Esercizi

$$ay'' + by' + cy = g(x)$$

polinomio

e^{rx}

$e^{rx}\cos\omega x$

$e^{rx}\,\mathrm{sen}\,\omega x$

$$ay'' + by' + cy = g_1(x) + g_2(x)$$

perché l'eq diff. è **LINEARE**

$$y = y_0 + \underbrace{\tilde{y}_1}_{} + \underbrace{\tilde{y}_2}_{}$$

soluz. particolare

$$\begin{cases} (\quad\quad) = g_1 \\ (\quad\quad) = g_2 \end{cases}$$

Es. $y'' - 6y' + 10y = (t+1) + \cos t$

$y = y(t)$

$y'' - 6y' + 10y = 0$

$\lambda^2 - 6\lambda + 10 = 0$ $\qquad \lambda = 3 \pm \sqrt{9-10}$
$= 3 \pm i$

$y_0(t) = e^{3t}(\alpha \cos t + \beta \sin t)$

$\tilde{y}_1$ soluzione particolare di

$y'' - 6y' + 10y = (t+1)$ $\quad$ (A)

$\tilde{y}_1(t) = At + B$ ...... si det. A e B imponendo che $\tilde{y}_1$ verifichi ⊛

$A = 1/10$ $\quad B = 8/50$ $\qquad \tilde{y}_1 = \frac{t}{10} + \frac{8}{50}$

$\tilde{y}_2$ soluzione particolare

$$y'' - 6y' + 10y = \cos t \qquad ⊙$$

$$\tilde{y}_2 = A\cos t + B\sin t \qquad 0 + 1i$$

A e B ...... $A = 1/3$ $\quad B = 0$

$$\tilde{y}_2 = \frac{1}{3}\cos t$$

$$y(t) = e^{3t}\left(\alpha \cos t + \beta \sin t\right) + \frac{t}{10} + \frac{8}{50} + \frac{1}{3}\cos t$$

<u>PC.</u>

$$\begin{cases} y'' - 6y' + 10y = (t+1) + \cos t \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$$

Es. $y'' - 2y' + 2y = e^t \operatorname{sen} t$

$y'' - 2y + 2y = 0$

$\lambda^2 - 2\lambda + 2 = 0 \qquad \lambda = 1 \pm \sqrt{1-2} =$

$= 1 \pm i$

$y_0(x) = e^t (\alpha \cos t + \beta \operatorname{sen} t)$.

$\tilde{y}(x) = t e^t (A \cos t + B \operatorname{sen} t)$

A e B .... da determinare

Es. Det. le soluzioni dell'eq. diff.

$$y'' - y' - Ky = e^{3x}$$

al variare di $K \in \mathbb{R}$

y_0 $\quad y'' - y' - Ky = 0$

$$\lambda^2 - \lambda - K = 0$$

$$\rightarrow \lambda = \frac{1 \pm \sqrt{1+4K}}{2} = \lambda_1, \lambda_2$$

1) $1 + 4K > 0 \qquad K > -1/4 \qquad \lambda_1, \lambda_2 \in \mathbb{R}$

$$y_0(x) = \alpha e^{\lambda_1 x} + \beta e^{\lambda_2 x}$$

2) $1 + 4K = 0$ $\qquad \lambda = \frac{1}{2}$

$$y_0(x) = \alpha e^{\frac{1}{2}x} + \beta x e^{\frac{1}{2}x}$$

3) $1 + 4K < 0$ $\qquad \lambda = \underbrace{\frac{1}{2}}_{\gamma} \pm \overbrace{\frac{i}{2}\sqrt{-1-4K}}^{\omega}$

$$y_0(x) = e^{\frac{1}{2}x}\left(\alpha \cos \frac{\sqrt{-1-4K}}{2} x + \beta \operatorname{sen} \frac{\sqrt{-1-4K}}{2}\right)$$

Soluzione particolare

$$\tilde{y}(x) = A e^{3x}$$

$$y'' - y' - Ky = e^{3x}$$

se 3 non è radice dell'eq. caratteristica

$$\tilde{y}(x) = A x e^{3x}$$

Quando $\lambda = 3$ è soluzione dell'eq. caratteristica?

$$\lambda = \frac{1 \pm \sqrt{1+4K}}{2} \stackrel{?}{=} 3$$

$$1 \pm \sqrt{1+4k} = 6$$

$$\sqrt{1+4k} = 5 \qquad 1+4k = 25$$

$$4k = 24 \qquad k=6$$

$$\underbrace{1 - \sqrt{1+4k}}_{<1} = 6 \quad \text{no!}$$

se $k=6$ $\quad \lambda=3$ è soluzione dell'eq. caratteristica

Quindi se $K \neq 6$

$$\tilde{y}(x) = A e^{3x}$$ e ritrovo A imponendo che $\tilde{y}$ sia soluzione

se $K = 6$

$$\tilde{y}(x) = A x e^{3x}$$ " " "

" "

$$y(x) = \underbrace{y_0(x)}_{\substack{K > -1/4 \\ K = -1/4 \\ K < -1/4}} + \underbrace{\tilde{y}(x)}_{\substack{K = 6 \\ K \neq 6}}$$

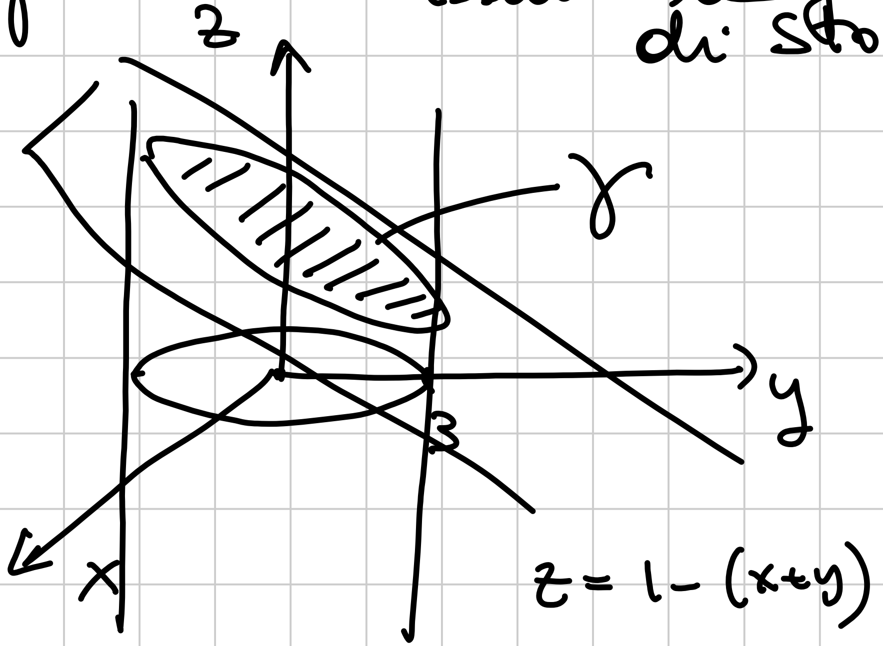
Es. Calcolare

$$\int_\gamma x^2 z\,dx + x y^2\,dy + z^2\,dz$$

usando la formula di Stokes

dove γ è definita da

$$\begin{cases} x+y+z=1 \\ x^2+y^2=9 \end{cases}$$



1) Senza "Stokes".

si parametrizza γ

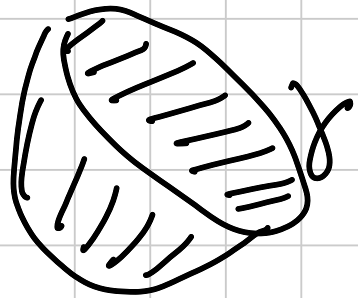
$$\begin{cases} x = 3\cos\theta \\ y = 3\operatorname{sen}\theta \\ z = 1 - (3\cos\theta + 3\operatorname{sen}\theta) \end{cases}$$

$\theta \in [0, 2\pi)$.

2) Con Stokes

$$\int_\gamma F_1\,dx + F_2\,dy + F_3\,dz =$$

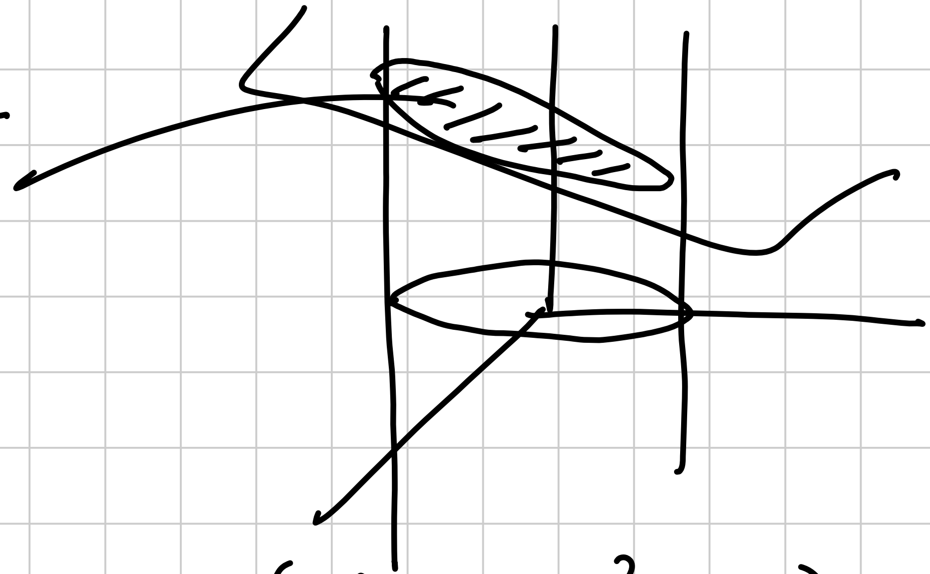
$$= \int_\Sigma \mathrm{rot}\,F \cdot N\,dS$$



dove Σ è una qualsiasi superficie che ha γ come bordo

nel mio caso prendo come Σ la superficie nel piano $x+y+z=1$ delimitata da γ

$$\begin{cases} z = 1-(x+y) \\ z = f(x,y) \\ x^2+y^2 \le 9 \end{cases} \to \Sigma$$



$$\int_\Sigma \operatorname{rot} F \cdot N \, dS$$

$$F = (x^2 z,\ xy^2,\ z^2)$$

$$\operatorname{rot} F = (0,\ x^2,\ y^2)$$

N / basta vettore normale $\quad z = f(x,y)$

$\hookrightarrow (-f_x,\ -f_y,\ 1)$

$$z = 1-(x+y) = f(x,y)$$

$(1, 1, 1)$ vettore normale alla superficie

$$\text{rot}\, F \cdot N = (0, x^2, y^2) \cdot (1, 1, 1) = x^2 + y^2$$

$$\int_{\Sigma} \text{rot}\, F \cdot N \, dS = \iint_{x^2+y^2 \le 9} (x^2+y^2)\, dx\, dy =$$

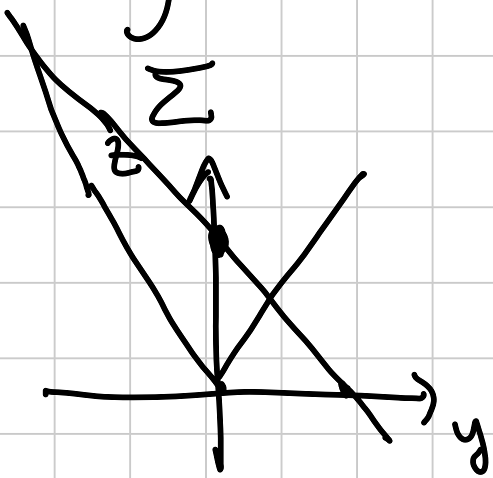
.... ~~polare~~.

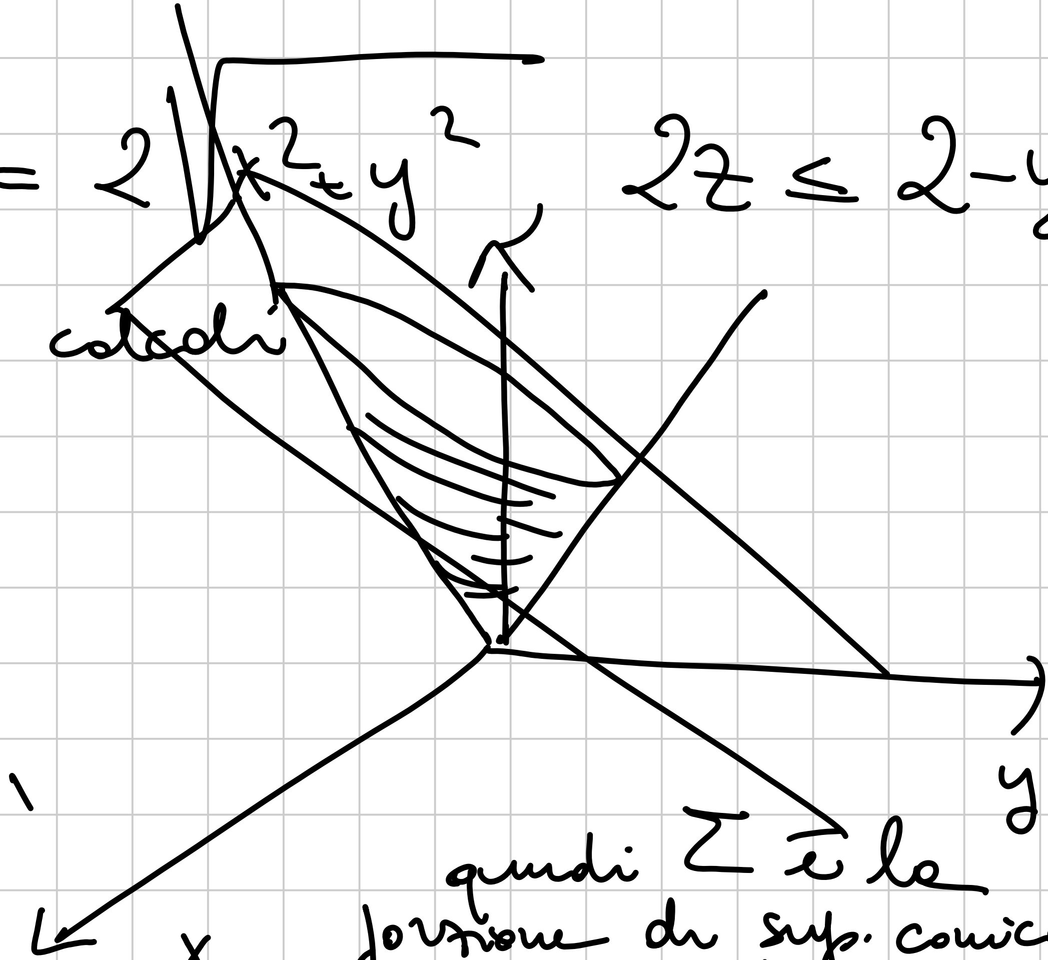
Es.

$$\Sigma = \left\{ (x,y,z) : \quad z = 2\sqrt{x^2+y^2}, \quad 2z \leq 2-y \right\}$$

si disegni Σ e si calcoli

$$\int_{\Sigma} |x| \, dS$$





quindi Σ è la porzione di sup. conica che sta sotto il piano.

$z = 2\sqrt{x^2+y^2}$ con $2z \le 2-y$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\,\mathrm{sen}\,\theta \\ z = 2\rho \end{cases}$$

$\theta \in [0, 2\pi)$

$$2(2\rho) \le 2 - \rho\,\mathrm{sen}\,\theta$$

$$4\rho + \rho\,\mathrm{sen}\,\theta \le 2$$

$$\rho\,(4 + \mathrm{sen}\,\theta) \le 2$$

$$0 \le \rho \le \frac{2}{4+\mathrm{sen}\,\theta}, \quad \theta \in [0, 2\pi)$$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho \operatorname{sen}\theta \\ z = 2\rho \end{cases} \qquad \theta \in [0, 2\pi)$$

$$0 \le \rho \le \frac{2}{4+\operatorname{sen}\theta}$$

$$\int_{\Sigma} |x|\, dS = \qquad \Big\lfloor \|\sigma_u \wedge \sigma_v\|$$

$$= \int_0^{2\pi}\int_0^{\frac{2}{4+\operatorname{sen}\theta}} \rho\,|\cos\theta| \cdot \|\sigma_u \wedge \sigma_v\|\, d\rho\, d\theta$$

$$=$$

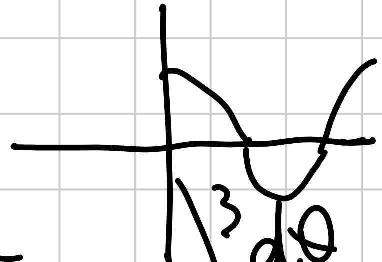
$$\begin{cases} x = \rho\cos\theta \\ y = \rho\,\mathrm{sen}\,\theta \\ z = 2\rho \end{cases}
\qquad
\begin{array}{cc}
\begin{pmatrix} \cos\theta & -\rho\,\mathrm{sen}\,\theta \\ \mathrm{sen}\,\theta & \rho\cos\theta \\ 2 & 0 \end{pmatrix} \\
\begin{array}{cc} G_u & G_v \end{array}
\end{array}$$

$$\| G_u \wedge G_v \| = \sqrt{A^2 + B^2 + C^2} = \sqrt{\rho^2 + 4\rho^2\cos^2\theta + 4\rho^2\,\mathrm{sen}^2\theta} = \rho\sqrt{5}$$

Quindi l'integrale diventa

$$= \sqrt{5} \int_0^{2\pi} \int_0^{\frac{2}{4+\sin\theta}} \rho \, |\cos\theta| \, \rho \, d\rho \, d\theta =$$

$$= \sqrt{5} \int_0^{2\pi} |\cos\theta| \, \frac{1}{3} \left(\frac{2}{4+\sin\theta} \right)^3 d\theta$$



$$= \frac{\sqrt{5}}{3} \int_0^{\pi/2} \cos\theta \left(\frac{2}{4+\sin\theta} \right)^3 d\theta - \frac{\sqrt{5}}{3} \int_{\pi/2}^{3/2\pi} \cos\theta \left(\frac{2}{4+\sin\theta} \right)^3 d\theta +$$

$$+ \int_{3/2\pi}^{2\pi} \cos\theta \left(\frac{2}{4+\sin\theta}\right)^3 d\theta$$

$$4+\sin\theta = t$$

$$dt = \cos\theta\, d\theta$$

$$\int \cos\theta \left(\frac{2}{4+\sin\theta}\right)^3 d\theta$$

$$= 8\int t^{-3}\, dt$$

$$= \text{.......}$$

Es. Data l'equazione

$$t\, e^{\frac{x}{y}} + t^2 \log(x+y) - \cos x - (t-1)y^2 = 0$$

a) verificare che $\forall t \in \mathbb{R}$ è definita implicitamente una funzione $y = g(x)$ in un intorno di $(0,1)$

b) det. per quali valori del parametro t la funzione $y = g(x)$ ha un p.to critico in $x=0$

$$f_t(x,y) = 0$$

$$\begin{cases} f_t(0,1) = 0 \\ \end{cases}$$

se volgeho le due condizioni vale Dini $\Longleftarrow \left\{ \frac{\partial f_t}{\partial y}(0,1) \neq 0 \right.$

$\exists \quad y = g(x)$

$$f_t(x,y) = t e^{\frac{x}{y}} + t^2 \log(x+y) - \cos x - (t-1)y^2$$

$$f_t(0,1) = t + t^2 \cancel{\log 1} - 1 - (t-1) = 0 \quad \forall t \in \mathbb{R}$$

$$\frac{\partial f}{\partial y} = t e^{\frac{x}{y}} \cdot \left(-\frac{x}{y^2}\right) + \frac{t^2}{x+y} - 2y(t-1)$$

$$\frac{\partial f}{\partial y}(0,1) = t \cdot 0 + \frac{t^2}{1} - 2(t-1) =$$

$$= t^2 - 2t + 2 = 0 \text{ ? no!}$$

$$\Delta = 1 - 2 < 0$$

Quindi $\underline{\underline{\forall t \in \mathbb{R}}}$ $\quad > 0$

$\neq 0$

$$\frac{\partial f}{\partial y}(0,1) \neq 0$$

$\Rightarrow \forall t \in \mathbb{R} \ \exists \ y = g(x)$ in

un intorno $(0,1)$

b) per quali t $g(x)$ ha un punto critico in $x=0$

$$g'(0) = 0 \qquad g'(0) = \frac{-f_x(0,1)}{f_y(0,1)} = 0$$

$$f_x(0,1) = t + t^2$$

$$f_x(x,y) = \frac{t e^{x/y}}{y} + \frac{t^2}{x+y} + \operatorname{sen} x$$

$$g'(0) = 0 \iff f_x(0,1) = 0 \iff t + t^2 = 0$$

$t(1+t)=0$

$t=0$

$t=-1$