Analisi 1, Paola Mannucci

(ATTENZIONE: in un compito non basta scrivere come nel seguito, vanno giustificati tutti i passaggi!)

Vicenza, dicembre 2011.

Limiti di successione e di funzione (da appelli)

1. Calcolare il limite seguente:

$$\lim_{n\to+\infty} \frac{5n! - \sqrt{5}n^{2+n^3}}{n^3 \log\left(1+\frac{1}{\sqrt{n}}\right) - 2n^2\log(1+n)}\, n^{-n^3}.$$

1. Usando le scale (e giustificando gli "o-piccolo" usati..)

$$\lim_{n\to+\infty} \frac{5n! - \sqrt{5}n^{2+n^3}}{n^3 \log\left(1+\frac{1}{\sqrt{n}}\right) - 2n^2\log(1+n)}\, n^{-n^3} = \lim_{n\to+\infty} \frac{-\sqrt{5}n^{2+n^3} n^{-n^3}}{n^3 \log\left(1+\frac{1}{\sqrt{n}}\right)} = \lim_{n\to+\infty} \frac{-\sqrt{5}n^2}{\frac{n^3}{\sqrt{n}}} = 0.$$

2. Per ogni valore di $\alpha \in \mathbb{R}$, determinare il seguente limite:

$$\lim_{x\to 0^+} \frac{2^x - \sin(\alpha x) - 1 + x^3 \sin\frac{1}{x}}{1-\cos(\sqrt{x}) - \frac{1}{2}\log(x+1)}.$$

(non svolto: si usa Mac-Laurin, scrivere $2^x = e^{x\log 2}$)
Risulta: $+\infty$ se $\alpha < \log 2$; $-\infty$ se $\alpha > \log 2$; $\frac{12}{5}\log^2 2$ se $\alpha = \log 2$.

3. Calcolare il limite della successione

$$a_n = \frac{1+\tan^3\left(\frac{1}{n}\right) - e^{\sin^3\left(\frac{1}{n}\right)}}{\frac{1}{n^{3+\alpha}}\left(e^{\sin^2\left(\frac{2}{n}\right)} - e^{\frac{1}{n^2}}\right)}$$

per $n \to +\infty$ al variare del parametro $\alpha \in \mathbb{R}$.
(Usando Mac-Laurin)

$$\lim_n \frac{1+\tan^3\left(\frac{1}{n}\right) - e^{\sin^3\left(\frac{1}{n}\right)}}{\frac{1}{n^{3+\alpha}}\left(e^{\sin^2\left(\frac{2}{n}\right)} - e^{\frac{1}{n^2}}\right)} = \lim_n \frac{\frac{3}{2}\frac{1}{n^5}}{\frac{1}{n^{3+\alpha}}\left(\frac{3}{n^2}\right)} = \lim_n \frac{1}{2} n^{\alpha},$$

perchè

$$\tan^3\left(\frac{1}{n}\right) = \left(\frac{1}{n} + \frac{1}{3}\frac{1}{n^3} + o\left(\frac{1}{n^3}\right)\right)^3 = \frac{1}{n^3} + \frac{1}{n^5} + o\left(\frac{1}{n^5}\right),$$

$$\sin^3\left(\frac{1}{n}\right) = \left(\frac{1}{n} - \frac{1}{6}\frac{1}{n^3} + o\left(\frac{1}{n^3}\right)\right)^3 = \frac{1}{n^3} - \frac{1}{2}\frac{1}{n^5} + o\left(\frac{1}{n^5}\right),$$

$$\sin^2\left(\frac{2}{n}\right) = \left(\frac{2}{n} + o\left(\frac{1}{n}\right)\right)^2 = \frac{4}{n^2} + o\left(\frac{1}{n^2}\right),$$

$$e^{\frac{1}{n^2}} = 1 + \frac{1}{n^2} + \frac{1}{2}\frac{1}{n^4} + o\left(\frac{1}{n^4}\right),$$

$$e^{\sin^3\left(\frac{1}{n}\right)} = 1+\sin^3\left(\frac{1}{n}\right)+\frac{1}{2}\sin^6\left(\frac{1}{n}\right)+o\left(\sin^6\left(\frac{1}{n}\right)\right) = 1+\frac{1}{n^3}-\frac{1}{2}\frac{1}{n^5}+o\left(\frac{1}{n^5}\right).$$

$$e^{\sin^2\left(\frac{2}{n}\right)} = 1+\sin^2\left(\frac{2}{n}\right)+o\left(\sin^2\left(\frac{2}{n}\right)\right) = 1+\frac{4}{n^2}+o\left(\frac{1}{n^2}\right).$$

Risulta: $+\infty$ se $\alpha > 0$; $\frac{1}{2}$ se $\alpha = 0$; 0 se $\alpha < 0$.

3. modificato Se si considera al posto dell' es. 3 l'esercizio seguente

$$\lim_n \frac{1+\tan^3\left(\frac{1}{n}\right)-e^{\sin^3\left(\frac{1}{n}\right)}}{\frac{1}{n^{3+\alpha}}\left(e^{\sin^2\left(\frac{1}{n}\right)}-e^{\frac{1}{n^2}}\right)} = \lim_n \frac{\frac{3}{2}\frac{1}{n^5}}{\frac{1}{n^{3+\alpha}}\left(-\frac{1}{3}\frac{1}{n^4}\right)} = \lim_n -\frac{9}{2}n^{2+\alpha},$$

usando Mac-Laurin si ha quanto sopra, perchè

$$\tan^3\left(\frac{1}{n}\right) = \left(\frac{1}{n}+\frac{1}{3}\frac{1}{n^3}+o\left(\frac{1}{n^3}\right)\right)^3 = \frac{1}{n^3}+\frac{1}{n^5}+o\left(\frac{1}{n^5}\right),$$

$$\sin^3\left(\frac{1}{n}\right) = \left(\frac{1}{n}-\frac{1}{6}\frac{1}{n^3}+o\left(\frac{1}{n^3}\right)\right)^3 = \frac{1}{n^3}-\frac{1}{2}\frac{1}{n^5}+o\left(\frac{1}{n^5}\right),$$

$$\sin^2\left(\frac{1}{n}\right) = \left(\frac{1}{n}-\frac{1}{6}\frac{1}{n^3}+o\left(\frac{1}{n^3}\right)\right)^2 = \frac{1}{n^2}-\frac{1}{3}\frac{1}{n^4}+o\left(\frac{1}{n^4}\right),$$

$$e^{\frac{1}{n^2}} = 1+\frac{1}{n^2}+\frac{1}{2}\frac{1}{n^4}+o\left(\frac{1}{n^4}\right),$$

$$e^{\sin^3\left(\frac{1}{n}\right)} = 1+\sin^3\left(\frac{1}{n}\right)+\frac{1}{2}\sin^6\left(\frac{1}{n}\right)+o\left(\sin^6\left(\frac{1}{n}\right)\right) = 1+\frac{1}{n^3}-\frac{1}{2}\frac{1}{n^5}+o\left(\frac{1}{n^5}\right).$$

$$e^{\sin^2\left(\frac{1}{n}\right)} = 1+\sin^2\left(\frac{1}{n}\right)+\frac{1}{2}\sin^4\left(\frac{1}{n}\right)+o\left(\sin^4\left(\frac{1}{n}\right)\right) = 1+\frac{1}{n^2}-\frac{1}{3}\frac{1}{n^4}+\frac{1}{2}\frac{1}{n^4}+o\left(\frac{1}{n^4}\right).$$

Risulta: $-\infty$ se $\alpha > -2$; $-\frac{9}{2}$ se $\alpha = -2$; 0 se $\alpha < -2$.

4. Calcolare il limite seguente al variare di $a \in \mathbb{R}$:

$$\lim_{x\to+\infty} \frac{\left(\frac{1}{x}\right)^{1/x}-2e^{1/x}+\cos\left(\frac{1}{x}\right)+\frac{a}{x}\log\left(\frac{1}{x}\right)}{\left(\sqrt{1+\sinh\left(\frac{1}{x}\right)}-\sqrt{1+\sin\left(\frac{1}{x}\right)}\right)^{1/3}}.$$

(Si usa Mac-Laurin, dopo la sost. $y = 1/x$)

$$L = \lim_{x\to+\infty} \frac{\left(\frac{1}{x}\right)^{1/x}-2e^{1/x}+\cos\left(\frac{1}{x}\right)+\frac{a}{x}\log\left(\frac{1}{x}\right)}{\left(\sqrt{1+\sinh\left(\frac{1}{x}\right)}-\sqrt{1+\sin\left(\frac{1}{x}\right)}\right)^{1/3}} = \lim_{y\to0^+} \frac{y^y-2e^y+\cos(y)+ay\log(y)}{\left(\sqrt{1+\sinh(y)}-\sqrt{1+\sin(y)}\right)^{1/3}}$$

dove

$$y^y = e^{y\log y} = 1+y\log y+\frac{1}{2}y^2\log^2 y+o(y^2\log^2 y) \quad (\text{ perchè } y\log y \to 0..),$$

$$\sqrt{1+\sinh(y)} = 1+\frac{1}{2}\sinh y-\frac{1}{8}\sinh^2 y+\frac{1}{16}\sinh^3 y+o(\sinh^3 y) = 1+\frac{1}{2}\left(y+\frac{1}{6}y^3\right)-\frac{1}{8}y^2+\frac{1}{16}y^3+o(y^3)$$

$$\sqrt{1+\sin(y)} = 1+\frac{1}{2}\sin y-\frac{1}{8}\sin^2 y+\frac{1}{16}\sin^3 y+o(\sin^3 y) = 1+\frac{1}{2}\left(y-\frac{1}{6}y^3\right)-\frac{1}{8}y^2+\frac{1}{16}y^3+o(y^3).$$

Quindi

$$L = \lim_{y\to 0^+} \frac{(1+a)y\log y - 2y + o(y)}{\left(\frac{1}{12}y^3 + \frac{1}{12}y^3 + o(y^3)\right)^{1/3}} = \lim_{y\to 0^+} \frac{(1+a)y\log y - 2y}{\frac{1}{\sqrt[3]{6}}y}$$

e risulta $-\infty$ se $a > -1$; $+\infty$ se $a < -1$; $-2\sqrt[3]{6}$ se $a = -1$.