Teorema della divergenza in $\mathbb{R}^2$ e in $\mathbb{R}^3$.

in $\mathbb{R}^2$

Def. di <u>divergenza</u> di un campo

$$F : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2$$

$$(x,y) \longrightarrow (F_1(x,y), F_2(x,y)) = F$$

$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$$

$$F(x,y) = (xy + \operatorname{sen} x, \quad y + e^x + y^2)$$

$$\operatorname{div} F = y + \cos x + 1 + 2y$$

Analogamente $F : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}^3$

$$(x,y,z) \to \big(F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)\big) \;!$$

$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

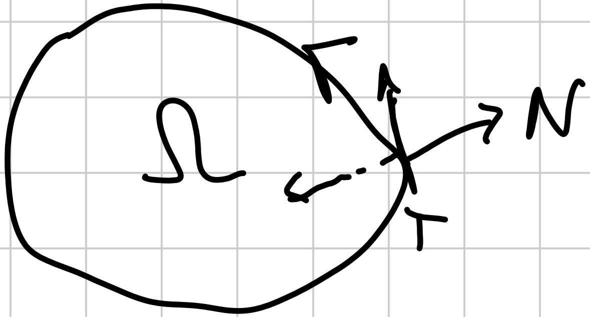
Teorema della divergenza in $\mathbb{R}^2$

$\Omega \subseteq \mathbb{R}^2$, regolare, $F : \Omega \to \mathbb{R}^2$, $F = (F_1, F_2)$

$F \in C^1(\Omega)$. Allora

$$\iint_\Omega \operatorname{div} F \, dx\,dy = \int_{\partial\Omega} F \cdot N \, ds = \int_{\partial\Omega^+} -F_2\,dx + F_1\,dy$$

dove N è la normale uscente da Ω.



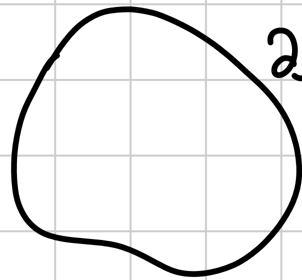
Oss. l'orientamento positivo di $\partial\Omega$ corrisponde a N normale esterna.

Dim.

$$\iint_\Omega F_{1x}\,dx\,dy + \iint_\Omega F_{2y}\,dx\,dy \overset{G.G.}{=} \int_{\partial\Omega^+} F_1\,dy - \int_{\partial\Omega^+} F_2\,dx$$

Ora dim. che

$$\int_{\partial\Omega^+} F_1\,dy - F_2\,dx \overset{?}{=} \int_{\partial\Omega} F\cdot N\,ds$$



$\partial\Omega$

γ param. de $\partial\Omega$ $\quad$ $\begin{cases} x = \gamma_1(t) \\ y = \gamma_2(t) \end{cases}$

$t \in [a;b]$

$$T = \frac{1}{\|\gamma'\|}(\gamma_1', \gamma_2')$$

$$N = \frac{1}{\|\gamma'\|}(\gamma_2', -\gamma_1')$$

$$\int_a^b \left(F_1(\gamma(t))\gamma_2' - F_2(\gamma(t))\gamma_1'\right) dt$$

$$\int_a^b \left(F_1 \frac{\gamma_2'}{\cancel{\|\gamma'\|}} + F_2 \frac{(-\gamma_1')}{\cancel{\|\gamma'\|}}\right) \cancel{\|\gamma'\|}\, dt$$

#

Oss.

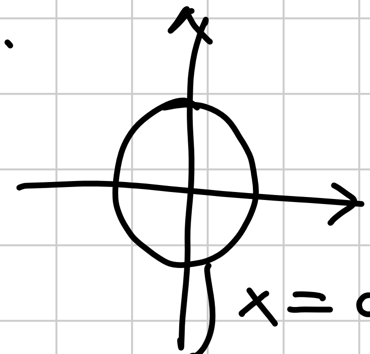
$$\int_{\partial\Omega} F \cdot N \, ds = \int_a^b \left(F_1 \, \gamma_2' + F_2 \left(-\gamma_1' \right) \right) dt$$

$(\gamma_2', -\gamma_1')$ vettore normale non versore.

rappresenta il flusso di F uscente da Ω

Esercizio $F(x,y) = \left(-2x^3y,\ -\frac{1}{2}x^4\right)$

Calcolare il flusso uscente dalla circonferenza di centro $C=(0,0)$ e raggio 1.



$$\text{Flusso} \doteq \int_{\partial\Omega} F \cdot N \, ds$$

$$\begin{cases} x = \cos\varphi = \gamma_1 \\ y = \operatorname{sen}\varphi = \gamma_2 \end{cases} \qquad \varphi \in [0, 2\pi)$$

1) con la def.

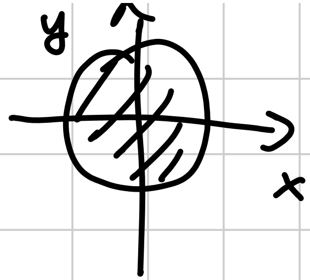
$$N = \frac{1}{\|\gamma'\|}\left(\gamma_2', -\gamma_1'\right) = \frac{1}{1}(\cos\varphi, \operatorname{sen}\varphi)$$

$$\int_{\partial\Omega} F \cdot N\, ds = \int_0^{2\pi} \left(F_1(\gamma) \cos\varphi + F_2(\gamma) \operatorname{sen}\varphi \right) d\varphi =$$

$$= \int_0^{2\pi} \left(-2 \cos^3\varphi \operatorname{sen}\varphi \cos\varphi + \left(-\frac{1}{2}\cos^4\varphi\right) \operatorname{sen}\varphi \right) d\varphi$$

$$= \int_0^{2\pi} -\frac{5}{2} \cos^4\varphi \operatorname{sen}\varphi \, d\varphi = +\frac{5}{2} \left. \frac{\cos^5\varphi}{5} \right|_0^{2\pi} = 0$$

i) Col teorema della divergenza.



$$\int_{\partial\Omega} F\cdot N\, ds = \iint_{\Omega} \operatorname{div} F \, dx\, dy$$

$$F = \left(-2x^3y, \ -\frac{1}{2}x^4\right)$$

$$\operatorname{div} F = -6x^2y$$

$$\iint_{\Omega} -6x^2y \, dx\, dy \overset{?}{=} 0$$

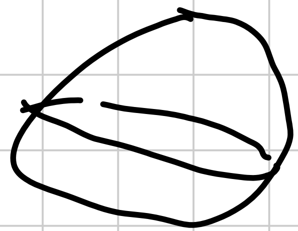
perché f dispari in y in Ω simmetrico rispetto a x

$\downarrow$ fare il conto esplicitamente.

Teorema della divergenza in $\mathbb{R}^3$

$F : \Omega \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ $F = (F_1, F_2, F_3)$, $F \in C^1(\Omega)$

$$\iiint_{\Omega} \operatorname{div} F \, dx\,dy\,dz = \int_{\partial\Omega} F \cdot N \, ds$$



dove N è la normale uscente da $\partial\Omega$.

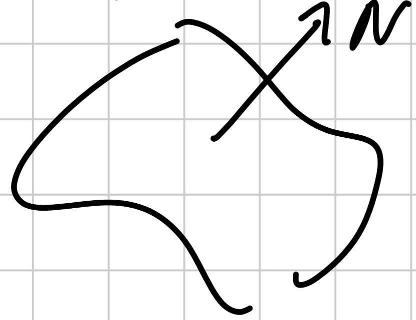
integrale di superficie

oss.

$$\int_{\partial\Omega} F \cdot N \, dS$$ rappresenta il flusso di F uscente da Ω.

$\Sigma = \partial\Omega$ $\quad \sigma(u,v) = \big(x(u,v),\ y(u,v),\ z(u,v)\big)$.

$$N = \frac{1}{\|\sigma_u \wedge \sigma_v\|}(A,\ B,\ C)$$



$(u,v) \in D$

$$\int_{\partial\Omega} F\cdot N\, dS = \iint_D F(\sigma)\cdot \underbrace{\frac{(A,\, B,\, C)}{\cancel{\|\sigma_u\wedge\sigma_v\|}}}_{N} \cdot \cancel{\|\sigma_u\wedge\sigma_v\|}\, du\, dv$$

$$= \iint_D F\cdot (A,\, B,\, C)\, du\, dv$$

non importa calcolarsi $\|\sigma_u\wedge\sigma_v\|$ perché si semplifica.

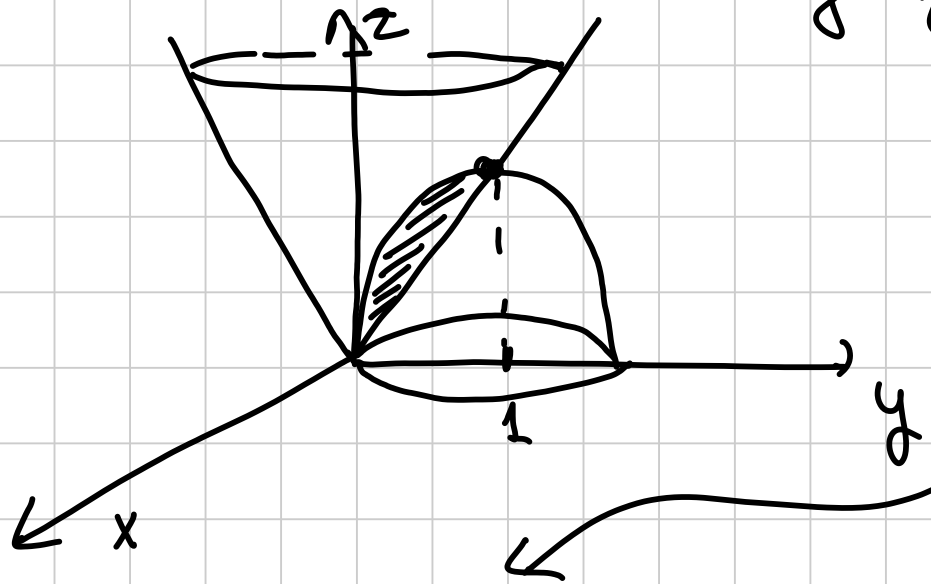
Esercizio

Calcolare il flusso del campo $F = (0, yz, x)$ uscente da

$$\Omega = \left\{ (x,y,z) \in \mathbb{R}^3 : \ x^2+y^2 \le z^2, \ x^2+y^2+z^2 \le 2y, \ z \ge 0 \right\}$$

con il teo. della divergenza.

$$x^2 + (y-1)^2 + z^2 \le 1$$



intersezione sul piano zy $(x=0)$

$$\begin{cases} z = y \\ (y-1)^2 + z^2 = 1 \end{cases} \Rightarrow y^2 + \cancel{1} - 2y + y^2 = \cancel{1}$$

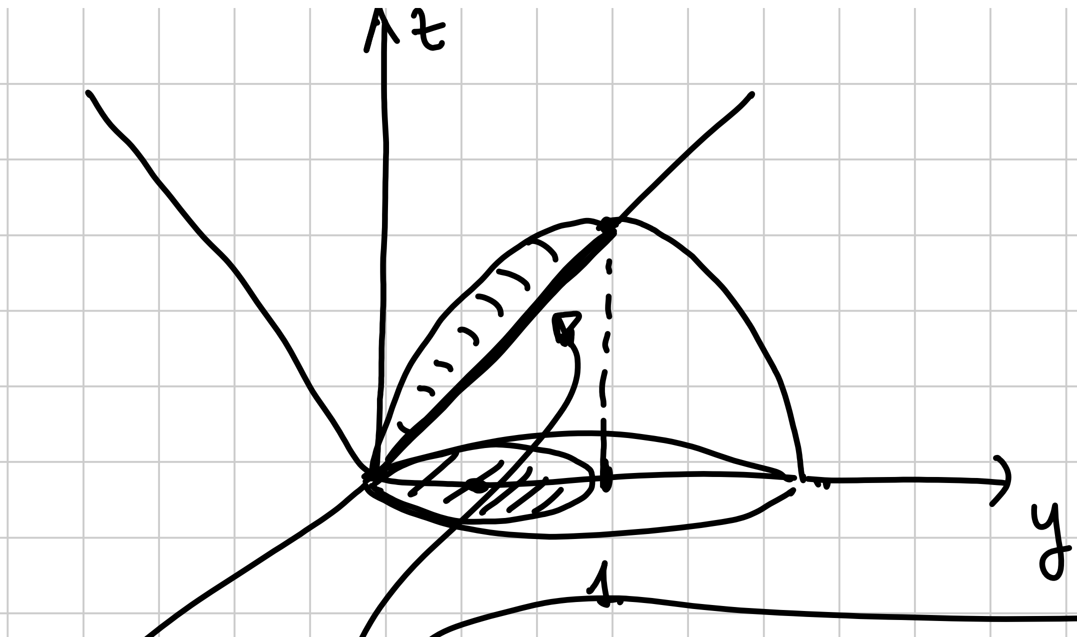
$$2y(y-1) = 0 \quad \begin{cases} y = 0 \\ y = 1 \end{cases}$$

$$\text{Flusso} = \int_{\partial\Omega} F \cdot N \, ds = \iiint_{\Omega} \operatorname{div} F \, dx\,dy\,dz =$$

$$F = (0, yz, x)$$

$$\operatorname{div} F = z$$

$$= \iiint_{\Omega} z \, dx\,dy\,dz$$



$x^2+y^2 \le z^2$ cono

$x^2+y^2+z^2 \le 2y$

$z^2 \le 2y - x^2 - y^2$

$z \le \sqrt{2y-x^2-y^2}$

$$\underbrace{\sqrt{x^2+y^2}}_{\text{cono}} \le z \le \sqrt{2y-x^2-y^2}$$

$(x,y) \in$ Circonferenza di centro $\left(0,\frac{1}{2}\right)$ e raggio $1/2$

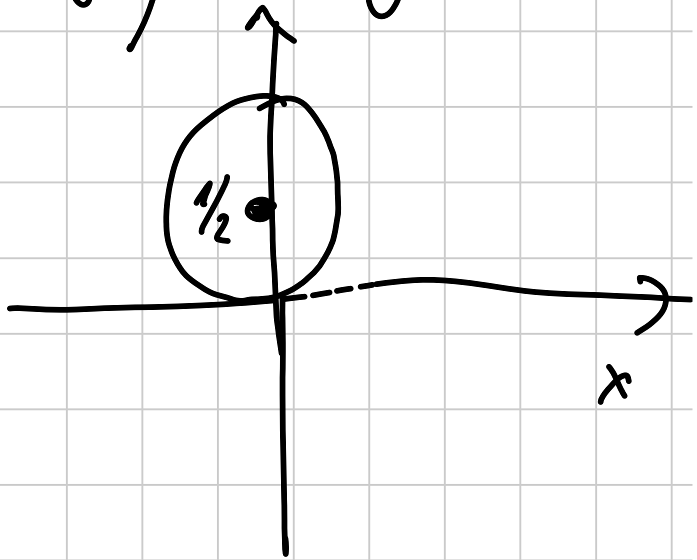
Quindi (x,y) sono tali che $x^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{4}$

$$\Omega=\left\{(x,y,z):\ \sqrt{x^2+y^2}\le z\le\sqrt{2y-x^2-y^2},\ x^2+\left(y-\frac{1}{2}\right)^2\le 1/4\right\}$$

formule di riduzione

$$\iiint_{\Omega} z\,dx\,dy\,dz=\iint_{x^2+\left(y-\frac{1}{2}\right)^2\le\frac{1}{4}}\left(\int_{\sqrt{x^2+y^2}}^{\sqrt{2y-x^2-y^2}} z\,dz\right)dx\,dy$$

$$= \iint_{x^2+\left(y-\frac{1}{2}\right)^2\le\frac{1}{4}} \frac{1}{2}\left(2y - x^2 - y^2 - x^2 - y^2\right)dx\,dy =$$



$$= \iint_{x^2+\left(y-\frac{1}{2}\right)^2\le\frac{1}{4}} \left(y - y^2 - x^2\right)dx\,dy$$

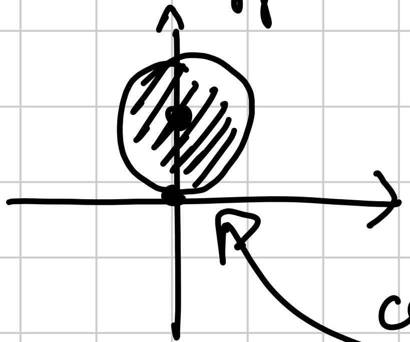
1

$$\begin{cases} x = \rho\cos\varphi \\ y - \frac{1}{2} = \rho\operatorname{sen}\varphi \end{cases}$$

$\varphi \in [0, 2\pi]$ $\rho \in [0, 1/2]$.

(2) $\begin{cases} x = \rho \cos\varphi \\ y = \rho \operatorname{sen}\varphi \end{cases}$

come si rappresenta D



con le coordinate polari centrate in (0,0)

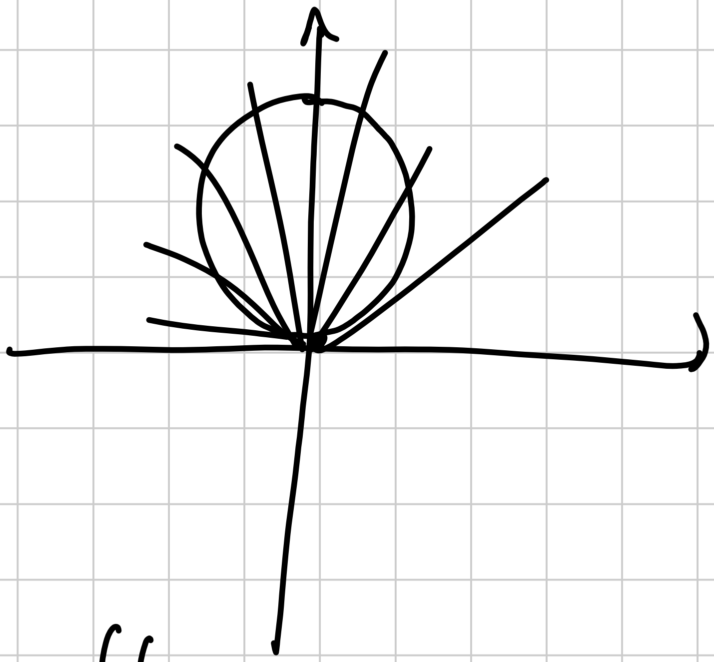
$$x^2 + \left(y - \frac{1}{2}\right)^2 \leq \frac{1}{4} \qquad x^2 + y^2 - y \leq 0$$

$$\rho^2 - \rho \operatorname{sen}\varphi \leq 0$$

$$0 \leq \rho \leq \operatorname{sen}\varphi$$

questo cerchio ha sempre $y \geq 0$

$$\Rightarrow \operatorname{sen}\varphi \geq 0 \Rightarrow \varphi \in [0, \pi]$$



$$D = \left\{ 0 \le \rho \le \operatorname{sen}\varphi ,\ \varphi \in [0,\pi] \right\}$$

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\operatorname{sen}\varphi \end{cases}$$

$$\iint_D (y - y^2 - x^2)\, dx\,dy = \int_0^{\pi}\int_0^{\operatorname{sen}\varphi} (\rho\operatorname{sen}\varphi - \rho^2)\,\rho\, d\rho\, d\varphi$$

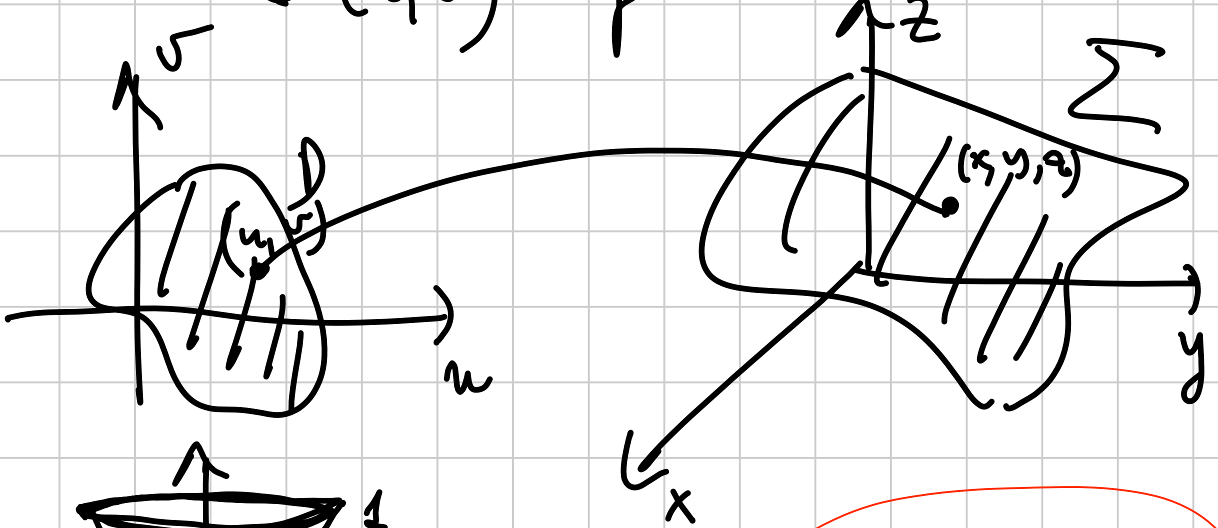
$$= \dots \frac{\pi}{32}$$

provare anche con $\begin{cases} x = \rho\cos\varphi \\ y = \frac{1}{2}\rho\operatorname{sen}\varphi \end{cases}$

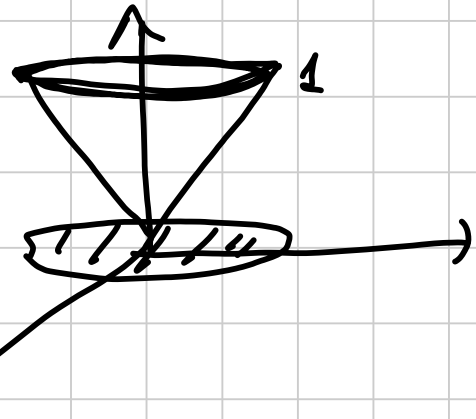
Bordo di una superficie

Σ superficie $\quad \sigma(u,v)$ parametrizzazione

$(u,v) \in D$

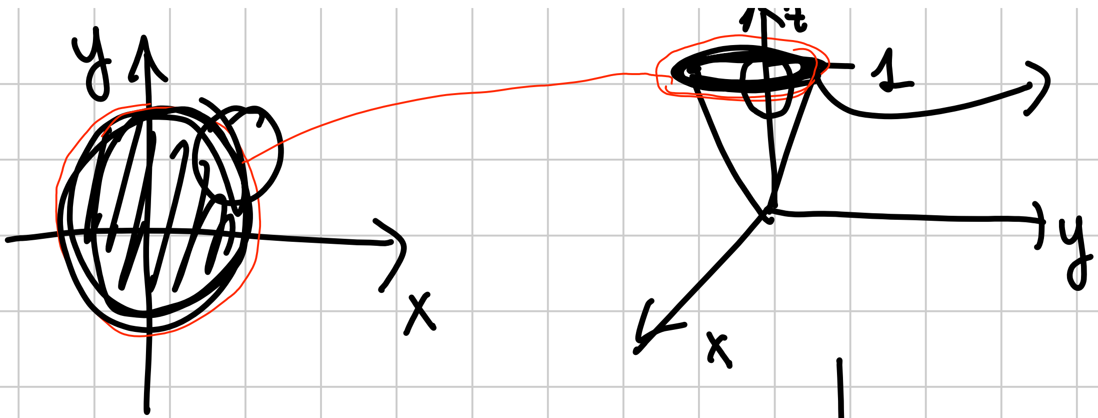


Cono troncato

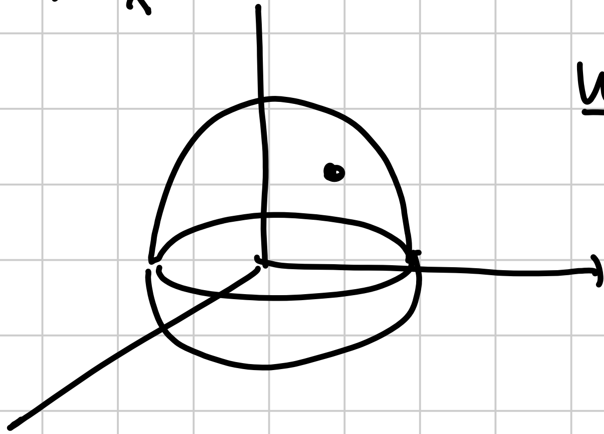


$$z = \sqrt{x^2+y^2}$$

$x^2+y^2 \leq 1$

$D =$ 

bordo

es. superficie sferica 

non ha bordo

Def. $P \in \Sigma$ è punto **INTERNO** se $\exists\, U$ di P e una parametrizzazione σ tale che U è immagine di σ (cioè $\exists\, D$ t.c. $\sigma(D) = U$)

Def. $\partial \Sigma =:$ bordo di Σ $= \{$ punti che non sono interni $\}$