{aligned}$$

Consider now an arbitrary weak pullback

$$\begin{array}{ccc} Z & \xrightarrow{h} & A \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & B. \end{array}$$

By weak universality, there is $t: X \longrightarrow Z$ such that $kt = f'$ and $ht = g'$. Hence

$$P_g\mathit{\exists}_f(\alpha) = \mathit{\exists}_{f'}P_{g'}(\alpha) = \mathit{\exists}_k\mathit{\exists}_t P_t P_h(\alpha) \le \mathit{\exists}_k P_h(\alpha). \qquad \square$$

2.20 Corollary. *If $P: \mathcal{C}^{\mathrm{op}} \longrightarrow$ **InfSL** is an elementary existential doctrine, then $P_{\mathrm{cx}}: {\mathcal{Prd}_P}^{\mathrm{op}} \longrightarrow$ **InfSL** is an existential m-variational doctrine, and the left adjoint functors $\mathit{\exists}_f$ satisfy the Beck-Chevalley Condition.*

2.21 Remark. Existential m-variational doctrines $P: \mathcal{C}^{\mathrm{op}} \longrightarrow$ **InfSL** are related to proper factorizations systems, see [HJ03]. Every such a doctrine determines a proper factorization system (E, M) in $\mathcal{C}$, see [FK72], where the monos in M are the comprehensions in $\mathcal{C}$ and the epis in E are ***surjective with respect to*** P, namely those arrows $f: A \longrightarrow B$ in $\mathcal{C}$ such that $\mathit{\exists}_f(\top_A) = \top_B$.

2.22 Proposition. *For an existential m-variational doctrine $P: \mathcal{C}^{\mathrm{op}} \longrightarrow$ **InfSL**, the unit of the adjunction in 2.17 $(N, \mathfrak{n}): P \longrightarrow P_{\mathrm{cx}}$ has a retraction $(M, \mathfrak{m}): P_{\mathrm{cx}} \longrightarrow P$ which is also right adjoint to the 1-arrow $(N, \mathfrak{n})$ in* **ED**.

Proof. By 2.17 the 1-arrow $N: P \longrightarrow P_{\mathrm{cx}}$ maps an object A in $\mathcal{C}$ to $(A, \top_A)$, an arrow $f: A \longrightarrow B$ in $\mathcal{C}$ to $[f] : (A, \top_A) \longrightarrow (B, \top_B)$; $\mathfrak{n}_A$ is the identity on the fibre $P(A)$ since $P_{\mathrm{cx}}(A, \top_A) = P(A)$. For the retraction, consider an arrow $[g] : (A, \alpha) \longrightarrow (B, \beta)$ in $\mathcal{Prd}_P$, so that $\alpha \le P_g(\beta)$ and $[h] = [g]$ when $\alpha \le P_{\langle g,h\rangle}(\delta_B)$. Let $\{\!|\alpha|\!\}: X \longrightarrow A$ and $\{\!|\beta|\!\}: Y \longrightarrow B$. Hence

$$\top_X \le P_{\{\!|\alpha|\!\}}(\alpha) \le P_{\{\!|\alpha|\!\}}(P_g(\beta))$$

ensures that $g\{\!|\alpha|\!\}$ factors as $\{\!|\beta|\!\}g'$. Similarly, for $[h] = [g]$, we obtain that $h\{\!|\alpha|\!\} = g\{\!|\alpha|\!\}$. In other words, the arrow $g': X \longrightarrow Y$ is uniquely determined by the class $[g]$. It is easy to see that assignment, mapping $[g]\colon (A,\alpha) \longrightarrow (B,\beta)$ to $g'\colon X \longrightarrow Y$, gives rise to a functor $M\colon \mathcal{Prd}_P \longrightarrow \mathcal{C}$ which preserves products. Since P is an existential m-variational doctrine, the fibre $P_{\text{cx}}(A,\alpha) = P_{\text{c}}(A,\alpha)$ is isomorphic to $P(X)$ via the functors

$$P_{\text{c}}(A,\alpha) \xrightarrow{P_{\{\!|\alpha|\!\}}} P(X) \xrightarrow{\exists_{\{\!|\alpha|\!\}}} P_{\text{c}}(A,\alpha)$$

as $\gamma = \exists_{\{\!|\alpha|\!\}}(P_{\{\!|\alpha|\!\}}(\gamma))$ by 2.20. As for the adjunction, it is immediate to see that A is isomorphic to $N(M(A))$ in $\mathcal{C}$. On the other hand, for (A,α) in $\mathcal{Prd}_P$, the comprehnsion of α provides an arrow $[\{\!|\alpha|\!\}]\colon (X,\top_X) \longrightarrow (A,\alpha)$ in P_{cx}. It is easy to see that they form an adjunction between $\mathcal{C}$ and $\mathcal{Prd}_P$. The conclusion follows since the fibres are isomorphic. □

2.23 Remark. The result in 2.22 can be read as a property of existential m-variational doctrines: they are 2-algebras for a 2-monad on **ED**.

Note that the arrow $\{\!|\alpha|\!\}\colon (X,\top_X) \longrightarrow (A,\alpha)$ for (A,α) in P_{cx} is such that

$$\exists_{[\{\!|\alpha|\!\}]}(\top_{(X,\top_X)}) = \alpha = \top_{(A,\alpha)}.$$

So it is monic and surjective with respect to P_{cx}, but may fail to have an inverse in P_{cx}.

In addition, consider that the 2-monad on **ED** is KZ as is the case for any completion, and the unit $P_{\text{cx}} \longrightarrow (P_{\text{cx}})_{\text{cx}}$ is left adjoint (in **ED**) to the multiplication $(P_{\text{cx}})_{\text{cx}} \longrightarrow P_{\text{cx}}$ which maps an object $((A,\alpha),\beta)$ with $\beta \leq \alpha$ in $P(A)$ to the object (A,β).

3 Categories of entire functional relations

As pointed out in [Kel92], the notion of elementary existential doctrine contains the logical data which allow describe relational composition as well as functionality and entirety.

3.1 Definition. Let $P\colon \mathcal{C}^{\text{op}} \longrightarrow$ **InfSL** be an elementary existential doctrine. Let ϕ be in $P(A\times B)$ and ψ in $P(B\times C)$. The ***relational composition of*** ϕ ***and*** ψ is

$$\exists_{\langle \text{pr}_1,\text{pr}_3\rangle}(P_{\langle \text{pr}_1,\text{pr}_2\rangle}(\phi) \wedge P_{\langle \text{pr}_2,\text{pr}_3\rangle}(\psi))$$

where pr_i are the projections from $A\times B\times C$. Also one says that ϕ is ***entire from*** A ***to*** B if $\top_A \leq \exists_{\text{pr}_1}(\phi)$, and that ϕ is ***functional from*** A ***to*** B when $P_{\langle \text{pr}_1,\text{pr}_2\rangle}(\phi) \wedge P_{\langle \text{pr}_1,\text{pr}_3\rangle}(\phi) \leq P_{\langle \text{pr}_2,\text{pr}_3\rangle}(\delta_B)$ in $P(A\times B\times B)$. The category $\mathcal{EF}_P$ of entire functional relations of P has objects those of $\mathcal{C}$; an arrow $\phi\colon A \longrightarrow B$ is a entire functional relation from A to B. They compose by relational composition with the δ_A as identities.

Note that, given an arrow $f\colon A \longrightarrow B$ in $\mathcal{C}$, its ***graph*** $P_{f\times \text{id}_B}(\delta_B)$ is a entire functional relation from A to B and this defines a *graph functor* from $G\colon \mathcal{C} \longrightarrow \mathcal{EF}_P$.

As a simple extension of a result in [Kel92] we have the following.

3.2 Theorem. *Let* $P\colon \mathcal{C}^{\rm op} \longrightarrow \textsf{InfSL}$ *be an elementary existential doctrine.*

(i) *The category* $\mathcal{EF}_P$ *has products.*

(ii) $\mathcal{EF}_P \equiv \mathcal{EF}_{P_{\rm x}}$

(iii) *The graph functor* $G\colon \mathcal{C} \longrightarrow \mathcal{EF}_P$ *preserves products. It is faithful exactly when* P *has comprehensive diagonals.*

(iv) *If* P *is an m-variational doctrine, the category* $\mathcal{EF}_P$ *is regular.*

Proof. (i) is a direct calculation which we leave to the reader.
(ii) is immediate since the definition of the category $\mathcal{EF}_P$ involves only projection arrows.
(iii) is obvious.
(iv) As an equalizer of $\phi, \psi\colon A \longrightarrow B$ in $\mathcal{EF}_P$, one considers the graph in $\mathcal{EF}_P$ of the comprehension $\{\!|\exists_{\mathrm{pr}_1}(\phi \wedge \psi)|\!\}\colon X \longrightarrow A$ in $\mathcal{C}$. The image of $\phi\colon A \longrightarrow B$ in $\mathcal{EF}_P$ is computed taking the graph in $\mathcal{EF}_P$ of the comprehension

$$\{\!|\exists_{\mathrm{pr}_2}(\phi)|\!\}\colon Y \longrightarrow B.$$

□

By the results in [Kel92] we know that the construction in 3.2 produces the ***regular completion*** of an elementary existential doctrine in the following sense.

3.3 Theorem. *The inclusion of the 2-category* **Reg** *of regular categories with regular functors and natural transformations into* **EED** *has a left biadjoint is computed as* $\mathcal{EF}_{P_{\rm cx}}$ *on an elementary existential doctrine* P.

3.4 Example. The regular completion $\mathcal{D}_{\rm reg/lex}$ of a category $\mathcal{D}$ with finite product and weak equalizers in [CV98] is equivalent to the regular completion $\mathcal{EF}_{(\Psi_{\mathcal{D}})_{\rm cx}}$ of the doctrine $\Psi_{\mathcal{D}}\colon \mathcal{D}^{\rm op} \longrightarrow \textsf{InfSL}$ of the weak subobjects of $\mathcal{D}$.

3.5 Proposition. *If* P *is a m-variational doctrine then* $\mathcal{EF}_P \equiv \mathcal{EF}_{P_{\rm cx}}$.

Proof. Applying $\mathcal{EF}$ to the retraction in 2.22, we obtain a retraction between $\mathcal{EF}_P$ and $\mathcal{EF}_{P_{\rm cx}}$. But in $\mathcal{EF}_{P_{\rm cx}}$ the arrow given by the graph of $[\{\!|\alpha|\!\}] : (X, \top_X) \longrightarrow (A, \alpha)$ is iso. So applying $\mathcal{EF}$ to the retraction produces an equivalence of categories. □

4 The construction from tripos to topos

The construction from tripos to topos, together with the notion of entire functional relation, involves also the notion of quotient. We review them briefly from [MR13b] and [Pit02].

4.1 Definition. Let $P\colon \mathcal{C}^{\rm op} \longrightarrow \textsf{InfSL}$ be an elementary doctrine, an object A in $\mathcal{C}$ and ρ in $P(A \times A)$, one says that ρ is a P***-equivalence relation on*** A if it satisfies

reflexivity: $\delta_A \leq \rho$;

***symmetry*:** $\rho \leq P_{\langle \mathrm{pr}_2, \mathrm{pr}_1 \rangle}(\rho)$, for $\mathrm{pr}_1, \mathrm{pr}_2 \colon A \times A \longrightarrow A$ the first and second projection, respectively;

***transitivity*:** $P_{\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle}(\rho) \wedge P_{\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle}(\rho) \leq P_{\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle}(\rho)$, for $\mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_3 \colon A \times A \times A \longrightarrow A$ the projections to the first, second and third factor, respectively.

4.2 Examples. (a) Given an elementary doctrine $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ and an object A in $\mathcal{C}$, the object δ_A is a P-equivalence relation on A.
(b) Given a first order theory $\mathscr{T}$ with equality predicate, consider the elementary doctrine $LT \colon \mathcal{V}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$. An LT-equivalence relation is a $\mathscr{T}$-provable equivalence relation.
(c) For a category $\mathcal{D}$ with products and pullbacks, consider the elementary doctrine $\mathrm{Sub}_{\mathcal{D}} \colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ of the subobjects of $\mathcal{D}$. A $\mathrm{Sub}_{\mathcal{D}}$-equivalence relation is an equivalence relation in $\mathcal{D}$.
(d) For a cartesian category $\mathcal{C}$ with products and weak equalizers, consider the elementary doctrine $\Psi_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ of the weak subobjects. A $\Psi_{\mathcal{C}}$-equivalence relation is a pseudo-equivalence relation in $\mathcal{C}$, see [CC82].

4.3 Definition. For an elementary doctrine $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, the ***elementary quotient completion*** P is the doctrine $\widehat{P} \colon \mathcal{Q}_P^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ where the category $\mathcal{Q}_P$ is determined as follows.

Objects: a pair (A, ρ) such that ρ is a P-equivalence relation on A.

Arrows: an arrow $[f] : (A, \rho) \longrightarrow (B, \sigma)$ is an equivalence class of arrows $f \colon A \longrightarrow B$ in $\mathcal{C}$ such that $\rho \leq P_{f \times f}(\sigma)$ in $P(A \times A)$ with respect to the relation determined by the condition that $\rho \leq P_{f \times g}(\sigma)$.

Composition is given by that of $\mathcal{C}$ on representatives, and identities are represented by identities of $\mathcal{C}$.

The doctrine $\widehat{P} \colon \mathcal{Q}_P^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is defined as

$$\widehat{P}(A, \rho) := \{\alpha \in P(A) \mid P_{\mathrm{pr}_1}(\alpha) \wedge \rho \leq P_{\mathrm{pr}_2}(\alpha)\}$$

where $\mathrm{pr}_1, \mathrm{pr}_2 \colon A \times A \longrightarrow A$ are the projections.

The elementary doctrine $\widehat{P}$ is the completion with respect to quotients of P. There are several details that one must check in order to verify the statements above, and we refer the interested reader to [MR13b].

4.4 Examples. The category of enumerated sets in [Erš73] is the category $\mathcal{Q}_P$ for $P \colon \mathcal{Rec}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ the doctrine on the category of finite powers of the natural numbers with recursive functions where $P(\mathbb{N}^k)$ is the powerset of $\mathbb{N}^k$ and P_f is given by inverse image for f a recursive function.

A similar example is the category $\mathcal{Equ}$ of equilogical spaces, see [BBS04, Ros15]. The doctrine $P \colon \mathcal{Top}_0^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is given on the category of T_0-spaces and continuous functions by taking $P(X, \tau)$ as the powerset of X and P_f is inverse image along f for f a continuous function.

Many other examples are provided by the construction of a category of "partial equivalence relations" on a partial combinatory algebra, see [Sco76]. They are obtained as categories of quotients $\mathcal{Q}_D$ from doctrines which are of the form $D = P_{\text{cx}}$. We should warn the reader that, although the name, these are a *different* categorical construction from $\mathcal{T}_P$ introduced by [Pit02], which we recall below.

We collect in the following statements a few properties of a elementary quotient completion from [MR13b].

4.5 Proposition. *For an elementary doctrine $P\colon \mathcal{C}^{\text{op}} \longrightarrow \mathsf{InfSL}$, the indexed poset $\widehat{P}\colon \mathcal{Q}_P^{op} \longrightarrow \mathsf{InfSL}$ is an elementary doctrine. Moreover*

(i) *If P is existential, then $\widehat{P}$ is existential and $\mathcal{Q}_P$ is regular.*

(ii) *If P is a variational doctrine, then $\widehat{P}$ is an m-variational doctrine.*

Recall from [Pit02] the construction of a category from a tripos. We state it in the case of an elementary existential doctrine $P\colon \mathcal{C}^{\text{op}} \longrightarrow \mathsf{InfSL}$ as the further structure is irrelevant for our discussion (and for the construction). We refer the reader to [MR15, Pas15a] for an analysis of that.

Given an elementary existential doctrine $P\colon \mathcal{C}^{\text{op}} \longrightarrow \mathsf{InfSL}$ the category $\mathcal{T}_P$ consists of

objects: pairs (A, ρ) such that ρ is in $P(A \times A)$ and satisfies symmetry and transitivity as in 4.1;

arrows: an arrow $\phi\colon (A, \rho) \longrightarrow (B, \sigma)$ is an object ϕ in $P(A \times B)$ such that

(i) $\phi \leq P_{\langle \text{pr}_1, \text{pr}_1 \rangle}(\rho) \wedge P_{\langle \text{pr}_2, \text{pr}_2 \rangle}(\sigma)$;

(ii) $P_{\langle \text{pr}_1, \text{pr}_2 \rangle}(\rho) \wedge P_{\langle \text{pr}_2, \text{pr}_3 \rangle}(\phi) \leq P_{\langle \text{pr}_1, \text{pr}_3 \rangle}(\phi)$ in $P(A \times A \times B)$ where the pr_i's are the projections from $A \times A \times B$;

(iii) $P_{\langle \text{pr}_1, \text{pr}_2 \rangle}(\phi) \wedge P_{\langle \text{pr}_2, \text{pr}_3 \rangle}(\sigma) \leq P_{\langle \text{pr}_1, \text{pr}_3 \rangle}(\phi)$ in $P(A \times B \times B)$ where the pr_i's are the projections from $A \times B \times B$;

(iv) $P_{\langle \text{pr}_1, \text{pr}_2 \rangle}(\phi) \wedge P_{\langle \text{pr}_1, \text{pr}_3 \rangle}(\phi) \leq P_{\langle \text{pr}_2, \text{pr}_3 \rangle}(\sigma)$ in $P(A \times B \times B)$ where the pr_i's are as in (iii);

(v) $P_{\langle \text{id}_A, \text{id}_A \rangle}(\rho) \leq \exists_{\text{pr}_1}(\phi)$ in $P(A)$ where the pr_i's are the projections from $A \times B$.

Composition $(A, \rho) \xrightarrow{\phi} (B, \sigma) \xrightarrow{\psi} (C, \tau)$ is defined as

$$\exists_{\langle \text{pr}_1, \text{pr}_3 \rangle}(P_{\langle \text{pr}_1, \text{pr}_2 \rangle}(\phi) \wedge P_{\langle \text{pr}_2, \text{pr}_3 \rangle}(\psi))$$

and identity is $(A, \rho) \xrightarrow{\rho} (A, \rho)$.

This constructions was called the ***exact completion*** of the elementary existential doctrine P in [MR15] for reasons which will become apparent in 4.9.

4.6 Examples. The main examples of this construction are localic toposes and realizability toposes obtained from a tripos, see [HJP80, Pit02, vO08].

It is immediate to check that

4.7 Theorem. *Given an elementary existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSL}$, the category $\mathcal{T}_P$ is equivalent to $\mathcal{EF}_{\widehat{P_{cx}}}$.*

The construction of the exact completion $\mathcal{A}_{\mathrm{ex/reg}}$ of a regular category $\mathcal{A}$ was produced by Freyd in a way that resembled logic, see [FS91]. Indeed it can be obtained as $\mathcal{EF}_{\widehat{\mathrm{Sub}_{\mathcal{A}}}}$, see [MR15] where the operation $\mathcal{EF}_{\widehat{(-)}}$ is written as $\mathcal{E}_{(-)}$. This is indeed an exact completion when performed on existential m-variational doctrines and we recall here its explicit description.

Given an elementary existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSL}$, the category $\mathcal{E}\chi_P = \mathcal{EF}_{\widehat{P}}$ consists of

objects: pairs (A, ρ) such that ρ is in $P(A \times A)$ and satisfies reflexivity, symmetry and transitivity as in 4.1;

arrows: $\phi\colon (A, \rho) \longrightarrow (B, \sigma)$ are objects ϕ in $P(A \times B)$ such that

(i) $P_{\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle}(\rho) \wedge P_{\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle}(\phi) \leq P_{\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle}(\phi)$ in $P(A \times A \times B)$ where the pr_i's are the projections from $A \times A \times B$;

(ii) $P_{\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle}(\phi) \wedge P_{\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle}(\sigma) \leq P_{\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle}(\phi)$ in $P(A \times B \times B)$ where the pr_i's are the projections from $A \times B \times B$;

(iii) $P_{\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle}(\phi) \wedge P_{\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle}(\phi) \leq P_{\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle}(\sigma)$ in $P(A \times B \times B)$ where the pr_i's are as in (iii);

(iv) $\top_A \leq \exists_{\mathrm{pr}_1}(\phi)$.

Composition $(A, \rho) \xrightarrow{\phi} (B, \sigma) \xrightarrow{\psi} (C, \tau)$ is defined as

$$\exists_{\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle}(P_{\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle}(\phi) \wedge P_{\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle}(\psi))$$

and identity is $(A, \rho) \xrightarrow{\rho} (A, \rho)$.

For reasons which will become apparent in 4.9 we refer to the construction $\mathcal{E}\chi_P$ as the ***exact completion*** of the existential m-variational doctrine P.

4.8 Examples. As already stated, the leading example of the above construction $\mathcal{E}\chi_P$ is the exact completion $\mathcal{A}_{\mathrm{ex/reg}}$ of a regular category $\mathcal{A}$, see [FS91, Car95, CV98]. It coincides with $\mathcal{E}\chi_{\mathrm{Sub}_{\mathcal{A}}}$ for the doctrine $\mathrm{Sub}_{\mathcal{A}}\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathsf{InfSL}$ of the subobjects of $\mathcal{A}$.

It follows from 3.4 that also the exact completion $\mathcal{D}_{\mathrm{ex/lex}}$ of a category $\mathcal{D}$ with finite products and weak equalizers is an example, since $\mathcal{D}_{\mathrm{ex/lex}} \equiv (\mathcal{D}_{\mathrm{reg/lex}})_{\mathrm{ex/reg}}$, see [CV98]. Explicitly, the exact completion $(\mathcal{D})_{\mathrm{ex/lex}}$ of the category $\mathcal{D}$ is $\mathcal{E}\chi_{\Psi_{\mathcal{D}}}$.

Other examples come from theories used in the formalization of constructive mathematics: the category of total setoids à la Bishop and functional relations based on the Minimalist Type Theory in [Mai09], which coincides with the exact completion $\mathcal{E}\mathcal{X}_{G^{\mathbf{mtt}}}$ where the doctrine $G^{\mathbf{mtt}}$ is defined as in [MR13b], or the category of total setoids à la Bishop and functional relations based on the Calculus of Constructions [Coq90], which coincides with the exact completion $\mathcal{E}\mathcal{X}_{G^{\mathrm{CoC}}}$ where the doctrine G^{CoC} is constructed from the Calculus of Constructions as $G^{\mathbf{mtt}}$ in [MR13b], and it forms a topos as mentioned in [BCP03].

Applying $\mathcal{E}\mathcal{F}$ to the 1-arrow $P_{\mathrm{x}} \longrightarrow P_{\mathrm{cx}}$, we see that the exact completion $\mathcal{E}\mathcal{X}_P$ is a full subcategory of $\mathcal{T}_P$, as one can also see directly comparing the two explicit constructions. Considering also the embedding $\mathcal{P}rd_P \longrightarrow \mathcal{Q}_{P_{\mathrm{cx}}}$, part of the 1-arrow from P_{cx} to $\widehat{P_{\mathrm{cx}}}$, we obtain the following diagram of embeddings of categories

$$\begin{array}{ccccc} & & \mathcal{P}rd_P & \xhookrightarrow[\text{full}]{} & \mathcal{Q}_{P_{\mathrm{cx}}} \\ & & & & \big\downarrow G \\ \mathcal{E}\mathcal{F}_P & \xhookrightarrow[\text{full}]{} & \mathcal{E}\mathcal{X}_P & \xhookrightarrow[\text{full}]{} & \mathcal{T}_P. \end{array}$$

The difference between the two constructions $\mathcal{E}\mathcal{X}_P$ and $\mathcal{T}_P$ is subtle; from [MR15] we know the following, where composing the left adjoint in 4.9 (i) and that in 2.17 produces that in 4.9 (ii). Let **ESD** be the 2-full 2-subcategory of **EED** on the existential m-variational doctrines whose 1-arrows preserve comprehensions.

4.9 Theorem. (i) *The 2-functor* **Xct** $\longrightarrow$ **ESD** *that takes an exact category* $\mathcal{C}$ *to the doctrine* $\mathrm{Sub}_{\mathcal{C}}$ *of its subobjects has a left biadjoint which associates the exact category* $\mathcal{E}\mathcal{X}_P$ *to an existential m-variational doctrine* P *in* **ESD**.

(ii) *The 2-functor* **Xct** $\longrightarrow$ **EED** *that takes an exact category to the elementary existential doctrine of its subobjects has a left biadjoint which associates the exact category* $\mathcal{T}_P$ *to an elementary existential doctrine* P.

It is clear that the difference depends on the way comprehensions are handled. Indeed, from [MR15] we know that:

4.10 Theorem. *For an existential variational doctrine* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow$ **InfSL**, *the inclusion of* $\mathcal{E}\mathcal{X}_P$ *into* $\mathcal{T}_P$ *is an equivalence of categories. Hence* $\mathcal{E}\mathcal{X}_P$ *is equivalent to* $\mathcal{E}\mathcal{X}_{P_{\mathrm{cx}}}$.

Now, to strengthen our analysis of such exact completions, recall from [MR15] the following.

4.11 Theorem. *Let* P *be an existential m-variational doctrine. The exact completion* $\mathcal{E}\mathcal{X}_P$ *is equivalent to* $(\mathcal{E}\mathcal{F}_P)_{\mathrm{ex/reg}}$.

Now from theorem 4.10 and theorem 4.11 we conclude

4.12 Theorem. *Let* P *be an existential variational doctrine. The exact completion* $\mathcal{E}\mathcal{X}_P$ *is equivalent to* $(\mathcal{E}\mathcal{F}_{P_{\mathrm{cx}}})_{\mathrm{ex/reg}}$.

5 Choice principles

In this section we review rules of choice which are instrumental to prove the main theorems of this paper.

The Rule of Unique Choice

The rule of unique choice allows to characterize those doctrines which coincide with the doctrine of the subobjects of a regular category.

5.1 Definition. An elementary existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSL}$ ***satisfies the Rule of Unique Choice*** (RUC) if, for every pair of objects A and B in $\mathcal{C}$, and every entire functional relation ϕ from A to B, there is an arrow $f\colon A \longrightarrow B$ in $\mathcal{C}$ such that

$$\top_A \leq P_{\langle \mathrm{id}_A, f\rangle}(\phi).$$

5.2 Example. The doctrine $\mathrm{Sub}_{\mathcal{A}}\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathsf{InfSL}$ of the subobjects of a regular category $\mathcal{A}$ satisfies (RUC).

Actually the example of the doctrine of the subobjects of any regular category is the main example of m-variational doctrines satisfying (RUC). Indeed from 4.4.4 and 4.9.4 of [Jac99] one can derive the following result.

5.3 Proposition. *Given an elementary existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSL}$, the following are equivalent:*

(i) *$\mathcal{C}$ is a regular category and P is the doctrine of its subobjects.*

(ii) *P has full comprehensions, comprehensive diagonals and satisfies* (RUC).

This agrees with the fact that the regular completion of an m-variational doctrine P adds exactly what is needed to satisfy (RUC). In particular if P already satisfies (RUC), the regular completion coincides with P itself.

5.4 Corollary. *Given a regular category $\mathcal{A}$, the regular completion $\mathcal{EF}_{\mathrm{Sub}_{\mathcal{A}}}$ of the doctrine* $\mathrm{Sub}_{\mathcal{A}}$ *of the subobjects of $\mathcal{A}$ is equivalent to $\mathcal{A}$.*

The Rule of Choice

The rule of choice allows to characterize the doctrines of the weak subobjects of categories with finite products and weak equalizers.

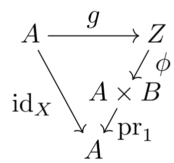
5.5 Definition. For an elementary existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSL}$, we say that P ***satisfies the Rule of Choice*** (RC) if, for every $\phi \in P(A \times B)$ such that

$$\top_A \leq \exists_{\mathrm{pr}_1}(\phi)$$

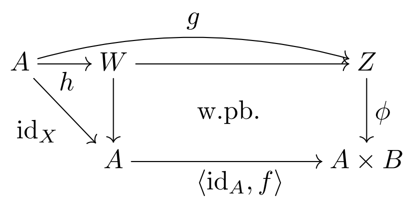
there is an arrow $f\colon A \longrightarrow B$ in $\mathcal{C}$ such that

$$\top_A \leq P_{\langle \mathrm{id}_A, f\rangle}(\phi).$$

5.6 Examples. (a) The doctrine $\Psi_{\mathcal{C}}$ based on a category $\mathcal{C}$ with finite limits in 2.3(b) satisfies (RC). For $[\phi]$ in $\Psi_{\mathcal{C}}(A \times B)$, where $\phi\colon Z \longrightarrow A \times B$ in $\mathcal{C}$, the condition $\top_A \leq \exists_{\mathrm{pr}_1}\phi$ in $\Psi_{\mathcal{C}}(A)$ means that there is a commutative diagram



in $\mathcal{C}$. In other words, there is an arrow $f := \mathrm{pr}_2\phi g$ such that, for some W and $h\colon A \longrightarrow W$,



where the square is a weak pullback which ensures the existence of h.
(b) The doctrine $\mathrm{Sub}_{\mathcal{A}}\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ of the subobjects on a regular category $\mathcal{A}$ satisfies (RC) if and only if regular epis split in $\mathcal{A}$.

5.7 Definition. For an elementary existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ we say that P ***satisfies the Extended Rule of Choice*** (ERC) if, for every $\phi \in P(B)$ and for every $g\colon B \longrightarrow A$ such that

$$\top_A \leq \exists_g(\phi).$$

there is an arrow $f\colon A \longrightarrow B$ in $\mathcal{C}$ such that $gf = \mathrm{id}_A$ and

$$\top_A \leq P_f(\phi).$$

5.8 Lemma. *Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an elementary existential doctrine with comprehensive diagonals. If P satisfies* (RC), *then it satisfies* (ERC).

Proof. Suppose $\phi \in P(B)$ and $g\colon B \longrightarrow A$ is such that $\top_A \leq \exists_g(\phi)$. By proposition 2.6

$$\exists_g(\phi) = \exists_{\mathrm{pr}_1}(P_{\mathrm{pr}_2}(\phi) \wedge P_{\langle \mathrm{pr}_1, g\mathrm{pr}_2\rangle}(\delta_A))$$

where pr_1 and pr_2 are the projections from $A \times B$. So, by (RC), there is $f\colon A \longrightarrow B$ in $\mathcal{C}$ such that

$$\begin{aligned} \top_A &\leq P_{\langle \mathrm{id}_A, f\rangle}(P_{\mathrm{pr}_2}(\phi) \wedge P_{\langle \mathrm{pr}_1, g\mathrm{pr}_2\rangle}(\delta_A)) \\ &= P_f(\phi) \wedge P_{\langle \mathrm{id}_A, gf\rangle}(\delta_A) \end{aligned}$$

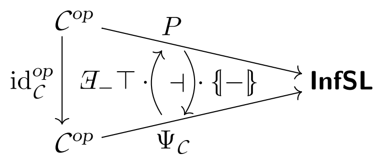
So $\top_A \leq P_f(\phi)$, and $\top_A \leq P_{\langle \mathrm{id}_A, gf\rangle}(\delta_A)$. Since P has comprehensive diagonals, the second inequality is equivalent to $\mathrm{id}_A = gf$ as required. □

The Rule of Choice was used in [MR16] to characterize those doctrines whose elementary quotient completion is the doctrine of the subobjects of an exact category. Here we use the Rule of Choice to characterize those m-variational doctrines which coincide with the doctrine of the weak subobjects of their base.

5.9 Theorem. *Let* $P\colon \mathcal{C}^{op} \longrightarrow \textsf{InfSL}$ *be an existential variational doctrine. The following are equivalent:*

(i) *P satisfies* (RC).

(ii) *The fibered adjunction*



is an equivalence in **ED**.

Proof. First of all, note that, for every β in $P(B)$, it is $\mathit{\exists}_{\{\!|\beta|\!\}}(\top_X) = \beta$ where $\{\!|\beta|\!\}\colon X \longrightarrow B$ since weak comprehensions are full. Note also that the hypothesis on P ensure that both $\{\!|-|\!\}$ and $\mathit{\exists}_{-}\top$ define arrows in **ED** (2.19, 2.10 and 2.15).
(ii)⇒(i) follows from 5.6(a).
(i)⇒(ii) Suppose P satisfies (RC). Consider $h\colon Z \longrightarrow B$; in the doctrine P, one has that $\top_Z \le P_h(\mathit{\exists}_h(\top_Z))$. So h factors through a(ny) weak comprehension $\{\!|\mathit{\exists}_h(\top_Z)|\!\}\colon X \longrightarrow B$ with respect to P. Consider now $\mathit{\exists}_h(\top_Z)$; let the following diagram be a weak pullback of h along $\{\!|\mathit{\exists}_h(\top_Z)|\!\}$.

$$\begin{array}{ccc} Y & \xrightarrow{\quad g \quad} & Z \\ {\scriptstyle k}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ X & \xrightarrow[\{\!|\mathit{\exists}_h(\top_B)|\!\}]{} & B \end{array}$$

By 2.19 it is

$$\top_X \le_X P_{\{\!|\mathit{\exists}_h(\top_Z)|\!\}}(\mathit{\exists}_h(\top_Z)) = \mathit{\exists}_k P_g(\top_Z) = \mathit{\exists}_k(\top_Y).$$

By 5.8, (ERC) yields that there exists $f\colon X \longrightarrow Y$ such that $kf = \mathrm{id}_X$ Hence

$$hgf = \{\!|\mathit{\exists}_h(\top_Z)|\!\}kf = \{\!|\mathit{\exists}_h(\top_Z)|\!\}.$$

So $\{\!|\mathit{\exists}_h(\top_Z)|\!\}$ factors through h. Thus $\{\!|\mathit{\exists}_h(\top_Z)|\!\}$ and h represent the same object in $\Psi_{\mathcal{C}}(B)$. It follows that the composition $\{\!|-|\!\}(\mathit{\exists}_{-}\top)$ is the identity natural transformation. □

The ϵ-operator

Here we introduce yet another rule connected with the epsilon operator introduced by Hilbert in classical logic, see [HB01a, HB01b]. It allows to characterize when the free full comprehensive doctrine P_{cx} with comprehensive diagonals of a given existential elementary doctrine P coincides with the doctrine of the weak subobjects of the base $\mathcal{Prd}_P$ of P_{cx}.

5.10 Definition. Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an elementary existential doctrine. An object B in $\mathcal{C}$ is equipped with an ***ϵ-operator*** if, for any object A in $\mathcal{C}$ and any α in $P(A \times B)$ there exists an arrow $\epsilon_\alpha\colon A \longrightarrow B$ such that

$$\mathit{\exists}_{\mathrm{pr}_1}(\alpha) = P_{\langle \mathrm{id}_A, \epsilon_\alpha \rangle}(\alpha)$$

holds in $P(A)$, where $\mathrm{pr}_1\colon A \times B \longrightarrow A$ is the first projection.

The definition is motivated by the fact that arrows of the form ϵ_α behave like Hilbert's epsilon terms [Bel93b, HB01a, HB01b].

Recall that, in a category $\mathcal{C}$ with terminal object 1, an object B is ***well pointed*** if there exists an arrow $1 \longrightarrow B$.

5.11 Lemma. *In an elementary existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, if B is equipped with an ϵ-operator, then B is well pointed.*

Proof. Take $\alpha := \top_{1\times B}$. Then $\epsilon_\alpha\colon 1 \longrightarrow B$. □

5.12 Definition. We say that an elementary existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is ***equipped with ϵ-operators*** if every object in $\mathcal{C}$ is equipped with an ϵ-operator.

5.13 Example. The doctrine LT presented in the examples 2.3 is equipped with ϵ-operators if and only if $\mathscr{T}$ is Hilbert's epsilon calculus [Pas17].

5.14 Example. Let ξ be an ordinal with a greatest element. Then $\mathcal{H} = (\xi, \geq)$ is a frame. Consider the doctrine $\mathcal{H}^{(-)}\colon \mathcal{Set}_*^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ on the category of non-empty sets as in 2.3(d). For a function α in $\mathcal{H}^{X\times Y}$ and a in X consider the set

$$I(a) = \left\{ b \in Y \,\middle|\, \alpha(a,b) = \textstyle\bigvee_{y\in Y} \alpha(a,y) \right\} \subseteq Y.$$

Clearly $I(a)$ is not empty. Thus, by the Axiom of Choice, there is a function $\epsilon_\alpha\colon X \longrightarrow Y$ with $\epsilon_\alpha(a) \in I(a)$. That function is such that, for every a in X

$$\alpha(a, \epsilon_\alpha(a)) = \bigvee_{y\in Y} \alpha(a,y)$$

proving that $\mathcal{H}$ satisfies the epsilon rule.

Other examples of elementary existential doctrine equipped with ϵ-operators are in [Pas16].

The Rule of Choice and ϵ-operators are related through the comprehension completion P_{c} of an elementary existential doctrine.

5.15 Theorem. *Let $P\colon \mathcal{C}^{op} \longrightarrow \mathsf{InfSL}$ be an elementary existential doctrine. The following are equivalent:*

(i) *P is equipped with ϵ-operators.*

(ii) *The free completion doctrine $P_{\mathrm{c}}\colon {\mathcal{G}_P}^{op} \longrightarrow \mathsf{InfSL}$ of P with full comprehensions satisfies* (RC).

(iii) *The doctrine $P_{\mathrm{cx}}\colon {\mathcal{P}rd_P}^{op} \longrightarrow \mathsf{InfSL}$ of P satisfies* (RC).

Proof. (i)⇒(ii): Let (A,α) and (B,β) be objects of $\mathcal{G}_P$. Consider ϕ in $P_{\mathrm{c}}((A,\alpha)\times(B,\beta))$ such that

$$\top_{(A,\alpha)} \leq \mathit{\exists}_{\mathrm{pr}_1}(\phi)$$

holds in $P_{\mathrm{c}}((A,\alpha))$. The two conditions in the doctrine P_{c} are translated in the doctrine P as

$$\phi \leq P_{\mathrm{pr}_1}(\alpha) \wedge P_{\mathrm{pr}_2}(\beta) \quad \text{ and } \quad \alpha \leq \mathit{\exists}_{\mathrm{pr}_1}(\phi).$$

So $\alpha = \mathit{\exists}_{\mathrm{pr}_1}(\phi)$. Also, since A is equipped with an ϵ-operator, there is an arrow $\epsilon_\phi\colon A \longrightarrow B$ such that $\mathit{\exists}_{\mathrm{pr}_1}(\phi) = P_{\langle \mathrm{id}_A, \epsilon_\phi\rangle}(\phi)$ holds in $P(A)$. But ϵ_ϕ determines an arrow in $\mathcal{G}_P$ from (A,α) to (B,β) since

$$\alpha = \mathit{\exists}_{\mathrm{pr}_1}(\phi) = P_{\langle \mathrm{id}_A, \epsilon_\phi\rangle}(\phi) \leq P_{\langle \mathrm{id}_A, \epsilon_\phi\rangle}(P_{\mathrm{pr}_1}(\alpha) \wedge P_{\mathrm{pr}_2}(\beta)) = \alpha \wedge P_{\epsilon_\phi}(\beta).$$

Thus $\alpha \leq P_{\epsilon_\phi}(\beta)$ and $\top_{(A,\alpha)} \leq (P_{\mathrm{c}})_{\langle \mathrm{id}_A, \epsilon_\phi\rangle}(\phi)$.
(ii)⇒(i): Suppose ϕ is in $P(A\times B)$. Since $\phi \leq P_{\mathrm{pr}_1}(\mathit{\exists}_{\mathrm{pr}_1}(\phi))$, one has that ϕ is in $P_{\mathrm{c}}((A, \mathit{\exists}_{\mathrm{pr}_1}(\phi)) \times (B, \top_B))$. Note that trivially

$$\top_{(A,\mathit{\exists}_{\mathrm{pr}_1}(\phi))} = \mathit{\exists}_{\mathrm{pr}_1}(\phi)$$

in $P_{\mathrm{c}}((A, \mathit{\exists}_{\mathrm{pr}_1}(\phi)))$. Since P_{c} satisfies (RC) there is an arrow $f\colon (A, \mathit{\exists}_{\mathrm{pr}_1}(\phi)) \longrightarrow (B, \top_B)$ in $\mathcal{G}_P$ such that $\top_{(A,\mathit{\exists}_{\mathrm{pr}_1}(\phi))} \leq (P_{\mathrm{c}})_{\langle \mathrm{id}_A, f\rangle}(\phi)$ holds in $P_{\mathrm{c}}((A, \mathit{\exists}_{\mathrm{pr}_1}(\phi)))$. Since $(P_{\mathrm{c}})_{\langle \mathrm{id}_A, f\rangle}(\phi) = P_{\langle \mathrm{id}_A, f\rangle}(\phi)$ it follows that

$$\mathit{\exists}_{\mathrm{pr}_1}(\phi) = P_{\langle \mathrm{id}_A, f\rangle}(\phi)$$

holds in $P(A)$.

(ii)⇔(iii): Immediate because the condition required to satisfy (RC) does not involve commutative diagrams in the base category. □

6 Applications

By combining 5.9 and 5.15 we get the main technical result.

6.1 Theorem. *Let $P\colon \mathcal{C}^{op} \longrightarrow \mathsf{InfSL}$ be an elementary existential doctrine. The following are equivalent:*

(i) *P is equipped with ϵ-operators.*

(ii) *The fibered adjunction*

$$\begin{array}{ccc} \mathcal{Prd}_P{}^{op} & \xrightarrow{\quad P_{cx}\quad} & \\ {\scriptstyle \mathrm{id}^{op}_{\mathcal{Prd}_P}}\big\downarrow & \exists_{-}(\top)\cdot \dashv \cdot \{\!|-|\!\} & \textsf{InfSL} \\ \mathcal{Prd}_P{}^{op} & \xrightarrow{\quad \Psi_{\mathcal{Prd}_P}\quad} & \end{array}$$

is an equivalence in **ED**.

In preparation to 6.2 we review some of the canonical functors which connect the various completions. For the rest of the section let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \textsf{InfSL}$ be an elementary existential doctrine.

Since $\mathcal{T}_P$ is equivalent to $\mathcal{EF}_{\widehat{P_{\mathrm{cx}}}}$, consider the composite

$$\begin{array}{ccccc} \mathcal{Prd}_P & \xrightarrow{\ \eta_1\ } & Q_{P_{\mathrm{cx}}} & & \\ {\scriptstyle G}\big\downarrow & & \big\downarrow{\scriptstyle G'} & \overset{K}{\searrow} & \\ \mathcal{EF}_{P_{\mathrm{cx}}} & \xrightarrow{\ \mathcal{EF}_{(\eta_1,\mathfrak{h})}\ } & \mathcal{EF}_{\widehat{P_{\mathrm{cx}}}} & \underset{J}{\equiv\equiv} & \mathcal{T}_P \end{array}$$

where $(\eta_1, \mathfrak{h})$ is the universal arrow into the elementary quotient completion in [MR13a] (under a different name) and the natural family of functors G was introduced in 3.2. The functor K maps an object (A, α) to an object $(A, \delta_A \wedge P_{\mathrm{pr}_1}(\alpha))$, and an arrow $f\colon (A, \alpha) \longrightarrow (B, \beta)$ in $\mathcal{Prd}_P$ to the graph

$$P_{f\times \mathrm{id}_B}(\delta_B) \wedge P_{\mathrm{pr}_1}(\alpha) = (P_{\mathrm{cx}})_{f\times \mathrm{id}_{(B,\beta)}}(\delta_{(B,\beta)}).$$

By the universal properties of the functors involved, the composition preserves finite limits. So one obtains the exact functor K^{ex}

$$\begin{array}{ccc} \mathcal{Prd}_P & \hookrightarrow & (\mathcal{Prd}_P)_{\mathrm{ex/lex}} \\ & {\scriptstyle K}\searrow & \big\downarrow{\scriptstyle K^{\mathrm{ex}}} \\ & & \mathcal{T}_P \end{array}$$

by the universal property of the exact completion.

Also note that the functor $\mathcal{EF}_{(\eta_1,\mathfrak{h})}\colon \mathcal{EF}_{P_{\mathrm{cx}}} \longrightarrow \mathcal{EF}_{\widehat{P_{\mathrm{cx}}}}$ is regular. So G can be extended to the regular completion $\mathcal{Prd}_{P\,\mathrm{reg/lex}}$ and the diagram

$$\begin{array}{ccccc} \mathcal{Prd}_P & \hookrightarrow & (\mathcal{Prd}_P)_{\mathrm{reg/lex}} & \hookrightarrow & (\mathcal{Prd}_P)_{\mathrm{ex/lex}} \\ & {\scriptstyle G}\searrow & \big\downarrow{\scriptstyle G^{\mathrm{reg}}} & {\scriptstyle K}\searrow & \big\downarrow{\scriptstyle K^{\mathrm{ex}}} \\ & & \mathcal{EF}_{P_{\mathrm{cx}}} & \xrightarrow{\ J\mathcal{EF}_{(\eta_1,\mathfrak{h})}\ } & \mathcal{T}_P \end{array} \qquad (1)$$

commute up to a natural iso.

We are ready to state the main result.

6.2 Theorem. *Let* $P\colon \mathcal{C}^{\rm op} \longrightarrow \textsf{InfSL}$ *be an elementary existential doctrine. The following are equivalent:*

(i) *P is equipped with ϵ-operators.*

(ii) *$G^{\rm reg}\colon (\mathcal{Prd}_P)_{\rm reg/lex} \longrightarrow \mathcal{EF}_{\mathcal{P}_{\rm cx}}$ is an equivalence.*

(iii) *$K^{\rm ex}\colon (\mathcal{Prd}_P)_{\rm ex/lex} \longrightarrow \mathcal{T}_P$ is an equivalence.*

Proof. (i)⇔(ii) Consider the left-hand triangle in (1) and replace the regular completion $(\mathcal{Prd}_P)_{\rm reg/lex}$ with its equivalent presentation via the other completions—squeezing it down.

$$\begin{array}{ccc} & & \mathcal{EF}_{(\Psi_{\mathcal{Prd}_P})_{\rm cx}} \\ & \nearrow & \Big\downarrow \mathcal{EF}_{(\exists_-(\top))} \\ \mathcal{Prd}_P & \xrightarrow[G]{} & \mathcal{EF}_{P_{\rm cx}} \end{array}$$

From 6.1, (i) holds if and only if the above diagram becomes part of a naturality diagram of equivalences in **ESD**

$$\begin{array}{ccc} P_{\rm cx} & \hookrightarrow & {\rm Sub}_{\mathcal{EF}_{P_{\rm cx}}} \\ \exists_-(\top) \uparrow \dashv \downarrow \{\!|-|\!\} & & {\rm Sub}_{\mathcal{EF}_{(\exists_-(\top))}} \uparrow \dashv \downarrow {\rm Sub}_{\mathcal{EF}_{\{\!|-|\!\}}} \\ \Psi_{\mathcal{Prd}_P} & \hookrightarrow & {\rm Sub}_{\mathcal{EF}_{(\Psi_{\mathcal{Prd}_P})_{\rm cx}}} \end{array}$$

since the adjunction on the left is an equivalence if and only if adjunction on the right exists (observe that $\mathcal{EF}_-$ can only be applied if $\{\!|-|\!\}$ is existential!) and is an equivalence.

(ii)⇔(iii): Similar to the previous part, this time consider the right-hand triangle in (1) and replace the exact completion $(\mathcal{Prd}_P)_{\rm ex/lex}$ and $\mathcal{T}_P$ with their equivalent presentations via the other completions

$$\begin{array}{ccc} \mathcal{EF}_{P_{\rm cx}} & \xhookrightarrow{\mathcal{EF}_{(\eta_1,\mathfrak{h})}} & \mathcal{EF}_{\widehat{P_{\rm cx}}} \\ \mathcal{EF}_{(\exists_-(\top))} \uparrow \dashv \downarrow \mathcal{EF}_{\{\!|-|\!\}} & & \mathcal{EF}_{\widehat{(\exists_-(\top))}} \uparrow \dashv \downarrow \mathcal{EF}_{\widehat{\{\!|-|\!\}}} \\ \mathcal{EF}_{(\Psi_{\mathcal{Prd}_P})_{\rm cx}} & \xrightarrow[\mathcal{EF}_{(\eta_1,\mathfrak{h})}]{} & \mathcal{EF}_{\widehat{(\Psi_{\mathcal{Prd}_P})_{\rm cx}}} \end{array}$$

where we applied 4.10. The conclusion follows immediately. □

The above theorem applied to the tripos-to-topos construction yields the following.

6.3 Corollary. *Let $P\colon \mathcal{C}^{op} \longrightarrow \mathsf{Heyt}$ be a tripos. The following are equivalent:*

(i) *P is equipped with ϵ-operators.*

(ii) *the functor $K^{\mathrm{ex}}\colon (\mathcal{Prd}_P)_{\mathrm{ex/lex}} \longrightarrow \mathcal{T}_P$ is part of an equivalence between the exact on lex completion $(\mathcal{Prd}_P)_{\mathrm{ex/lex}}$ and the tripos-to-topos $\mathcal{T}_P$ of P.*

6.4 Examples. An application of 6.3 is the localic topos obtained from the tripos $\mathcal{H}^{(-)}\colon (\mathcal{Set}_*)^{\mathrm{op}} \longrightarrow \mathsf{InfSL}$ in example 5.14.

An application of 6.3 with relevance in logic is provided by the doctrine introduced in 2.3 where the theory $\mathscr{T}$ is exactly Peano Arithmetic together with Hilbert's ϵ-operator, already studied in [Tai10], and which inspired the ϵ-operators in the present paper.

From most triposes P on $\mathcal{C}$ we can easily obtain a tripos whose base has only pointed objects in such a way that the two corresponding toposes are equivalent. To this purpose the following lemma might be useful. Given an elementary existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSL}$, let $\mathcal{C}_*$ be the full subcategory of $\mathcal{C}$ on the pointed objects and $P_*\colon \mathcal{C}_*^{\mathrm{op}} \longrightarrow \mathsf{InfSL}$ the restriction of P.

6.5 Lemma. *If $P\colon \mathcal{C}^{op} \longrightarrow \mathsf{Heyt}$ is a tripos, then $P_*\colon \mathcal{C}_*^{op} \longrightarrow \mathsf{InfSL}$ is also a tripos.*

Proof. Immediate. □

We conclude our paper by observing that the ϵ-operators are not preserved by the tripos-to-topos construction.

6.6 Theorem. *The topos built from the tripos $\mathcal{H}^{(-)}\colon \mathcal{Set}_*^{op} \longrightarrow \mathsf{InfSL}$ in example 5.14 is not equipped with ϵ-operators while the doctrine $\mathcal{H}^{(-)}$ is.*

Proof. Suppose the doctrine of the subobjecs of the topos $\mathcal{T}_{\mathcal{H}^{(-)}}$ is equipped with ϵ-operators. It follows from [Bel93b] that it satisfies also (AC). Therefore the topos is boolean by Diaconescu's theorem, see [MM92]. But the global sections of the subobject classifier are $\mathcal{H}$ which is not boolean. □

6.7 Remark. From [Bel93a] it follows that any tripos equipped with ϵ-operators satisfies a weak form of excluded middle, whilst it does not necessarily satisfies the full form.

7 Concluding remarks

We have characterized the triposes P for which the universal arrow from the exact completion of their category of predicates $\mathcal{Prd}_P$ to $\mathcal{T}_P$ is an equivalence as those equipped with ϵ-operators. An example of a non-boolean topos whose tripos is equipped with ϵ-operators is given as a localic topos.

These results constitute an application to the study of the tripos-to-topos construction of the investigations on exact completions relativized to suitable doctrines performed in [MR15] and generalized to other quotient completions in [MR13a]. A major benefit of relativizing exact completions to suitable doctrines is the possibility of viewing various notions of exact completion as instances of a single, more general completion. This reveals that it is indeed the choice of the doctrine that yields different regular completions, hence different notions of exact completion.

In particular, inspired by results in [MR16] about the notion of elementary quotient completion, in this paper we observed how common choice principles in proof theory, when expressed in the language of doctrines, correspond to categorical equivalences between appropriate completions.

In future work we intend to apply these results to study models of Heyting arithmetics. In particular, examples of triposes equipped with ϵ-operators should provide models witnessing that the underlying logic of Heying arithmetics with Hilbert's ϵ-operator is not necessarily classical.

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