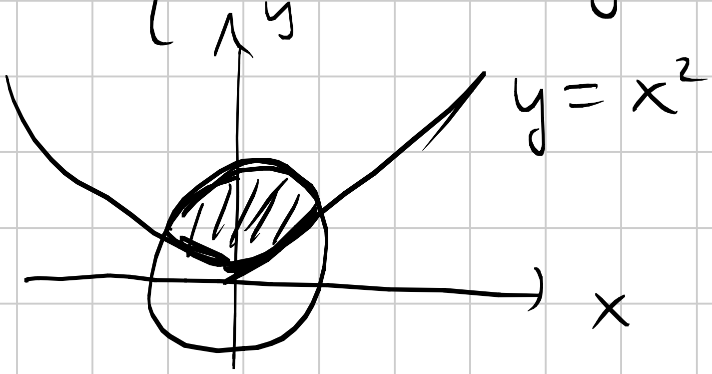
Canale 2   2 ottobre

Topologia (continua)

Intersezione e unione di aperti è aperto e analogo per i chiusi.

$$E = \{ (x,y) : \ y \geq x^2 \ \text{e} \ x^2 + y^2 \leq 1 \}$$



$f : \mathbb{R}^n \to \mathbb{R}$ continua

$$E = \{ x \in \mathbb{R}^n : f(x) > 0 \}$$
$$\{ f(x) < 0 \}$$
$$\{ f(x) \neq 0 \}$$

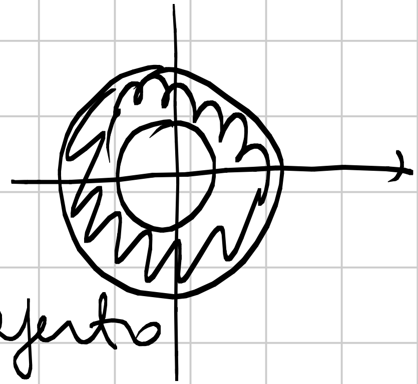
— aperto

$$E = \{ x \in \mathbb{R}^n : f(x) \geq 0 \}$$
$$\{ f \leq 0 \}$$
$$\{ f = 0 \}$$

— chiuso

$E = \{ (x,y) : 1 < x^2+y^2 < 2 \}$

senza bordo



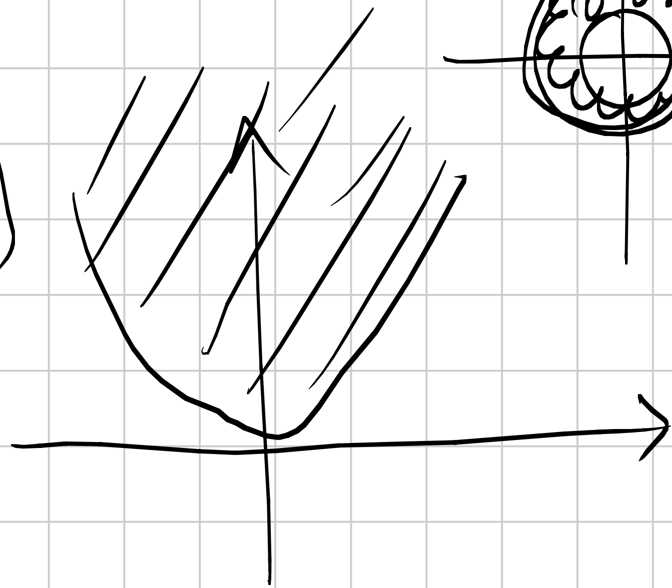
aperto

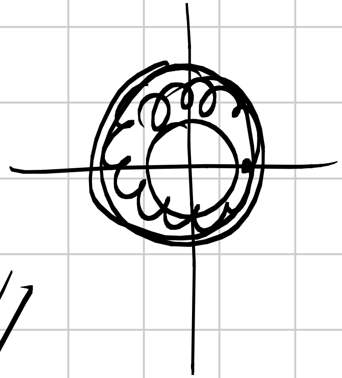
$E = \{ (\quad) : 1 \leq x^2+y^2 \leq 2 \}$ chiuso

$E = \{ (\quad) : 1 < x^2+y^2 \leq 2 \}$ né aperto né chiuso.

Def. $E \subseteq \mathbb{R}^n$ è limitato se esiste un intorno sferico che lo contiene

$\{(x,y) : y \geq x^2\}$



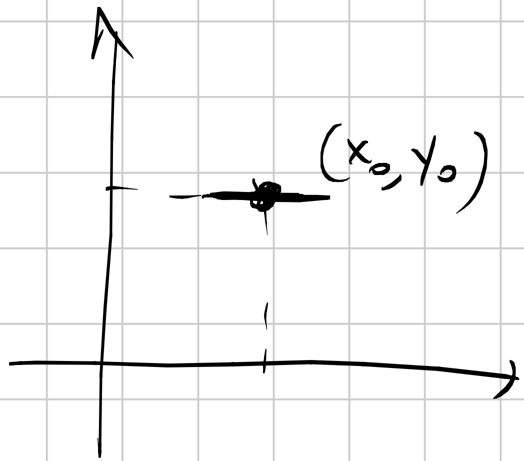


Teo. di Weierstrass

$f : E \subseteq \mathbb{R}^n \to \mathbb{R}$ continua, E chiuso e limitato. Allora f ha massimo e minimo in E.

Derivate parziali di f

$f : \mathbb{R}^2 \to \mathbb{R} \qquad f(x, y)$



$$\lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} =: \frac{\partial f}{\partial x}(x_0, y_0) = f_x(x_0, y_0)$$

$$\lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h} =: \frac{\partial f}{\partial y}(x_0, y_0) \; \left(= f_y(x_0, y_0)\right)$$

$$\left(f_x(x_0, y_0),\ f_y(x_0, y_0) \right) = \nabla f\,(x_0, y_0)$$

$$= \text{grad} f\,(\quad)$$

In generale

$$f(x_1, x_2, \dots x_n)$$

$$\frac{\partial f}{\partial x_i}(x_1, x_2, \dots x_n) =$$

$$= \lim_{h \to 0} \frac{f(x_1, x_2, \dots, x_i + h, \dots x_n) - f(x_1, x_2, \dots x_i, \dots x_n)}{h}$$

$$\nabla f(x_1, \dots x_n) = \left(f_{x_1}(x_1 \dots x_n), f_{x_2}(\dots), \dots, f_{x_n}(\dots) \right)$$

$$f(x,y) = \operatorname{sen}(xy) + e^x \cdot y^2$$

$$f_x(x,y) = \cos(xy) \cdot y + e^x y^2$$

$$f_y(x,y) = \cos(xy)\, x + e^x \cdot 2y$$

$$\nabla f(1,0) = (0, 1)$$

Def. f è **derivabile** in (x_0, y_0) se $\exists$ le sue derivate parziali in (x_0, y_0)

f derivabile in (x_0, y_0) $\not\Rightarrow$ f continua in (x_0, y_0)

Es. $f(x_1, x_2, x_3, x_4) = \log(x_3 x_1) + x_2^2 x_1$

$$f_{x_1} = \frac{1}{x_3 x_1} \cdot x_3 + x_2^2$$

$$f_{x_2} = 2 x_2 x_1$$

$$f_{x_3} = \frac{1}{x_3 x_1} x_1$$

$$f_{x_4} = 0$$

Derivate successive

$f : \mathbb{R}^2 \to \mathbb{R}$ $\qquad f(x,y)$

$f_x(x,y)$ — $f_{xx}(x,y)$, $f_{xy}(x,y)$

$f_y(x,y)$

$f_{yx}(x,y)$

$f_{yy}(x,y)$

$$D^2 f(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

matrice Hessiana (2×2)

Teorema di Schwarz

$f: A \subseteq \mathbb{R}^2 \to \mathbb{R}$

$f(x,y)$ derivabile due volte in A

Se $f_{xy}(x,y)$ e $f_{yx}(x,y)$ sono continue in A allora $f_{xy}(x,y) = f_{yx}(x,y)$

<u>Es.</u> $f(x,y) = xy^3$

$f_x = y^3$ $\qquad f_{xx} = 0$

$\qquad\qquad\quad f_{xy} = 3y^2$

$f_y = 3xy^2$

$f_{yx} = 3y^2 \qquad f_{yy} = 6xy$

$$D^2 f(x,y) = \begin{pmatrix} 0 & 3y^2 \\ 3y^2 & 6xy \end{pmatrix}$$

$$D^2 f(1,2) = \begin{pmatrix} 0 & 12 \\ 12 & 12 \end{pmatrix}$$

- Dico che $f \in C^1(A)$ se f è derivabile e le derivate parziali sono continue

• Dico che $f \in C^2(A)$ se f è due volte derivabile e tutte le derivate seconde sono continue in A.

es. $f(x,y,z) = x y^3 z^2$

$f_x = y^3 z^2$

$f_y = 3 x y^2 z^2$

$f_z = 2 x y^3 z$

$f_{xx} = 0$

$f_{xy} = 3 y^2 z^2$

$f_{xz} = 2 y^3 z$

$f_{yx} = 3 y^2 z^2$

$f_{yy} = 6 x y z^2$

$$f_{zx} = 2y^3 z$$

$$f_{zy} = 6xy^2 z$$

$$f_{zz} = 2xy^3$$

$$f_{yz} = 6xy^2 z$$

$$D^2 f(x,y,z) = \begin{pmatrix} 0 & 3y^2z^2 & 2y^3z \\ 3y^2z^2 & 6xyz^2 & 6xy^2z \\ 2y^3z & 6xy^2z & 2xy^3 \end{pmatrix}$$

$$D^2 f(1,1,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Funzioni differenziabili

$f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ $\quad \underline{x}_0 \in A$

è **differenziabile** in $\underline{x}_0$ se

1) f è derivabile in $\underline{x}_0$ $\left(\exists\, f_{x_i}(\underline{x}_0) \ \forall i \right)$

2) $$\lim_{\underline{h} \to 0} \frac{f(\underline{x}_0 + \underline{h}) - f(\underline{x}_0) - \nabla f(\underline{x}_0) \cdot \underline{h}}{|\underline{h}|} = 0$$

$$\left(f(\underline{x}_0 + \underline{h}) - f(\underline{x}_0) = \nabla f(\underline{x}_0) \cdot \underline{h} + o(|\underline{h}|)_{\ \underline{h} \to 0} \right)$$

se $n = 1$

$$f(x_0+h) = f(x_0) + f'(x_0) \cdot h + o(h) \quad h \to 0$$

f differenziabile in $\underline{x}_0 \Rightarrow f$ continua in $\underline{x}_0$

Def di f differenziabile in 2 variabili

$\underline{f(x,y)}$ $\quad \underline{h} = (x-x_0, y-y_0)$

$$f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) +$$

$$+ f_y(x_0,y_0)(y-y_0) + o\left(\sqrt{(x-x_0)^2+(y-y_0)^2}\right)$$

$x \to x_0$
$y \to y_0$

$$\nabla f(x_0,y_0)\cdot \underline{h}$$

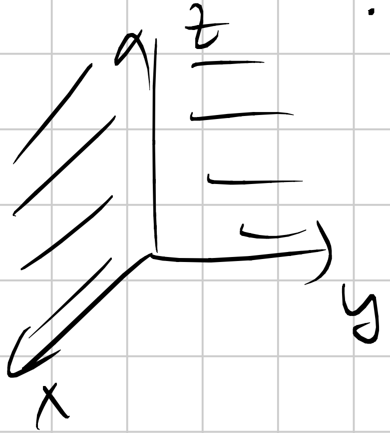
$$\boxed{z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

eq. piano tg al grafico di f in $(x_0, y_0, f(x_0,y_0))$.

$$z = ax + by + c$$

Es. Disegnare piani in $\mathbb{R}^3$!

$$z = 3x - y + 1$$



Condizione sufficiente di differenziabilità

$f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, $\underline{x}_0 \in A$. Se

$f_{x_1}(\underline{x})$, $f_{x_2}(\underline{x})$, ..., $f_{x_n}(\underline{x})$

sono continue in $\underline{x}_0 \Rightarrow f$ è differenziabile in

$\underline{x}_0$.

<u>es.</u> $f(x,y) = xy^2$ $\quad A = \mathbb{R}^2$

$f_x(x,y) = y^2$

$f_y(x,y) = 2xy$

sono continue in $\mathbb{R}^2$

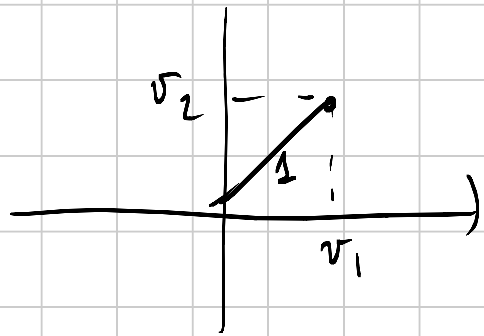
$\Rightarrow f$ è differenziabile su tutto $\mathbb{R}^2$.

Derivate direzionali

$f : \mathbb{R}^n \to \mathbb{R}$ $\quad \underline{v} \in \mathbb{R}^n$ $\quad |\underline{v}| = 1$ versore

$$\lim_{t\to 0} \frac{f(\underline{x}_0 + t\underline{v}) - f(\underline{x}_0)}{t} =: D_v f(\underline{x}_0)$$

in $\mathbb{R}^2$ $\quad \underline{v} = (v_1, v_2)$



se $\underline{v} = (1, 0, \dots, 0) \Rightarrow D_v f(\underline{x}_0) = \frac{\partial f}{\partial x_1}(\underline{x}_0)$.

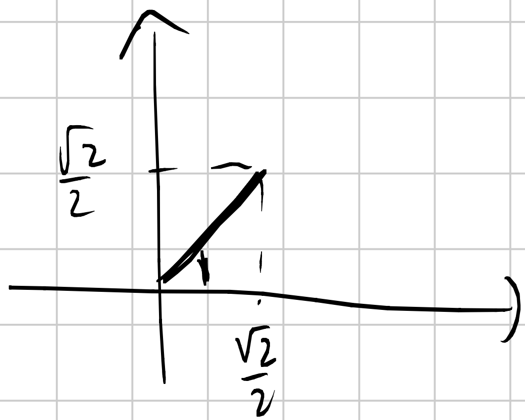
Formula del gradiente (per calcolare $D_{\underline{v}} f(\underline{x}_0)$)

f è differenziabile in $\underline{x}_0$

$\underline{v}$ versore assegnato

$$D_{\underline{v}} f(\underline{x}_0) = \nabla f(\underline{x}_0) \cdot \underline{v}$$

Es. $f(x,y) = \operatorname{sen}(xy)$



$$\underline{v} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$D_{\underline{v}} f(1,1)$

$f_x = \cos(xy)\cdot y$

$f_y = \cos(xy)\cdot x$

sono continue
quindi
f è differenziabile

$D_v f(1,1) = \nabla f(1,1)\cdot \underline{v}$

$f_x(1,1) = \cos 1$

$f_y(1,1) = \cos 1$

$= (\cos 1, \cos 1)\cdot\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$= \cos 1 \cdot \frac{\sqrt{2}}{2} + \cos 1 \frac{\sqrt{2}}{2} = \frac{\cancel{2}\sqrt{2}}{\cancel{2}}\cos 1 = \sqrt{2}\cos 1.$