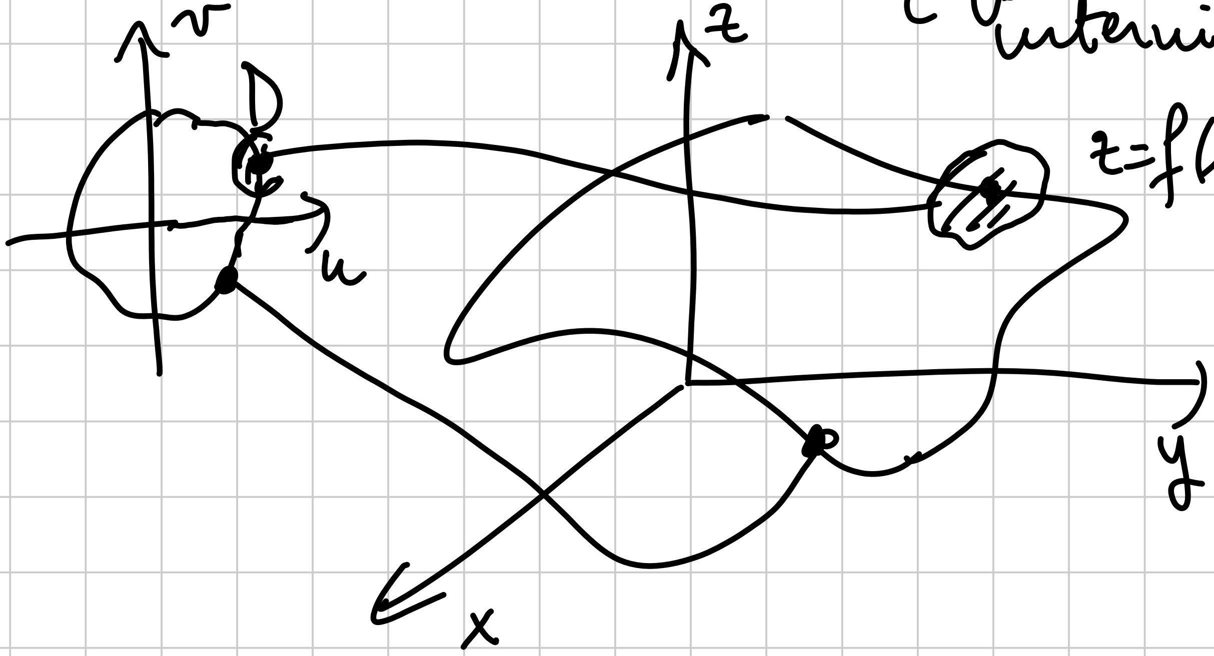
Formula di Stokes

Σ 　 6

$\partial\Sigma$ = bordo di Σ = { punti di Σ che non sono interni }



$z=f(x,y)$ es. se Σ è una superficie cartesiana

$z = f(x,y)$

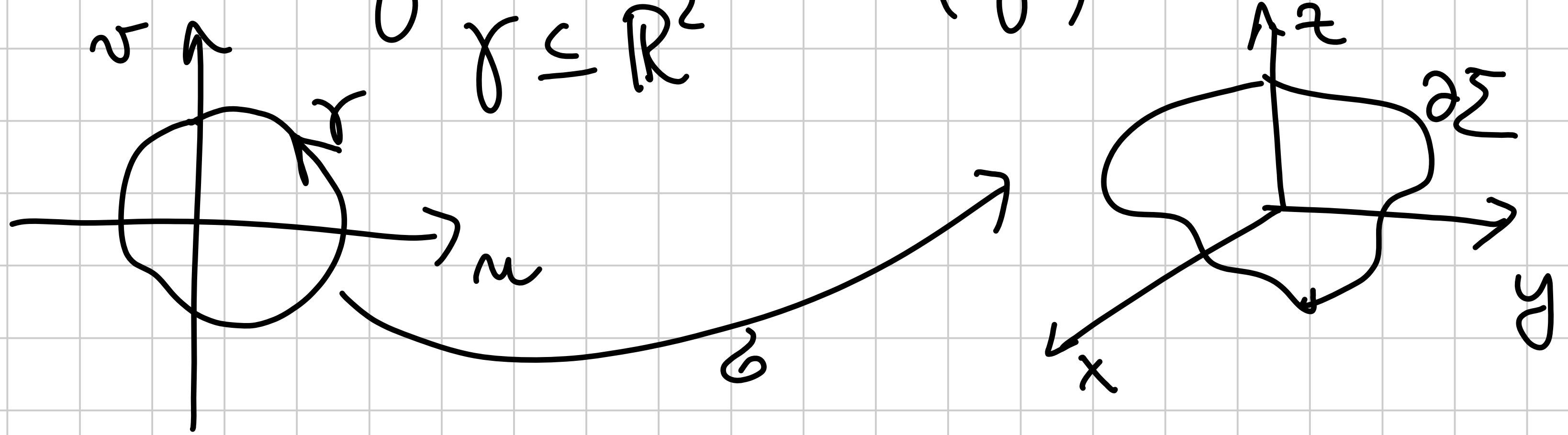
$(x,y) \in D$

se $(x,y) \in \partial D \Longrightarrow (x,y,f(x,y)) \in \partial\Sigma$.

Def. <u>Orientazione di</u> $\partial\Sigma$.

Prendo γ t.c. $\sigma(\gamma) = \partial\Sigma$

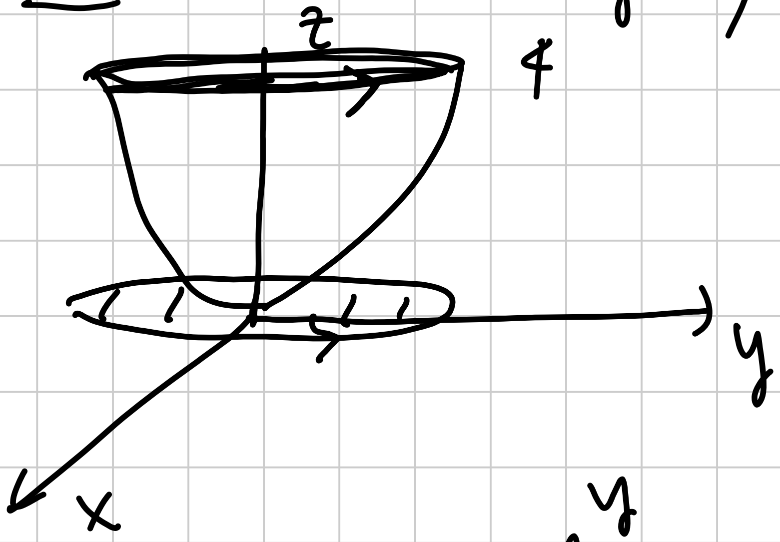
$\gamma \subseteq \mathbb{R}^2$



γ^+ quindi l'orientazione di γ mi induce un'orientazione di $\partial\Sigma$.

$\gamma^+ \Longrightarrow \partial\Sigma^+$ bordo di Σ orientato positivamente

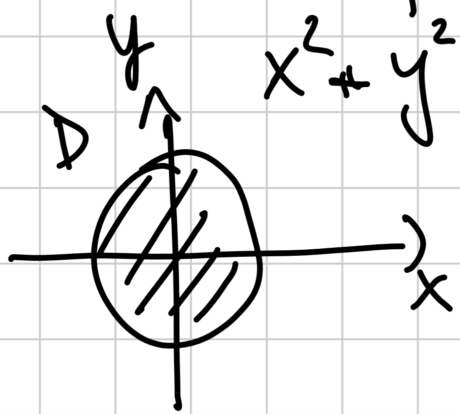
<u>es.</u> $z = x^2 + y^2$, $z \in [0, 4]$



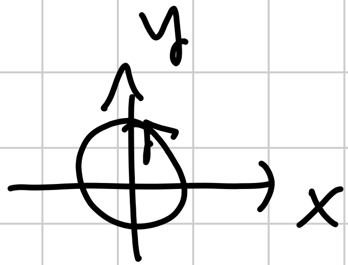
$z = f(x, y) = x^2 + y^2$

$x^2 + y^2 \leq 4$

$\partial D = \{x^2 + y^2 = 4\}$



se prendo



Rotore di un campo **vettoriale**

$$F : A \subseteq \mathbb{R}^3 \to \mathbb{R}^3$$

$$(x,y,z) \longrightarrow \left(F_1(x,y,z),\ F_2(x,y,z),\ F_3(x,y,z) \right)$$

$$\text{rot}\, F = \left(F_{3y} - F_{2z},\ F_{1z} - F_{3x},\ F_{2x} - F_{1y} \right)$$

$$\omega = F_1\,dx + F_2\,dy + F_3\,dz$$

ω chiuso in A se $\text{rot}\, F = 0$ — Campo irrotazionale es. $F = (x^2 z, y, yz)$ def su $\mathbb{R}^3$

$$\operatorname{rot} F = (z - 0, x^2 - 0, 0 - 0)$$
$$= (z, x^2, 0)$$

Formula di Stokes

oss. in $\mathbb{R}^2$, già vista (Gauss-Green scritte in forma compatta).

$\operatorname{rot} F = F_{2x} - F_{1y}$ $\quad$ $F = (F_1, F_2)$

- in $\mathbb{R}^3$

$\sigma : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ parametrizzazione di superficie Σ con bordo $\partial\Sigma$.

$F : A \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ campo vettoriale di classe C^1 t.c. $\Sigma \subseteq A$, $F = (F_1, F_2, F_3)$. Allora

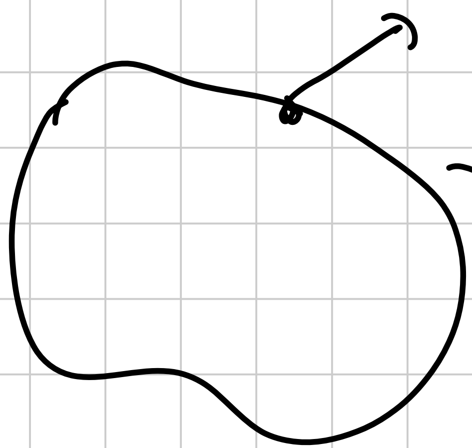
$$\int_{\Sigma} (\text{rot} F \cdot N) \, dS = \int_{\partial\Sigma^+} F_1 \, dx + F_2 \, dy + F_3 \, dz$$

dove N è la normale a Σ e $\partial\Sigma^+$ è orientato positivamente.

$$\int_{\partial\Sigma^+} F_1\,dx + F_2\,dy + F_3\,dz = \text{"circuitazione" di } F \text{ lungo } \partial\Sigma$$

lavoro

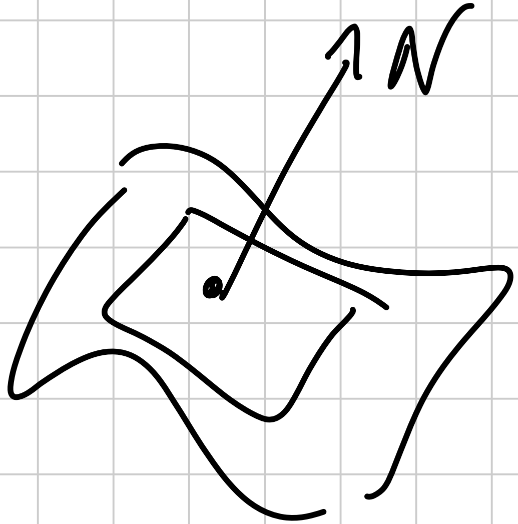
F



$\gamma \subseteq \mathbb{R}^3$

la circuitazione è il lavoro compiuto da una forza F per spostare un p.to su una curva chiusa γ.

Si considera una superficie Σ che ha γ come bordo e si calcola $\int_\Sigma (\mathrm{rot}\,F \cdot N)\, ds$.



OSS.

Σ $\quad \sigma(u, v) \quad \sigma_u \quad \sigma_v$

$$N = (A, B, C) \frac{1}{\sqrt{A^2 + B^2 + C^2}}$$

$$\int_{\Sigma} (\operatorname{rot} F \cdot N)\, dS = \iint_{D} \operatorname{rot} F(G) \cdot \frac{(A, B, C)}{\cancel{\sqrt{A^2+B^2+C^2}}} \cdot \cancel{\sqrt{A^2+B^2+C^2}}$$

$$\| G_u \wedge G_v \|$$

$$= \iint_{D} (\operatorname{rot} F) \cdot (A, B, C)\, du\, dv$$

<u>Esercizio</u> Σ di parametrizzazione

$$\sigma(u,v) = (u-v,\ u+v,\ u^2+v^2)$$

$$(u,v) \in D = \{(u,v) : u^2+v^2 \le 1\}$$

Calcolare $\displaystyle\int_{\partial\Sigma^+} x^2 z\,dx + y\,dy - yz\,dz$ con la formula di Stokes

$$F = (x^2 z,\ y,\ -yz)$$

$$\operatorname{rot} F = (-z,\ x^2,\ 0)$$

$$= \int_{\partial\Sigma} \operatorname{rot} F \cdot N\, dS$$

Calcolo N, anzi basta A, B, C

$$J = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 2u & 2v \end{pmatrix}$$

$$A = \det\begin{pmatrix} 1 & 1 \\ 2u & 2v \end{pmatrix} = 2(v-u)$$

$$B = \det\begin{pmatrix} 2u & 2v \\ 1 & -1 \end{pmatrix} = -2(u+v)$$

$$C = \det\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = 2$$

$$\int_{\partial\Sigma} \operatorname{rot} F \cdot N \, ds = \iint_D \left(-z, x^2, 0\right) \cdot \left(2(v-u), -2(u+v), 2\right) du\, dv =$$

$$z = u^2 + v^2$$
$$x = u - v$$

$$= 2 \iint_D \left[-\left(u^2+v^2\right)\left(v-u\right) - \left(u-v\right)^2 \left(u+v\right) \right] du\,dv$$

$$D = \left\{ u^2 + v^2 \leq 1 \right\}$$

$$u = \rho \cos\varphi$$
$$v = \rho \sin\varphi$$
$$\rho \in [0,1]$$
$$\varphi \in [0, 2\pi]$$

$$= 2 \int_0^{2\pi} \int_0^1 \left[-\rho^2 \left(\rho \sin\varphi - \rho\cos\varphi\right) - \left(\rho\cos\varphi - \rho\sin\varphi\right)^2 \cdot \left(\rho\cos\varphi + \rho\sin\varphi\right) \right] \rho \, \cancel{d\rho\, d\varphi =}$$

$= \text{fine} \ldots\ldots = 0$

Esplicitamente

$$\int_{\partial\Sigma^+} x^2 z\,dx + y\,dy - yz\,dz$$

$$z = u^2+v^2 = \frac{(x+y)^2}{4} + \frac{(y-x)^2}{4}$$

$$\begin{cases} x = u-v \\ y = u+v \\ z = u^2+v^2 \le 1 \end{cases} \qquad u^2+v^2 \le 1$$

$x+y = u-\cancel{v}+u+\cancel{v}$

$u = \frac{x+y}{2}$

$x-y = (u-v)-(u+v)$

$= -2v$

$v = \frac{y-x}{2}$

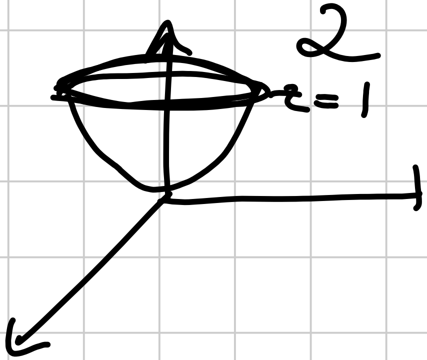
$$= \frac{2x^2 + 2y^2}{4} = \frac{x^2+y^2}{2}$$

$u^2+v^2 \leq 1$

$\Downarrow$

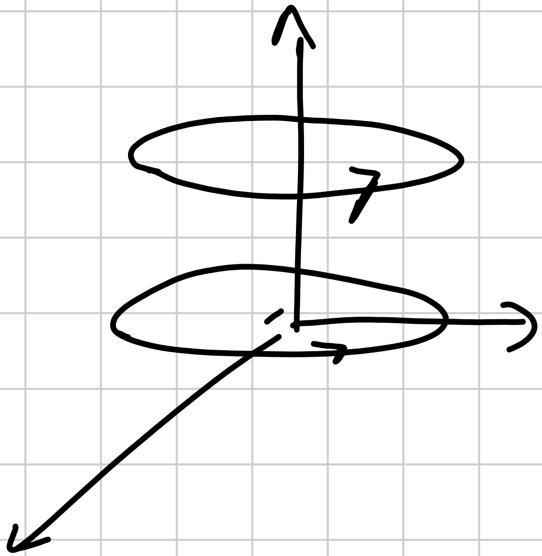
$z \leq 1$

$$\boxed{z = \frac{x^2+y^2}{2}}$$



$$\partial\Sigma = \left\{ z = 1, \quad x^2+y^2 = 2 \right\}$$

parametrizzando



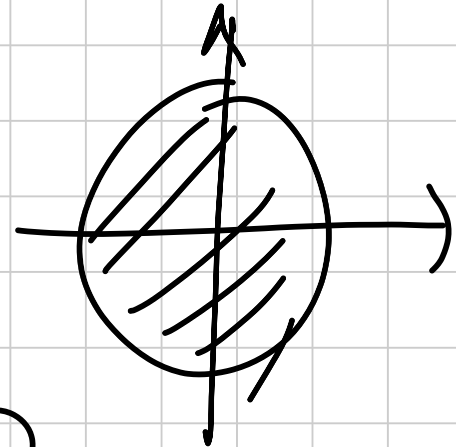
$$\begin{cases} x = \sqrt{2}\cos\varphi \\ y = \sqrt{2}\,\mathrm{sen}\,\varphi \\ z = 1 \end{cases} \quad \partial\Sigma^+ \quad \varphi \in [0, 2\pi] \Longrightarrow$$

Oss. su integrale precedente:

$$\iint_D \left[(u^2+v^2)(u-v) - (u-v)^2(u+v)\right] du\,dv$$

$$= \iint_D (u-v)\left[\cancel{u^2} + v^2 - \cancel{u^2} + v^2\right] du\, dv =$$

$$= \iint_D 2v^2 (u-v)\, du\, dv$$



$$= 0$$

Es. Dato $F = (x, 2y, -3z)$

calcolare la circuitazione lungo la linea intersezione delle superfici $\gamma : \begin{cases} z = xy \\ x^2 + y^2 = 1 \end{cases}$

$$\int_\gamma F_1\,dx + F_2\,dy + F_3\,dz = \text{circuitazione}$$

$$= \int_\Sigma \underbrace{\text{rot} F}_{=0} \cdot N\,ds = 0$$

$$\text{rot} F = (0, 0, 0)$$

<u>Es.</u> Esame 2004

Calcolare flusso di $F = (y, x, -2z)$ uscente dalla superficie

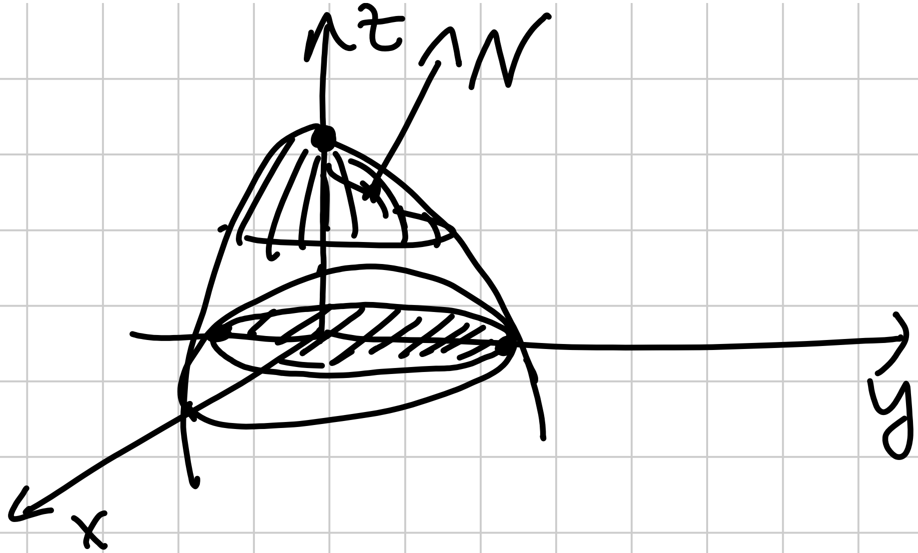
$$\Sigma = \{(x,y,z) : \ z = 1 - x^2 - y^2, \ \ 9x^2 + y^2 \le 1\}$$

orientata in modo che la normale N abbia la terza componente positiva.

Σ è sup. scritta in forma cartesiana

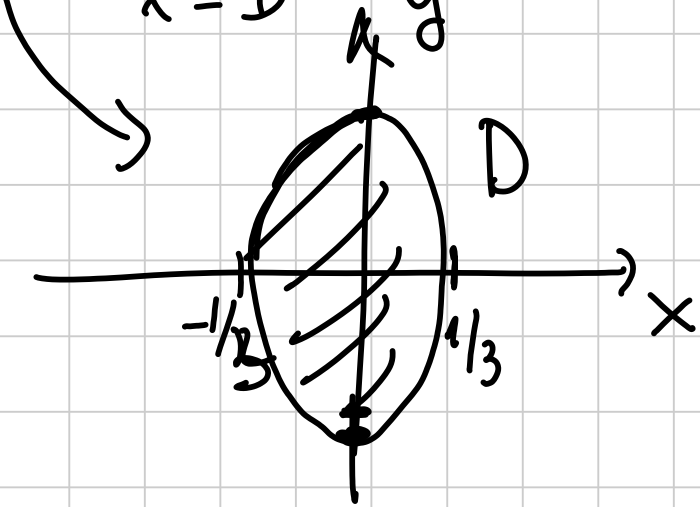
$$z = f(x,y), \qquad (x,y) \in D$$

$$z = 1 - (x^2 + y^2) \qquad D = \{ 9x^2 + y^2 \le 1 \}$$



$z = 1 - y^2$

$x = 0$



$$\underline{\text{Flusso}} = \int_{\Sigma} \bar{F} \cdot N \, dS$$

con la definizione

$z = f(x, y)$

$$N = \frac{1}{\sqrt{1+f_x^2+f_y^2}}\left(-f_x, -f_y, \underset{>0}{1}\right)$$

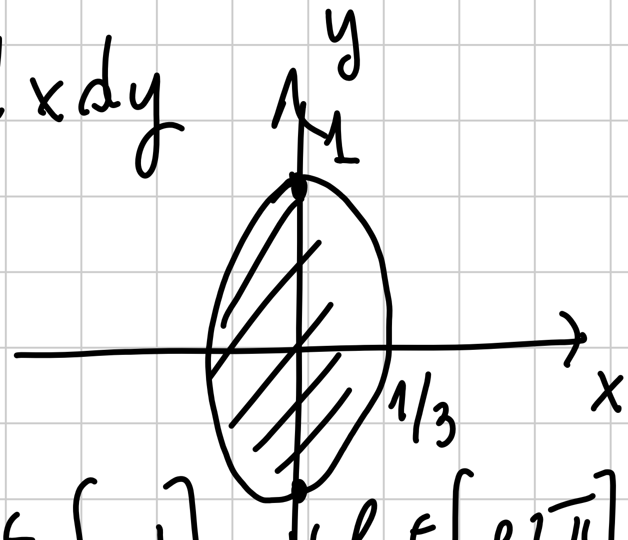
$$\int_\Sigma F \cdot N \, ds = \iint_D (y, x, -2z) \cdot (-f_x, -f_y, 1) \, dx\,dy$$

$$z = 1 - x^2 - y^2 = f(x,y)$$

$$\left| \begin{array}{l} f_x = -2x \\ f_y = -2y \end{array} \right.$$

$$= \iint_D 2xy + x2y - 2(1-x^2-y^2) \, dx\,dy$$

$$= \iint_D 4xy - 2(1-x^2-y^2)\,dx\,dy$$



$$\begin{cases} x = \frac{1}{3}\rho\cos\varphi \\ y = \rho\sin\varphi \end{cases} \qquad \rho \in [0,1], \; \varphi \in [0, 2\pi]$$

la matrice Jacobiana ha det $\frac{\rho}{3}$

$$= \int_0^1 \int_0^{2\pi} \left(- - - - \right) \frac{\rho}{3}\, d\rho\, d\varphi =$$

$$= \dots = -\frac{26}{54}\pi .$$

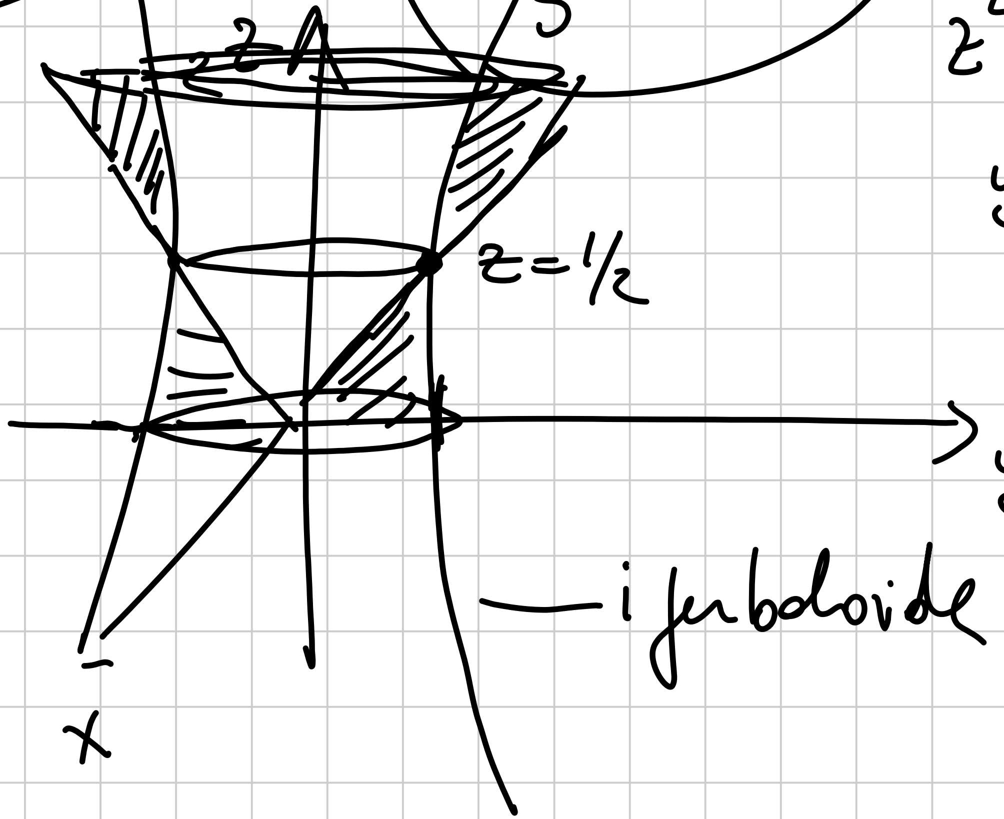
Es. Calcolare volume della regione compresa tra le superficie

$$x^2+y^2=1+z^2 \quad \text{e} \quad \frac{x^2}{5}+y^2=z^2$$

$z \in [0, 2]$.

$x=0$

$y^2-z^2=1$



$x=0$

$z^2=\frac{y^2}{5}$

$y>0$

$z=\frac{y}{\sqrt{5}}$

punto di intersezione $\quad x=0 \quad \begin{cases} z^2 = \dfrac{y^2}{5} \\ y^2 - z^2 = 1 \end{cases}$

$5z^2 - z^2 = 1$

$4z^2 = 1 \qquad z = 1/2$

$$V(S) = \int_0^{1/2} \left(\iint_{\cdots} dx\,dy \right) + \int_{1/2}^{2} \left(\iint dx\,dy \right)$$