INDUCTIVE AND COINDUCTIVE TOPOLOGICAL GENERATION WITH CHURCH'S THESIS AND THE AXIOM OF CHOICE

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Abstract. In this work we consider an extension $\mathbf{MF}_{cind}$ of the Minimalist Foundation **MF** for predicative constructive mathematics with the addition of inductive and coinductive topological definitions sufficient to generate Sambin's Positive topologies, namely Martin-Löf- Sambin formal topologies (used to describe pointfree formal open subsets) equipped with a Positivity relation (used to describe pointfree formal closed subsets). In particular the intensional level of $\mathbf{MF}_{cind}$, called $\mathbf{mTT}_{cind}$, is defined by extending with coinductive topological definitions another theory $\mathbf{mTT}_{ind}$ extending the intensional level **mTT** of **MF** with the sole addition of inductive topological definitions. In previous work we have shown that $\mathbf{mTT}_{ind}$ is consistent with adding simultaneously formal Church's Thesis, **CT**, and the Axiom of Choice, **AC**, via an interpretation in Aczel's **CZF +REA**, namely Constructive Zermelo-Fraenkel Set Theory with the Regular Extension Axiom. Our aim is to show the expectation that the addition of coinductive topological definitions to $\mathbf{mTT}_{ind}$ does not increase its consistency strength by reducing the consistency of $\mathbf{mTT}_{cind}$+**CT**+**AC** to the consistency of **CZF +REA** through various interpretations. We reach our goal in two ways. One way consists in first interpreting $\mathbf{mTT}_{cind}$+**CT**+**AC** in the theory extending **CZF** with the **Union Regular Extension Axiom**, $\mathbf{REA}_{\bigcup}$, a strengthening of **REA**, and the Axiom of Relativized Dependent Choice, **RDC**. The theory $\mathbf{CZF} + \mathbf{REA}_{\bigcup} + \mathbf{RDC}$ is then interpreted in $\mathbb{MLS}^*$, a version of Martin-Löf's type theory with Palmgren's superuniverse $\mathbb{S}$. The last step consists in interpreting $\mathbb{MLS}^*$ back into **CZF +REA**. The alternative way consists in first interpreting $\mathbf{mTT}_{cind}$+**AC** + **CT** directly in a version of Martin-Löf's type theory with Palmgren's superuniverse extended with **CT**, which is then interpreted back to **CZF +REA**. A key benefit of the first way is that the theory $\mathbf{CZF} + \mathbf{REA}_{\bigcup} + \mathbf{RDC}$ also supports the intended set-theoretic interpretation of the extensional level of $\mathbf{MF}_{cind}$. Finally, all the theories considered to reach our goals, except $\mathbf{mTT}_{cind}$+**AC** + **CT**, are shown to be of the same proof-theoretic strength.

Key words and phrases: Realizability, Axiom of Choice, Church's thesis, Point-free topology, constructive set theory.

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1. Introduction

This paper extends results from previous papers [IMMS18, MMR21] and provides a further contribution to the project of establishing what extensions of the intensional level of the Minimalist Foundation in [Mai09] are consistent with Formal Church's thesis and the Axiom of Choice.

The Minimalist Foundation, for short **MF**, was first ideated in [MS05] and completed in [Mai09]. It was introduced to serve as a constructive foundation compatible with the most relevant constructive and classical foundations.

The reason is that still today no foundation for constructive mathematics has affirmed itself as the standard reference as it happened with Zermelo-Fraenkel axiomatic set theory for classical mathematics. The many existing foundations for constructive mathematics all adopt intuitionistic logic for their logical reasoning but they differ in the accepted set-theoretic principles. For example, their consistency status with principles such as the formal Church's thesis **CT** and the Axiom of Choice may be very different. Examples vary from Heyting arithmetic in all finite types, which is constructively consistent with both principles simultaneously (cf. [Tv88]), to foundations for Brouwer's intuitionistic mathematics in [Tv88] which are inconsistent with **CT**, to foundations like Aczel's Constructive Set Theory, for short **CZF**, which is constructively consistent with **CT** together with some restricted forms of the Axiom of Choice (see [Rat06, 9.2,10.1]), whereas the combination of **CT** and the full axiom of choice together with **CZF** gives rise to inconsistency.

Moreover, another peculiarity of the Minimalist Foundation **MF** in [Mai09] is that of being a two-level foundation, as required in [MS05]. Indeed, **MF** is equipped with an extensional level, called **emTT**, formulated in a language close to that of everyday mathematical practice and interpreted via a quotient model in a further intensional level, called **mTT**, designed as a type-theoretic base for a proof-assistant.

The notion of two-level constructive foundation was introduced in [MS05], motivated by the need to founding constructive mathematics on a theory consistent with the axiom of choice and Church's thesis **CT** (via an extension of Kleene realizability interpretation of intuitionistic arithmetics) including a way to formalize usual desirable extensional set-theoretic principles, such as extensional equality of functions, which are inconsistent with the axiom of choice and **CT** (see also [Tv88]).

Only recently, in [IMMS18], it was shown that the intensional level **mTT** of **MF** is consistent with the axiom of choice **AC** and the formal Church's thesis **CT**, expressed as follows: **AC** states that from any total relation we can extract a type-theoretic function

$$(\mathbf{AC})\ (\forall x \in A)\,(\exists y \in B)\,R(x,y) \to (\exists f \in (\Pi x \in A)\,B)\,(\forall x \in A)\,R(x, \mathsf{Ap}(f,x))$$

with A and B generic collections and $R(x,y)$ any relation, while **CT** (see also [Tv88]) states that from any total relation on natural numbers we can extract a (code of a) recursive function by using the Kleene predicate T and the extracting function U

$$(\mathbf{CT})\ (\forall f \in \mathsf{N} \to \mathsf{N})(\exists e \in \mathsf{N})\,(\forall x \in \mathsf{N})\,(\exists z \in \mathsf{N})\,(T(e,x,z) \wedge \mathsf{Ap}(f,x) =_{\mathsf{N}} U(z)).$$

Furthermore, the whole **MF** turned out to be not only constructive but also predicative in the strict sense of Feferman, as first shown in [MM16].

Another example of two-level foundation is the theory $\mathbf{MF}_{ind}$ proposed in [MMR21]. $\mathbf{MF}_{ind}$ extends **MF** with all the inductive definitions necessary to formalize the inductive topological methods developed in [CSSV03, CMS13] within Formal Topology which is a predicative and constructive approach to topology put forward by P. Martin-Löf and G.

Sambin in [Sam87]. Indeed, it was expected that **MF** is not able to formalize all such inductive methods, due to its minimality.

In particular, in [MMR21] we showed that the intensional level $\mathbf{mTT}_{ind}$ of $\mathbf{MF}_{ind}$ is consistent with **AC** and **CT** by extending the realizability semantics used in [IMMS18, Rat93, GR94, Rat03]. A major improvement of the realizability semantics in [MMR21] was its formalization in a constructive theory such as the (generalized) predicative set theory **CZF+REA**, namely Aczel's constructive Zermelo-Fraenkel set theory extended with the regular extension axiom **REA** (see [Acz86]). Instead, the semantics in [IMMS18] was formalized in Feferman's predicative theory of non-iterative fixpoints $\widehat{ID}_1$ which is governed by classical logic. Both semantics extend Kleene realizability semantics of intuitionistic arithmetic and hence they validate **CT** and **AC**.

This paper aims to further extend $\mathbf{MF}_{ind}$ by also including the coinductive topological methods introduced by P. Martin-Löf and G. Sambin in [Sam03, Sam19] to describe both open and closed subsets of a point-free topology, later developed as Positive Topology in [Sam03]. Instances of such inductive and coinductive definitions seem necessary to provide a constructive and predicative version of results like the coreflection of the category of locales in that of open locales in [MV04], or the embedding of the category of locales in that of positive topologies in [CS18].

To meet our purpose, here we build a two-level extension $\mathbf{MF}_{cind}$ of $\mathbf{MF}_{ind}$, by enriching both levels of $\mathbf{MF}_{ind}$ with Martin-Löf and Sambin's coinductive definitions necessary to generate Sambin's Basic Topologies in [Sam03, Sam19], namely inductively generated basic covers (used to describe pointfree formal open subsets) enriched with a positivity relation (used to describe pointfree formal closed subsets),

Then, we prove two main results concerning $\mathbf{MF}_{cind}$. First, we show that $\mathbf{MF}_{cind}$ is a two-level theory in the sense of [MS05] by showing that its intensional level, called $\mathbf{mTT}_{cind}$, is consistent with **AC** and **CT**. Second, we show the expectation that the addition of coinductive topological definitions to $\mathbf{mTT}_{ind}$ does not increase its consistency strength by reducing the consistency of $\mathbf{mTT}_{cind}$**+CT+AC** to the consistency of **CZF +REA** through various interpretations. We reach this goal in two ways.

One way consists in first interpreting $\mathbf{mTT}_{cind}$**+CT+AC** in the theory extending **CZF** with the **Union Regular Extension Axiom**, $\mathbf{REA}_{\bigcup}$, a strengthening of **REA**, as well as with the Axiom of Relativized Dependent Choice, **RDC**. The theory $\mathbf{CZF} + \mathbf{REA}_{\bigcup} + \mathbf{RDC}$ is then interpreted in $\mathbb{MLS}^*$, a version of Martin-Löf's type theory with Palmgren's superuniverse $\mathbb{S}$. The last step consists in interpreting $\mathbb{MLS}^*$ back into **CZF +REA**.

The alternative way consists in first interpreting $\mathbf{mTT}_{cind}$$+\mathbf{AC} + \mathbf{CT}$ directly in a version of Martin-Löf's type theory with Palmgren's superuniverse extended with **CT**, which is then interpreted in **CZF +REA** by extending the realizability semantics in [MMR21].

A key benefit of the first way is that the intermediate theory $\mathbf{CZF} + \mathbf{REA}_{\bigcup} + \mathbf{RDC}$ also supports the intended set-theoretic interpretation of the extensional level of $\mathbf{MF}_{cind}$.

Finally, all the theories considered to reach our goals, except $\mathbf{mTT}_{cind}$$+\mathbf{AC} + \mathbf{CT}$, are shown to be of the same proof-theoretic strength.

We leave it to future work to establish the consistency strength of $\mathbf{mTT}_{ind}$ and $\mathbf{mTT}_{cind}$, given that it is still an open problem to establish that of **mTT** itself.

Another future goal would be to apply the realizability interpretations presented here to build predicative variants of Hyland Effective Topos as in [MM21] but in a constructive meta-theory such as **CZF +REA**.

Last but not least there is the obvious question of generalizing our result by extending **MF** with more general coinductive definitions such as those of [Lin89, Rat03] including streams and systems, or strictly positive coinductive types in [AAG05] or other coinductive definitions like those applied in [vdBM07, Cur18, BT21]. The theory $\mathbf{CZF}+\mathbf{REA}_{\bigcup}+\mathbf{RDC}$ is very capacious in that it allows for the interpretation of all of these types as sets. It is the ideal axiomatic theory for a very general treatment of inductive and coinductive definitions in its class and set forms, delineating the necessary conditions for such definitions to give rise to sets rather than classes (see [AR10] chapters 12 and 13). But it is not clear how to turn such extensions into proper two-level foundations according to [MS05], by identifying an intensional type theoretic version of such coinductive definitions which is preserved by the syntactic quotient completion in [Mai09]. Actually it is not clear how to define such generic coinductive types within intensional type theory with well-behaved rules (see for example [BG16]). Moreover, the current encodings of coinductive types in [AAG05, BCS15] are performed by employing extensional principles like extensional equality of functions which is not validated in our realizability interpretation by its inconsistency with $\mathbf{AC}+\mathbf{CT}$.

2. The extension $\mathbf{MF}_{cind}$ with inductive and coinductive topological definitions.

Here we introduce an extension of the Minimalist Foundation **MF**, called $\mathbf{MF}_{cind}$, equipped with the inductive and coinductive topological definitions necessary to generate Sambin's Basic Topologies and Positive Topologies in [Sam03] within the approach of Martin-Löf - Sambin's Formal Topology in [Sam87]. In particular, the notion of Basic Topology constitutes an enrichment of the notion of *basic cover* in [BS06] (used to describe open subsets of a point-free topology) with a *positivity relation* used to describe closed subsets in a primitive way.

In more detail, we recall that the notion of **basic cover** in [BS06] aims to represent complete suplattices in a predicative and constructive way by an infinitary relation indicated with the notation

$$a \lhd V$$

between elements a of a set A, thought of as *basic opens*, and subsets V of A, meaning that *the basic open a is covered by the union of basic opens in the subset V*.

The elements of the complete suplattices represented by the basic cover, and called *formal open subsets*, are the fixpoints of the *closure operator*

$$\lhd(-) : \mathcal{P}(A) \;\longrightarrow\; \mathcal{P}(A)$$

defined by putting $\lhd(V) \equiv \{\ x \in A \ \mid\ x \lhd V\ \}$. These suplattices are *complete* with respect to families of subsets *indexed over a set.*

Then any basic cover $a \lhd V$ gives rise to a *Basic Topology* as in [Sam03], if it is paired with a **positivity relation**

$$a \ltimes V$$

between elements a of a set A and subsets V of A, meaning that *there is a point in the basic open a whose basic neighbourhoods are all in V* (see [Sam03]). In more detail, the positivity relation is required to satisfy the following *compatibility condition* with the basic cover

$$\frac{a \ltimes V \qquad a \lhd U}{\exists_{x\in A}\ (\ x \ltimes V\ \ \&\ x \,\epsilon\, U\ )}$$

and to induce an *interior operator*

$$\ltimes(-) : \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$$

defined by putting $\ltimes(V) \equiv \{ x \in A \mid x \ltimes V \}$. Then, the fixpoints of this operator form the suplattice of formal closed subsets.

A basic topology whose basic cover satisfies a convergence property is renamed as *positive topology* in [Sam19] [1], because the resulting complete suplattice of $\lhd$-fixpoints actually forms a *locale* enriched with a suplattice of formal closed subsets. A positive topology is also a formal topology in the sense of [Sam87] if the positivity relation $a \ltimes A$ defines a positivity predicate $\mathsf{Pos}(a)$ meaning that *the basic open a is inhabited by a point.* Therefore the notion of positive topology appears to be a strengthening of the notion of formal topology in [Sam87] with the modification of just requiring the suplattice of formal opens to form a locale rather than an open locale as in [Sam87].

Classically, a positivity relation can be associated to any basic cover in the form

$$a \ltimes V \equiv \neg a \lhd \neg V$$

but constructively one needs to add a primitive operator (see [Sam19]).

A powerful method to generate basic and positive topologies is to pair the inductive generation of basic covers introduced in [CSSV03] with the coinductive generation of the positivity relation introduced by Martin-Löf and Sambin in [Sam03]. Since **MF** cannot formalize all the inductively generated basic covers, like for example the inductive topology of the Baire space (see [MMR21]), it is clear that to formalize such inductive and coinductive topological methods we need to build an extension of **MF** like $\mathbf{MF}_{cind}$ that we are going to describe now.

By extension of **MF** we actually mean *a two-level extension*, namely we extend both the intensional level **mTT** of **MF** and its extensional level **emTT** to form a two-level theory satisfying the requirements of a constructive foundation in [MS05].

In [MMR21] we built a two-level theory, called $\mathbf{MF}_{ind}$, by extending both levels of **MF** with the inductive definitions introduced in [CSSV03] necessary to generate basic covers in [BS06, CMS13].

Knowing that the generation of basic topologies consists in enriching the inductive generation of basic covers in [CSSV03] with the coinductive generation of positivity relations, we simply extend both levels of $\mathbf{MF}_{ind}$ with Martin-Löf-Sambin's coinductive positivity relations [Sam03].

Now, we define in detail the two-level structure of $\mathbf{MF}_{cind}$ by first describing its extensional level $\mathbf{emTT}_{cind}$ and then its intensional level $\mathbf{mTT}_{cind}$.

2.1. The extensional level $\mathbf{emTT}_{cind}$ of $\mathbf{MF}_{cind}$.

The extensional level $\mathbf{emTT}_{cind}$ of $\mathbf{MF}_{cind}$ is defined as the extension of the extensional level $\mathbf{emTT}_{ind}$ of $\mathbf{MF}_{ind}$ in [MMR21] with the rules generating positivity relations by coinduction.

[1] Positive topologies were called "balanced formal topologies" in [Sam03].

In $\mathbf{emTT}_{cind}$ we still keep the four kinds of types of **emTT**, namely **collections**, **sets**, **propositions** and **small propositions** according to the following subtyping relations:

$$\begin{array}{ccc} \textbf{small propositions} & \hookrightarrow & \textbf{sets} \\ \big\downarrow & & \big\downarrow \\ \textbf{propositions} & \hookrightarrow & \textbf{collections} \end{array}$$

where collections include the power-collection $\mathcal{P}(A)$ (which is not a set!) of any set A and small propositions are defined as those propositions closed under intuitionistic connectives, and quantifiers and equalities restricted to sets.

In addition to the rules of **emTT**, in $\mathbf{emTT}_{ind}$ and hence in $\mathbf{emTT}_{cind}$, we have new primitive *small propositions*

$$a \triangleleft_{I,C} V \ prop_s$$

expressing that *the basic open a is covered by the union of basic opens in V* for any a element of a *set A*, V subset of A, assuming that *the basic cover is generated by a family of (open) subsets of A indexed on a family of sets $I(x)$ set $[x \in A]$ and represented by*

$$C(x, j) \in \mathcal{P}(A) \ [x \in A, j \in I(x)]$$

which we call *axiom-set* and indicate with the abbreviation I, C.

As for **emTT** and $\mathbf{emTT}_{ind}$, in $\mathbf{emTT}_{cind}$ we can can define a *subset membership* relation

$$a \,\epsilon\, V$$

between a subset $V \in \mathcal{P}(A)$ and an element a of a set A as in [Mai09], which is different from the *primitive* typing membership $a \in A$ which is a judgement.

We also adopt the convention of writing ϕ *true* for a proposition ϕ instead of $\mathsf{true} \in \phi$ as in [Mai09].

Recall that the rules of $\mathbf{emTT}_{ind}$ in [MMR21] were obtained by extending those of **emTT** with the addition of the rules defining the basic cover $a \triangleleft_{I,C} V \ prop_s$ inductively generated from an axiom-set I, C.

The rules of $\mathbf{emTT}_{cind}$ are obtained by extending those of $\mathbf{emTT}_{ind}$ with the following ones defining positivity relations by coinduction:

Rules of coinductive Positivity relations in $\mathbf{emTT}_{cind}$

$$\text{F- Pos}\ \frac{\begin{array}{l} A\ set \quad I(x)\ set\ [x \in A] \quad C(x,j) \in \mathcal{P}(A)\ \ [x \in A, j \in I(x)] \\ V \in \mathcal{P}(A) \qquad a \in A \end{array}}{a \ltimes_{I,C} V\ \ prop_s}$$

$$\text{crf- Pos}\ \frac{\begin{array}{l} A\ set \quad I(x)\ set\ [x \in A] \quad C(x,j) \in \mathcal{P}(A)\ \ [x \in A, j \in I(x)] \\ V \in \mathcal{P}(A) \qquad a \in A \qquad\qquad a \ltimes_{I,C} V\ true \end{array}}{a \,\epsilon\, V\ true}$$

$$\text{ax-mon-Pos}\ \frac{\begin{array}{l} A\ set \quad I(x)\ set\ [x \in A] \quad C(x,j) \in \mathcal{P}(A)\ \ [x \in A, j \in I(x)] \\ a \in A \qquad i \in I(a) \qquad\qquad V \in \mathcal{P}(A) \qquad a \ltimes_{I,C} V\ true \end{array}}{\exists_{y\in A}\ (y \,\epsilon\, C(a,i)\ \&\ y \ltimes_{I,C} V\ )\ true}$$

$$\text{cind-Pos}\ \frac{\begin{array}{l} A\ set \quad I(x)\ set\ [x \in A] \quad C(x,j) \in \mathcal{P}(A)\ \ [x \in A, j \in I(x)] \\ a \in A \qquad V \in \mathcal{P}(A) \\ P(x)\ prop\ [x \in A] \qquad \mathsf{split}(V,P)\ true \qquad P(a)\ true \end{array}}{a \ltimes_{I,C} V\ true}$$

where

$$\mathsf{split}(V,P)\ \equiv\ \ \forall_{x\in A}\ (\ P(x)\ \rightarrow\ (\ x\varepsilon V\ \ \&\ \forall_{z\in I(x)}\ \exists_{y\in A}\ (\ y \,\epsilon\, C(x,z)\ \&\ P(y)\ )\ )\ )$$

The coinductive generation of positivity relations is crucial to provide a predicative and constructive representation of results concerning applications of the theory of positivity relations and basic topologies to that of locales, namely the definition of closed sublocales in [Vic07], the relation between formal topology and Brouwer's principles in [KS19, Sam19] and the connections between locales, open locales and positive topologies in [MV04, CS18]. More in detail, in $\mathbf{emTT}_{cind}$ we can formalize the embedding of the category of locales represented as inductively generated formal covers into the category of positive topologies in [CS18], by embedding any inductive formal cover $\lhd_{I,C}$ as the positive topology with the same basic cover enriched with the coinductive positivity relation $\ltimes_{I,C}$. Moreover, in $\mathbf{emTT}_{cind}$ we can formalize the coreflection in [MV04] of the category of locales represented as inductively generated formal covers into its subcategory of open locales represented as inductively generated formal topologies by coreflecting any convergent basic cover $\lhd_{I,C}$ inductively generated from an axiom-set I, C, into the inductively generated formal topology $(A, \lhd_{I^+,C^+}, \mathsf{Pos})$ whose positivity predicate Pos is defined in terms of the coinductive positivity relation $\ltimes_{I,C}$ as

$$\mathsf{Pos}(a)\ \equiv\ a \ltimes_{I,C} A$$

and its basic cover $\lhd_{I^+,C^+}$ is inductively generated by a new axiom-set $I^+(x)\ set\ [x \in A]$ and $C^+(x,j) \in \mathcal{P}(A)\ [x \in A, j \in I^+(x)]$ extending the given one by adding for each basic open a an index k such that

$$C^+(a,k)\ \equiv\ \{\ x \in A\ \mid\ \mathsf{Pos}(a)\ \}.$$

In view of interpreting $\mathbf{emTT}_{cind}$ in the intensional level $\mathbf{mTT}_{cind}$, as noted in [MMR21] for basic covers, it is crucial that basic topologies defined by $\lhd_{I,C}$ and $\ltimes_{I,C}$ on a quotient set base B/R can be equivalently presented by a cover and a positivity relation on the set

B itself which behaves like $\lhd_{I,C}$ and $\ltimes_{I,C}$, respectively, but in addition it considers as equal opens those elements which are related by R.

In order to properly show this fact, we recall a correspondence between subsets of B/R and subsets of B in [MMR21]:

Definition 2.1. *In* $\mathbf{emTT}_{cind}$*, given a quotient set* B/R*, for any subset* $W \in \mathcal{P}(B/R)$ *we define*

$$\mathsf{es}(W) \equiv \{\ b \in B \ \mid\ [b]\,\epsilon\, W\ \}$$

and given any $V \in \mathcal{P}(B)$ *we define* $\mathsf{es}^{-}(V) \equiv \{\ z \in B/R \ \mid\ \exists_{b\in B}\ (\ b\,\epsilon\, V \ \wedge\ z =_{B/R} [b]\ )\}$.

Definition 2.2. *Given an axiom set represented by a set* $A \equiv B/R$ *with* $I(x)$ *set* $[x \in A]$ *and* $C(x,j) \in \mathcal{P}(A)\ \ [x \in A, j \in I(x)]$*, we define a new axiom set as follows:*

$$A^R \equiv B \qquad\qquad I^R(x) \equiv I([x]) + \ (\Sigma y \in B)\ R(x,y) \qquad\qquad \textit{for } x \in B$$

where $C^R(b,j)$ *is the formalization of*

$$C^R(b,j) \equiv \begin{cases} \mathsf{es}(\,C([b],j)\,) & \textit{if } j \in I([b]) \\ \{\,\pi_1(j)\,\} & \textit{if } j \in (\Sigma y \in B)\ R(b,y) \end{cases}$$

for $b \in B$ *and* $j \in I^R(x)$.
We then call $\lhd^R_{I,C}$ *and* $\ltimes^R_{I,C}$ *the inductive basic cover and the coinductive positivity relation generated from this axiom set, respectively.*

From [MMR21] we know that:

Lemma 2.3. *For any axiom set in* $\mathbf{emTT}_{ind}$ *represented by a set* $A \equiv B/R$ *with* $I(x)$ *set* $[x \in A]$ *and* $C(x,j) \in \mathcal{P}(A)\ \ [x \in A, j \in I(x)]$*, the suplattice of fixpoints defined by* $\lhd_{I,C}$ *is isomorphic to that defined by* $\lhd^R_{I,C}$ *by means of an isomorphism of suplattices.*

Analogously we can show that

Lemma 2.4. *For any axiom set in* $\mathbf{emTT}_{ind}$ *represented by a set* $A \equiv B/R$ *with* $I(x)$ *set* $[x \in A]$ *and* $C(x,j) \in \mathcal{P}(A)\ \ [x \in A, j \in I(x)]$*, the suplattice of fixpoints defined by* $\ltimes_{I,C}$ *is isomorphic to that defined by* $\ltimes^R_{I,C}$ *by means of an isomorphism of suplattices.*

Proof. Just observe that for any subset W of B/R which is a fixpoint for $\ltimes_{I,C}$ the subset $\mathsf{es}(W)$ is a fixpoint for $\ltimes^R_{I,C}$ by cind-Pos. Conversely, for any subset V of B which is a fixpoint for $\lhd^R_{I,C}$ the subset $\mathsf{es}^{-}(V)$ is a fixpoint for $\ltimes_{I,C}$ always by cind-Pos. □

2.2. **The intensional level** $\mathbf{mTT}_{cind}$**.** Here we define the intensional level $\mathbf{mTT}_{cind}$ of $\mathbf{MF}_{cind}$ as an extension of the intensional level $\mathbf{mTT}$ of $\mathbf{MF}$ capable of interpreting the extensional level $\mathbf{emTT}_{cind}$. We actually describe $\mathbf{mTT}_{cind}$ as an extension of $\mathbf{mTT}_{ind}$ in [MMR21] with the rules generating a positivity relation by coinduction, as done for $\mathbf{emTT}_{cind}$ with respect to $\mathbf{emTT}_{ind}$.

As in $\mathbf{mTT}$ in [Mai09] and in $\mathbf{mTT}_{ind}$, in $\mathbf{mTT}_{cind}$ we have the same four kinds of types as in $\mathbf{emTT}_{cind}$ with the difference that in $\mathbf{mTT}_{cind}$ power-collections of sets are replaced by a *collection of small propositions* $\mathsf{prop_s}$ and function collections $A \to \mathsf{prop_s}$ for any set A. Such collections are enough to interpret power-collections of sets in $\mathbf{emTT}_{cind}$ within a quotient model of dependent extensional types built over $\mathbf{mTT}_{cind}$, as shown in [Mai09] when interpreting $\mathbf{emTT}$ in $\mathbf{mTT}$.

In addition to small proposition constructors of **mTT**, in $\mathbf{mTT}_{cind}$ and hence in $\mathbf{mTT}_{ind}$ we have new small propositions $a \triangleleft_{I,C} V\ prop_s$ with corresponding new proof-term constructors associated to them in order to represent a proof-relevant version of inductively generated basic covers so that judgements asserting that some proposition is true in $\mathbf{emTT}_{cind}$ are turned into judgements of $\mathbf{mTT}_{cind}$ producing a proof-term of the corresponding proposition.

Moreover, in $\mathbf{mTT}_{cind}$ as in **mTT** and in $\mathbf{mTT}_{ind}$, the universe of small propositions is defined in the version à la Russell [2].

When expressing the rules of inductive basic covers we used the abbreviation

$$a \,\epsilon\, V \qquad \text{to mean} \qquad \mathsf{Ap}(\, V \,,\, a \,)$$

for any set A, any small propositional function $V \in A \to \mathsf{prop_s}$ and any element $a \in A$.

Formally, $\mathbf{mTT}_{cind}$ is obtained by extending the rules of $\mathbf{mTT}_{ind}$ in [MMR21] with the following rules defining positivity relations by coinduction in the form of axioms with no equality rules:

Axioms of coinductive Positivity relations in $\mathbf{mTT}_{cind}$

$$\text{F-Pos}\ \frac{\begin{array}{l} A\ set \quad I(x)\ set\ [x \in A] \quad C(x,j) \in A \to \mathsf{prop_s}\ [x \in A, j \in I(x)] \\ V \in A \to \mathsf{prop_s} \qquad a \in A \end{array}}{a \ltimes_{I,C} V\ prop_s}$$

$$\text{crf-Pos}\ \frac{\begin{array}{l} A\ set \quad I(x)\ set\ [x \in A] \quad C(x,j) \in A \to \mathsf{prop_s}\ \ [x \in A, j \in I(x)] \\ V \in A \to \mathsf{prop_s} \qquad a \in A \qquad q \in a \ltimes_{I,C} V \end{array}}{\mathsf{ax_1}(a,q) \in a \,\epsilon\, V}$$

$$\text{ax-mon-Pos}\ \frac{\begin{array}{l} A\ set \quad I(x)\ set\ [x \in A] \quad C(x,j) \in A \to \mathsf{prop_s}\ [x \in A, j \in I(x)] \\ V \in A \to \mathsf{prop_s} \qquad a \in A \qquad i \in I(a) \\ q \in a \ltimes_{I,C} V \end{array}}{\mathsf{ax_2}(a,i,q) \in \exists_{y \in A}\ (\ y \epsilon C(a,i)\ \&\ y \ltimes_{I,C} V\ )}$$

$$\text{cind-Pos}\frac{\begin{array}{l} A\ set \quad I(x)\ set\ [x \in A] \quad C(x,j)\ \in A\ \to \mathsf{prop_s}\ [x \in A, j \in I(x)] \\ P(x)\ prop\ [x \in A] \qquad V \in A\ \to\ \mathsf{prop_s} \\ a \in A \qquad m \in P(a) \\ q_1(x,z) \in V(x)\ [x \in A, z \in P(x)] \\ q_2(x,j,z) \in \exists_{y \in A}(P(y)\ \&\ y \,\epsilon\, C(x,j))\ [x \in A, j \in I(x), z \in P(x)] \end{array}}{\mathsf{ax_3}(a,m,q_1,q_2) \in a \ltimes_{I,C} V}$$

As noted in [MMR21] for the cover relation, the positivity relation preserves extensional equality of subsets represented as small propositional functions:

Lemma 2.5. *For any axiom set in $\mathbf{mTT}_{cind}$ represented by a set A with $I(x)$ set $[x \in A]$ and $C(x,j) \in A\ \to \mathsf{prop_s}\ \ [x \in A, j \in I(x)]$, for any propositional functions $V_1 \in A\ \to \mathsf{prop_s}$*

[2] A version of **mTT** with the universe of small propositions à la Tarski can be found in [MM16].

and $V_2 \in A \rightarrow \mathsf{props_s}$*, we can derive for a proof-term* q

$$q \in V_1 =_{ext} V_2 \;\rightarrow\; \mathsf{Pos}_{I,C}(a, V_1) =_{ext} \mathsf{Pos}_{I,C}(a, V_2)$$

where for any small propositional functions W_1 *and* W_2 *on a set* A *we use the following abbreviation* $W_1 =_{ext} W_2 \equiv \forall_{x \in A}\; (\; W_1(x) \;\leftrightarrow\; W_2(x)\;)$*.*

Recall that the interpretation of **emTT** in **mTT** in [Mai09], as well as that of $\mathbf{emTT}_{ind}$ in $\mathbf{mTT}_{ind}$ in [MMR21], interprets a set A as an *extensional quotient* defined in **mTT** as a set A^J of **mTT**, called *support*, equipped with an equivalence relation $=_{A^J}$ over A^J, as well as families of sets are interpreted as families of extensional sets preserving the equivalence relations in their telescopic contexts. Now, Lemma 2.5 suggests that we can interpret a coinductive positivity relation on a set A of $\mathbf{emTT}_{cind}$ within $\mathbf{mTT}_{cind}$ as a coinductive positivity relation of $\mathbf{mTT}_{cind}$ on the support A^J by enriching the interpretation of the axiom-set in $\mathbf{mTT}_{cind}$ with the equivalence relation $=_{A^J}$ in a similar way to definition 2.2 as follows:

Definition 2.6. *For any axiom set in* $\mathbf{mTT}_{cind}$ *represented by a set* A *with* $I(x)$ *set* $[x \in A]$ *and* $C(x,j) \in A \rightarrow \mathsf{props_s}$ $[x \in A, j \in I(x)]$ *and for any given equivalence relation* $x =_A y \in \mathsf{props_s}$ $[x \in A, y \in A]$ *turning* A *into an extensional set as well as the family of set* $I(x)$ *set* $[x \in A]$ *and propositional functions* $C(x,j) \in A \rightarrow \mathsf{props_s}$ $[x \in A, j \in I(x)]$ *into an extensional family of sets and extensional propositional functions preserving* $=_A$ *according to the definitions in* [Mai09]*, we define a new axiom set as follows*

$$A^{=_A} \;\equiv\; A \qquad\qquad I^{=_A}(x) \;\equiv\; I(x) + (\Sigma y \in A)(x =_A y) \qquad\qquad \textit{for } x \in A$$

where $C^{=_A}(a,j)$ *is the formalization of*

$$C^{=_A}(a,j) \;\equiv\; \begin{cases} C(a,j) & \textit{if } j \in I(a) \\ \{\, \pi_1(j)\,\} & \textit{if } j \in (\Sigma y \in A)(a =_A y) \end{cases}$$

for $a \in A$ *and* $j \in I^{=_A}(x)$*.*
We then call $\lhd^{=_A}_{I,C}$ *and* $\ltimes^{=_A}_{I,C}$ *respectively the inductive basic cover and the coinductive positivity relation generated from this axiom set.*

We are now ready to interpret $\mathbf{emTT}_{cind}$ in the quotient model over $\mathbf{mTT}_{cind}$ built as in [Mai09] by extending the interpretation of $\mathbf{emTT}_{ind}$ in $\mathbf{mTT}_{ind}$ in [MMR21] as follows:

Proposition 2.1. *The interpretation of* **emTT** *in* **mTT** *in* [Mai09] *extends to an interpretation of* $\mathbf{emTT}_{cind}$ *in* $\mathbf{mTT}_{cind}$ *by interpreting an inductive basic cover* $a \lhd_{I,C} V$ *and a coinductive positivity relation* $a \ltimes_{I,C} V$ *for* $a \in A$ *and* $V \in \mathcal{P}(A)$ *respectively as the inductive basic cover* $\lhd^{=_{A^J}}_{I^J,C^J}$ *and the coinductive positivity relation* $\ltimes^{=_{A^J}}_{I^J,C^J}$ *over the support* A^J *of the interpretation of* A*.*

Proof. The given interpretation coincides with the interpretation of $\mathbf{emTT}_{ind}$ in $\mathbf{mTT}_{ind}$ given in [MMR21] for what regards inductive basic covers. To check its correctness for what regards coinductive positivity relations, just observe that $\ltimes^{=_{A^J}}_{I^J,C^J}$ is an extensional proposition over the extensional set interpreting A and over the interpretation of $\mathcal{P}(A)$ in the sense of [Mai09]. $\square$

3. The Martin-Löf's type theories $\mathbf{MLtt}_{cind}$ and $\mathbb{MLS}_{ind}$

In order to derive the consistency of $\mathbf{mTT}_{cind}$ with $\mathbf{AC} + \mathbf{CT}$ from the consistency of $\mathbf{CZF} + \mathbf{REA}$ we introduce two auxiliary theories $\mathbf{MLtt}_{cind}$ and $\mathbb{MLS}_{ind}$ with the idea of reducing the consistency of $\mathbf{mTT}_{cind}$ with $\mathbf{AC} + \mathbf{CT}$ to the consistency of $\mathbf{MLtt}_{cind}$ or $\mathbb{MLS}_{ind}$ with just $\mathbf{CT}$. The crucial point is that both $\mathbf{MLtt}_{cind}$ and $\mathbb{MLS}_{ind}$ are intensional Martin-Löf's type theories governed by Curry-Howard's interpretation of propositions-as-sets which validates the axiom of choice $\mathbf{AC}$ internally. This idea was already implemented to obtain similar results both in [IMMS18] and in [MMR21].

The theory $\mathbf{MLtt}_{cind}$ is an extension of the theory $\mathbf{MLtt}_{ind}$ in [MMR21] with rules generating coinductive positivity relations, which model those of $\mathbf{mTT}_{cind}$.

Instead, the theory $\mathbb{MLS}_{ind}$ includes Palmgren's superuniverse $\mathcal{S}$ in [Pal98] which is used to interpret small propositions of $\mathbf{mTT}_{cind}$. By contrast, the fragments of Martin-Löf's type theory $\mathbf{MLtt}_{cind}$ here, $\mathbf{MLtt}_1$ in [IMMS18] and $\mathbf{MLtt}_{ind}$ in [MMR21] just contain a single universe (whilst closed under some primitive inductive definitions in $\mathbf{MLtt}_{ind}$ and furthermore under some coinductive definitions in $\mathbf{MLtt}_{cind}$) to interpret small propositions of the considered extensions of $\mathbf{mTT}$. Moreover, as done for the universe in the theory $\mathbf{MLtt}_{ind}$ in [MMR21], we close the superuniverse under inductive covers rather than well-founded sets as in [Rat03]. The presence of the superuniverse in $\mathbb{MLS}_{ind}$ is crucial in defining the interpretation of the coinductive positivity relations of $\mathbf{mTT}_{cind}$ thanks to the closure of the superuniverse under universe operators and the axiom of choice.

3.1. The theory $\mathbf{MLtt}_{cind}$.

The theory $\mathbf{MLtt}_{cind}$ is obtained by extending $\mathbf{MLtt}_{ind}$ in [MMR21] (which has one universe U_0 à la Tarski) with the following axioms for coinductive positivity relations:

Coinductive positivity relations in $\mathbf{MLtt}_{cind}$

$$\text{F-Pos}\ \frac{\begin{array}{l} s \in \mathsf{U}_0 \quad i(x) \in \mathsf{U}_0\ [x \in \mathsf{T}(s)] \quad c(x,j) \in \mathsf{T}(s) \to \mathsf{U}_0\ [x \in \mathsf{T}(s), j \in \mathsf{T}(i(x))] \\ v \in \mathsf{T}(s) \to \mathsf{U}_0 \qquad a \in \mathsf{T}(s) \end{array}}{a \mathbin{\widehat{\ltimes}}_{s,i,c} v \in \mathsf{U}_0}$$

$$\text{crf-Pos}\ \frac{\begin{array}{l} s \in \mathsf{U}_0 \quad i(x) \in \mathsf{U}_0\ [x \in \mathsf{T}(s)] \quad c(x,j) \in \mathsf{T}(s) \to \mathsf{U}_0\ [x \in \mathsf{T}(s), j \in \mathsf{T}(i(x))] \\ v \in \mathsf{T}(s) \to \mathsf{U}_0 \qquad a \in \mathsf{T}(s) \qquad q \in a \ltimes_{s,i,c} v \end{array}}{\mathsf{ax}_1(a,q) \in a \,\epsilon\, v}$$

$$\text{ax-mon-Pos}\ \frac{\begin{array}{l} s \in \mathsf{U}_0 \quad i(x) \in \mathsf{U}_0\ [x \in \mathsf{T}(s)] \quad c(x,j) \in \mathsf{T}(s) \to \mathsf{U}_0\ [x \in \mathsf{T}(s), j \in \mathsf{T}(i(x))] \\ v \in \mathsf{T}(s) \to \mathsf{U}_0 \qquad a \in \mathsf{T}(s) \qquad j \in \mathsf{T}(i(a)) \\ q \in a \ltimes_{s,i,c} v \end{array}}{\mathsf{ax}_2(a,i,q) \in (\Sigma y \in \mathsf{T}(s))\ (\ y \,\epsilon\, c(a,j)\ \times\ y \ltimes_{s,i,c} v)}$$

$$\text{cind-Pos}\ \frac{\begin{array}{l} s \in \mathsf{U}_0 \quad i(x) \in \mathsf{U}_0\ [x \in \mathsf{T}(s)] \quad c(x,j) \in \mathsf{T}(s) \to \mathsf{U}_0\ [x \in \mathsf{T}(s), j \in \mathsf{T}(i(x))] \\ P(x)\ type\ [x \in \mathsf{T}(s)] \qquad v \in \mathsf{T}(s)\ \to\ \mathsf{U}_0 \\ a \in \mathsf{T}(s) \qquad m \in P(a) \\ q_1(x,z) \in \mathsf{T}(v(x))\ [x \in \mathsf{T}(s), z \in P(x)] \\ q_2(x,j,z) \in (\Sigma y \in \mathsf{T}(s))(\ y \,\epsilon\, c(x,j) \times P(y))\ [x \in \mathsf{T}(s), j \in \mathsf{T}(i(x)), z \in P(x)] \end{array}}{\mathsf{ax}_3(m,q_1,q_2) \in a \ltimes_{s,i,c} v}$$

where $a \ltimes_{s,i,c} v$ is a shorthand for $\mathsf{T}(a \widehat{\ltimes}_{s,i,c} v)$ and we used the abbreviation $t \,\epsilon\, s$ for $\mathsf{T}(\mathsf{Ap}(s,t))$.

We can interpret the theory $\mathbf{mTT}_{cind}$ (which has a universe of small propositions à la Russell) in $\mathbf{MLtt}_{cind}$ (which has a universe à la Tarski) as follows:

(1) We first consider a pre-syntax of $\mathbf{mTT}_{cind}$ consisting of pre-types and of pre-terms with the upshot that pre-terms include also all the pre-types contained in the universe of small propositions prop_s;
(2) We define by mutual recursion on the complexity of the pre-syntax two interpretation functions; the first $\|-\|_t$ maps each pre-term of $\mathbf{mTT}_{cind}$ to a pre-term of $\mathbf{MLtt}_{cind}$, the second $\|-\|_T$ maps each pre-type of $\mathbf{mTT}_{cind}$ to a pre-type of $\mathbf{MLtt}_{cind}$; these interpretations are defined in the obvious way by simply following the proposition-as-types interpretation and having care of interpreting pre-types seen as pre-terms in code-terms corresponding to the relative pre-type interpretations.

The pre-type prop_s will be interpreted as the universe U_0, and types and terms introduced in the positivity coinduction rules of $\mathbf{mTT}_{cind}$ are interpreted in the obvious way using the corresponding types and terms in $\mathbf{MLtt}_{cind}$. Judgements of $\mathbf{mTT}_{cind}$ are interpreted in judgements of $\mathbf{MLtt}_{cind}$ in the obvious way, by using $\|-\|_t$ or $\|-\|_T$ when the role of a small proposition in a judgement is that of a term or that of a type, respectively. For example if φ is a small proposition in $\mathbf{mTT}_{cind}$ the judgement $a \in \varphi$ is translated into $\|a\|_t \in \|\varphi\|_T$, while the judgement $\varphi \in \mathsf{prop}_s$ is translated into $\|\varphi\|_t \in \mathsf{U}_0$.

Proposition 3.1. *We can interpret* $\mathbf{mTT}_{cind}+\mathbf{AC}+\mathbf{CT}$ *into* $\mathbf{MLtt}_{cind}+\mathbf{CT}$ *according to the above interpretation so that the consistency of* $\mathbf{mTT}_{cind}+\mathbf{AC}+\mathbf{CT}$ *is reduced to the consistency of* $\mathbf{MLtt}_{cind}+\mathbf{CT}$.

3.2. **The theory** $\mathbb{MLS}_{ind}$**.** We here briefly describe the theory $\mathbb{MLS}_{ind}$ obtained by extending the first-order fragment of intensional Martin-Löf's type theory in [NPS90] with a superuniverse $\mathcal{S}$ à la Tarski closed under *inductive covers* besides the usual first-order type constructors and universe constructors as in [Pal98].

In accordance with Curry-Howard's proposition-as-sets interpretation, which is a peculiarity of Martin-Löf's type theory, as done for $\mathbf{MLtt}_{ind}$ in [MMR21] and contrary to $\mathbf{mTT}_{ind}$ as well as to $\mathbf{mTT}_{cind}$, we strengthen the elimination rule of inductive basic covers to act towards sets depending on their proof-terms according to inductive generation of types in Martin-Löf's type theory.

To this purpose we add to $\mathbb{MLS}_{ind}$ the code

$$a \,\widehat{\lhd}_{s,i,c}\, v \in \mathcal{S} \qquad \text{for } a \in \mathsf{T}(s) \text{ and } v \in \mathsf{T}(s) \;\to\; \mathcal{S}$$

meaning that *the element a of a small set $\mathsf{T}(s)$ represented by the code $s \in \mathcal{S}$ is covered by the subset v* represented by a small propositional function from $\mathsf{T}(s)$ to the (large) set of small propositions identified with $\mathcal{S}$ by the propositions-as-sets correspondence.

Moreover, we use $\mathsf{axcov}(s, i, c)$ to denote collectively the following judgements

$$s \in \mathcal{S} \quad i(x) \in \mathcal{S}\ [x \in \mathsf{T}(s)] \quad c(x, j) \in \mathsf{T}(s) \to \mathcal{S}\ [x \in \mathsf{T}(s), j \in \mathsf{T}(i(x))]$$

Then, the precise rules of inductive basic covers in $\mathbb{MLS}_{ind}$ are obtained by those of $\mathbf{MLtt}_{ind}$ in [MMR21] by replacing U_0 with $\mathcal{S}$ as follows:

Rules of inductively generated basic covers in $\mathbb{MLS}_{ind}$

$$\text{F-}\lhd \;\frac{\mathsf{axcov}(s, i, c) \qquad a \in \mathsf{T}(s) \qquad v \in \mathsf{T}(s) \to \mathcal{S}}{a \,\widehat{\lhd}_{s,i,c}\, v \in \mathcal{S}}$$

$$\text{rf-}\lhd \;\frac{\mathsf{axcov}(s, i, c) \qquad a \in \mathsf{T}(s) \qquad v \in \mathsf{T}(s) \to \mathcal{S} \qquad r \in a \,\epsilon\, v}{\mathsf{rf}(a, r) \in a \lhd_{s,i,c} v}$$

$$\text{tr-}\lhd \;\frac{\begin{array}{l}\mathsf{axcov}(s, i, c) \qquad a \in \mathsf{T}(s) \qquad j \in \mathsf{T}(i(a)) \qquad v \in \mathsf{T}(s) \to \mathcal{S} \\ r \in (\Pi z \in \mathsf{T}(s))(z \,\epsilon\, c(a, j) \to z \lhd_{s,i,c} v)\end{array}}{\mathsf{tr}(a, j, r) \in a \lhd_{s,i,c} v}$$

$$\text{ind-}\lhd \frac{\begin{array}{l}\mathsf{axcov}(s, i, c) \qquad v \in \mathsf{T}(s) \to \mathcal{S} \qquad P(x, u)\ type\ [x \in \mathsf{T}(s), u \in x \lhd_{s,i,c} v] \\ a \in \mathsf{T}(s) \qquad m \in a \lhd_{s,i,c} v \\ q_1(x, w) \in P(x, \mathsf{rf}(x, w))\ [x \in \mathsf{T}(s), w \in x \,\epsilon\, v] \\ q_2(x, h, k, f) \in P(x, \mathsf{tr}(x, h, k)) \\ \qquad [x \in \mathsf{T}(s), h \in \mathsf{T}(i(x)), \\ \qquad\ k \in (\Pi z \in \mathsf{T}(s))(z \,\epsilon\, c(x, h) \to z \lhd_{s,i,c} v), \\ \qquad\qquad f \in (\Pi z \in \mathsf{T}(s))(\Pi u \in z \,\epsilon\, c(x, h))\, P(z, \mathsf{Ap}(\mathsf{Ap}(k, z), u))]\end{array}}{\mathsf{ind}(m, q_1, q_2) \in P(a, m)}$$

$$\text{C}_1\text{-ind-}\lhd\ \frac{\begin{array}{l}\mathsf{axcov}(s,i,c) \qquad v \in \mathsf{T}(s) \to \mathcal{S} \qquad P(x,u)\ type\ [x \in \mathsf{T}(s), u \in x \lhd_{s,i,c} v] \\ a \in \mathsf{T}(s) \qquad r \in a \,\epsilon\, v \\ q_1(x,w) \in P(x, \mathsf{rf}(x,w))\ [x \in \mathsf{T}(s), w \in x \,\epsilon\, v] \\ q_2(x,h,k,f) \in P(x, \mathsf{tr}(x,h,k)) \\ \qquad [x \in \mathsf{T}(s), h \in \mathsf{T}(i(x)), \\ \qquad\quad k \in (\Pi z \in \mathsf{T}(s))(z \,\epsilon\, c(x,h) \to x \lhd_{s,i,c} v), \\ \qquad\qquad f \in (\Pi z \in \mathsf{T}(s))(\Pi u \in z \,\epsilon\, c(x,h))\, P(z, \mathsf{Ap}(\mathsf{Ap}(k,z),u))]\end{array}}{\mathsf{ind}(\mathsf{rf}(a,r), q_1, q_2) = q_1(a,r) \in P(a, \mathsf{rf}(a,r))}$$

$$\text{C}_2\text{-ind-}\lhd\ \frac{\begin{array}{l}\mathsf{axcov}(s,i,c) \qquad v \in \mathsf{T}(s) \to \mathcal{S} \qquad P(x,u)\ type\ [x \in \mathsf{T}(s), u \in x \lhd_{s,i,c} v] \\ a \in \mathsf{T}(s) \qquad j \in \mathsf{T}(i(a)) \qquad r \in (\Pi z \in \mathsf{T}(s))(z \,\epsilon\, c(a,j) \to z \lhd_{s,i,c} v) \\ q_1(x,w) \in P(x, \mathsf{rf}(x,w))\ [x \in \mathsf{T}(s), w \in x \,\epsilon\, v] \\ q_2(x,h,k,f) \in P(x, \mathsf{tr}(x,h,k)) \\ \qquad [x \in \mathsf{T}(s), h \in \mathsf{T}(i(x)), \\ \qquad\quad k \in (\Pi z \in \mathsf{T}(s))(z \,\epsilon\, c(x,h) \to z \lhd_{s,i,c} v), \\ \qquad\qquad f \in (\Pi z \in \mathsf{T}(s))(\Pi u \in z \,\epsilon\, c(x,h)) P(z, \mathsf{Ap}(\mathsf{Ap}(k,z),u))]\end{array}}{\mathsf{ind}(\mathsf{tr}(a,j,r), q_1, q_2) = q_2(a,j,r,\lambda z.\lambda u.\mathsf{ind}(\mathsf{Ap}(\mathsf{Ap}(r,z),u), q_1, q_2)) \in P(a, \mathsf{tr}(a,j,r))}$$

It is worth stressing that universes within the superuniverse are *not necessarily closed under inductive covers.*

Definition 3.1. *A crucial deviation from the ordinary versions of Martin-Löf's type theory is that for* $\mathbb{MLS}_{ind}$*, and also for* $\mathbf{MLtt}_{cind}$*, we postulate just the replacement rule repl)*

$$repl)\ \frac{\begin{array}{l} c(x_1,\ldots,x_n) \in C(x_1,\ldots,x_n) \ \ [x_1 \in A_1,\ \ldots,\ x_n \in A_n(x_1,\ldots,x_{n-1})] \\ a_1 = b_1 \in A_1 \ \ldots\ a_n = b_n \in A_n(a_1,\ldots,a_{n-1})\end{array}}{c(a_1,\ldots,a_n) = c(b_1,\ldots,b_n) \in C(a_1,\ldots,a_n)}$$

in place of the usual congruence rules which would include the ξ-rule in accordance with the rules of **mTT** *in* [Mai09] *(see* [IMMS18] *for further details on this point).*

We employ this restriction in $\mathbf{mTT}_{cind}$, $\mathbf{MLtt}_{cind}$ and in $\mathbb{MLS}_{ind}$ because the realizability semantics we present in the next sections, based on the original Kleene realizability in [Tv88], does not validate the ξ-rule[3] of lambda-terms

$$\xi\ \frac{c = c' \in C\ [x \in B]}{\lambda x^B.c = \lambda x^B.c' \in (\Pi x \in B)C}$$

which is instead valid in [NPS90].

Moreover, observe that the lack of the ξ-rule does not affect the possibility of adopting **mTT** as the intensional level of a two-level constructive foundation as intended in [MS05], considered that the term equality rules of $\mathbf{mTT}_{cind}$ suffice to interpret the extensional level $\mathbf{emTT}_{cind}$ including extensionality of functions, by means of a quotient model as that introduced in [Mai09] and studied abstractly in [MR13a, MR13b, MR15].

Now note that we can interpret $\mathbf{mTT}_{cind}$ without Positivity relations, namely $\mathbf{mTT}_{ind}$ in [MMR21], within $\mathbb{MLS}_{ind}$ by following the same strategy adopted for interpreting $\mathbf{mTT}_{cind}$ in $\mathbf{MLtt}_{cind}$ namely by defining a pair of functions $\|-\|_t$ and $\|-\|_T$ as for proposition 3.1.

[3]Notice that a trivial instance of the ξ-rule is derivable from repl) when c and c' do not depend on x^B.

Now we show that in $\mathbb{MLS}_{ind}$ we can define positivity relations of $\mathbf{MLtt}_{cind}$ where the universe U_0 is substituted with the superuniverse $\mathcal{S}$. In this way we can interpret in $\mathbb{MLS}_{ind}$ not only $\mathbf{mTT}_{ind}$ but also $\mathbf{mTT}_{cind}$.

To increase readability and to avoid an heavy use of codes, we assume the following conventions: we call any type A a *small set* if $A \equiv T(c)$ for some $c \in \mathcal{S}$. Similarly, we say that V is a subset of A and we write $V \subseteq A$ for any propositional function $V \in A \ \rightarrow \mathcal{S}$ derivable in $\mathbb{MLS}_{ind}$. In addition, given two subsets V, W of A, we write $V \subseteq W$ if we have a proof-term p for which we can derive $p \in \forall_{x \in A} \ (\ V(x) \ \rightarrow \ W(x) \ )$. We also adopt the abbreviation $\{ \ x \in A \ \mid \ \phi(x) \ \}$ to indicate a propositional function $\phi \in A \ \rightarrow \mathcal{S}$.

Moreover, given a family of small sets $B(x)$ *type* $[x \in A]$ on a small set A we denote by $U(A, B(x))$ the universe in $\mathcal{S}$ containing them.

Finally, we also call *axiom-set* the judgements

$$A \ type \quad I(x) \ type \ [x \in A] \quad C(x,j) \ type \ [x \in A, j \in I(x)]$$

if and only if they are derived from an axiom-set $\mathsf{axcov}(s,i,c)$ in the sense that $A = T(s)$, $I(x) = T(i(x))$ and $C(x,j) = T(c(x,j))$ for $x \in A$ and $j \in I(x)$.

Theorem 3.2. *In $\mathbb{MLS}_{ind}$ we can define Coinductive Positivity relations satisfying the instances of the rules in $\mathbf{MLtt}_{cind}$ at the beginning of subsection 3.1 where the universe U_0 is substituted with the superuniverse $\mathcal{S}$.*

Proof. We adapt here an argument in the appendix of [MV04] originally due to Thierry Coquand. For sake of readability and thanks to the notational conventions just established, we will proceed as if we were working with a superuniverse à la Russell.

For any axiom-set A *type* with $I(x)$ *type* $[x \in A]$ and $C(x,j)$ *type* $[x \in A, j \in I(x)]$, made all of small sets and small families, and for any fixed subset $V \subseteq A$ and $a \in A$ the positivity relation amounts to be defined as

$$a \ltimes V \equiv a \, \epsilon \, W_{\max}(V) \in \mathcal{S}$$

where $W_{max}(V) \in \mathcal{S}$ is the maximal fixpoint of an operator $\tau \in \mathcal{P}_{int}(V) \rightarrow \mathcal{P}_{int}(V)$ on the *intensional* representation of the powerset of the subset V

$$\mathcal{P}_{int}(V) \equiv \Sigma_{x \in A} V(x) \ \rightarrow \mathcal{S}$$

defined by setting

$$\tau(X) \ \equiv \ \{ \ x \in A \ \mid \ x \, \epsilon \, X \ \& \ \forall_{i \in I(x)} \ \exists_{y \in A} \ (\ y \, \epsilon \, C(x,i) \ \& \ y \, \epsilon \, X \ ) \ \}$$

for any subset X of A within V which preserves extensional equality of subsets as $=_{ext}$ of Lemma 2.5 [4].

Now, observe that there exists a universe $U_{I,C}$ in the superuniverse $\mathcal{S}$ containing all the components of the axiom-set. Then the biggest fixed point can be defined in $\mathbb{MLS}_{ind}$ as an element of $A \ \rightarrow \ \mathcal{S}$ as follows

$$W_{max}(V) \ \equiv \ \{ \ x \in A \ \mid \ \exists_{\ Z \in A \ \rightarrow \ U_{I,C}} \ (\ x \, \epsilon \, Z \ \& \ (\ Z \subseteq \tau(Z) \ \ \& \ Z \subseteq V \ ) \ ) \ \}$$

since $\mathcal{S}$ contains $U_{I,C}$ as well as Z and V.

[4]The operator τ is clearly monotone and hence in an impredicative classical foundation, by Tarski fixpoint theorem, it admits a maximal fixed point $W_{max}(V)$ defined as *the union of all the subsets Y of A within V such that $Y \subseteq \tau(Y)$* after noting that such a family of subsets is not empty, since the empty subset $\emptyset$ satisfies the condition trivially.

This is really the biggest fixed point since for any fixed subset Y of A within V such that $Y \subseteq \tau(Y)$ and any fixed $\overline{a} \in A$ such that $\overline{a} \,\epsilon\, Y$, we have that $\overline{a} \,\epsilon\, W_{max}(V)$ holds, namely that $Y \subseteq W_{max}(V)$.

To this purpose by an application of the so called *axiom of choice* of Martin-Löf's type theory to the formalization of $Y \subseteq \tau(Y)$

$$\forall_{x\in A} \; (\; x \,\epsilon\, Y \;\rightarrow\; \forall_{j\in I(x)} \; \exists_{w\in A} \; (\; w \,\epsilon\, C(x,j) \;\&\; w \,\epsilon\, Y \;)\;)$$

we derive the existence of a choice function f_x for any $x \in A$ as follows

$$\forall_{x\in A} \; (\; x \,\epsilon\, Y \;\rightarrow\; \exists_{f_x \in I(x)\rightarrow A} \; \forall_{j\in I(x)} \; (\; f_x(j) \,\epsilon\, C(x,j) \;\&\; f_x(j) \,\epsilon\, Y \;)\;)$$

Then, we define by induction on natural numbers the following sequence of subsets of A:

$$\begin{array}{rcl} X_0 & \equiv & \{\; w \in A \;\mid\; w =_A \overline{a} \;\} \\ X_{n+1} & \equiv & X_n \cup \{\; w \in \; A \mid \exists_{x\in A} \; \exists_{j\in I(x)} (\; (\; x \,\epsilon\, X_n \;\&\; w =_A f_x(j) \;\} \end{array}$$

Observe that the subset of A defined as

$$X^* \equiv \bigcup_{n\in \mathsf{Nat}} X_n$$

is in $U_{I,C}$, since $U_{I,C}$ is closed under all first-order type constructors, and $X^* \;\subseteq\; Y$.

Finally observe that $X^* \subseteq \tau(X^*)$ is true in $\mathbb{MLS}_{ind}$ and hence $X^* \subseteq W_{\max}(V)$ so that we can conclude $\overline{a} \;\ltimes V \equiv \overline{a} \,\epsilon\, W_{\max}(V) \in \mathcal{S}$ as claimed.

By following the same strategy adopted for Proposition 3.1 we can conclude:

Corollary 3.2. $\mathbf{mTT}_{cind}$ *as well as* $\mathbf{MLtt}_{cind}$ *can be interpreted in* $\mathbb{MLS}_{ind}$ *by interpreting coinductive positivity relations as in Proposition 3.2. In particular, the consistency of* $\mathbf{mTT}_{cind} + \mathbf{AC} + \mathbf{CT}$ *is reduced to the consistency of* $\mathbb{MLS}_{ind} + \mathbf{CT}$.

4. Inductive and coinductive definitions in Aczel's CZF

In the following we shall introduce several inductively defined classes, and, moreover, we have to ensure that such classes can be formalized in **CZF**.

Definition 4.1. *We define an* inductive definition *to be a class of ordered pairs. If* Φ *is an inductive definition and* $\langle x, a\rangle \in \Phi$ *then we write*

$$\frac{x}{a}\;\Phi$$

and call $\frac{x}{a}$ *an* (inference) step *of* Φ, *with set* x *of* premisses *and* conclusion a. *For any class* Y, *let*

$$\Gamma_\Phi(Y) \;=\; \big\{a \mid \exists x \big(x \subseteq Y \;\wedge\; \frac{x}{a}\;\Phi\,\big)\big\}.$$

The class Y *is* Φ-closed *if* $\Gamma_\Phi(Y) \subseteq Y$. *Note that* Γ *is monotone; i.e. for classes* Y_1, Y_2, *whenever* $Y_1 \subseteq Y_2$, *then* $\Gamma(Y_1) \subseteq \Gamma(Y_2)$.

We define the class inductively defined by Φ *to be the smallest* Φ*-closed class.*

The main result about inductively defined classes states that this class, denoted $\mathsf{I}(\Phi)$, always exists.

Theorem 4.2. (**CZF**) (Class Inductive Definition Theorem) *For any inductive definition* Φ *there is a smallest* Φ*-closed class* $\mathsf{I}(\Phi)$.

Proof. [Acz86] section 4.2 or [AR01] Theorem 5.1 or [AR10] Theorem 12.1.1. □

A similar result can be obtained for the dual notion of largest Φ-closed class. However, we need to enlist a choice principle.

Definition 4.3. *The* Relativized Dependent Choices Axiom, **RDC**, *is the following scheme. For arbitrary formulae ϕ and ψ, whenever*

$$\forall x\big[\phi(x) \rightarrow \exists y\big(\phi(y) \wedge \psi(x,y)\big)\big]$$

and $\phi(b_0)$, then there exists a function f with domain ω such that $f(0) = b_0$ and

$$(\forall n \in \omega)\big[\phi(f(n)) \wedge \psi(f(n), f(n+1))\big].$$

Theorem 4.4. $(\mathbf{CZF} + \mathbf{RDC})$ (Class Co-Inductive Definition Theorem)

Let Φ be an inductive definition on a class S, i.e. Φ is a subclass of $Pow(S) \times S$, where $Pow(S)$ is the class of all subsets of S.

Moreover, let Δ_Φ be defined by

$$\Delta_\Phi(Y) \quad = \quad \big\{a \mid \forall x\,\big(\frac{x}{a}\;\Phi \rightarrow \exists u(u \in x \wedge u \in Y\,)\big)\big\}.$$

Then Δ_Φ is monotone, too.

(i) There is a largest Φ-closed class (denoted by $\mathsf{Co\text{-}I}(\Phi)$).
(ii) Suppose $\{x \mid \frac{x}{a}\;\Phi\}$ is a set for each $a \in S$. Then there is a largest Δ_Φ closed class (denoted by $\mathsf{Co\text{-}I}(\Delta_\Phi)$). It is also known as the largest Φ-progressive class.

Proof. (i) follows from [Acz88], Theorem 6.5 (classically) or [Rat04] Theorem 5.17 or [AR10] Theorem 13.1.3 in combination with Proposition 13.1.2.

(ii) follows from [AR10] Theorem 13.1.5 in combination with Proposition 13.1.2. □

We are mostly interested in the cases when $\mathsf{I}(\Phi)$, $\mathsf{Co\text{-}I}(\Phi)$, and $\mathsf{Co\text{-}I}(\Delta_\Phi)$ are sets. This requires some stronger axioms, though.

Definition 4.5. *A set A is a* regular set *if it is transitive and inhabited such that whenever $a \in A$ and $R \subseteq A \times A$ is a relation satisfying $\forall x \in A\, \exists y \in A\; xRy$ then there exists a set $c \in A$ such that*

$$\forall x \in a\, \exists y \in c\, xRy \quad \wedge \quad \forall y \in c\, \exists x \in a\, xRy.$$

The set A is said to be strongly regular or $\bigcup$-regular if it is regular and $\forall a \in A\; \bigcup a \in A$.

REA *is the assertion that every set is contained in a regular set.* $\mathbf{REA}_{\bigcup}$ *is the assertion that every set is contained in a $\bigcup$-regular set.*

Theorem 4.6. $(\mathbf{CZF} + \mathbf{REA})$ *Let Φ be an inductive definition such that Φ is a set, then $\mathsf{I}(\Phi)$ is a set.*

Proof. [Acz86] or [AR01] Theorem 5.7 or [AR10] Theorem 12.2.4. □

Theorem 4.7. $(\mathbf{CZF} + \mathbf{REA}_{\bigcup} + \mathbf{RDC})$ *Let Φ be an inductive definition such that Φ is a set. Then $\mathsf{Co\text{-}I}(\Phi)$ and $\mathsf{Co\text{-}I}(\Delta_\Phi)$ are sets.*

Proof. This follows from [AR10] Theorems 13.2.3 and 13.2.4 in combination with [AR10] Proposition 13.1.2. More precisely, [AR10] Theorem 13.2.3 uses the notion of an **RRS** strongly regular set and assumes the axiom that every set is contained in an **RRS** strongly regular set. In the presence of **RDC**, however, one can show that every $\bigcup$-regular set which contains ω as an element is already an **RRS** strongly regular set. The latter follows basically by the same argument as in the proof of [AR10] Proposition 13.1.2. □

5. Realizability interpretations for $\mathbf{MLtt}_{cind}$ and $\mathbb{MLS}_{ind}$

5.1. **A realizability interpretation of $\mathbf{MLtt}_{cind}$ with CT in $\mathbf{CZF}+\mathbf{REA}_{\cup}+\mathbf{RDC}$.** Here we are going to describe a realizability model of $\mathbf{MLtt}_{cind}$ with $\mathbf{CT}$ extending that in [MMR21] in the constructive theory $\mathbf{CZF}+\mathbf{REA}_{\cup}+\mathbf{RDC}$.

As per usual in set theory, we identify the natural numbers with the finite ordinals, i.e. $\mathbb{N}:=\omega$. To simplify the treatment we will assume that **CZF** has names for all (meta) natural numbers. Let $\overline{n}$ be the constant designating the n^{th} natural number. We also assume that **CZF** has function symbols for addition and multiplication on $\mathbb{N}$ as well as for a primitive recursive bijective pairing function $\mathfrak{p}:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ and its primitive recursive inverses $\mathfrak{p}_0$ and $\mathfrak{p}_1$, that satisfy $\mathfrak{p}_0(\mathfrak{p}(n,m))=n$ and $\mathfrak{p}_1(\mathfrak{p}(n,m))=m$. We also assume that **CZF** is endowed with symbols for a primitive recursive length function $\ell:\mathbb{N}\to\mathbb{N}$ and a primitive recursive component function $(-)_{-}:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ determining a bijective encoding of finite lists of natural numbers by means of natural numbers. **CZF** should also have a symbol T for Kleene's T-predicate and the result extracting function U. Let $P(\{e\}(n))$ be a shorthand for $\exists m(T(e,n,m)\wedge P(U(m)))$. Further, let $\mathfrak{p}(n,m,k):=\mathfrak{p}(\mathfrak{p}(n,m),k)$, $\mathfrak{p}(n,m,k,h):=\mathfrak{p}(\mathfrak{p}(n,m,k),h)$, etc. We will use $\mathfrak{p}^n_i$ (for $0<i<n$) to denote the ith component function for which $\mathfrak{p}^n_i(\mathfrak{p}(m_0,...,m_{n-1}))=m_i$ for every $m_0,...,m_{n-1}\in\mathbb{N}$. A similar convention will be adopted for application of partial recursive functions: Let $\{e\}(a,b):=\{\{e\}(a)\}(b)$, $\{e\}(a,b,c):=\{\{e\}(a,b)\}(c)$ etc. We use $a,b,c,d,e,f,n,m,l,k,q,r,s,v,j,i$ as metavariables for natural numbers.

We first need to introduce some abbreviations:

(1) n_0 is $\mathfrak{p}(0,0)$, n_1 is $\mathfrak{p}(0,1)$ and n is $\mathfrak{p}(0,2)$.
(2) $\widetilde{\Sigma}(a,b)$ is $\mathfrak{p}(1,\mathfrak{p}(a,b))$, $\widetilde{\Pi}(a,b)$ is $\mathfrak{p}(2,\mathfrak{p}(a,b))$ and $+(a,b)$ is $\mathfrak{p}(3,\mathfrak{p}(a,b))$.
(3) $\mathsf{list}(a)$ is $\mathfrak{p}(4,a)$ and $\mathsf{id}(a,b,c)$ is $\mathfrak{p}(5,\mathfrak{p}(a,b,c))$.
(4) $a\widetilde{\triangleleft}_{c,d,e}b$ is $\mathfrak{p}(6,\mathfrak{p}(a,b,c,d,e))$.
(5) $\widetilde{\mathsf{rf}}(a,r)$ is $\mathfrak{p}(7,\mathfrak{p}(a,r))$.
(6) $\widetilde{\mathsf{tr}}(a,j,r)$ is $\mathfrak{p}(8,\mathfrak{p}(a,j,r))$.
(7) $a\widetilde{\ltimes}_{c,d,e}b$ is $\mathfrak{p}(9,\mathfrak{p}(a,b,c,d,e))$.

Recall that in intuitionistic set theories ordinals are defined as transitive sets all of whose members are transitive sets, too. Unlike in the classical case, one cannot prove that they are linearly ordered but they are perfectly good as a scale along which one can iterate various processes. The trichotomy of 0, successor, and limit ordinal, of course, has to be jettisoned. As usual, we use lowercase Greek letters as metavariables for ordinals.

Definition 5.1. *By transfinite recursion on ordinals (cf.* [AR10]*, Proposition 9.4.4) we define simultaneously two relations* $\mathsf{Set}_\alpha(n)$ *and* $n\,\varepsilon_\alpha\,m$ *on* $\mathbb{N}$ *in* $\mathbf{CZF}+\mathbf{REA}_{\cup}+\mathbf{RDC}$.

In the following definition we use the shorthand $\mathsf{Fam}_\alpha(e,k)$ *to convey that* $\mathsf{Set}_\alpha(k)$ *and* $\forall j(j\,\varepsilon_\alpha\,k\to\mathsf{Set}_\alpha(\{e\}(j)))$ *and we shall write* $\mathsf{Set}_{\in\alpha}(n)$ *for* $\exists\beta\in\alpha(\mathsf{Set}_\beta(n))$*,* $n\,\varepsilon_{\in\alpha}\,m$ *for* $\exists\beta\in\alpha(n\,\varepsilon_\beta\,m)$ *and* $\mathsf{Fam}_{\in\alpha}(e,k)$ *for* $\exists\beta\in\alpha(\mathsf{Fam}_\beta(e,k))$.

(1) $\mathsf{Set}_\alpha(\mathsf{n}_j)$ *if* $j=0$ *or* $j=1$*, and* $m\,\varepsilon_\alpha\,\mathsf{n}_j$ *if* $m<j$*;*
(2) $\mathsf{Set}_\alpha(\mathsf{n})$ *holds, and* $m\,\varepsilon_\alpha\,\mathsf{n}$ *if* $m\in\mathbb{N}$*.*

(3) *If* $\mathsf{Fam}_{\in\alpha}(e,k)$*, then* $\mathsf{Set}_\alpha(\widetilde{\Pi}(k,e))$ *and* $\mathsf{Set}_\alpha(\widetilde{\Sigma}(k,e))$*;*
if $\mathsf{Fam}_{\in\alpha}(e,k)$*, then*

(a) $n\,\varepsilon_\alpha\,\widetilde{\Pi}(k,e)$ *if there exists* $\beta\in\alpha$ *such that* $\mathsf{Fam}_\beta(e,k)$ *and* $(\forall i\,\varepsilon_\beta\,k)\,\{n\}(i)\,\varepsilon_\beta\,\{e\}(i)$.[5]

(b) $n\,\varepsilon_\alpha\,\widetilde{\Sigma}(k,e)$ *if there exists* $\beta\in\alpha$ *such that* $\mathsf{Fam}_\beta(e,k)$ *as well as* $\mathfrak{p}_0(n)\,\varepsilon_\beta\,k$ *and* $\mathfrak{p}_1(n)\,\varepsilon_\beta\,\{e\}(\mathfrak{p}_0(n))$.

(4) *If there exists* $\beta\in\alpha$ *such that* $\mathsf{Set}_\beta(n)$ *and* $\mathsf{Set}_\beta(m)$, *then* $\mathsf{Set}_\alpha(+(n,m))$, *and* $i\,\varepsilon_\alpha+(n,m)$ *if there exists* $\beta\in\alpha$ *such that* $\mathsf{Set}_\beta(n)$, $\mathsf{Set}_\beta(m)$ *and*

$$[\mathfrak{p}_0(i)=0\wedge\mathfrak{p}_1(i)\,\varepsilon_\beta\,n]\ \vee\ [\mathfrak{p}_0(i)=1\wedge\mathfrak{p}_1(i)\,\varepsilon_\beta\,m].$$

(5) *If there exists* $\beta\in\alpha$ *such that* $\mathsf{Set}_\beta(n)$, *then* $\mathsf{Set}_\alpha(\mathsf{list}(n))$, *and* $i\,\varepsilon_\alpha\,\mathsf{list}(n)$ *if there exists* $\beta\in\alpha$ *such that* $\mathsf{Set}_\beta(n)$ *and* $\forall j\,[j<\ell(i)\to(i)_j\,\varepsilon_\beta\,n]$.

(6) *If* $\mathsf{Set}_{\in\alpha}(n)$, *then* $\mathsf{Set}_\alpha(\mathsf{id}(n,m,k))$, *and* $s\,\varepsilon_\alpha\,\mathsf{id}(n,m,k)$ *if there exists* $\beta\in\alpha$ *such that* $\mathsf{Set}_\beta(n)$, $m\,\varepsilon_\beta\,n$ *and* $s=m=k$.

(7) *Let* $\beta\in\alpha$. *Suppose that the following conditions (collectively called* $*_\beta$*) are satisfied:*

(a) $\mathsf{Set}_\beta(s)$,

(b) $a\,\varepsilon_\beta\,s$,

(c) $\mathsf{Fam}_\beta(v,s)$,

(d) $\mathsf{Fam}_\beta(i,s)$ *and*

(e) $\forall x\forall y[x\,\varepsilon_\beta s\ \wedge\ y\,\varepsilon_\beta\{i\}(x)\to\mathsf{Fam}_\beta(\{c\}(x,y),s)]$.

Then $\mathsf{Set}_\alpha(a\widetilde{\lhd}_{s,i,c}v)$

For $\beta\in\alpha$ *satisfying* $*_\beta$, *let* V_β *be the smallest subset of* $\mathbb{N}$ *satisfying the following conditions:*

(a) *if* $z\,\varepsilon_\beta s$ *and* $r\,\varepsilon_\beta\{v\}(z)$ *then* $\mathfrak{p}(z,\widetilde{\mathsf{rf}}(z,r))\in V_\beta$;

(b) *if* $r\in\mathbb{N}$, $z\,\varepsilon_\beta\,s$, $j\,\varepsilon_\beta\{i\}(z)$ *and* $(\forall u\,\varepsilon_\beta s)\,(\forall t\,\varepsilon_\beta\{c\}(z,j,u))\ \mathfrak{p}(u,\{r\}(u,t))\in V_\beta$ *then* $\mathfrak{p}(z,\widetilde{\mathsf{tr}}(z,j,r))\in V_\beta$.

The existence of the set V_β *is guaranteed by the fact that* **REA** *holds, i.e., Theorem 4.6. We define* $q\,\varepsilon_\alpha\,a\widetilde{\lhd}_{s,i,c}v$ *by* $\exists\beta\in\alpha\,[\,*_\beta\ \wedge\ \mathfrak{p}(a,q)\in V_\beta]$.

(8) *Let* $\beta\in\alpha$. *Suppose that the conditions* $*_\beta$ *as in* (7) *are satisfied, then* $\mathsf{Set}_\alpha(a\widetilde{\ltimes}_{s,i,c}v)$. *For* $\beta\in\alpha$ *satisfying* $*_\beta$, *let* W_β *be the largest subset of* $\mathbb{N}$ *satisfying the following conditions:*

(a) *if* $z\,\varepsilon_\beta\,s$ *and* $\mathfrak{p}(z,q)\in W_\beta$, *then* $\mathfrak{p}_0(q)\,\varepsilon_\beta\,\{v\}(z)$;

(b) *if* $z\,\varepsilon_\beta\,s$, $j\,\varepsilon_\beta\,\{i\}(z)$ *and* $\mathfrak{p}(z,q)\in W_\beta$, *then*

$$\mathfrak{p}_0(\{\mathfrak{p}_1(q)\}(j))\,\varepsilon_\beta\,s\wedge$$
$$\mathfrak{p}_0(\mathfrak{p}_1(\{\mathfrak{p}_1(q)\}(j)))\,\varepsilon_\beta\{\{c\}(z,j)\}((\mathfrak{p}_0(\{\mathfrak{p}_1(q)\}(j)))\wedge$$
$$\mathfrak{p}(\mathfrak{p}_0(\{\mathfrak{p}_1(q)\}(j)),\mathfrak{p}_1(\mathfrak{p}_1(\{\mathfrak{p}_1(q)\}(j))))\in W_\beta$$

The existence of the set W_β *is guaranteed by* $\mathbf{REA}_\cup+\mathbf{RDC}$ *(see Theorem 4.7). We define* $q\,\varepsilon_\alpha\,a\widetilde{\ltimes}_{s,i,c}v$ *as* $\exists\beta\in\alpha\,[\,*_\beta\ \wedge\ \mathfrak{p}(a,q)\in W_\beta]$.

As done in [MMR21] one can prove the following crucial lemma.

Lemma 5.1. ($\mathbf{CZF}+\mathbf{REA}_\cup+\mathbf{RDC}$)

[5]We use the obvious shorthand $(\forall i\,\varepsilon_\beta\,k)\ldots$ for $\forall i[i\,\varepsilon_\beta\,k\to\ldots]$; also employed henceforth.

- *For all* $m \in \mathbb{N}$, *if* $\mathsf{Set}_\alpha(m)$ *and* $\alpha \subseteq \rho$, *then* $\mathsf{Set}_\rho(m)$.
- *For all* $m \in \mathbb{N}$, *if* $\mathsf{Set}_\alpha(m)$, *then for all* ρ *such that* $\mathsf{Set}_\rho(m)$,

$$\forall i \in \mathbb{N}(i \,\varepsilon_\alpha\, m \leftrightarrow i \,\varepsilon_\rho\, m).$$

Definition 5.2. *We define in* $\mathbf{CZF} + \mathbf{REA}_\cup + \mathbf{RDC}$ *the formula* $\mathsf{Set}(n)$ *as* $\exists\alpha(\mathsf{Set}_\alpha(n))$ *and* $x \,\varepsilon\, y$ *as* $\exists\alpha(x \,\varepsilon_\alpha\, y)$.

Theorem 5.3. *Consistency of the theory* $\mathbf{CZF} + \mathbf{REA}_\cup + \mathbf{RDC}$ *implies the consistency of the theory* $\mathbf{MLtt}_{cind}$ *extended with the formal Church thesis* $\mathbf{CT}$.

Proof. We outline a realizability semantics in $\mathbf{CZF} + \mathbf{REA}_\cup + \mathbf{RDC}$. In what follows $\mathbf{p}$ will be a code for the primitive recursive pairing function $\mathfrak{p}$ introduced just before Definition 5.1, that is $\{\mathbf{p}\}(n, m) = \mathfrak{p}(n, m)$. Moreover $\mathbf{p}_1$ and $\mathbf{p}_2$ will be codes for $\mathfrak{p}_0$ and $\mathfrak{p}_1$. In the same way we fix codes $\mathbf{p}^n$ and $\mathbf{p}^n_i$ representing the encoding of n-tuples and their projections, respectively.

Every pre-term is interpreted as a $\mathcal{K}_1$-applicative term (that is, a term built with numerals, variables and Kleene application) as it is done in [MMR21].

We must notice that in introducing codes for sets in the universe in Definition 5.1 we took account of dependencies by means of natural numbers representing recursive functions; however every pre-term depending on variables will be interpreted as a $\mathcal{K}_1$-applicative term having the same free variables (we identify the variables of $\mathbf{MLtt}_{cind}$ with those in $\mathbf{CZF} + \mathbf{REA}_\cup + \mathbf{RDC}$). For these reasons, whenever a term s in $\mathbf{MLtt}_{cind}$ depends on terms $t_1, \ldots, t_n$ in context, its interpretation will depend on the interpretations of $t_1, \ldots, t_n$ bounded with adequate Λ operators. The variables which will be bounded by these Λs will be the ones used in the rule where the term s is introduced. This abuse of notation allows us to avoid heavy fully-annotated terms in the syntax.

In order to complete the interpretation of pre-terms we only need to interpret the new pre-terms of $\mathbf{MLtt}_{cind}$ as follows.

(1) $(a \ltimes_{s,i,c} v)^I := \{\mathbf{p}\}(9, \{\mathbf{p}^5\}(a^I, v^I, s^I, \Lambda x.i^I, \Lambda x.\Lambda y.c^I))$
(2) $\mathsf{ax}_1(a, q)^I := \{\mathbf{p}_0\}(q^I)$
(3) $\mathsf{ax}_2(a, j, q)^I := \{\{\mathbf{p}_1\}(q^I)\}(j^I)$
(4) $\mathsf{ax}_3(a, m, q_1, q_2)^I := \{\mathbf{p}\}(q_1^I[a^I/x, m^I/z], \Lambda j.q_2^I[a^I/x, m^I/z])$

If τ is a $\mathcal{K}_1$-applicative term and $A = \{x|\, \phi\}$ is a class, we will define $\tau \in A$ as an abbreviation for $\phi[\tau/x]$.

The interpretation of types, contexts and judgements is exactly as in [MMR21].
In particular, we interpret pre-types into the language of $\mathbf{CZF} + \mathbf{REA}_\cup + \mathbf{RDC}$ as definable subclasses of $\mathbb{N}$ as follows:

(1) $\mathsf{N}_0^I := \{x \in \mathbb{N}|\, \bot\}$.
(2) $\mathsf{N}_1^I := \{x \in \mathbb{N}|\, x = 0\}$.
(3) $\mathsf{N}^I := \{x \in \mathbb{N}|\, x = x\}$.
(4) $((\Sigma y \in A)B)^I := \{x \in \mathbb{N}|\, \mathfrak{p}_0(x) \in A^I \wedge \mathfrak{p}_1(x) \in B^I[\mathfrak{p}_0(x)/y]\}$.
(5) $((\Pi y \in A)B)^I := \{x \in \mathbb{N}|\, \forall y \in \mathbb{N}\,[y \in A^I \rightarrow \{x\}(y) \in B^I]\}$.
(6) $(A + B)^I := \{x \in \mathbb{N}|\, [\mathfrak{p}_0(x) = 0 \wedge \mathfrak{p}_1(x) \in A^I] \vee [\mathfrak{p}_0(x) = 1 \wedge \mathfrak{p}_1(x) \in B^I]\}$.
(7) $(\mathsf{List}(A))^I := \{x \in \mathbb{N}|\, \forall i \in \mathbb{N}\,[i < \ell(x) \rightarrow (x)_i \in A^I]\}$.
(8) $(\mathsf{Id}(A, a, b))^I := \{x \in \mathbb{N}|\, x = a^I \wedge a^I = b^I \wedge a^I \in A^I\}$.
(9) $U_0^I := \{x|\, \mathsf{Set}(x)\}$.
(10) $\mathsf{T}(a)^I := \{x|\, x \,\varepsilon\, a^I\}$.

Pre-contexts are interpreted as conjunctions of set-theoretic formulas as follows:
(1) $[\,]^I$ is the formula $\top$;
(2) $[\Gamma, x \in A]^I$ is the formula $\Gamma^I \wedge x^I \in A^I$.
Validity of judgements J in $\mathbf{CZF} + \mathbf{REA}_\cup + \mathbf{RDC}$ under the foregoing interpretation is defined as follows:
(1) $A\, type\, [\Gamma]$ holds if $\mathbf{CZF} + \mathbf{REA}_\cup + \mathbf{RDC} \vdash \Gamma^I \to \forall x\, (x \in A^I \to x \in \mathbb{N})$
(2) $A = B\, type\, [\Gamma]$ holds if $\mathbf{CZF} + \mathbf{REA}_\cup + \mathbf{RDC} \vdash \Gamma^I \to \forall x\, (x \in A^I \leftrightarrow x \in B^I)$
(3) $a \in A\ [\Gamma]$ holds if $\mathbf{CZF} + \mathbf{REA}_\cup + \mathbf{RDC} \vdash \Gamma^I \to a^I \in A^I$
(4) $a = b \in A\ [\Gamma]$ holds if $\mathbf{CZF} + \mathbf{REA}_\cup + \mathbf{RDC} \vdash \Gamma^I \to a^I \in A^I \wedge a^I = b^I$,
where x is a fresh variable.

The encoding of lambda-abstraction in terms of $\mathcal{K}_1$-applicative terms can be chosen (see [IMMS18]) in such a way that if a and b are terms and x is a variable which is not bounded in a, then the terms $(\, a[b/x]\,)^I$ and $a^I[\,b^I/x^I]$ coincide.

The rules relative to positivity relations in $\mathbf{MLtt}_{cind}$ are satisfied by the realizability interpretation and Theorem 4.7 plays a crucial role for the validity of the rule cind-Pos.

□

5.2. **A realizability interpretation of $\mathbb{MLS}_{ind}$ with CT in $\mathbf{CZF} + \mathbf{REA}$.**

Here we are going to describe a realizability model of $\mathbb{MLS}_{ind}$ with $\mathbf{CT}$ in the constructive theory $\mathbf{CZF} + \mathbf{REA}$. The interpretation is analogous to that of $\mathbf{MLtt}_{ind}$ in $\mathbf{CZF} + \mathbf{REA}$ in [MMR21], but one needs to take care of the universe constructor $\mathbf{u}(a, (x)b)$ for small universes; note that the latter are not required to be closed under inductive and co-inductive types. Such universes can be modelled following the construction in Definition 5.1 but omitting the clauses (7) and (8). Whereas the transfinite recursion of Definition 5.1 has to run through all ordinals, rendering the model a proper class, one can show in $\mathbf{CZF} + \mathbf{REA}$ that there exist ordinals ρ where this recursion without the clauses (7) and (8) comes to a halt, that is, no new types are generated at later stages γ when $\rho \in \gamma$. To find such ρ, choose a regular set R that contains all the relevant types and let ρ be $\{\xi \mid \xi \in R\}$. As a result, small universes can be modelled via sets in one fell swoop and then their sets and the elementhood relation between them become part of the big inductive definition of the superuniverse at level α. In other words, the small universes $U(k, e, A, (Ba)a \in A)$ are already defined in their entirety before one starts to define the large superuniverse as a class.

To be a bit more specific, given $k \in \mathbb{N}$, a set $A \subseteq \mathbb{N}$, $e \in \mathbb{N}$ for which $\{e\}(a)$ is defined for all $a \in A$ and a family $(B_a)_{a\in A}$ of subsets of $\mathbb{N}$ such that $\{e\}(a) = \{e\}(a')$ entails $B_a = B_{a'}$, let

$$\mathbb{U}(k, e, A, (B_a)_{a\in A})$$

be the small universe defined by the clauses (1)–(6) of Definition 5.1 plus an initial clause to the effect that $\mathbb{U}(k, e, A, (B_a)_{a\in A}) \models \mathsf{Set}(k)$, $\mathbb{U}(k, e, A, (B_a)_{a\in A}) \models \mathsf{Set}(\{e\}(a))$ for $a \in A$, $\mathbb{U}(k, e, A, (B_a)_{a\in A}) \models a\,\varepsilon\, k$ iff $a \in A$, and $\mathbb{U}(k, e, A, (B_a)_{a\in A}) \models m\,\varepsilon\,\{e\}(a)$ iff $m \in A_a$, whenever $a \in A$; here we use $\mathbb{U}(k, e, A, (B_a)_{a\in A}) \models \theta$ to express that θ holds in the sense of $\mathbb{U}(k, e, A, (B_a)_{a\in A})$.
To define the model for $\mathbb{MLS}_{ind}$ we proceed as in Definition 5.1, keeping clauses (1)–(7), but replacing clause (8) as follows.
(8') Suppose $\mathsf{Fam}_{\in\alpha}(k, e)$. Then

$$\mathsf{Set}_\alpha(\widetilde{\mathsf{u}}(k, e)),$$

where $\widetilde{\mathsf{u}}(k, e) := \mathfrak{p}(10, \mathfrak{p}(k, e))$.

Let $A := \{x \mid x \,\varepsilon_\beta\, k$ for some $\beta \in \alpha\}$ and $B_x := \{v \mid v \,\varepsilon_\beta\, \{e\}(x)$ for some $\beta \in \alpha\}$. Since A and the sets B_a are determined by k and e, we just write $\mathbb{U}(k, e)$ for the small universe $\mathbb{U}(k, e, A, (B_x)_{x \in A})$. We also want to inject $\mathbb{U}(k, e)$ into the model for $\mathbb{MLS}_{ind}$. This is effected by the following postulations.

(a) $d \,\varepsilon_\alpha\, \widetilde{\mathsf{u}}(k, e)$ iff $\mathbb{U}(k, e) \models \mathsf{Set}(d)$.

(b) Moreover, if $\mathbb{U}(k, e) \models \mathsf{Set}(d)$, then $\mathsf{Set}_\alpha(d)$, and $x \,\varepsilon_\alpha\, d$ iff $\mathbb{U}(k, e) \models x \,\varepsilon\, d$.

The interpretation of type theory is then carried out in the same vein as in [MMR21]. One only has to add the interpretation of terms and types involving the universe constructors and the superuniverse. We follow here the syntax in [Pal98], although we use $\mathcal{S}$ for the superuniverse and T for its decoding constructor:

(1) the universe constructor terms are interpreted as $(\mathbf{u}(a, (x)b))^I =^{def} \{\mathbf{p}\}(10, \{\mathbf{p}\}(a^I, \Lambda x.b^I))$ and their corresponding universe types are interpreted as
$$(\mathsf{U}(\mathsf{T}(a), (x)\mathsf{T}(b)))^I := \{x \mid x \,\varepsilon\, \{\mathbf{p}\}(10, \{\mathbf{p}\}(a^I, \Lambda x.b^I))\};$$ [6]

(2) the superuniverse is interpreted as $\mathcal{S}^I := \{x \mid \mathsf{Set}(x)\}$ while its decodings are interpreted as $(\mathsf{T}(a))^I := \{x \mid x \,\varepsilon\, a^I\}$;

(3) the interpretation of terms $t(a, (x)b, c)$ representing the decoding in the superuniverse (which is itself still a code) of a code c in the universe generated by the family b on a, is simply defined as c^I, while the interpretation of decodings relative to the universe $\mathsf{U}(\mathsf{T}(a), (x)\mathsf{T}(b))$ is defined as
$$(\mathsf{T}(\mathsf{T}(a), (x)\mathsf{T}(b), c))^I := \{x \mid x \,\varepsilon\, c^I\}$$

Hence we have the following theorem.

Theorem 5.4. *Consistency of the theory* $\mathbf{CZF} + \mathbf{REA}$ *implies the consistency of the theory* $\mathbb{MLS}_{ind}$ *extended with the formal Church thesis* $\mathbf{CT}$.

Corollary 5.5. *Consistency of the theory* $\mathbf{CZF} + \mathbf{REA}$ *implies the consistency of the theory* $\mathbf{mTT}_{cind}$ *extended with the formal Church thesis* $\mathbf{CT}$ *and the axiom of choice* $\mathbf{AC}$.

Proof. This is a consequence of Corollary 3.2 and Theorem 5.4. □

Moreover, we know from Theorem 4.6 in [MMR21] that $\mathbf{CZF} + \mathbf{REA}$ and $\mathbf{MLtt}_{ind}$ have the same proof-theoretic strength. Since $\mathbf{MLtt}_{ind}$ is a subsystem of $\mathbf{MLtt}_{cind}$ which can be interpreted in $\mathbb{MLS}_{ind}$ as in Corollary 3.2, from Theorem 5.4 it follows that $\mathbb{MLS}_{ind}$ and $\mathbf{CZF} + \mathbf{REA}$ have the same proof-theoretical strength, as well as $\mathbf{MLtt}_{ind}$ and $\mathbf{MLtt}_{cind}$.

Corollary 5.6. *The following theories share the same proof-theoretic strength.*

(1) $\mathbf{CZF} + \mathbf{REA}$
(2) $\mathbf{MLtt}_{ind}$
(3) $\mathbf{MLtt}_{cind}$
(4) $\mathbb{MLS}_{ind}$.

Remark 5.7. It is worth noting that the above theorems still hold if we replace inductively generated formal topologies with the closure of **mTT**-sets with well founded trees in [NPS90], also called W-types, which we simply call W-sets when added to **mTT**. Indeed we can prove that such a extension $\mathbf{mTT} + W$**-sets** can be interpreted in the extension $\mathbb{MLS}_W$ of the

[6] We do not need to interpret universe types $\mathsf{U}(A, (x)B)$ for arbitrary pretypes A and B since universe types can be defined in $\mathbb{MLS}_{ind}$ only in the case in which A and B are (decodings of terms) in the superuniverse $\mathcal{S}$.

first-order fragment of intensional Martin-Löf's type theory in [NPS90] with a superuniverse $\mathcal{S}$ à la Tarski closed under well founded trees, i.e. W-types, besides the usual first-order type constructors and universe constructors as in [Pal98]. Then we can prove that $\mathbb{MLS}_W$ is consistent with **CT** and hence $\mathbf{mTT} + W$**-sets** is consistent with $\mathbf{AC} + \mathbf{CT}$. The proof that $\mathbb{MLS}_W$ is consistent with **CT** can be obtained by building a realizability interpretation as that in this section which is in turn based on that of section 5.1. This is obtained by removing clause (7) in section 5.1 and substituting it with clause (7'), after having introduced the abbreviations $\mathsf{w}(a, b) := \mathfrak{p}(11, \mathfrak{p}(a, b))$ and $\widetilde{\mathsf{sup}}(a, b) := \mathfrak{p}(12, \mathfrak{p}(a, b))$, as follows:

(7') Let $\beta \in \alpha$. Suppose that the following conditions are satisfied:
  (a) $\mathsf{Set}_\beta(a)$,
  (b) $\mathsf{Fam}_\beta(b, a)$.
  Then $\mathsf{Set}_\alpha(\mathsf{w}(a, b))$.
  For $\beta \in \alpha$ satisfying $\mathsf{Set}_\beta(a)$ and $\mathsf{Fam}_\beta(b, a)$, let H_β be the smallest subset of $\mathbb{N}$ satisfying the following condition: if $c \,\varepsilon_\beta\, a$ and $\forall z \,\varepsilon_\beta \{b\}(c)\ (\{d\}(z) \in H_\beta)$, then $\widetilde{\mathsf{sup}}(c, d) \in H_\beta$. The existence of the set H_β is guaranteed by the axiom **REA** and we define $h \,\varepsilon_\alpha\, \mathsf{w}(a, b)$ by $\exists \beta \in \alpha\, [\mathsf{Set}_\beta(a) \wedge \mathsf{Fam}_\beta(b, a) \wedge h \in H_\beta]$.
  The interpretations of elimination terms *wrec* (see p.99 pf [NPS90]) relative to W-types are obtained as codes $\mathbf{n}_q$ of recursive functions depending primitively recursively on the parameters in q satisfying

$$\{\mathbf{n}_q\}(\widetilde{\mathsf{sup}}(c, d)) \simeq \{\Lambda x.\Lambda y.\Lambda z.q^I\}(c, d, \Lambda v.\{\mathbf{n}_q\}(\{d\}(v)))$$

6. CZF with large set existence axioms and the superuniverse in type theory

The axioms **REA** and $\mathbf{REA}_{\bigcup}$ of the previous section section can be viewed as constructive analogs of regular cardinals (see [RL03, AR10]).[7] Further strengthenings of the notion of regularity lead to weakly inaccessible, inaccessible and Mahlo sets (see [RGP98, CR02] and [AR10] chapter 18).[8] In a constructive environment such as **CZF**, however, the existence of such sets does not entail the enormous proof-theoretic strength they engender in the classical context. Indeed, the existence of weakly inaccessible sets does not add more strength to $\mathbf{CZF} + \mathbf{REA}$. In more detail, if one adds to **CZF** the axiom **wINACC** asserting that every set is contained in a weakly inaccessible set then $\mathbf{CZF} + \mathbf{wINACC}$ possesses a recursive realizability interpretation in $\mathbf{CZF} + \mathbf{REA}$. This follows basically from [Rat03, Theorem 4.7], when one changes the interpreting theory classical theory **KPi** therein to the theory $\mathbf{CZF} + \mathbf{REA}$ by deploying a constructivization of the techniques of [Rat93, GR94] (details will be provided after Definition 6.7). Furthermore, there is a close connection between superuniverses in type theory (see [Pal98, RGP98, Rat00b, Rat00a, Rat01]) and the axiom **wINACC** in set theory.

Definition 6.1. We say that a set is *weakly inaccessible* if it is regular and a model of **CZF**. A set is **inaccessible** if it is regular and a model of $\mathbf{CZF} + \mathbf{REA}$.

[7] **ZF** alone, though, cannot prove the existence of regular sets containing ω. This follows from [RL03, Corollary 7.1].

[8] The terminology varies between different papers. Inaccessible sets are called set-inaccessible in [RGP98] while weakly inaccessible sets are called inaccessible in [CR02, AR01, AR10]. The terminology in the present paper is the same as used in [Rat17]. On the basis of **ZFC**, though, the notions coincide.

The formalization of the notion of inaccessibility in Definition 6.1 is somewhat syntactic in that it requires a satisfaction predicate for formulae interpreted over a set. An alternative and more 'algebraic' characterization can be given as follows.

Definition 6.2. Let $\Omega := \{x : x \subseteq \{0\}\}$. Ω is the class of truth values with 0 representing falsity and $1 = \{0\}$ representing truth. Classically one has $\Omega = \{0, 1\}$ but intuitionistically one cannot conclude that those are the only truth values.

For $a \subseteq \Omega$ define

$$\bigwedge a \quad = \quad \{x \in 1 : (\forall u \in a) x \in u\}.$$

A class B is $\bigwedge$-*closed* if for all $a \in B$, whenever $a \subseteq \Omega$, then $\bigwedge a \in B$.

For sets a, b let ${}^a b$ be the class of all functions with domain a and with range contained in b. Let $\mathbf{mv}({}^a b)$ be the class of all sets $r \subseteq a \times b$ satisfying $\forall u \in a\, \exists v \in b\, \langle u, v\rangle \in r$. A set c is said to be *full in* $\mathbf{mv}({}^a b)$ if $c \subseteq \mathbf{mv}({}^a b)$ and

$$\forall r \in \mathbf{mv}({}^a b)\, \exists s \in c\, s \subseteq r.$$

The expression $\mathbf{mv}({}^a b)$ should be read as the collection of *multi-valued functions* from a to b.

Proposition 6.3. (**CZF**) *A set I is weakly inaccessible if and only if the following are satisfied:*
(1) *I is a regular set,*
(2) *$\omega \in I$,*
(3) *$(\forall a \in I) \bigcup a \in I$,*
(4) *I is $\bigwedge$-closed,*
(5) *$(\forall a, b \in I)(\exists c \in I)\, \big[c$ is full in $\mathbf{mv}({}^a b)\big]$.*

Proof. See [Rat00b], Proposition 3.4. □

Remark 6.4. *Note that* $\mathbf{CZF} + \mathbf{REA}_{\bigcup}$ *is a subtheory of* $\mathbf{CZF} + \mathbf{wINACC}$.

Viewed classically and in the presence of the axiom of choice, weakly inaccessible sets give rise to strongly inaccessible cardinals, i.e. regular cardinals $\kappa > \omega$ such that $2^\rho < \kappa$ forall $\rho < \kappa$.

Let V_α denote the αth level of the von Neumann hierarchy.

Proposition 6.5. (**ZFC**) *If I is weakly inaccessible set then $I = V_\kappa$ for some strongly inaccessible cardinal κ.*

Proof. This is a consequence of the proof of [RGP98], Corollary 2.7. □

The next result from [Rat03] shows that the strength of **wINACC** is quite modest when based on constructive set theory.

Theorem 6.6. $\mathbf{CZF} + \mathbf{REA}$ *and* $\mathbf{CZF} + \mathbf{wINACC}$ *have the same proof-theoretic strength as the subsystem of second order arithmetic with $\mathbf{\Delta}^1_2$-comprehension and bar induction.*

Proof. [Rat03] Theorem 4.7. □

Indeed, [Rat03] basically furnishes interpretations between $\mathbf{CZF} + \mathbf{REA}$ and $\mathbf{CZF} + \mathbf{wINACC}$ and a version of type theory $\mathbb{MLS}^*$ with a superuniverse as we shall argue below.

Definition 6.7. *The type theory $\mathbb{MLS}^*$ has the following ingredients:*[9]

[9]Regarding the exact formalization of a superuniverse $\mathbb{S}$ and the universe operator $\mathbf{U}$ see [Rat00b].

- $\mathbb{MLS}^*$ *demands closure under the usual type constructors* $\Pi, \Sigma, +, I, \mathbb{N}, \mathbb{N}_0, \mathbb{N}_1$ *(but not the W-type).*
- $\mathbb{MLS}^*$ *has a superuniverse* $\mathbb{S}$ *which is closed under* $\Pi, \Sigma, +, I, \mathbb{N}, \mathbb{N}_0, \mathbb{N}_1$ *and the W-type and the universe operator* $\mathbf{U}$*.*
- $\mathbb{MLS}^*$ *has a type* $\mathbf{V}$ *of iterative sets over* $\mathbb{S}$ *(see* [Acz78]*).*
- *The universe operator* $\mathbf{U}$ *takes a type* A *in* $\mathbb{S}$ *and a family of types* $B : A \to \mathbb{S}$ *and produces a universe* $\mathbf{U}(A, B)$ *in* $\mathbb{S}$ *which contains* A *and* $B(x)$ *for all* $x \in A$*, and is closed under* $\Pi, \Sigma, +, I, \mathbb{N}, \mathbb{N}_0, \mathbb{N}_1$ *(but not the W-type).*

The type $\mathbf{V}$ then enables one to perform an interpretation of $\mathbf{CZF} + \mathbf{wINACC} + \mathbf{RDC}$ in $\mathbb{MLS}^*$ à la Aczel (see [Acz78, Acz82, Acz86]), using the techniques of [RGP98] section 4. The universe operator is crucial for modelling the weakly inaccessible sets in the manner of [RGP98] section 4. Note that **RDC** is modelled by this interpretation, too, as follows from [Acz82].

Conversely, $\mathbb{MLS}^*$ has a recursive realizability interpretation in $\mathbf{CZF} + \mathbf{REA}$. This uses a constructive modification of the techniques of [Rat93, GR94] sections 4 and 5. Although the latter articles give interpretations in versions of classical Kripke-Platek set theories (**KP** and **KPi**) the interpretations are constructively valid once one jettisons the unnecessary trichotomy of dividing ordinals into 0, successor and limit cases, as it was done in our previous paper [MMR21] in section 4. The recursive realizability interpretation of the universe operator follows in the same way as in Definition 4.12, Theorem 4.13 and Lemma 5.7 of [Rat93, GR94]. Instead of using an admissible set as in [Rat93, GR94] Lemma 5.7 to show that the inductive definition of a universe gives rise to a set one employs a regular set A containing all the parameters and uses recursion on the ordinals in the rank of A to build up the set that models the universe.

So the upshot of the foregoing is the following result, mainly extracted from [Rat03].

Theorem 6.8 ([Rat03])**.** *The theories* $\mathbf{CZF} + \mathbf{REA}$*,* $\mathbf{CZF} + \mathbf{wINACC} + \mathbf{RDC}$ *and* $\mathbb{MLS}^*$ *are of the same strength. More precisely, they can be mutually interpreted in each other via the above interpretations.*

The type $\mathbf{V}$ plays a crucial role in the interpretation of set theory in type theory à la Aczel. However, it doesn't add any proof-theoretic strength. This phenomenon was first observed in [Rat93, GR94] section 6. Let $\mathbb{MLS}$ be $\mathbb{MLS}^*$ without the type $\mathbf{V}$. Since the interpretation of the theory **IRA** of [Rat93, GR94] Definition 6.5 is also an interpretation in $\mathbb{MLS}$ we have the following result by [Rat93, GR94] Theorem 6.13.

Theorem 6.9. *The theories* $\mathbf{CZF} + \mathbf{REA}$*,* $\mathbf{CZF} + \mathbf{wINACC}$*,* $\mathbb{MLS}$ *and* $\mathbb{MLS}^*$ *are of the same proof-theoretic strength.*

Finally, let us address the issue of adding Church's thesis, **CT**, to various type theories. Of course, for this the ξ-rule has to be dropped in the same way as in [IMMS18] and [MMR21] as explained in Definition 3.1. So, in future, if we write $\mathbf{ML} + \mathbf{CT}$ where **ML** is a system of type theory, it is understood that this is type theory without the ξ-rule.

From Theorems 6.8 and 6.9 (or rather their proofs) it follows that the modified type theories with **CT** are consistent and, moreover, that they have the same strength as their cousins with the ξ-rule. The main reason for the type theories with **CT** being consistent is that we use recursive realizability interpretations for the type theories as in [MMR21]. On the other hand, the reason for the fact that the strength does not drop is that for the interpretation of set theory (or **IRA**) in type theory the ξ-rule doesn't matter at all.

Therefore we conclude:

Theorem 6.10. *The following theories share the same proof-theoretic strength.*
(1) $\mathbf{CZF} + \mathbf{REA}$
(2) $\mathbf{CZF} + \mathbf{wINACC} + \mathbf{RDC}$
(3) $\mathbb{MLS}$
(4) $\mathbb{MLS}^*$
(5) $\mathbb{MLS} + \mathbf{CT}$
(6) $\mathbb{MLS}^* + \mathbf{CT}$

As a consequence of Proposition 3.1, Theorem 5.3, Remark 6.4 and Theorem 6.9 we obtain again the following result.

Corollary 6.11. *Consistency of the theory* $\mathbf{CZF} + \mathbf{REA}$ *implies the consistency of the theory* $\mathbf{mTT}_{cind}$ *extended with the formal Church thesis* $\mathbf{CT}$ *and axiom of choice* $\mathbf{AC}$*.*

Remark 6.12. In $\mathbf{mTT}_{cind}$ we added coinductive predicates by inserting primitive proof-terms in an axiomatic way justified by the fact that such predicates become proof-irrelevant within $\mathbf{emTT}_{cind}$. Indeed we are not aware of explicit well-behaved rules for coinductive types to be added to Martin-Löf's type theory or to the Calculus of Constructions (as mentioned in [BG16]) and hence to the intensional level of $\mathbf{MF}$.

On the other hand, current encodings of coinductive types like those in [AAG05, BCS15] are perfomed within versions of type theory validating extensional equality of functions which can not be validated in our Kleene realizability interpretation for its inconsistency with $\mathbf{AC} + \mathbf{CT}$.

What we can currently say is that coinductive types in [AAG05] are consistent with Formal Church thesis $\mathbf{CT}$ over the extensional level $\mathbf{emTT}$ of $\mathbf{MF}$ extended with well founded trees, called W-sets in $\mathbf{MF}$-terminology. Indeed we can encode coinductive types, or better coinductive sets in $\mathbf{MF}$-terminology, following [AAG05] within $\mathbf{emTT} + W\text{-}\mathbf{sets}$ because this theory validates the assumptions needed for the enconding in [AAG05] due to the fact that $\mathbf{emTT}$ extends the first order version of extensional Martin-Loef's type theory and validates $0 \neq 1$ as shown in [Mai09]. Moreover, a two-level extension of $\mathbf{MF}$ extended with W-sets can be provided as described in [Bre15]. Then, a Kleene realizability interpretation for $\mathbf{mTT} + W\text{-}\mathbf{sets} + \mathbf{AC} + \mathbf{CT}$ can be built as that for $\mathbf{mTT}_{ind}$ in [MMR21], see remark 5.7. By composing such an interpretation with the interpretation of $\mathbf{emTT}$ in $\mathbf{mTT}$ in [Mai09] extended to $W\text{-}\mathbf{sets}$ in [Bre15], we obtain a realizability interpretation for $\mathbf{emTT} + \mathbf{W\text{-}sets} + \mathbf{CT}$, which guarantees that the encoded coinductive sets are consistent with $\mathbf{CT}$. Such an interpretation does not validate $\mathbf{AC}$ since $\mathbf{AC}$ is constructively incompatible with $\mathbf{emTT}$ as shown in [Mai09].

Conclusions. We have shown that the intensional level $\mathbf{mTT}_{cind}$ of the extension $\mathbf{MF}_{cind}$ of the Minimalist Foundation $\mathbf{MF}$ in [Mai09] is consistent with $\mathbf{AC} + \mathbf{CT}$ in two different ways. In both ways we show this by extending Kleene realizability interpretation of intuitionistic arithmetics in a constructive theory whose consistency strength is that of $\mathbf{CZF} + \mathbf{REA}$. A key benefit of the first way is that the intermediate theory $\mathbf{CZF} + \mathbf{REA}_{\bigcup} + \mathbf{RDC}$ also supports the intended set-theoretic interpretation of the extensional level $\mathbf{emTT}_{cind}$ of $\mathbf{MF}_{cind}$.

This work lets us conclude that the extension $\mathbf{MF}_{cind}$ of $\mathbf{MF}$ with all the inductive and coinductive methods developed in the field of Formal Topology constitutes a two-level foundation in the sense of [MS05]. Moreover, we confirmed the expectation that the addition of coinductive topological definitions to $\mathbf{mTT}_{ind}$ to form $\mathbf{mTT}_{cind}$ does not increase its consistency strength.

Finally, all the theories used to reach our goal, except $\mathbf{mTT}_{cind}$ and $\mathbf{mTT}_{ind}$, have shown of the same proof-theoretic strength.

We leave it to future work to establish the consistency strength of $\mathbf{mTT}_{cind}$ and $\mathbf{mTT}_{ind}$ given that it is still an open problem to establish that of **mTT** itself.

Another future goal would be to apply the realizability interpretations presented here to build predicative variants of Hyland's Effective Topos as in [MM21] but in a constructive meta-theory such as **CZF** +**REA**.

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References

[AAG05] M. G. Abbott, T. Altenkirch, and N. Ghani. Containers: Constructing strictly positive types. *Theoretical Computer Science*, 342(1):3–27, 2005.

[Acz78] P. Aczel. The type theoretic interpretation of constructive set theory. In A. MacIntyre, L. Pacholski, and J. Paris, editors, *Logic Colloquium '77*, pages 55–66. North Holland, 1978.

[Acz82] P. Aczel. The type theoretic interpretation of constructive set theory: Choice principles. In A.S. Troelstra and D. van Dalen, editors, *The L.E.J. Brouwer Centenary*, pages 1–40. North Holland, 1982.

[Acz86] P. Aczel. The type theoretic interpretation of constructive set theory: Inductive definitions. In R.B. Marcus et al., editor, *Logic, Methodology and Philosophy of Science VII*, pages 17–49. North Holland, 1986.

[Acz88] P. Aczel. *Non-well-founded sets*, volume 14 of *CSLI Lecture Notes*. Center for the study of language and information, 1988.

[AR01] P. Aczel and M. Rathjen. Notes on constructive set theory. Mittag-Leffler Technical Report No.40, 2001.

[AR10] P. Aczel and M. Rathjen. Constructive set theory. Available at http://www1.maths.leeds.ac.uk/~rathjen/book.pdf, 2010.

[BCS15] B.Ahrens, P. Capriotti, and R. Spadotti. Non-wellfounded trees in homotopy type theory. In Thorsten Altenkirch, editor, *13th International Conference on Typed Lambda Calculi and Applications, TLCA 2015, July 1-3, 2015, Warsaw, Poland*, volume 38 of *LIPIcs*, pages 17–30. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2015.

[BG16] H. Basold and H. Geuvers. Type theory based on dependent inductive and coinductive types. In Martin Grohe, Eric Koskinen, and Natarajan Shankar, editors, *Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science, LICS '16, New York, NY, USA, July 5-8, 2016*, pages 327–336. ACM, 2016.

[Bre15] Luca Bressan. An extension of the Minimalist Foundation with well founded trees. Master's thesis, Dep. of Mathematics, University of Padova, supervised by M.E. Maietti, 2015.

[BS06] G. Battilotti and G. Sambin. Pretopologies and a uniform presentation of sup-lattices, quantales and frames. In *Special Issue: Papers presented at the 2nd Workshop on Formal Topology (2WFTop 2002)*, volume 137 of *Annals of Pure and Applied Logic*, pages 30–61, (2006).

[BT21] U. Berger and H. Tsuiki. Intuitionistic fixed point logic. *Annals of Pure and Applied Logic*, 172(3):102903, 56, 2021.

[CMS13] F. Ciraulo, M.E. Maietti, and G. Sambin. Convergence in formal topology: a unifying notion. *Journal of Logic and Analysis*, 5, 2013.

[CR02] L. Crosilla and M. Rathjen. Inaccessible set axioms may have little consistency strength. *Annals of Pure and Applied Logic*, 115:33–70, (2002.

[CS18] F. Ciraulo and G. Sambin. Embedding locales and formal topologies into positive topologies. *Archive for Mathematical Logic*, 57(7-8):755–768, 2018.

[CSSV03] T. Coquand, G. Sambin, J. Smith, and S. Valentini. Inductively generated formal topologies. *Annals of Pure and Applied Logic*, 124(1-3):71–106, 2003.

[Cur18] G. Curi. Abstract inductive and co-inductive definitions. *The Journal of Symbolic Logic*, 83(2):598–616, 2018.

[GR94] E. Griffor and M. Rathjen. The strength of some Martin-Löf type theories. *Archive for Mathematical Logic*, 33(5):347–385, 1994.

[IMMS18] H. Ishihara, M.E. Maietti, S. Maschio, and T. Streicher. Consistency of the intensional level of the Minimalist Foundation with Church's thesis and axiom of choice. *Archive for Mathematical Logic*, 57(7-8):873–888, 2018.

[KS19] T. Kawai and G. Sambin. The principle of pointfree continuity. *Logical Methods in Computer Science*, 15(1), 2019.

[Lin89] I. Lindström. A construction of non-well-founded sets within Martin-Löf's type theory. *The Journal of Symbolic Logic*, 54(1):57–64, 1989.

[Mai09] M.E. Maietti. A minimalist two-level foundation for constructive mathematics. *Annals of Pure and Applied Logic*, 160(3):319–354, 2009.

[MM16] M.E. Maietti and S. Maschio. A predicative variant of a realizability tripos for the Minimalist Foundation. *IfColog Journal of Logics and their Applications*, 3(4):595–668, 2016.

[MM21] M.E. Maietti and S. Maschio. A predicative variant of Hyland's Effective Topos. *J. Symb. Log.*, 86(2):433–447, 2021.

[MMR21] M.E. Maietti, S. Maschio, and M. Rathjen. A realizability semantics for inductive formal topologies, Church's Thesis and Axiom of Choice. *Logical Methods in Computer Science*, 17(2), 2021.

[MR13a] M.E. Maietti and G. Rosolini. Elementary quotient completion. *Theory and Applications of Categories*, 27(17):445–463, 2013.

[MR13b] M.E. Maietti and G. Rosolini. Quotient completion for the foundation of constructive mathematics. *Logica Universalis*, 7(3):371–402, 2013.

[MR15] M.E. Maietti and G. Rosolini. Unifying exact completions. *Applied Categorical Structures*, 23(1):43–52, 2015.

[MS05] M.E. Maietti and G. Sambin. Toward a minimalist foundation for constructive mathematics. In L. Crosilla and P. Schuster, editor, *From Sets and Types to Topology and Analysis: Practicable Foundations for Constructive Mathematics*, number 48 in Oxford Logic Guides, pages 91–114. Oxford University Press, 2005.

[MV04] M.E. Maietti and S. Valentini. A structural investigation on formal topology: coreflection of formal covers and exponentiability. *The Journal of Symbolic Logic*, 69:967–1005, 2004.

[NPS90] B. Nordström, K. Petersson, and J. Smith. *Programming in Martin Löf's Type Theory.* Clarendon Press, Oxford, 1990.

[Pal98] E. Palmgren. *On universes in type theory*, page 191 – 204. Oxford Logic Guides. Oxford University Press, 1998.

[Rat93] M. Rathjen. The strength of some Martin-Löf type theories. Preprint, Department of Mathematics, Ohio State University,available at http://www1.maths.leeds.ac.uk/∼rathjen/typeOHIO.pdf, 1993.

[Rat00a] M. Rathjen. The strength of Martin-Löf type theory with a superuniverse. part I. *Archive for Mathematical Logic*, 39(1):1–39, 2000.

[Rat00b] M. Rathjen. The superjump in Martin-Löf type theory. In P. Pudlak S. Buss, P. Hajek, editor, *Logic Colloquium '98*, volume 13 of *Lecture Notes in Logic*, pages 363–386. Association for Symbolic Logic, 2000.

[Rat01] M. Rathjen. The strength of Martin-Löf type theory with a superuniverse. part II. *Archive for Mathematical Logic*, 40(3):207–233, 2001.

[Rat03] M. Rathjen. The anti-foundation axiom in constructive set theories. In G. Mints and R. Muskens, editors, *Games, Logic, and Constructive Sets*, pages 87–108. CSLI Publications Stanford, 2003.

[Rat04] M. Rathjen. Predicativity, circularity, and anti-foundation. In G. Link, editor, *One hundred years of Russell's paradox*, volume 6 of *de Gruyter Series in Logic and its Applications*, pages 191–219. de Gruyter, 2004.

[Rat06] M. Rathjen. Realizability for Constructive Zermelo-Fraenkel set theory. In J. Väänänen and V. Stoltenberg-Hansen, editors, *Logic Colloquium '03*, volume 24 of *Lecture Notes in Logic*, pages 228–314. A.K. Peters, Wellesley, 2006.

[Rat17] M. Rathjen. Proof theory of constructive systems: Inductive types and univalence. In G. Jäger and W. Sieg, editors, *Feferman on foundations*, volume 13 of *Outstanding Contributions to Logic*, pages 385–419. Springer, 2017.

[RGP98] M. Rathjen, E. Griffor, and E. Palmgren. Inaccessibility in constructive set theory and type theory. *Annals of Pure and Applied Logic*, 94:181–200, 1998.

[RL03] M. Rathjen and R. Lubarsky. On the regular extension axiom and its variants. *Mathematical Logic Quarterly*, 49(5):1–8, 2003.

[Sam87] G. Sambin. Intuitionistic formal spaces - a first communication. *Mathematical logic and its applications*, pages 187–204, 1987.

[Sam03] G. Sambin. Some points in formal topology. *Theoretical Computer Science*, 305:347–408, 2003.

[Sam19] G. Sambin. Dynamics in foundations: what does it mean in the practice of mathematics? In Stefania Centrone, Deborah Kant, and Deniz Sarikaya, editors, *Reflections on the Foundations of Mathematics. Univalent Foundations, Set Theory and General Thoughts*, volume 407 of *Synthese Library. Studies in Epistemology, Logic, Methodology, and Philosophy of Science*, pages 455–494. Springer, 2019.

[Tv88] A. S. Troelstra and D. van Dalen. Constructivism in mathematics, an introduction, vol. I and II. In *Studies in logic and the foundations of mathematics*. North-Holland, 1988.

[vdBM07] B. van den Berg and F. De Marchi. Models of non-well-founded sets via an indexed final coalgebra theorem. *The Journal of Symbolic Logic*, 72(3):767–791, 2007.

[Vic07] S. Vickers. Sublocales in formal topology. *The Journal of Symbolic Logic*, 72(2):463–482, 2007.