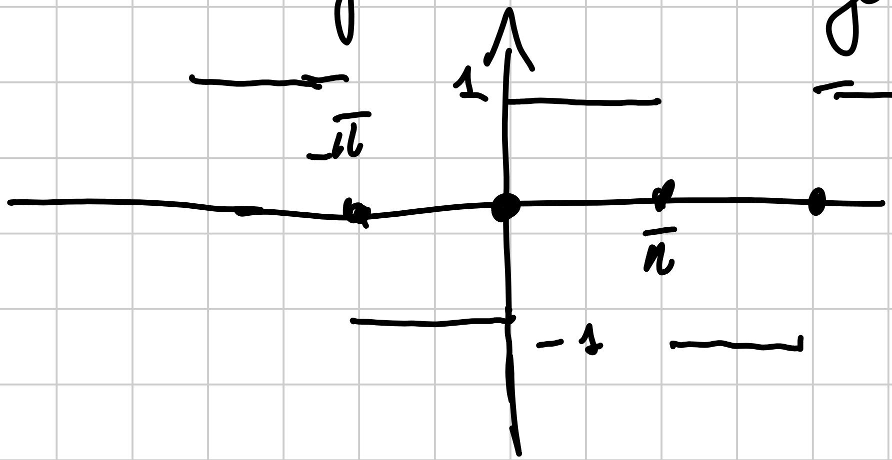
Esercizi

1) Trovare la serie di Fourier associata a

$$f(x) = \begin{cases} 1 & x \in (0,\pi] \\ -1 & x \in [-\pi, 0) \\ 0 & x = 0 \end{cases}$$

e periodica di periodo 2π



f regolare a tratti

$f(x)$ è dispari $\qquad a_n = 0$

$$S(x) = \sum_{n=1}^{+\infty} b_n \operatorname{sen} nx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{f(x) \cdot \operatorname{sen} nx}_{\text{pari}} \, dx =$$

$f(x) = 1 \quad x \in (0, \pi]$.

$$= \frac{2}{\pi} \int_0^{\pi} \operatorname{sen} nx \, dx =$$

$$= \frac{2}{\pi}\left(-\frac{1}{n}\cos nx \Big/_{0}^{\pi}\right) = \frac{2}{\pi n}\left(-(-1)^{n} + 1\right)$$

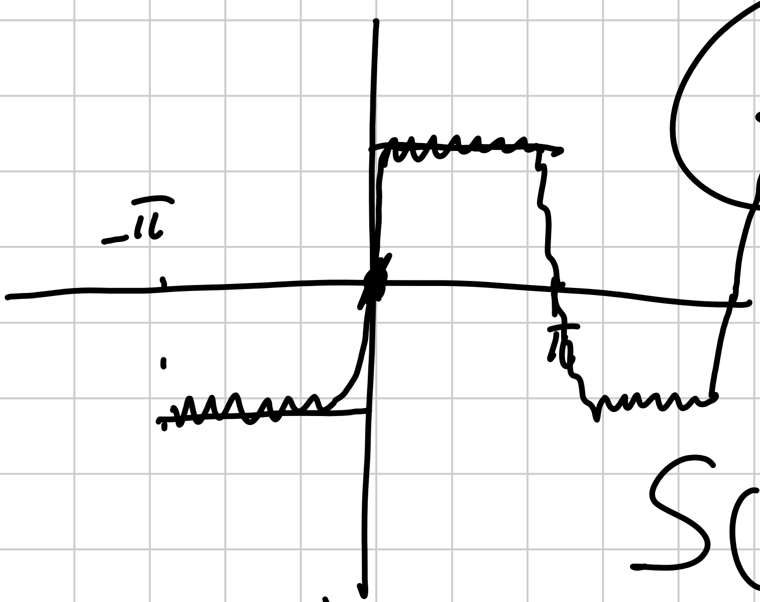
$$= \left.\begin{cases} 0 & n \text{ pari} \\ \dfrac{4}{\pi n} & n \text{ dispari} \end{cases}\right\} b_n$$

$$S(x) = \frac{4}{\pi}\sum \frac{\operatorname{sen}((2n+1)x)}{(2n+1)}$$

serie di Fourier associata a $f(x)$

dove f è continua $\quad S(x) = f(x)$

$$S(x) = \frac{f(x^+) + f(x^-)}{2}$$



$x = 0$ f è discontinua

$$S(0) = 0$$

$$f(0^+) + f(0^-) = 1 - 1 = 0$$

Es. 8/2/2010

coefficienti di Fourier di

$f(x) = \left(x - \frac{\pi}{2}\right)^2$ in $(0, \pi]$

dispari in $[-\pi, 0)$ e periodica di

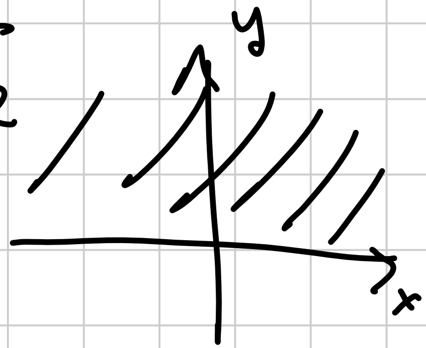
periodo 2π.

!

Esercizio

$$\omega = \underbrace{\left(y^2 e^{xy^2} + \log y\right)}_{F_1} dx + \underbrace{\left(2xy\, e^{xy^2} + x\, g(y)\right)}_{F_2} dy$$

1) dominio $\quad A = \{ (x,y) : y > 0 \}$



2) Trovare condizioni su $g(y)$ affinché ω sia una forma chiusa $\quad (F_{1y} = F_{2x})$

$$F_{1y} = 2y\, e^{xy^2} + y^2 e^{xy^2} \cdot 2xy + \frac{1}{y}$$

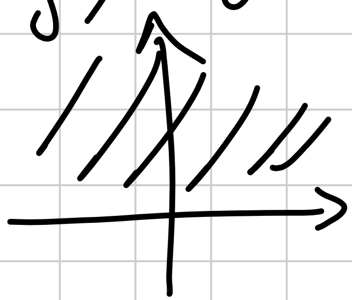
$$F_{2x} = 2y e^{xy^2} + 2xy e^{xy^2} \cdot y^2 + g(y)$$

$$F_{1y} = F_{2x} \quad \forall (x,y) \in A \iff g(y) = \frac{1}{y}$$

$$\omega = \left(y^2 e^{xy^2} + \lg y\right) dx + \left(2xy e^{xy^2} + \frac{x}{y}\right) dy$$

2) ω è esatta in A? sì!

perché chiusa in un dominio semplicemente connesso.



3) Trovare la funzione potenziale $U(x,y)$:

$$U_x = y^2 e^{xy^2} + \log y$$

$$U_y = 2xy\, e^{xy^2} + \frac{x}{y} \xrightarrow{F_2}$$

$$U(x,y) = e^{xy^2} + x \log y + h(y)$$

$$U_y = e^{xy^2} \cdot \cancel{2xy} + \cancel{\frac{x}{y}} + h'(y) =$$

$$= \cancel{2xy\, e^{xy^2}} + \cancel{\frac{x}{y}}$$

$$h'(y) = 0 \qquad h(y) = C$$

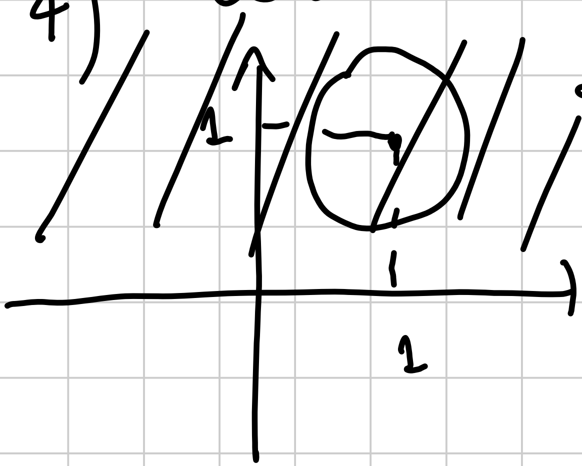
$U_x = F_1$

$U_y = F_2$

$$U(x,y) = e^{xy^2} + x\log y + C$$

4) Calcolare $\int_\gamma \omega$ dove γ è la curva def. da

$$\left\{(x,y) : (x-1)^2 + (y-1)^2 = \frac{1}{4}\right\}$$



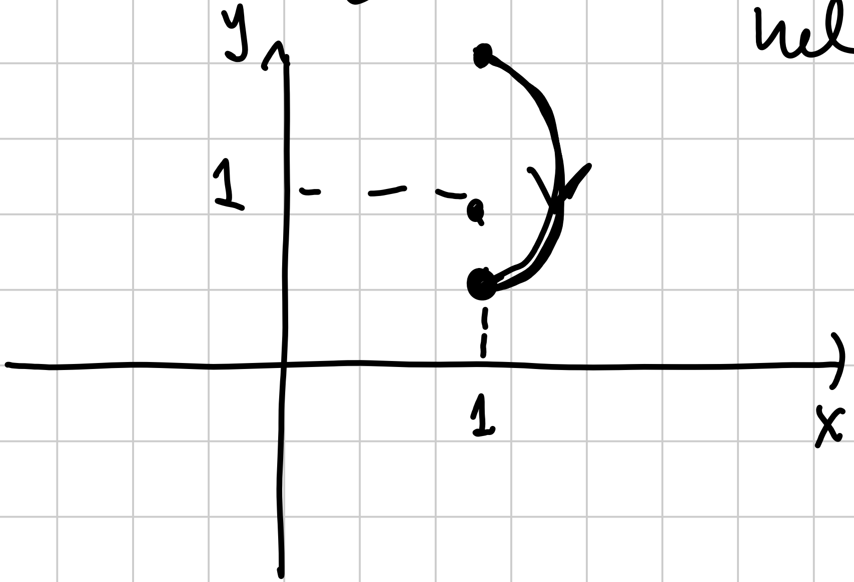
percorsa in senso antiorario

$$\int_\gamma \omega = 0$$

$\int_\gamma \omega$ dove γ è la curva def. da

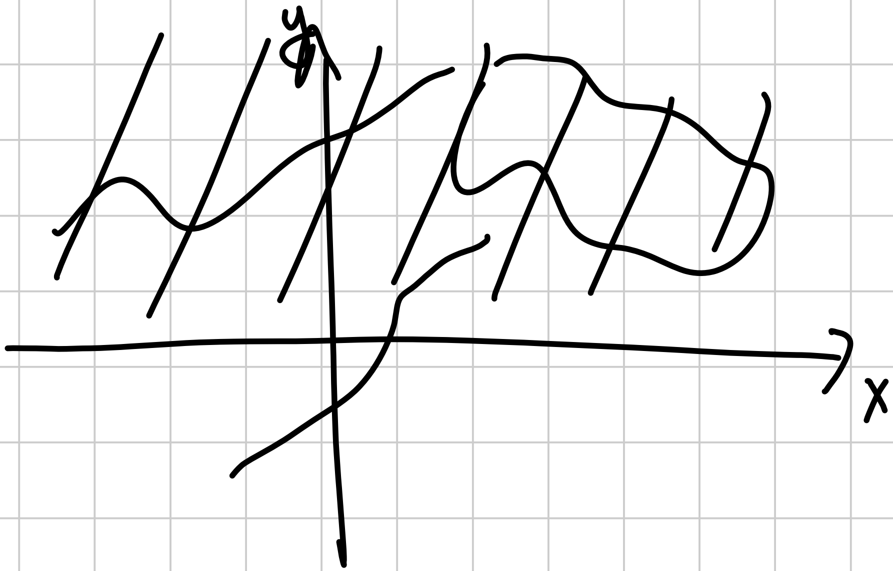
$\left\{ (x,y) : (x-1)^2 + (y-1)^2 = \frac{1}{4}, \ x > 1 \right\}$

nel verso delle y decrescenti.



$$\int_\gamma \omega = U\left(1, \frac{1}{2}\right) - U\left(1, \frac{3}{2}\right)$$

$= \cdots$



$$\int_\gamma \omega$$

Esercizio   Si consideri l'equazione

$$f(x,y) = -x e^{y} + 2y + K = 0 \qquad K \in \mathbb{R}$$

Determinare K affinché in un intorno di $(0,1)$ si possa applicare Dini, cioè $\exists$ una funzione del tipo $y = g(x)$ def. implicitamente da $f = 0$.

$x = h(y)$

Hp. • $f(x,y) \in C^{1}(\mathbb{R}^{2})$ o.k.

• $f(0,1) = 0$ $f(0,1) = 2 + K = 0$

$\parallel$

$\cdot \; f_y(0,1) \neq 0$ $\Downarrow$ $\boxed{K=-2}$

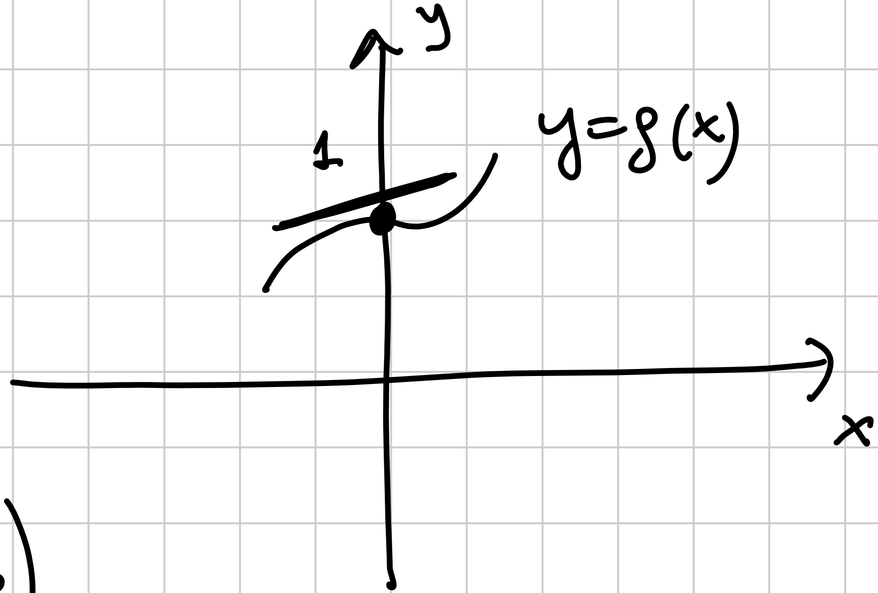
$f_y(x,y) = -xe^y + 2$

$f_y(0,1) = 2 \neq 0$

per $K=-2$ $\exists$ $y = g(x)$ def. implicitamente da $f(x,y)=0$ in un intorno di $(0,1)$.

$$f(x,y) = -xe^y + 2y - 2 = 0$$

Det. la retta tg al grafico della funzione $y = g(x)$ nel p.to $x=0$



$$y = y_0 + g'(x_0)(x - x_0)$$

$$y = 1 + g'(0)\,x$$

$$f(x,y) = -xe^{y} + 2y - 2$$

$$g'(0) = -\frac{f_x(0,1)}{f_y(0,1)} = \frac{e}{2}$$

$$f_x(x,y) = -e^y \qquad f_x(0,1) = -e$$

$$f_y(x,y) = -x e^y + 2 \qquad f_y(0,1) = 2$$

$$\boxed{y = 1 + \frac{e}{2} x}$$

retta tangente al grafico di $g(x)$ in $x = 0$.

$$g'(0) = \frac{e}{2}$$

$$g''(0) = \,?$$

per es.: scrivere lo sviluppo di Taylor della $g(x)$ in un intorno di $x=0$

fino al 2° ordine.

$$g(x) = \underbrace{y_0}_{g(x_0)} + g'(0)(x - x_0) + g''(0)\frac{(x-x_0)^2}{2} + o((x-x_0)^2)$$

$x \to x_0$

$$g(x) = 1 + \frac{e}{2}x + g''(0)\frac{x^2}{2} + o(x^2)$$

$x \to 0$

$$g'(x) = -\frac{f_x(x, g(x))}{f_y(x, g(x))}$$

$$g''(x) = -\left(\frac{(f_{xx} + f_{xy}\,g')f_y - f_x\,(f_{xy} + f_{yy}\,g')}{f_y^2}\right)$$

$$g''(0) = -\,\frac{-e\left(\frac{e}{\cancel{2}}\right)\cancel{2} + e(-e)}{4} = +\frac{e^2}{2}$$

$$f_x = -e^y \qquad f_{xx} = 0 \qquad f_{yy} = -xe^y$$

$$f_y = -xe^y + 2 \qquad f_{xy} = -e^y$$

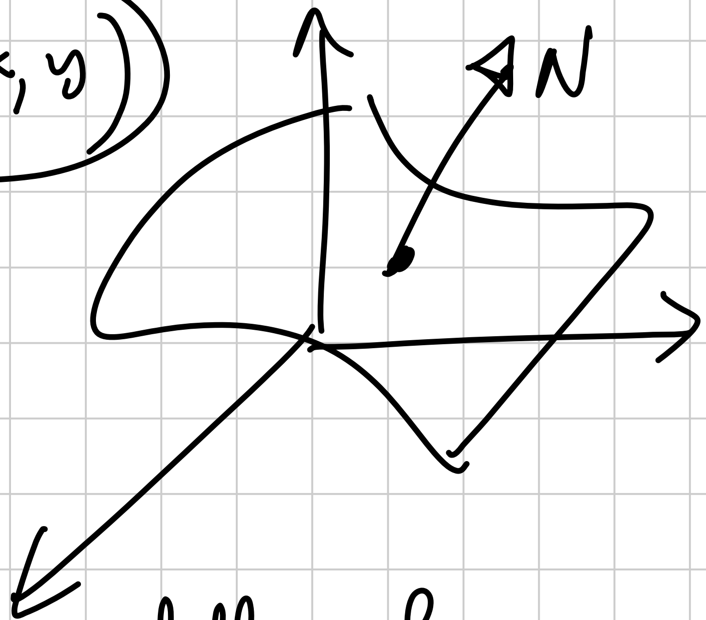
$$f_{xx}(0,1) = 0 \qquad f_{xy}(0,1) = -e \qquad f_{yy}(0,1) = 0$$

OSS. $f(x,y,z)=0$ se vale Dini in (x_0,y_0,z_0)

$z = h(x,y)$

$\Vert$

grafico



versore normale della superficie def dal grafico della funzione h in (x_0, y_0, z_0)

$$\mathcal{N}(x_0, y_0, z_0) = \left(\frac{-h_x}{\sqrt{1+h_x^2+h_y^2}}, \frac{-h_y}{\sqrt{}}, 1 \right)$$

superficie def. in forma cartesiana.

$$f(x,y,z)=0$$

$$z = h(x,y)$$

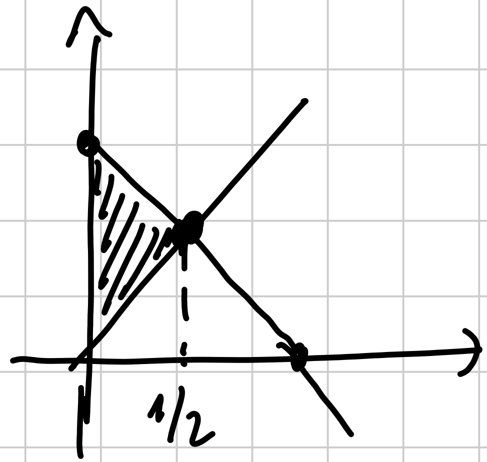
- $h_x^{(x_0,y_0)} = -\dfrac{f_x}{f_z}(x_0,y_0,z_0)$
- $h_y^{(x_0,y_0)} = -\dfrac{f_y}{f_z}(x_0,y_0,z_0)$

piano tangente ad una superficie combinazione

$$z = z_0 + h_x(x_0, y_0)(x - x_0) + h_y(x_0, y_0)(y - y_0)$$

Esercizio

$$\iint_D \frac{(y-x)(3y-x)}{4}\,dx\,dy$$



$$D = \left\{ (x,y) \in \mathbb{R}^2 : \begin{array}{l} x > 0 \\ y \leq 1-x \\ y \geq x \end{array} \right\}$$

1) D è normale rispetto all'asse y

$y = x \qquad y = 1 - x \qquad x = 1 - x$

$x = 1/2$

$$D = \left\{ x \in [0, 1/2] \, , \quad x \leq y \leq 1 - x \right\}$$

e poi formule di riduzione.

2) Con cambiamento di variabili

$$\begin{cases} u = y - x \\ v = 3y - x \end{cases}$$

$y = u + x$

$v = 3u + 3x - x$

$2x = v - 3u$

$$y = u + \frac{v - 3u}{2} =$$

$$= \frac{2u + v - 3u}{2} =$$

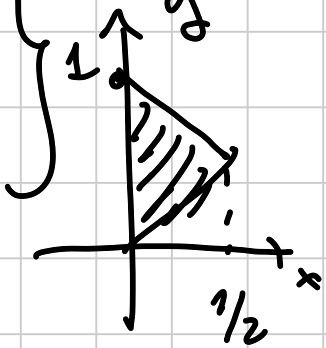
$$\begin{cases} x = \dfrac{v - 3u}{2} \\ y = \dfrac{v - u}{2} \end{cases}$$

$$= \frac{v-u}{2}$$

$$J_\psi(u,v) = \det\begin{pmatrix} -\frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = -\frac{3}{4} + \frac{1}{4} = \quad = -\frac{1}{2}$$

$$I = \iint_D \frac{(y-x)(3y-x)}{4}\, dx\,dy = \iint_{\tilde{D}} \frac{u \cdot v}{4} \cdot \underbrace{\frac{1}{2}}_{du\,dv}$$

$$D = \left\{ x > 0, \quad y \le 1-x, \quad y \geq x \right\}$$

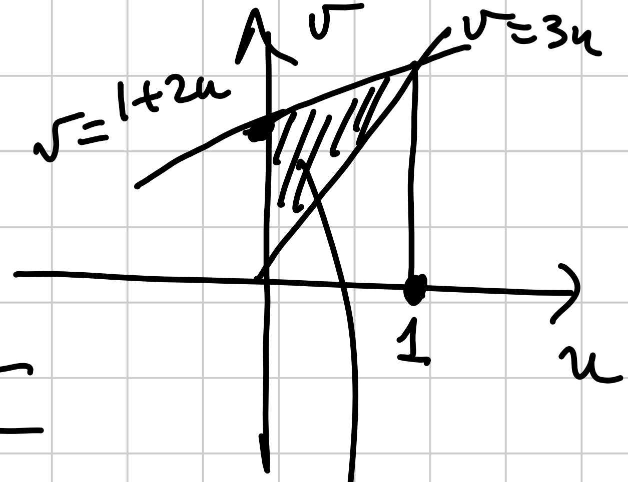


$$\begin{cases} u = y - x \\ v = 3y - x \end{cases} \longrightarrow \quad x = \frac{v-3u}{2} \qquad y = \frac{v-u}{2}$$

$\cdot \ y \geq x \Rightarrow u \geq 0$

$\cdot \ x > 0 \qquad \dfrac{v-3u}{2} > 0 \qquad v > 3u$

segue la retta



$\cdot \ y \leq 1 - x \qquad \dfrac{v-u}{2} \leq 1 - \dfrac{v-3u}{2}$

$v - u \leq 2 - v + 3u$

$2v \leq 2 + 4u$

$v \leq 1 + 2u$

$\tilde{D}$

intersezione due rette

$v = 1 + 2u$

$v = 3u$

$3u = 1 + 2u \qquad u = 1$

$$\tilde{D} = \left\{ u \in [0, 1] , \quad 3u \leq v \leq 1 + 2u \right\}$$