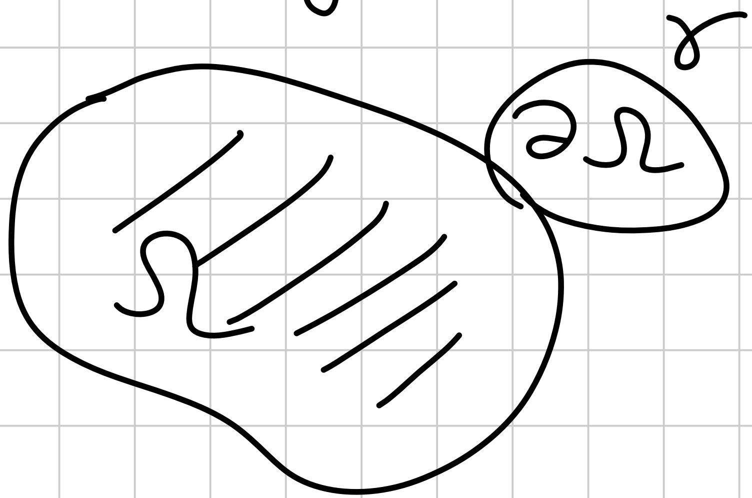
Formule di Gauss-Green nel piano.

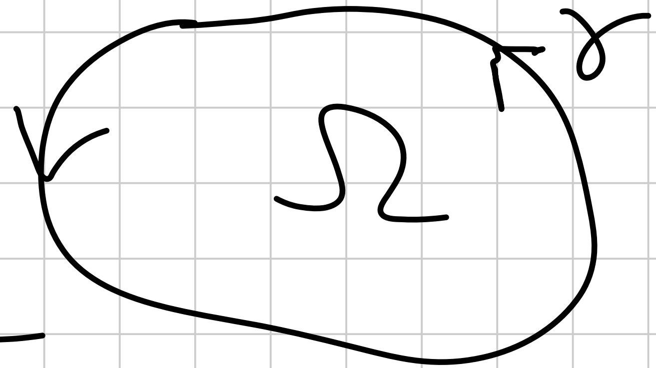
$\Omega \subseteq \mathbb{R}^2$ t.c. $\partial\Omega$ curva di Jordan

$$\int_{\partial\Omega} f\,ds \doteq \int_{\gamma} f\,ds$$



Orientamento

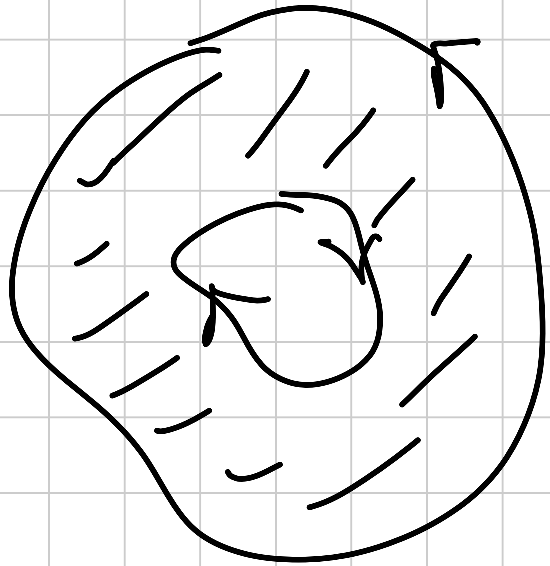
Orientamento positivo di una curva γ in $\mathbb{R}^2$
è quello che
percorrendo la curva
lascia il dominio Ω
sulla sinistra



γ^+

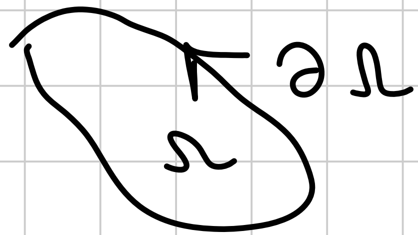
$$\int_\gamma f\, ds$$

$$\int_{\gamma^+} \omega$$



Formule di Gauss-Green

$\Omega \subseteq \mathbb{R}^2$, semplice rispetto a x,
$f(x,y) \in C^1(\Omega)$



- $$\iint_\Omega f_x(x,y)\,dx\,dy = \int_{\partial\Omega^+} f\,dy$$

  integrale di 2ª specie

- Ω semplice rispetto a y

  $$\iint_\Omega f_y(x,y)\,dx\,dy = -\int_{\partial\Omega^+} f\,dx$$

In forma compatta $F = (F_1, F_2)$

$$\iint_{\Omega} (F_{2x} - F_{1y}) \, dx\, dy = \int_{\partial\Omega^+} F_1 \, dx + F_2 \, dy$$

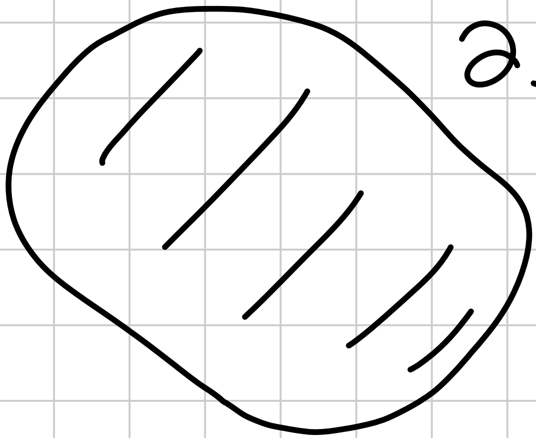
Applicare le due formule precedenti prima a F_1 e poi a F_2

Teorema di Stokes nel piano

Conseguenze

$\omega = F_1\,dx + F_2\,dy$ chiusa $(F_{1y} = F_{2x})$

Ω semplicemente connesso

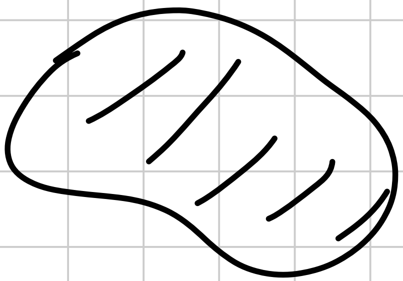


$\partial\Omega$

$$\iint_{\Omega} (F_{2x} - F_{1y})\,dx\,dy = 0$$

una delle formule di G-G

anche $\displaystyle\int_{\partial\Omega^+} F_1\,dx + F_2\,dy = 0$



γ $\Rightarrow$ $\displaystyle\int_{\gamma} F_1\,dx + F_2\,dy = 0$ $\forall \gamma$ chiusa

$\Rightarrow \omega$

$$\int_\gamma F_1\,dx + F_2\,dy = 0 \quad \forall \gamma \text{ curva chiusa}$$

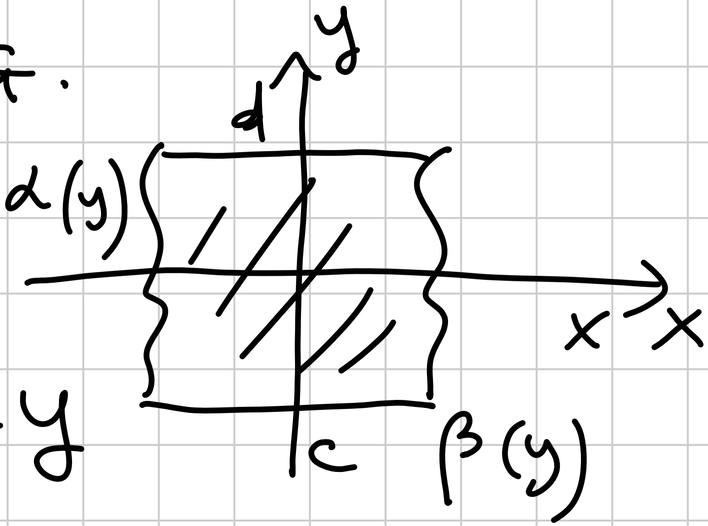
$\Longrightarrow$ $\omega = F_1\,dx + F_2\,dy$ è **esatta**

abbiamo dim. che

ω chiusa + Ω semplic. connesso $\Longrightarrow$ ω è **esatta**

Dim. della forma di G-G.

Ω semplice rispetto a x



$$\iint_{\Omega} f_x(x,y)\,dx\,dy = \int_{\partial\Omega^+} f\,dy$$

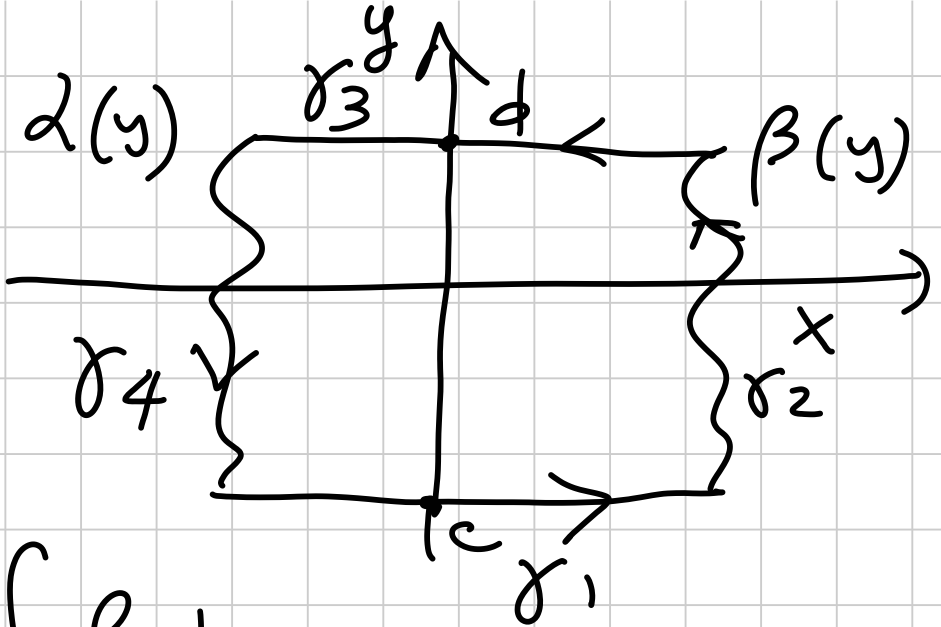
$$\Omega = \{ y \in [c,d],\ \alpha(y) \leq x \leq \beta(y) \}$$

$$\iint_{\Omega} f_x(x,y)\,dx\,dy = \int_c^d \left(\int_{\alpha(y)}^{\beta(y)} f_x(x,y)\,dx \right) dy =$$

$$= \int_c^d \left[f(\beta(y), y) - f(\alpha(y), y) \right] dy$$

Ora calcolo

$$\int_{\partial\Omega^+} f\,dy = \underbrace{\cancel{\int_{\gamma_1} f\,dy}}_{=0} + \int_{\gamma_2} f\,dy + \underbrace{\cancel{\int_{\gamma_3} f\,dy}}_{=0} + \int_{\gamma_4} f\,dy$$

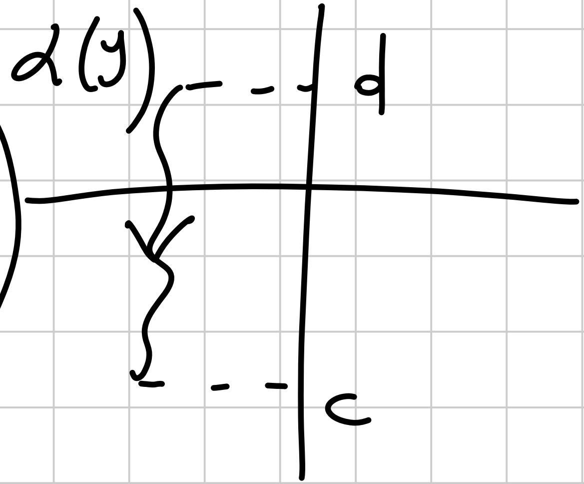


γ_1 e γ_3 sono segmenti del tipo $y =$ costante

$$\Rightarrow \quad \gamma_2 : \begin{cases} x = \beta(y) \\ y \in [c, d] \end{cases}$$

$$\int_{\gamma_2} f\,dy = \int_c^d f(\beta(y), y)\,dy$$

$$\int_{\gamma_4} f\,dy = -\int_c^d f(\alpha(y),y)\,dy$$



$$-\gamma_4 : \begin{cases} x = \alpha(y) \\ y \in [c,d] \end{cases}$$

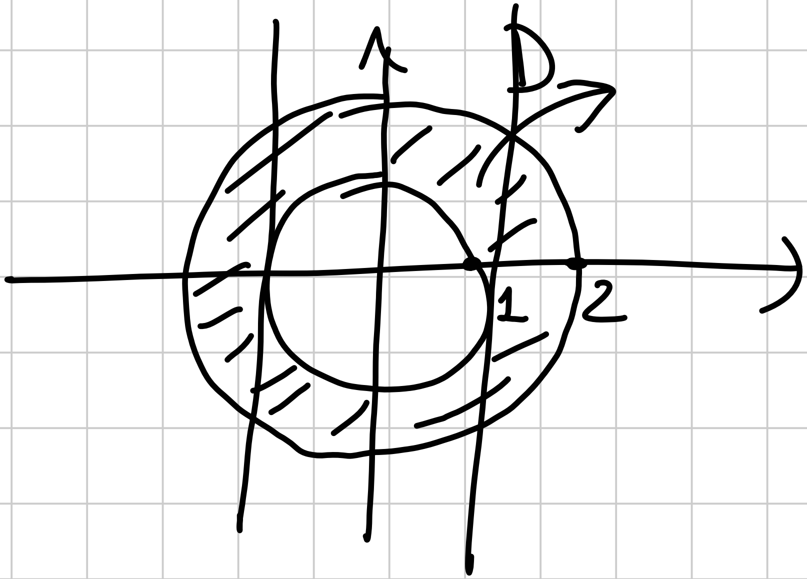
$$\int_{-\gamma_4} f\,dy = \int_c^d f\left(\alpha(y), y\right)dy$$

$$\int_{\partial\Omega^+} f\,dy = \underbrace{\int_c^d f(\beta(y),y)\,dy}_{\int_{\gamma_2} f\,dy} - \underbrace{\int_c^d f(\alpha(y),y)\,dy}_{\int_{\gamma_4} f\,dy}$$

quindi $\displaystyle\iint_{\Omega} f_x \, dx\,dy = \int_{\partial\Omega^+} f\,dy$ #.

OSS. le formule di G.-G. valgono per domini più generali, per esempio domini **regolari** (unione di domini semplici con tagli paralleli agli assi cartesiani)

Es. $\iint_D x^2(1+y)\,dx\,dy$ con G.G.



$$\iint_D f_x\,dx\,dy = \int_{\partial D^+} f\,dy$$

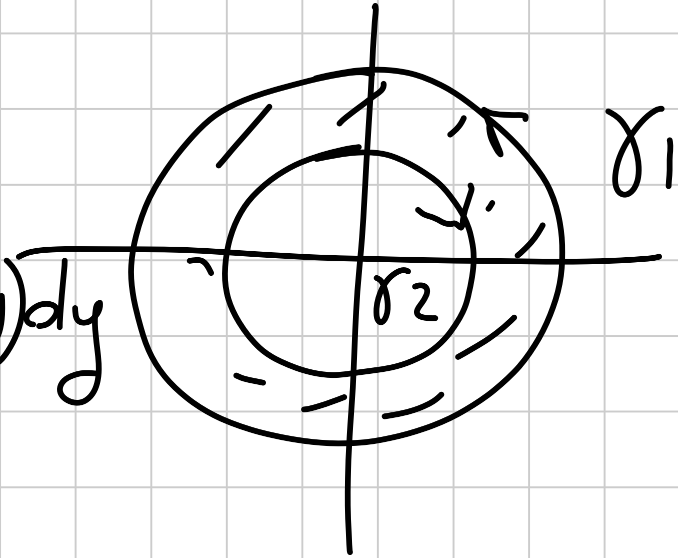
Voglio che $f_x = x^2(1+y)$

prendo come $f(x,y) = \frac{x^3}{3}(1+y)$

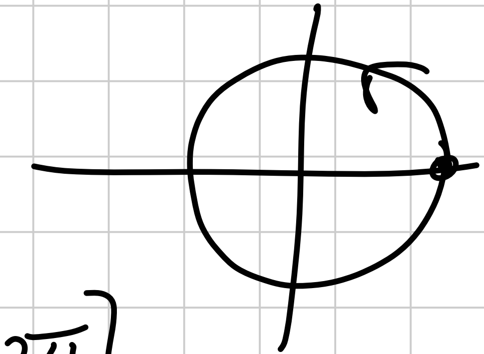
e Applico G.-G. a tale $f(x,y)$

$$\iint_D x^2(1+y)\,dx\,dy = \int_{\partial D^+} \frac{x^3}{3}(1+y)\,dy =$$

$$= \int_{\gamma_1} \frac{x^3}{3}(1+y)\,dy + \int_{\gamma_2} \frac{x^3}{3}(1+y)\,dy$$



$$\gamma_1 \quad \begin{cases} x = 2\cos\varphi \\ y = 2\,\mathrm{sen}\,\varphi \end{cases} \quad \varphi \in [0, 2\pi]$$



$$-\gamma_2 \quad \begin{cases} x = \cos\varphi \\ y = \mathrm{sen}\,\varphi \end{cases} \quad \varphi \in [0, 2\pi]$$

$$\int_{\gamma_1} \frac{x^3}{3}(1+y)\,dy = \int_0^{2\pi} \frac{8\cos^3\varphi}{3}(1+2\,\mathrm{sen}\,\varphi) \cdot 2\cos\varphi\, d\varphi$$

$$\int_{\gamma_2} \frac{x^3}{3}(1+y)\,dy = -\int_0^{2\pi} \frac{\cos^3\varphi}{3}(1+\mathrm{sen}\,\varphi)\cos\varphi\, d\varphi$$

Applicazione di G-G: calcolo di aree di domini in $\mathbb{R}^2$. (regolari)

$$|\Omega| \doteq \iint_\Omega 1 \, dx\,dy \qquad F = (F_1, F_2)$$

$$\iint_\Omega \underbrace{(F_{2x} - F_{1y})}_{=1} \, dx\,dy = \int_{\partial\Omega^+} F_1\,dx + F_2\,dy$$

Scelgo $F = (F_1, F_2)$ in maniera che

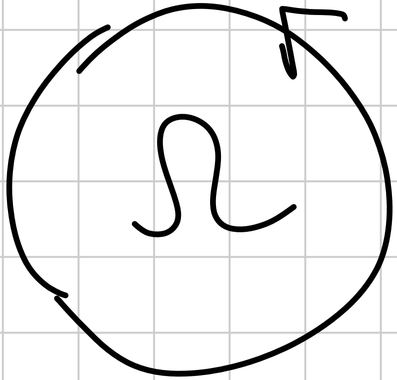
$$F_{2x} - F_{1y} = 1$$

1) $F = (0, x)$ 2) $F = (-y, 0)$

3) $\bar{F} = \frac{1}{2}(-y, \ x)$ $\qquad F_{2x} - F_{1y} = 1$

G. G. su $F = (0, x)$

$$|\Omega| = \iint_{\Omega} 1 \, dx \, dy = \int_{\partial\Omega^+} x \, dy$$



$$|\Omega| = \int_{\partial\Omega^+} x \, dy$$

Analogamente

$F = (-y, 0)$

$$|\Omega| = -\int_{\partial\Omega^+} y \quad dx$$

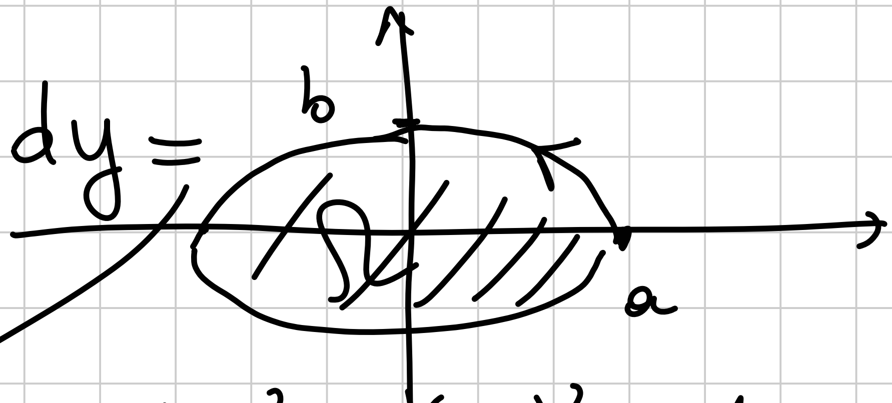
Analogamente

$$F = \frac{1}{2}(-y, x)$$

$$\boxed{|\Omega| = \frac{1}{2}\int_{\partial\Omega^+} -y\,dx + x\,dy}$$

Esempio Area dell'ellisse di semiassi a e b

$$|\Omega| = \frac{1}{2}\int_{\partial\Omega^+} -y\,dx + x\,dy =$$



$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

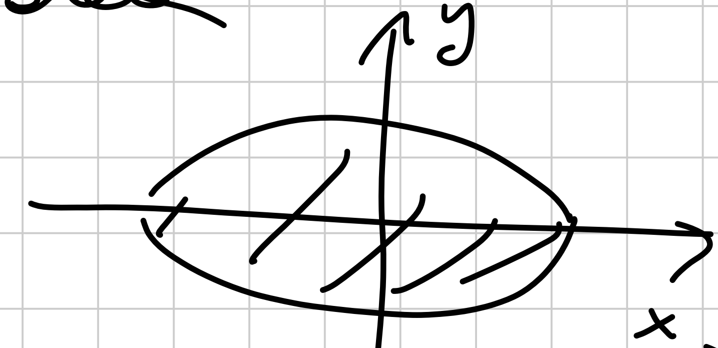
$$\begin{cases} x = a\cos\varphi \\ y = b\,\mathrm{sen}\,\varphi \end{cases}$$

$$\varphi \in [0, 2\pi).$$

$$= \frac{1}{2}\int_0^{2\pi} \left(-b\,\mathrm{sen}\,\varphi\,(-a\,\mathrm{sen}\,\varphi) + a\cos\varphi \cdot b\cos\varphi\right) d\varphi$$

$$= \frac{1}{2}\int_0^{2\pi} ab\,d\varphi = \frac{1}{\cancel{2}}\,ab\cdot\cancel{2}\pi = \pi a\cdot b$$

Con la definizione di area



$$|\Omega| = \iint_\Omega 1\,dx\,dy$$

$$\Omega = \left\{ x \in [-a, a],\ -\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)} \le y \le \sqrt{b^2\left(1-\frac{x^2}{a^2}\right)} \right\}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

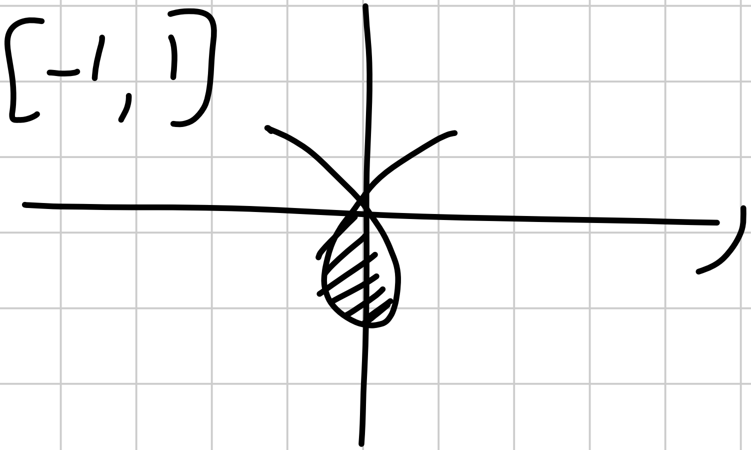
$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

provare anche così

Per casa

Trovare area del dominio racchiuso dalla curva:

$$\gamma \begin{cases} x(t) = t^3 - t \\ y(t) = t^2 - 1 \end{cases} \qquad t \in [-1, 1]$$

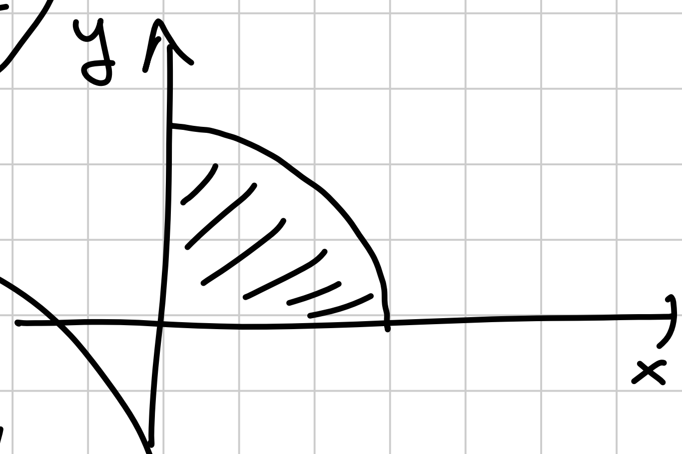


$$\int_{\partial\Omega^+} \left(\qquad \right) = \frac{8}{15}$$

Esercizio

Calcolare $\displaystyle\iint_D \frac{x^2 y}{\sqrt{x^2+y^2}}\,dx\,dy$ dove

con G.-G.



si usa

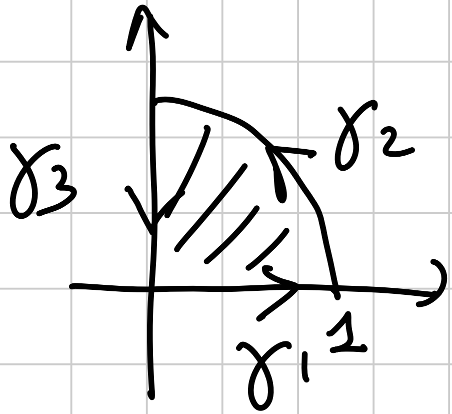
$$\iint_D f_y \,dx\,dy = -\int_{\partial\Omega^+} f\,dx$$

$$f(x,y) = x^2\sqrt{x^2+y^2} \Rightarrow$$

$$f_y = \frac{x^2 y}{\sqrt{x^2+y^2}}$$

Quindi si calcola (per G.-G.)

$$-\int_{\partial D^+} x^2\sqrt{x^2+y^2}\,dx$$



$$\int_{\partial D^+} x\sqrt{x^2+y^2}\,dx = \int_{\gamma_1} + \int_{\gamma_2} + \underbrace{\cancel{\int_{\gamma_3}}}_{=0}$$

$$= \cdots = -\frac{1}{4} + \frac{1}{3}$$

(col segno meno).