$$g(x) = \begin{cases} \dfrac{e^x - 1}{x^a} & \text{se } x > 0 \\ 1 + bx + x^2 & \text{se } x \le 0 \end{cases}$$

(Compito 13/12/12)

1) $a > 0$, $b \in \mathbb{R}$ continua in $x_0 = 0$

2) $a > 0$, $b \in \mathbb{R}$ derivabile in $x_0 = 0$

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} 1 + bx + x^2 = 1$$

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} \frac{e^x - 1}{x^a} = \lim_{x \to 0^+} \frac{e^x - 1}{x} \cdot \frac{x}{x^a}$$

$$= \lim_{x \to 0^+} \underbrace{\boxed{\frac{e^x - 1}{x}}}_{1} \cdot \boxed{x^{1-a}} = \begin{cases} 0 & \text{se } a < 1 \\ \boxed{1 \quad \text{se } a = 1} \\ +\infty & \text{se } a > 1 \end{cases}$$

$$\lim_{x \to 0^+} x^{1-a} = \begin{cases} 0 & 1 - a > 0 \\ 1 & 1 - a = 0 \\ +\infty & 1 - a < 0 \end{cases}$$

g è continua in $x_0 = 0$ sse $a = 1$

2) se $x \le 0$ $\quad g(x) = 1 + bx + x^2$

$g'_-(0) = b$

$a = 1$

$$g'_+(0) = \lim_{x \to 0^+} \frac{g(x) - g(0)}{x} =$$

$$\lim_{x \to 0^+} \frac{\frac{e^x - 1}{x} - 1}{x} = \lim_{x \to 0^+} \frac{e^x - 1 - x}{x^2} = (*)$$

$$= \lim_{x \to 0^+} \frac{\cancel{1} + \cancel{x} + o(x) - \cancel{1} - \cancel{x}}{x^2} \quad ??$$

$$(*) = \lim_{x \to 0^+} \frac{\cancel{1} + \cancel{x} + \frac{\cancel{x^2}}{2} + \frac{o(x^2)}{x^2} - \cancel{1} - \cancel{x}}{\cancel{x^2}} = \frac{1}{2} = g'_+(0)$$

g è derivabile in $x_0 = 0$ se soltanto se $a = 1$ e $b = \frac{1}{2}$.

Compito del 8/01/13

1) $g(x) = \begin{cases} (1+\sin x)^{\frac{3}{x}} & \text{se } x>0 \\ x^2+ax+b & \text{se } x \le 0 \end{cases}$

1) continuità in $x_0 = 0$

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} x^2+ax+b = b$$

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \left[(1+\sin x)^{\frac{1}{\sin x}}\right]^{\frac{\sin x}{x}\cdot 3}$$

$\frac{\sin x}{x} \to 1$

$t = \sin x$

$\lim_{t\to 0} (1+t)^{\frac{1}{t}} = e$

$$= e^3$$

$$\boxed{b = e^3}$$

2) derivabilità: supponiamo $b = e^3$

$$g'_-(0) = \frac{d}{dx}\left(x^2+ax+b\right)\Big|_{x=0} = 2x+a\Big|_{x=0} = a$$

$$g'_+(0) = \lim_{x\to 0^+} \frac{(1+\sin x)^{\frac{3}{x}} - e^3}{x} =$$

$$= \lim_{x\to 0^+} \frac{e^{\frac{3}{x}\log(1+\sin x)} - e^3}{x} =$$

$$= \lim_{x\to 0^+} \frac{e^3\left[e^{\left(\frac{3}{x}\log(1+\sin x) - 3\right)} - 1\right]}{x} =$$

$$= \lim_{x\to 0^+} e^3 \cdot \frac{e^{\left[\frac{3}{x}\log(1+\sin x)-3\right]}-1}{\frac{3}{x}\log(1+\sin x)-3} \cdot$$

$$\cdot \left[\frac{3}{x}\log(1+\sin x) - 3\right]\frac{1}{x}$$

$\to$ $\lim_{x\to 0^+}\left[\frac{3}{x}\log(1+\sin x)\right] = 3$ infatti

$$\lim_{x\to 0^+} \frac{3\sin x}{x}\,\frac{\log(1+\sin x)}{\sin x} = 3$$

$\frac{\sin x}{x} \to 1$

$\sin x = t$

$$\lim_{t\to 0} \frac{\log(1+t)}{t} = 1$$

$\to$ $t = \frac{3}{x}\log(1+\sin x) - 3$

$$\lim_{t\to 0} \frac{e^t - 1}{t} = 1$$

$$\lim_{x\to 0^+} \cdot \left[\frac{3}{x}\log(1+\sin x) - 3\right]\frac{1}{x} =$$

$$= 3\lim_{x\to 0^+} \underbrace{\frac{\log(1+\sin x) - x}{x^2}}_{\to \frac{1}{2}} =$$

$$= \frac{3}{2}$$

infatti:

$\log(1+\sin x)$

$$\log(1+t) = t + \frac{t^2}{2} + o(t^2)$$

$$\log(1+\sin x) = \sin x + \frac{\sin^2 x}{2} + o(\sin^2 x)$$

$$\lim_{x \to 0^+} \frac{\log(1+\sin x) - x}{x^2} =$$

$$= \lim_{x \to 0^+} \frac{\sin x + \frac{1}{2}\sin^2 x + o(\sin^2 x) - x}{x^2} =$$

$$= \lim_{x \to 0^+} \frac{\cancel{x} - \frac{x^3}{6} + o(x^4) + \frac{1}{2}\sin^2 x + o(\sin^2 x) - \cancel{x}}{x^2}$$

$$= \lim_{x \to 0^+} \underbrace{-\frac{x}{6}}_{0} + \underbrace{\frac{o(x^2)}{x^2}}_{0} + \underbrace{\frac{1}{2}\left(\frac{\sin x}{x}\right)^2}_{1} + \underbrace{\frac{o(x^2)}{x^2}}_{\to 0} = \frac{1}{2}$$

$$\underbrace{\frac{o(\sin^2 x)}{x^2}}_{0} = \underbrace{\frac{\sin^2 x}{x^2}}_{1} \cdot \underbrace{\frac{o(\sin^2 x)}{\sin^2 x}}_{0}$$

$$o(\sin^2 x) = o(x^2)$$

g è derivabile sse $b = e^3$ e $a = \frac{3}{2} e^3$

2) $f(x) = (x+2)\left[3 - \log(x+2)\right]$

1) $\mathrm{dom}\, f = \{ x \in \mathbb{R} : x+2>0 \}$
$= (-2, +\infty)$

2) $A = \{$ punti di acc. di $\mathrm{dom}\, f \} =$
$= [-2, +\infty) \cup \{+\infty\}$

$\lim_{x \to x_0} f(x) \qquad \forall x_0 \in A$

f è continua in $\mathrm{dom}\, f \Rightarrow \forall x_0 \in \mathrm{dom}\, f$

$\lim_{x \to x_0} f(x) = f(x_0)$

$\lim_{x \to -2^+} (x+2)\left[3 - \log(x+2)\right] =$

$= \underbrace{\lim_{x \to -2^+} (x+2) \cdot 3}_{\to 0} - \underbrace{\lim_{x \to -2^+} (x+2) \log(x+2)}_{} = 0$

$t = x+2$

$\lim_{t \to 0^+} t \log t = 0$

$\lim_{x \to +\infty} (x+2)\left[3 - \log(x+2)\right] = -\infty$

3) Asintoti :

non ci sono asintoti verticali

non ci sono asintoti orizzontali

obliqui ?

$\lim_{x \to +\infty} \frac{(x+2)\left[3 - \log(x+2)\right]}{x} = -\infty$

$\frac{x+2}{x} \to 1 \qquad 3 - \log(x+2) \to -\infty$

$\Rightarrow$ non ci sono asintoti obliqui

4) $f(x) = (x+2)\,[3 - \log(x+2)]$

$B = \operatorname{dom} f$

$\underline{f'(x)} = [3 - \log(x+2)] + \cancel{(x+2)}\left(-\frac{1}{\cancel{(x+2)}}\right) =$

$= 3 - \log(x+2) - 1 =$

$= \underbrace{2 - \log(x+2)}$

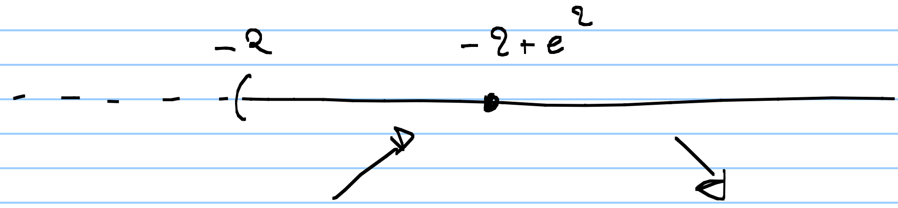
5) $f'(x) > 0 \qquad 2 - \log(x+2) > 0$

$\log(x+2) < 2$

$e^{\log(x+2)} < e^2$

$(x+2) < e^2$

$x < e^2 - 2$



f è strett. crescente in $(-2, -2+e^2)$

f è strett. decrescente in $(-2+e^2, +\infty)$

6) max / min rel / ass.

$f'(x) = 0 \iff x = e^2 - 2$ max rel e assoluto!

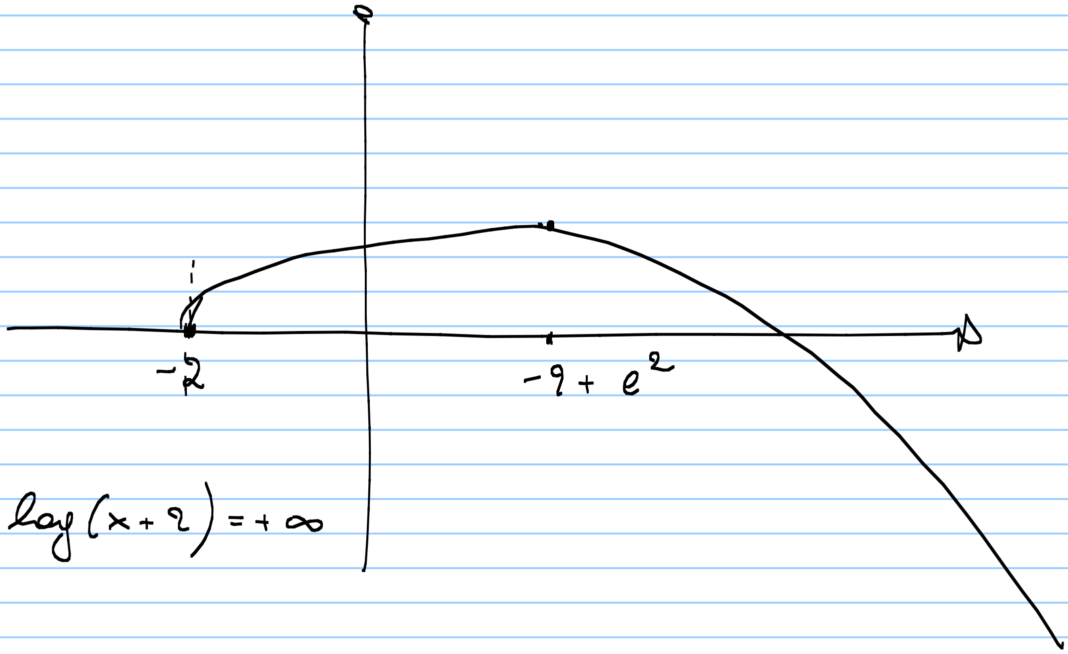
7) $f''(x) = \frac{d}{dx}\left(2 - \log(x+2)\right) =$

$= -\frac{1}{x+2} > 0 \iff \frac{1}{x+2} < 0$

$\iff x < -2$ $\boxed{\text{max}}$ $\operatorname{dom} f = (-2, +\infty)$

$\Rightarrow f$ è concava in $(-2, +\infty)$

8)



$$\lim_{x \to -2^+} 2 - \log(x+2) = +\infty$$

3) 1)

2)

3) $x^7 + x^6 + 1 = 0$

$$x^7 + x^6 + 1 \Big|_{-2} = (-2)^7 + (-2)^6 + 1 = -63 < 0$$

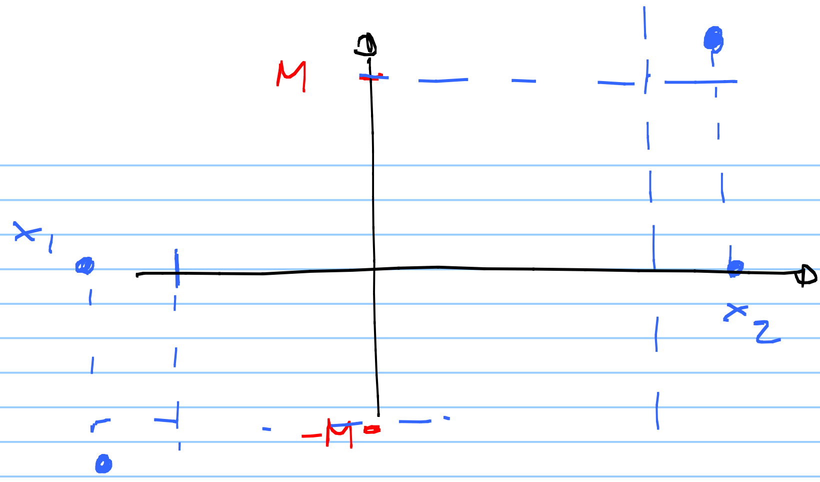
$$x^7 + x^6 + 1 \Big|_{x=1} = 3 > 0$$

applicare il teo di $\exists$ za degli zeri

$\Rightarrow \exists$ una sol. in $(-2, 1)$

In alternativa: $\lim_{x \to -\infty} x^7 + x^6 + 1 = -\infty$

$\lim_{x \to +\infty} x^7 + x^6 + 1 = +\infty$



per il teo di ∃ zeri ⇒ ∃ almeno una soluzione.

Compito del 28/06/13

1) $$g(x) = \begin{cases} \dfrac{e^{-2x} - 1}{x^{\sqrt{a}}} & \text{se } x > 0 \\ b + cx + x^2 & \text{se } x \le 0 \end{cases}$$

$a > 0$, $b, c \in \mathbb{R}$

1) g sia continua in $x_0 = 0$

$$\lim_{x \to 0^-} b + cx + x^2 = \boxed{b}$$

$$\lim_{x \to 0^+} \frac{e^{-2x} - 1}{x^{\sqrt{a}}} = \lim_{x \to 0^+} \frac{-2x}{x^{\sqrt{a}}} \cdot \frac{e^{-2x} - 1}{-2x} =$$

$$\lim_{t \to 0} \frac{e^t - 1}{t} \quad t = -2x \quad \to 1$$

$$= -2 \lim_{x \to 0^+} x^{1-\sqrt{a}} = \begin{cases} 0 & \text{se } 1 - \sqrt{a} > 0 \\ -2 & \text{se } 1 - \sqrt{a} = 0 \\ -\infty & \text{se } 1 - \sqrt{a} < 0 \end{cases}$$

$1 - \sqrt{a} > 0 \quad 1 > \sqrt{a} \quad 0 < a < 1$

$$= \begin{cases} 0 & \text{se } 0 < a < 1 \\ -2 & \text{se } a = 1 \\ -\infty & \text{se } a > 1 \end{cases}$$

g è continua se $0 < a < 1$ e $b = 0$

oppure se $a = 1$ e $b = -2$

2) derivabilità

i) $0 < a < 1 \quad b = 0$ ii) $a = 1$ e $b = -2$

i)

$$g(x) = \begin{cases} \dfrac{e^{-2x} - 1}{x^{\sqrt{a}}} & \text{se } x > 0 \\ cx + x^2 & \text{se } x \leq 0 \end{cases}$$

$$g'_-(0) = \frac{d}{dx}\left(cx + x^2\right)\Big|_{x=0} = c + 2x\Big|_{x=0} = c$$

$$g'_+(0) = \lim_{x \to 0^+} \frac{\dfrac{e^{-2x} - 1}{x^{\sqrt{a}}}}{x} = \lim_{x \to 0^+} \frac{e^{-2x} - 1}{x^{1+\sqrt{a}}}$$

$$= \lim_{x \to 0^+} \frac{e^{-2x} - 1}{-2x} \cdot \frac{-2x}{x^{1+\sqrt{a}}} =$$

$$= \lim_{x \to 0^+} (-2) \underbrace{\boxed{\frac{e^{-2x} - 1}{-2x}}}_{\downarrow \; 1} \cdot \underbrace{\boxed{x^{-\sqrt{a}}}}_{\downarrow \; +\infty} = +\infty$$

g non è derivabile

ii) $a = 1 \quad b = -2$

$$g(x) = \begin{cases} \dfrac{e^{-2x} - 1}{x} & \text{se } x > 0 \\ -2 + cx + x^2 & \text{se } x \leq 0 \end{cases}$$

$$g'_-(0) = \frac{d}{dx}\left(-2 + cx + x^2\right)\Big|_{x=0} = c + 2x\Big|_{x=0} = c$$

$$g_+'(0) = \lim_{x \to 0^+} \frac{\frac{e^{-2x}-1}{x} + 2}{x} =$$

$$= \lim_{x \to 0^+} \frac{e^{-2x} - 1 + 2x}{x^2} \quad (*)$$

$$e^t = 1 + t + \frac{t^2}{2} + o(t^2)$$

$$e^{-2x} = 1 - 2x + \frac{(-2x)^2}{2} + \underbrace{o(x^2)}_{= o((-2x)^2)}$$

$$(*) \lim_{x \to 0^+} \frac{\cancel{1} - \cancel{2x} + \frac{\overset{2}{\cancel{4}}x^2}{\cancel{2}} + o(x^2) - \cancel{1} + \cancel{2x}}{x^2} =$$

$$= 2$$

g è derivabile se e solo se
$a = 1 \quad b = -2 \quad c = 2$

2) $f(x) = \arcsin\left(\dfrac{|x+2|}{x^2+1}\right)$

$\operatorname{dom} f = \left\{ x \in \mathbb{R} : \underbrace{-1 \leq \dfrac{|x+2|}{x^2+1} \leq 1} \right\}$

$$\begin{cases} \dfrac{|x+2|}{x^2+1} \leq 1 \\ \dfrac{|x+2|}{x^2+1} \geq -1 \end{cases}$$

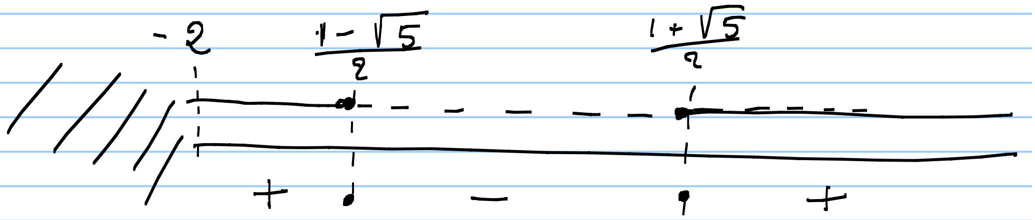
•) $x + 2 \geq 0 \quad (x \geq -2)$

$\dfrac{x+2}{x^2+1} \leq 1 \qquad\qquad \dfrac{x+2-x^2-1}{x^2+1} \leq 0$

$\dfrac{x^2-x-1}{x^2+1} \geq 0$

$N \geq 0 \quad \dfrac{1 \pm \sqrt{1+4}}{2} = \dfrac{1 \pm \sqrt{5}}{2} \qquad x < \dfrac{1-\sqrt{5}}{2} \ \vee \ x > \dfrac{1+\sqrt{5}}{2}$

$D > 0 \qquad \forall x$



$-2 < \dfrac{1-\sqrt{5}}{2} \qquad\qquad -4 < 1 - \sqrt{5}$

$\sqrt{5} < 5$ vero

$\dfrac{x+2}{x^2+1} \geq -1 \qquad\qquad \dfrac{x+2+x^2+1}{x^2+1} \geq 0$

$\dfrac{x^2+x+3}{x^2+1} \geq 0 \qquad\qquad N \geq 0 \quad \forall x$

$\downarrow \ \forall x \ (\geq -2) \qquad\qquad D > 0 \quad \forall x$

Quindi se $x \geq -2$

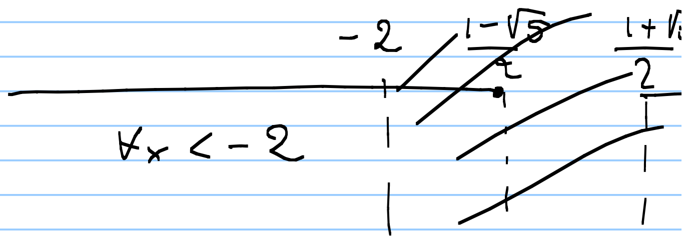
$$-2 \leq x \leq \frac{1-\sqrt{5}}{2} \quad \text{o} \quad x > \frac{1+\sqrt{5}}{2}$$

•) $x < -2$

$$\begin{cases} -\dfrac{x+2}{x^2+1} \leq 1 \\[2ex] -\dfrac{x+2}{x^2+1} \geq -1 \end{cases}$$

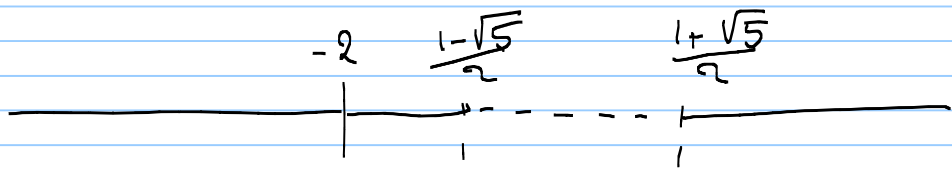
$\dfrac{x+2}{x^2+1} \geq -1$ $\quad \forall x$ (vedi sopra!)

$\dfrac{x+2}{x^2+1} \leq 1$



$\boxed{\forall x < -2}$

dom f



$\parallel$

$$\left(-\infty, \frac{1-\sqrt{5}}{2}\right] \cup \left[\frac{1+\sqrt{5}}{2}, +\infty\right)$$

- $A = \text{dom} f \cup \{+\infty\} \cup \{-\infty\}$

$\lim_{x \to x_0} f(x) = \qquad \forall x_0 \in A$

$\forall x_0 \in \text{dom} f \ (\subset A) \Rightarrow \lim_{x \to x_0} f(x) = f(x_0)$

perché f è continua

$$\lim_{x \to +\infty} \arcsin \underbrace{\frac{|x+2|}{x^2+1}}_{\to 0} = 0$$

$$\lim_{x\to-\infty} \arcsin \underbrace{\frac{|x+2|}{x^2+1}}_{\to 0} = 0$$

3) Asintoti

- non ci sono asintoti verticali
- $y=0$ è un asintoti orizzontale sia a $+\infty$ che a $-\infty$
- non ci sono asintoti obliqui

4) $B = \{x\in\mathbb{R} : f \text{ è derivabile in } x\}$

$$f(x) = \arcsin\left(\frac{|x+2|}{x^2+1}\right)$$

•) arcsin è derivabile in $(-1,1)$

$$-1 < \frac{|x+2|}{x^2+1} < 1$$

•) $|x|$ è derivabile in $\mathbb{R}\setminus\{0\}$

$$x+2 \neq 0$$

se $\boxed{-1 < \frac{|x+2|}{x^2+1} < 1}$ e $x\neq -2$

allora per i teoremi su algebra delle derivate e composizione di derivate e f è deribile.

se $\left(x < \frac{1-\sqrt{5}}{2} \text{ o } x > \frac{1+\sqrt{5}}{2}\right)$ e $x\neq-2$

$x>-2$

$$f'(x) = \frac{1}{\sqrt{1-\left(\frac{x+2}{x^2+1}\right)^2}} \cdot \frac{x^2+1-(x+2)2x}{(x^2+1)^2} =$$

$$= \frac{1}{\sqrt{1-\left(\frac{x+2}{x^2+1}\right)^2}} \cdot \frac{-x^2-4x+1}{(x^2+1)^2}$$

$x < -2$

$$f(x) = \arcsin\left(-\frac{x+2}{x^2+1}\right) =$$

$$= -\arcsin\left(\frac{x+2}{x^2+1}\right)$$

$$f'(x) = -\frac{1}{\sqrt{1-\left(\frac{x+2}{x^2+1}\right)^2}} \frac{-x^2-4x+1}{(x^2+1)^2}$$

$$\underbrace{x=-2}_{\downarrow\ \emptyset} \quad , \quad \underbrace{x = \frac{1-\sqrt{5}}{2} \quad , \quad x = \frac{1+\sqrt{5}}{2}}$$

$$f'_-(\cdot) \qquad f'_+(\cdot)$$

Osservazione!

Teorema $\quad f: I \to \mathbb{R} \quad x_0$ sia interno ad I

f sia continua in I, derivabile in $I \setminus \{x_0\}$.

Se $\lim_{x \to x_0} f'(x) = \ell \in \mathbb{R}$ $\quad \left(\begin{array}{l} \lim_{x \to x_0^+} = \ell \\ \lim_{x \to x_0^-} = \ell \end{array}\right)$

Allora f è derivabile

Se $\lim_{x \to x_0^+} f'(x) = \ell_1$ e $\lim_{x \to x_0^-} f'(x) = \ell_2$

$$\ell_1 \neq \ell_2$$

$\Rightarrow$ allora f non è derivabile.

Esempio: $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

f è continua in $\mathbb{R}$

è derivabile in $\mathbb{R} \setminus \{0\}$

$$\lim_{x \to 0} f'(x) = \lim_{x \to 0} \underbrace{2x \sin\frac{1}{x}}_{\downarrow\, 0} - \underbrace{\cancel{x^2}\left(\cos\frac{1}{x}\right)\left(\frac{1}{\cancel{x^2}}\right)}_{\downarrow\, \nexists}$$

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} x \sin\frac{1}{x} = 0$$

---

derivabilità in $x = -2$

$$\lim_{x \to -2^+} \frac{1}{\sqrt{1 - \left(\frac{x+2}{x^2+1}\right)^2}} \frac{-x^2 - 4x + 1}{(x^2+1)^2} = \frac{1}{5}$$

$$\lim_{x \to -2^-} \left(- \frac{1}{\sqrt{1 - \left(\frac{x+2}{x^2+1}\right)^2}} \frac{-x^2 - 4x + 1}{(x^2+1)^2} \right) = -\frac{1}{5}$$

$\Rightarrow f$ non è derivabile in $x = -2$

$$B = (-\infty, -2) \cup \left(-2, \frac{1-\sqrt{5}}{2}\right) \cup \left(\frac{1+\sqrt{5}}{2}, +\infty\right)$$

5) monotonia

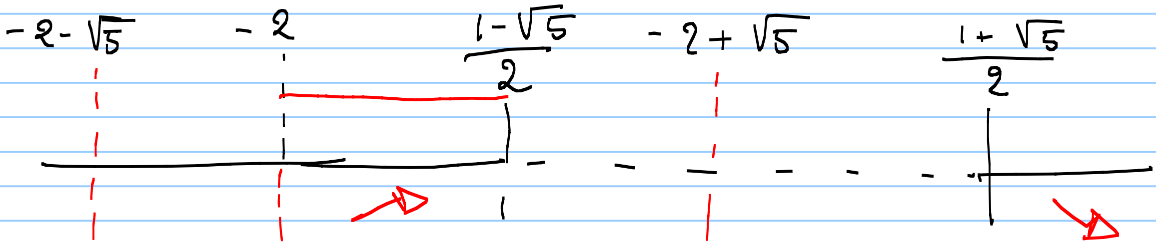
$f'(x) > 0$

$x > -2$ $\quad \underbrace{\frac{1}{\sqrt{1 - \left(\frac{x+2}{x^2+1}\right)^2}}}_{>0} \frac{-x^2 - 4x + 1}{\underbrace{(x^2+1)^2}_{>0}} > 0$

$\Leftrightarrow \quad -x^2-4x+1>0$

$\Leftrightarrow \quad \boxed{x^2+4x-1<0}$

$x_{1,2} = -2 \pm \sqrt{4+1} = -2 \pm \sqrt{5}$



$-2+\sqrt{5} > \frac{1-\sqrt{5}}{2}$

$-4+2\sqrt{5} > 1-\sqrt{5} \qquad 3\sqrt{5} > 5$

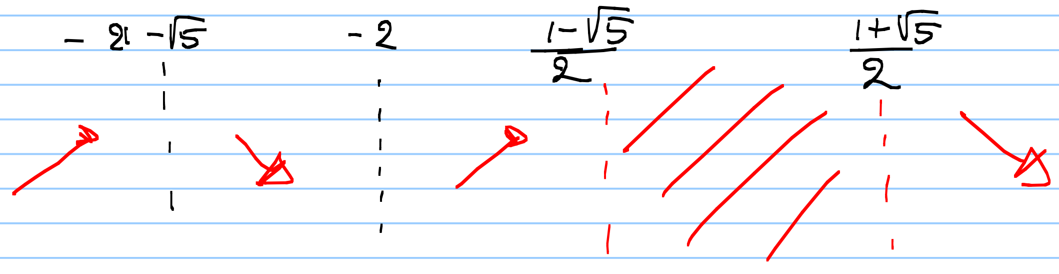
$9\cdot 5 > 25$ sì

• $x<-2$

$$f'(x) = -\frac{1}{\sqrt{1-\left(\frac{x+2}{x^2+1}\right)^2}} \cdot \frac{-x^2-4x+1}{(x^2+1)^2} > 0$$

$\Leftrightarrow \quad x^2+4x-1>0$

$x < -2-\sqrt{5} \qquad (x > -2+\sqrt{5})$

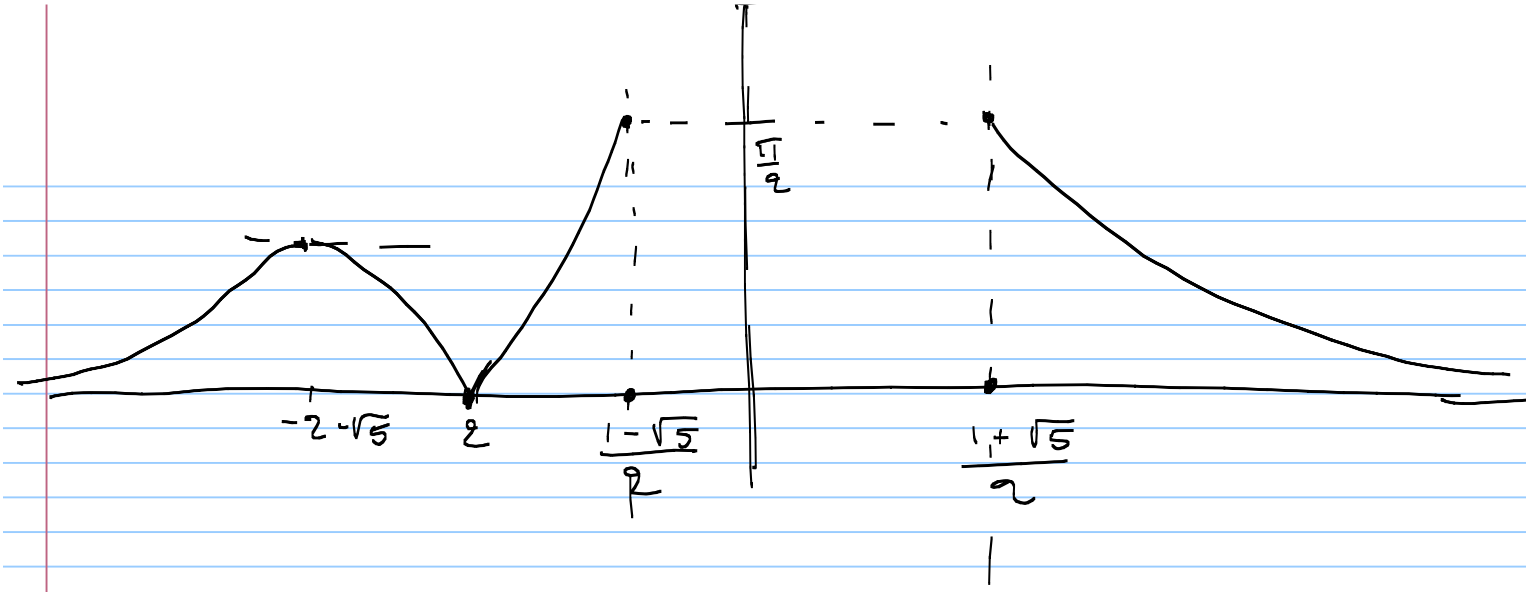


strettamente crescente in $(-\infty, -2-\sqrt{5}]$

e in $\left[-2, \frac{1-\sqrt{5}}{2}\right]$

strettamente decrescente in $[-2-\sqrt{5}, -2]$

e in $\left[\frac{1+\sqrt{5}}{2}, +\infty\right)$



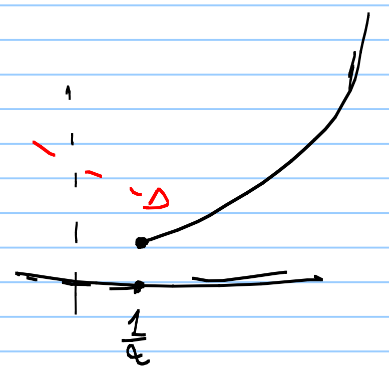
-2-√5
2
π/2
(1-√5)/2
(1+√5)/2

3) $h(x) = x^7 + 6x^6 - x$ invertibile in $\left(\frac{1}{2}, +\infty\right)$

$7x^6 + 36x^5 - 1 > 0$ (o < 0)

$$\underbrace{7\cdot\frac{1}{2^6} + \underbrace{36\,\frac{1}{2^5}}_{>1} - 1}_{>0} > 0$$

$42x^5 + 36\cdot 5x^4 > 0$



se $x > \frac{1}{2}$

$\Rightarrow x^7 + 6x^6 - x$ è invertibile in $\left(\frac{1}{2}, +\infty\right)$

•) $h\big|_{\left(\frac{1}{2}, \infty\right)}$ $\qquad (h^{-1})'(6) = \dfrac{1}{h'(h^{-1}(6))}$

$h^{-1}(6) = x$ $\qquad 6 = h(x)$

$\uparrow$

$\left(\frac{1}{2}, +\infty\right)$ $\qquad 6 = x^7 + 6x^6 - 1$

$x = 1$

$$(h^{-1})'(6) = \frac{1}{7 + 36 - 1} = \frac{1}{42}$$

Compito del 6/09/13

1) $$g(x) = \begin{cases} \dfrac{1 - \cos x^a}{x^4} & \text{se } x > 0 \\ x^2 + b & \text{se } x \leq 0 \end{cases}$$

$a > 0$, $b \in \mathbb{R}$

1) continuità:

$$\lim_{x \to 0^-} x^2 + b = b = g(0)$$

$$\lim_{x \to 0^+} \frac{1 - \cos x^a}{x^4} = \lim_{x \to 0^+} \frac{1 - \cos x^a}{x^{2a}} \cdot \frac{x^{2a}}{x^4}$$

$$= \frac{1}{2} \lim_{x \to 0^+} x^{2a-4} = \begin{cases} 0 & \text{se } a > 2 \\ \frac{1}{2} & \text{se } a = 2 \\ +\infty & \text{se } a < 2 \end{cases}$$

g è continua in $x_0 = 0 \iff a > 2$ e $b = 0$
o $a = 2$ e $b = \frac{1}{2}$

2) derivabilità:

$$g'_-(0) = 2x\Big|_{x=0} = 0$$

$$g'_+(0) = \lim_{x \to 0^+} \frac{g(x) - g(0)}{x - 0} =$$

i) $a > 2$

$$\lim_{x \to 0^+} \frac{\frac{1 - \cos x^a}{x^4}}{x} = \lim_{x \to 0^+} \frac{1 - \cos x^a}{x^5} =$$

$$= \lim_{x \to 0^+} \underbrace{\frac{1 - \cos x^a}{x^{2a}}}_{\to \frac{1}{2}} \cdot x^{2a-5} = \begin{cases} 0 & \text{se } a > \frac{5}{2} \\ \frac{1}{2} & \text{se } a = \frac{5}{2} \\ +\infty & \text{se } a > \frac{5}{2} \end{cases}$$

ii) $a = 2$

$$\lim_{x \to 0^+} \frac{\frac{1 - \cos x^2}{x^4} - \frac{1}{2}}{x} =$$

$$\lim_{x \to 0^+} \frac{1 - \cos x^2 - \frac{1}{2}x^4}{x^5} =$$

$$= \lim_{x \to 0^+} \frac{\cancel{1} - \left(\cancel{1} - \cancel{\frac{x^4}{2}} + \frac{1}{4!}x^8 + o(x^{10})\right) - \cancel{\frac{1}{2}x^4}}{x^5}$$

$$= 0$$

Quindi g é derivabile sse $a > \frac{5}{2}$ e $b = 0$

o $a = 2$ e $b = \frac{1}{2}$

2) $f(x) = \frac{|x|}{(x-2)^2} e^{(x-2)}$

1) $\operatorname{dom} f = \{x \in \mathbb{R} : x \neq 2\} = (-\infty, 2) \cup (2, +\infty)$

2) $A = \{x \in \mathbb{R} : x$ è di accumulazione per $\operatorname{dom} f\}$

$A = \mathbb{R}^*$

$\forall x_0 \in \operatorname{dom} f$, f è continua in x_0 (perché composizione, prodotto, ecc..., di funzioni continue).

Quindi $\lim_{x \to x_0} f(x) = f(x_0) \quad \forall x_0 \in \operatorname{dom} f$.

$$\lim_{x \to +\infty} \frac{|x|}{(x-2)^2} e^{(x-2)} = \lim_{x \to +\infty} \underbrace{\boxed{\frac{|x|}{(x-2)}}}_{\to 1} \underbrace{\boxed{\frac{e^{(x-2)}}{x-2}}}_{\to +\infty} =$$

$= +\infty$

$$\lim_{x \to -\infty} \underbrace{\frac{|x|}{(x-2)^2}}_{\to 0} \underbrace{e^{(x-2)}}_{\to 0} = 0$$

$$\lim_{x \to 2} \underbrace{\boxed{\frac{|x|}{(x-2)^2}}}_{\to +\infty} \underbrace{e^{(x-2)}}_{\to 1} = +\infty$$

3) Asintoti: $y = 0$ è un asintoto orizzontale a $-\infty$

$x = 2$ è un asintoto verticale

Potrebbe esserci un asintoto obliquo a $+\infty$:

$$\lim_{x \to +\infty} \frac{|x|}{(x-2)^2} e^{(x-2)} \cdot \frac{1}{x} = +\infty$$

non ci sono asintoti obliqui.

4) $f(x) = \frac{|x|}{(x-2)^2} e^{(x-2)}$

$B = \{x \in \mathbb{R} : f$ è derivabile in $x\}$

f è derivabile $\forall x \neq 2$ e $x \neq 0$ per i teoremi sull'algebra delle derivate e sulla derivazione delle funzioni composte.

$\forall x \neq 0$ e $\forall x \neq 2$ si ha

$$f(x) = \begin{cases} \dfrac{x}{(x-2)^2} e^{(x-2)} & \text{se } x \geq 0 \\[2ex] \dfrac{-x}{(x-2)^2} e^{(x-2)} & \text{se } x < 0 \end{cases}$$

se $x > 0$:

$$\frac{x}{(x+2)^2} e^{(x+2)}$$

$$f'(x) = \left[\frac{(x-2)^2 - 2x(x-2)}{(x-2)^4} + \frac{x}{(x-2)^2} \right] e^{(x-2)}$$

$$= \frac{(1+x)(x-2) - 2x}{(x-2)^3} e^{(x-2)}$$

$$= \frac{x^2 - x - 2 - 2x}{(x-2)^3} e^{(x-2)} =$$

$$= \frac{x^2 - 3x - 2}{(x-2)^3} e^{(x-2)}$$

se $x < 0$ $\quad f(x) = \dfrac{-x}{(x-2)^2} e^{(x-2)}$

$$f'(x) = -\frac{x^2 - 3x - 2}{(x-2)^3} e^{(x-2)}$$

Studiamo la derivabilità in $x_0 = 0$

$$\lim_{x \to 0^+} f'(x) = \frac{1}{4} e^{-2}$$

$$\lim_{x \to 0^-} f'(x) = -\frac{1}{4} e^{-2}$$

Quindi f non è derivabile in $x = 0$.

$$B = \{x \in \mathbb{R} : x \neq 0 \text{ e } x \neq 2\}$$

5) Studiamo la monotonia di f

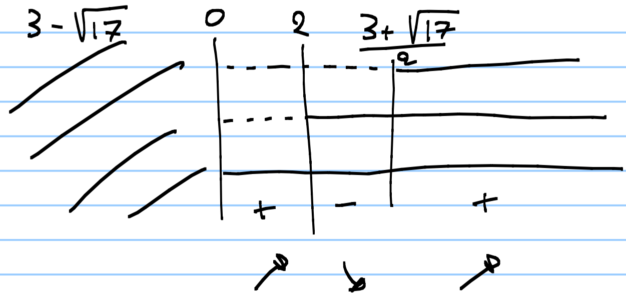
se $x > 0$: $\quad f'(x) = \dfrac{x^2 - 3x - 2}{(x-2)^3}\, e^{x-2} > 0$

$x^2 - 3x - 2 > 0 \qquad x_{1,2} = \dfrac{3 \pm \sqrt{9+8}}{2} = \dfrac{3 \pm \sqrt{17}}{2}$

$(x^2 - 3x - 2) > 0$

$(x-2)^3 > 0$

$e^{x-2} > 0$



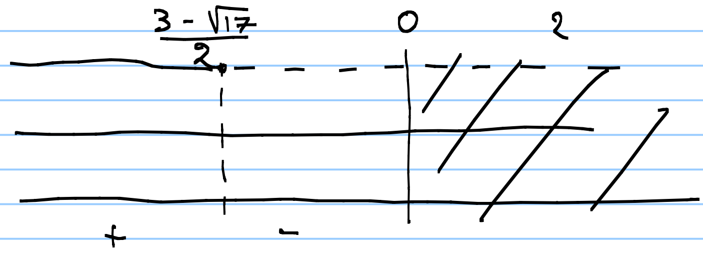
se $x < 0$ $\quad f'(x) = -\dfrac{(x^2 - 3x - 2)}{(x-2)^3}\, e^{(x-2)} > 0$

$\Leftrightarrow \quad \dfrac{(x^2 - 3x - 2)}{(x-2)^3}\, e^{(x-2)} < 0$

$x^2 - 3x - 2 > 0$

$x - 2 > 0$

$e^{x-2} > 0$



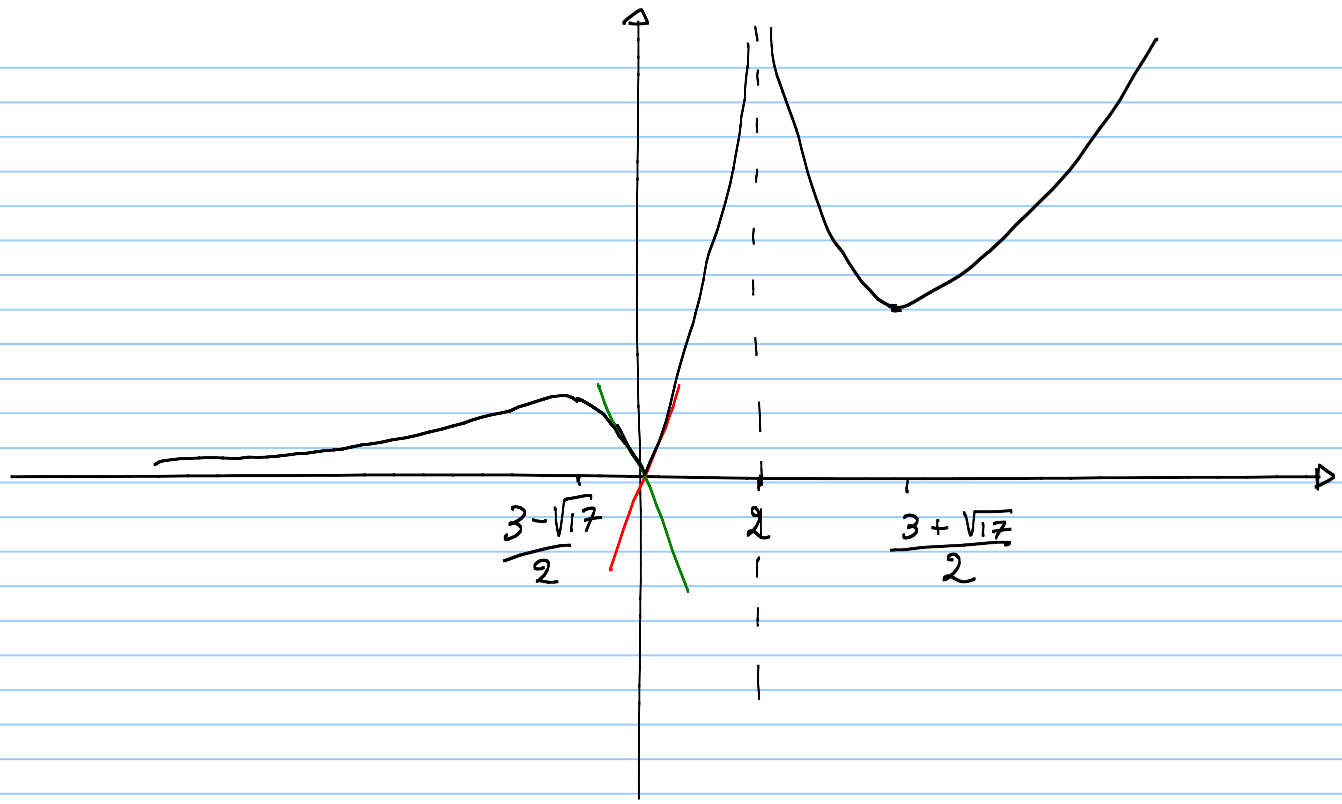
$f' < 0 \qquad f' > 0$

f è decrescente in $\left(-\infty, \dfrac{3-\sqrt{17}}{2}\right]$

e in $\left[2, \dfrac{3+\sqrt{17}}{2}\right]$

f è crescente in $\left[\dfrac{3-\sqrt{17}}{2}, 2\right]$

e in $\left[\dfrac{3+\sqrt{17}}{2}, +\infty\right)$



$\frac{3-\sqrt{17}}{2}$
2
$\frac{3+\sqrt{17}}{2}$