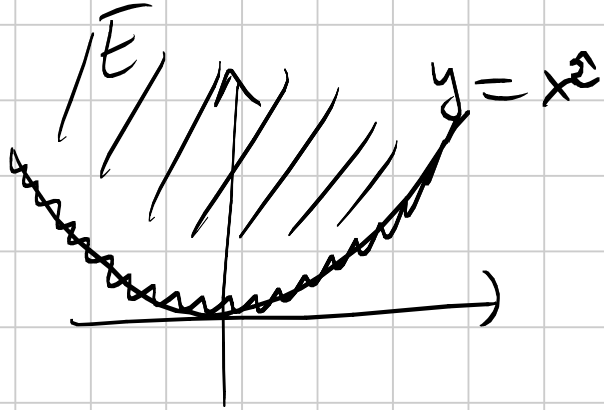
Canale 1, 2 Ottobre

Topologia (continua...)



$$E = \{ (x,y) : y > x^2 \}$$

$$\bar{E} = \{ (x,y) : y \geq x^2 \}$$

$$f(x,y) = y - x^2 > 0$$

$$f(x,y) = y - x^2 \geq 0$$

$f : \mathbb{R}^n \to \mathbb{R}$ continua

$$E = \{ \underline{x} \in \mathbb{R}^n : f(\underline{x}) > 0 \}$$
$$\{ \quad : f(\underline{x}) < 0 \}$$
$$\{ \quad : f(\underline{x}) \neq 0 \}$$

sono aperti

$$E = \{ \underline{x} \in \mathbb{R}^n : f(\underline{x}) \geq 0 \}$$
$$\{ \quad f(\underline{x}) \leq 0 \}$$
$$\{ \quad : f(\underline{x}) = 0 \}$$

sono chiusi

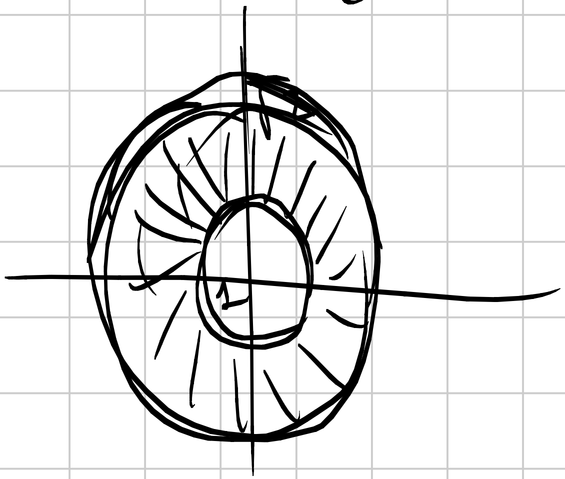
oss. Intersezione o Unione di aperti è aperto

Intersezione o Unione di chiusi è chiuso

es. $E = \{ (x,y) : \quad 1 \leq x^2+y^2 \leq 2 \}$

$x^2+y^2 \leq 2$

$x^2+y^2 \geq 1$



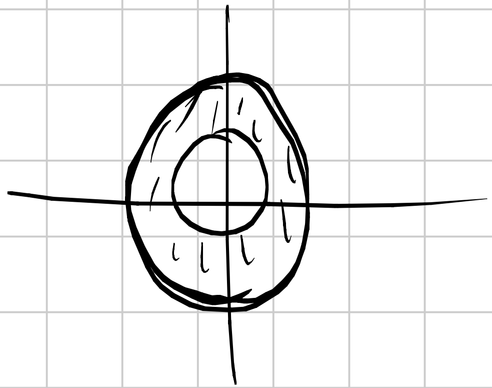
chiuso

es. $E = \{ (x,y) : 1 < x^2+y^2 < 2 \}$

aperto

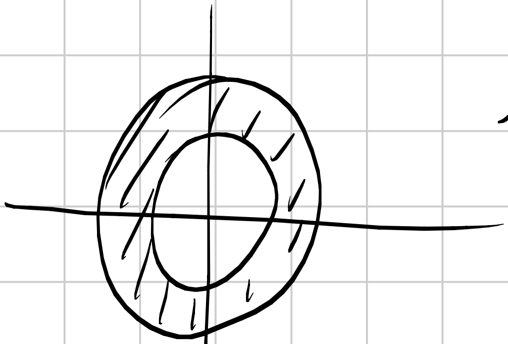
es. $E = \{ (x,y) : 1 < x^2+y^2 \leq 2 \}$

nè aperto
nè chiuso

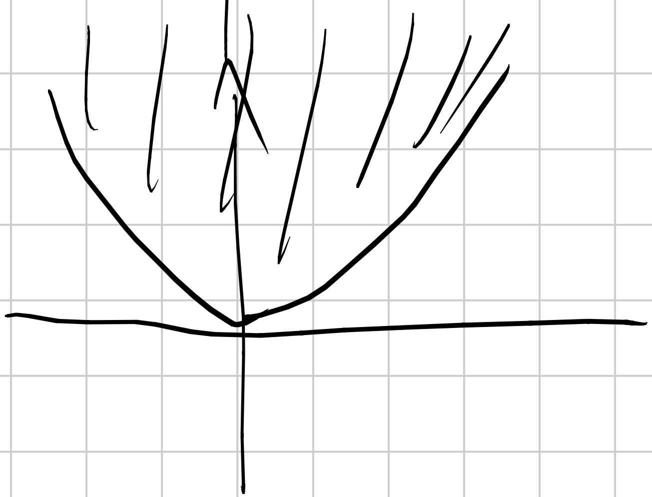


Def. $E \subseteq \mathbb{R}^n$ è limitato se esiste un intorno sferico che lo contiene

Es.



limitato



$\{ y > x^2 \}$

illimitato

Teorema di Weierstrass

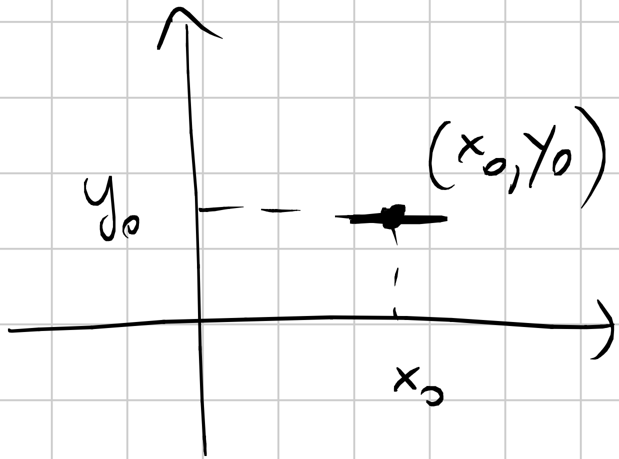
$E \subseteq \mathbb{R}^n$ chiuso e limitato

$f : E \subseteq \mathbb{R}^n \to \mathbb{R}$ continua

Allora f ha massimo e minimo in E.

Derivate parziali di f

$f(x, y)$



$$\lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h} =: \frac{\partial f}{\partial x}(x_0, y_0)$$
$$= f_x(x_0, y_0)$$

$$\lim_{h \to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h} =: \frac{\partial f}{\partial y}(x_0, y_0)$$
$$= f_y(x_0, y_0)$$

Def: f è <u>derivabile</u> in (x_0, y_0) se $\exists\ f_x(x_0, y_0),\ f_y(x_0, y_0)$.

$\underline{n = 1}$ $f(x)$ derivabile $\Rightarrow$ f continua e con la retta tg. al grafico

$n > 1$ $f(x)$ derivabile $\not\Rightarrow$ f continua

$$\left(f_x(x_0, y_0),\ f_y(x_0, y_0) \right) = \nabla f(x_0, y_0)$$

es. $f(x,y) = x^5 y^2 + 8$

$$f_x(x,y) = 5x^4y^2$$
$$f_y(x,y) = 2x^5y$$

$$\nabla f(1,2) = \left(f_x(1,2),\ f_y(1,2) \right) =$$
$$= (20,\ 4)$$

$$f : \mathbb{R}^n \to \mathbb{R} \qquad \underline{x} = (x_1, x_2, \ldots\ldots x_n)$$

$$f_{x_i}(x_1, \dots x_n) =$$

$$= \lim_{h \to 0} \frac{f(x_1, x_2, \dots x_i + h, \dots x_n) - f(x_1, x_2, \dots x_i, \dots x_n)}{h}$$

$$\nabla f(x_1, \dots x_n) = \left(f_{x_1}(x_1 \dots x_n), f_{x_2}(x_1 \dots x_n), \dots \dots f_{x_n}(x_1 \dots x_n) \right)$$

<u>es.</u> $f(x_1, x_2, x_3, x_4) = \operatorname{sen}(x_1 x_3) + e^{x_2} x_1$

$$f_{x_1} = \cos(x_1 x_3) \cdot x_3 + e^{x_2}$$

$$f_{x_2} = x_1 e^{x_2}$$

$$f_{x_3} = \cos(x_1 x_3) \cdot x_1$$

$$f_{x_4} = 0$$

$$Df(1,0,1,2) = \big(f_{x_1}(1,0,1,2), f_{x_2}(1,0,1,2),$$

$$f_{x_3}(1,0,1,2), f_{x_4}(1,0,1,2) \big) =$$

$$= (\cos 1 + 1, 1, \cos 1, 0)$$

Derivate successive

$f(x,y)$

$f_x(x,y)$ $\qquad$ $f_{xx}(x,y)$

$\qquad\qquad\quad$ $f_{xy}(x,y)$

$f_y(x,y)$ $\qquad$ $f_{yx}(x,y)$

$\qquad\qquad\quad$ $f_{yy}(x,y)$

derivate parziali di ordine 2

$$D^2 f(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$ matrice Hessiana di f

es. $f(x,y) = x y^3$

$f_x = y^3$ — $f_{xx} = 0$, $f_{xy} = 3y^2$

$f_y = 3xy^2$ — $f_{yx} = 3y^2$, $f_{yy} = 6xy$

$$D^2 f = \begin{pmatrix} 0 & 3y^2 \\ 3y^2 & 6xy \end{pmatrix}$$

$$f_{xy} = f_{yx}$$

Teo. di Schwarz $f: A \subseteq \mathbb{R}^2 \to \mathbb{R}$

$f(x,y)$ derivabile due volte in A e

f_{xy} e f_{yx} sono continue in A

Allora $f_{xy}(x,y) = f_{yx}(x,y) \quad \forall (x,y) \in A.$

ex. in $\mathbb{R}^3$

$$f(x,y,z) = x y^3 z^2$$

$$f_x(x,y,z) = y^3 z^2$$

$$f_y = 3 x y^2 z^2$$

$$f_z = 2 x y^3 z$$

$$f_{xx} = 0$$
$$f_{xy} = 3y^2z^2$$
$$f_{xz} = 2y^3z$$

$$f_{yx} = 3y^2z^2$$
$$f_{yy} = 6xyz^2$$
$$f_{yz} = 6xy^2z$$

$$f_{zx} = 2y^3z$$
$$f_{zz} = 2xy^3$$
$$f_{zy} = 6xy^2z$$

$$D^2 f(x,y,z) = \begin{pmatrix} 0 & 3y^2z^2 & 2y^3z \\ 3y^2z^2 & 6xyz^2 & 6xy^2z \\ 2y^3z & 6xy^2z & 2xy^3 \end{pmatrix}$$

$$D^2 f(0,1,1) = \begin{pmatrix} 0 & 3 & 2 \\ 3 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

Notazione f definita in $A \subseteq \mathbb{R}^n$

$f \in C^1(A)$ se f è derivabile e le derivate parziali sono continue in A

$f \in C^2(A)$ se f è derivabile due volte e tutte le derivate seconde sono continue in A

Funzioni differenziabili

$f : \mathbb{R}^n \to \mathbb{R}$ $\quad \underline{x}_0$ del dominio

f è differenziabile in $\underline{x}_0$ se

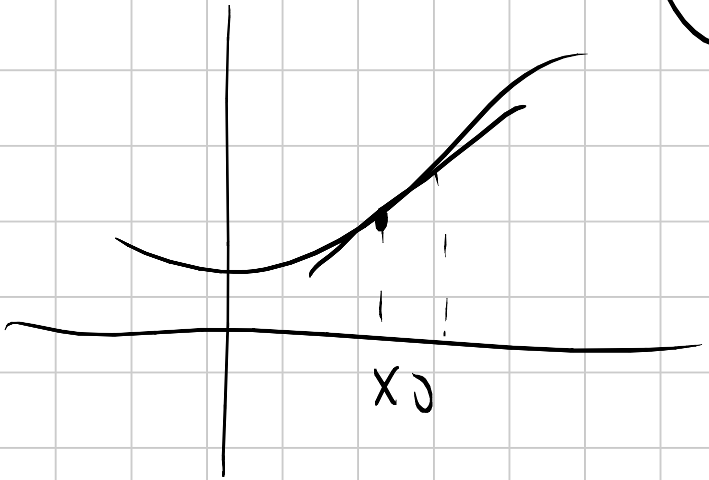
1) f è derivabile in $\underline{x}_0$ ($\exists$ cioè le derivate parziali)

2) $$\lim_{h \to 0} \frac{f(\underline{x}_0 + \underline{h}) - f(\underline{x}_0) - \nabla f(\underline{x}_0) \cdot \underline{h}}{h} = 0$$

$$f(\underline{x}_0 + \underline{h}) = f(\underline{x}_0) + \nabla f(\underline{x}_0) \cdot \underline{h} + o(|\underline{h}|) \quad \underline{h} \to 0$$

$n=1$

$$f(x_0+h) = \underbrace{f(x_0) + f'(x_0)\cdot h} + \underset{h\to 0}{o(h)}$$



In due variabili $\underline{h} = (\overset{h_1}{x-x_0},\ \overset{h_2}{y-y_0})$ —

$$f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)\cdot(x-x_0) + f_y(x_0,y_0)\cdot(y-y_0)$$
$$+\ o\left(\sqrt{(x-x_0)^2+(y-y_0)^2}\right).$$

$x \to x_0$

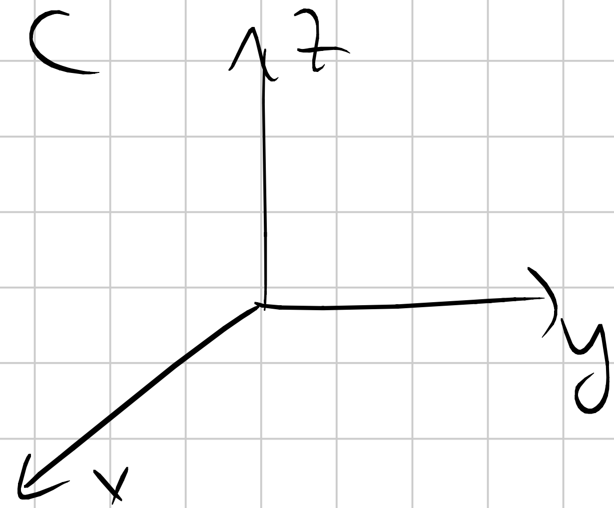
$y \to y_0$

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

eq. piano tangente al grafico di f nel p.to $(x_0, y_0, f(x_0, y_0))$.

$$z = ax + by + c$$

$$z = x - 2y + 1$$



Condizione sufficiente di differenziabilità

$f: \mathbb{R}^n \to \mathbb{R}$ derivabile in $\underline{x}_0$.

Se $f_{x_1}, f_{x_2}, \dots f_{x_n}$ continue in $\underline{x}_0$.

$\Rightarrow$ f è differenziabile in $\underline{x}_0$.

es. $f(x,y) = x^2 y$ $\quad A = \mathbb{R}^2$

$f_x = 2xy$

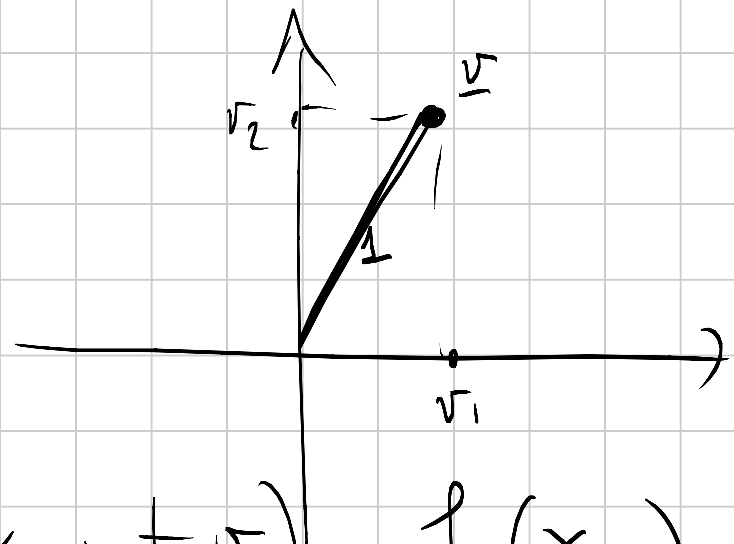
$f_y = x^2$

$\Rightarrow$ f è differenziabile in tutto $\mathbb{R}^2$.

Derivate direzionali

$f : \mathbb{R}^n \to \mathbb{R}$ $\quad \underline{x}_0$ del dominio

$\underline{v} \in \mathbb{R}^n : \quad |v| = 1 \quad$ versore



in $\mathbb{R}^2 \quad \underline{v} = (v_1, v_2)$

$$D_v f(\underline{x}_0) = \lim_{t \to 0} \frac{f(\underline{x}_0 + t\underline{v}) - f(\underline{x}_0)}{t}$$

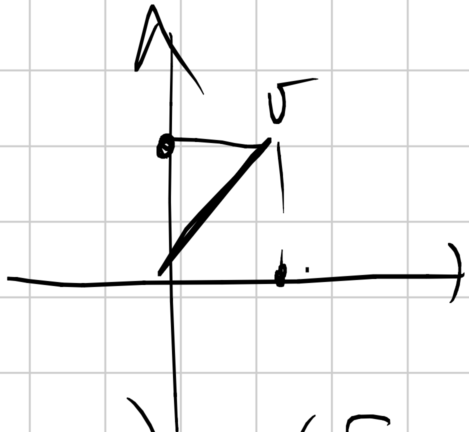
Per calcolarle:

Formula del gradiente

Se f è differenziabile in $\underline{x}_0$ Allora

$$D_v f(\underline{x}_0) = \nabla f(\underline{x}_0) \cdot \underline{v}$$

es. $f(x,y) = \operatorname{sen}(xy)$



$$\underline{v} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$$D_v f(1,1) = \left(f_x(1,1), f_y(1,1)\right) \cdot \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$= \ldots$

$= \sqrt{2}\cos 1$