Lezione del 11/11

Def. $\lim_{x \to c} f(x) = \ell \iff \forall \{x_n\}\ x_n \to c$
$f(x_n) \to \ell$

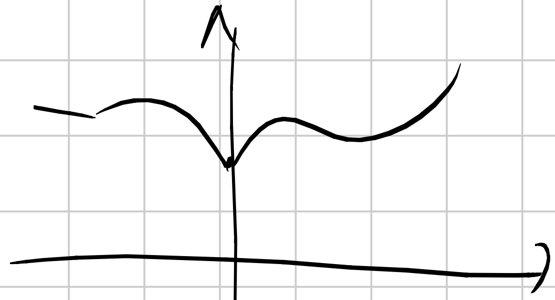
$\lim_{x \to 102} \frac{\operatorname{sen} x}{x} = \frac{\operatorname{sen} 102}{102}$

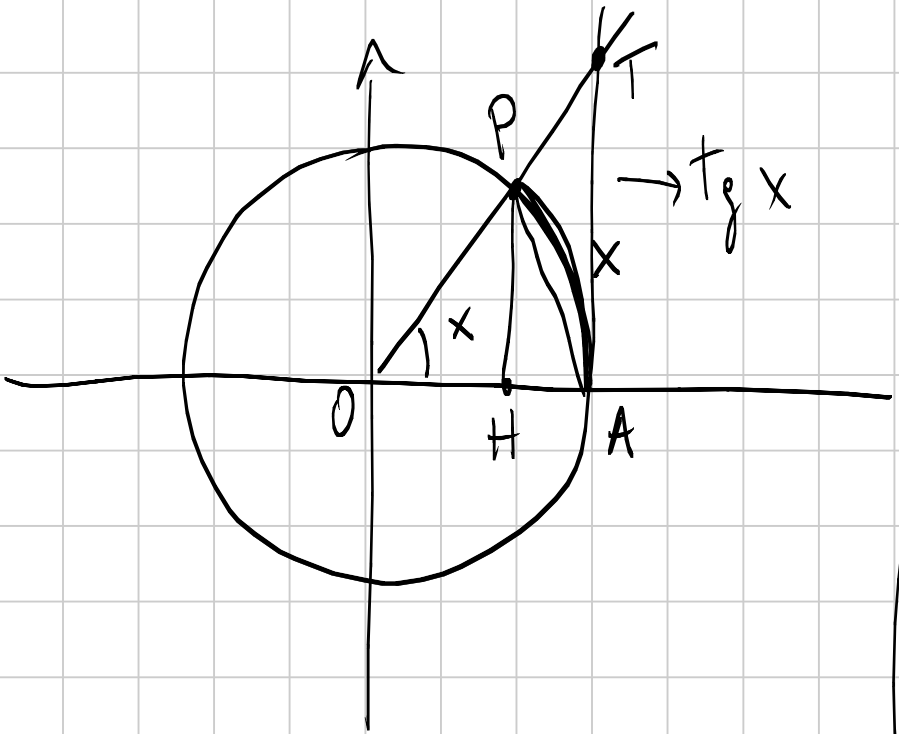
Limiti notevoli

- $\lim_{x \to 0} \frac{\operatorname{sen} x}{x} = 1$

Dim. $\frac{\operatorname{sen}(-x)}{-x} = \frac{-\operatorname{sen} x}{-x} = \frac{\operatorname{sen} x}{x}$ è pari

$$\lim_{x \to 0^+} \frac{\operatorname{sen} x}{x}$$





$TA = \operatorname{tg} x$

$PH = \operatorname{sen} x$

OPA Triangolo

OPA settore circolare

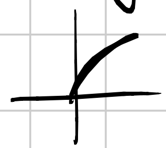
OTA Triangolo

area triangolo OPA $\leq$ area settore OPA $\leq$ area triangolo OTA

$$\frac{1 \cdot \operatorname{sen} x}{\cancel{2}} \leq \frac{x \cdot 1}{\cancel{2}} \leq \frac{1 \cdot \operatorname{tg} x}{\cancel{2}}$$

$$\operatorname{sen} x \le x \le \operatorname{tg} x$$

divido per sen x



$\operatorname{sen} x > 0$

$$1 \le \frac{x}{\operatorname{sen} x} \le \frac{1}{\cos x}$$

$$\cos x \le \frac{\operatorname{sen} x}{x} \le 1$$

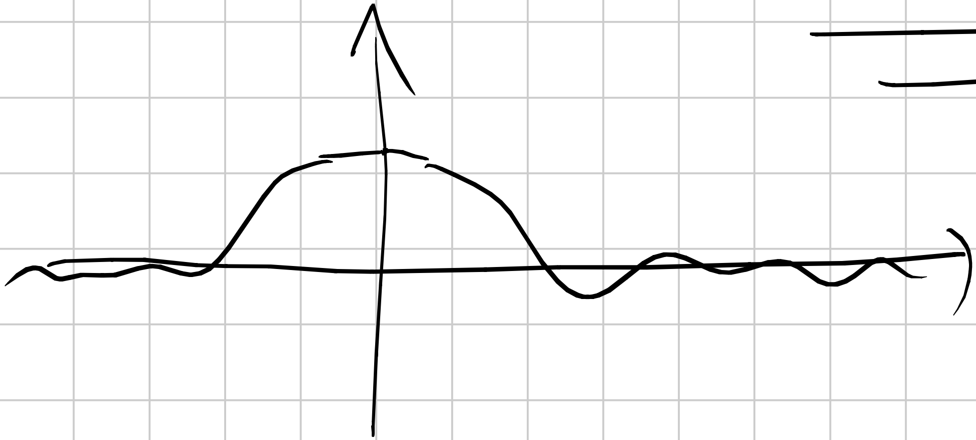
$$\cos x \xrightarrow{x \to 0^+} 1 \qquad \frac{\operatorname{sen} x}{x} \longrightarrow 1 \qquad 1 \xrightarrow{x \to 0^+} 1$$

$$\lim_{x \to 0} \frac{\operatorname{sen} x}{x} = 1$$

oss. $f(x) = \dfrac{\operatorname{sen} x}{x}$ def. in $\mathbb{R} \setminus \{0\}$

estendere f per continuità

posso prolungare f con una funzione continua su tutto $\mathbb{R}$



$$\tilde{f}(x) = \begin{cases} \dfrac{\operatorname{sen} x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

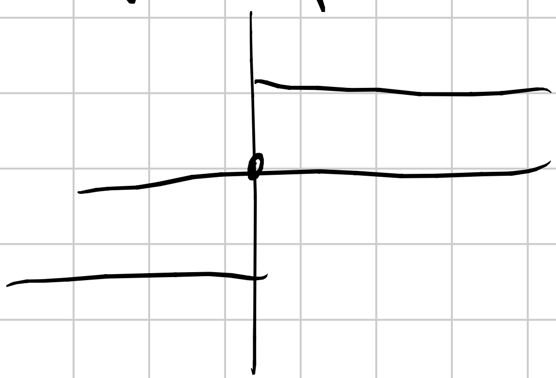
$\tilde{f}$ è continua su tutto $\mathbb{R}$

- è continua $x \neq 0$
- in $x = 0$ $\quad \lim_{x \to 0} \tilde{f}(x) = \tilde{f}(0)$

$$\lim_{x \to 0} \frac{\operatorname{sen} x}{x} = 1$$

Non posso sempre prolungare per continuità una funzione

$f(x) = \operatorname{sgn} x$



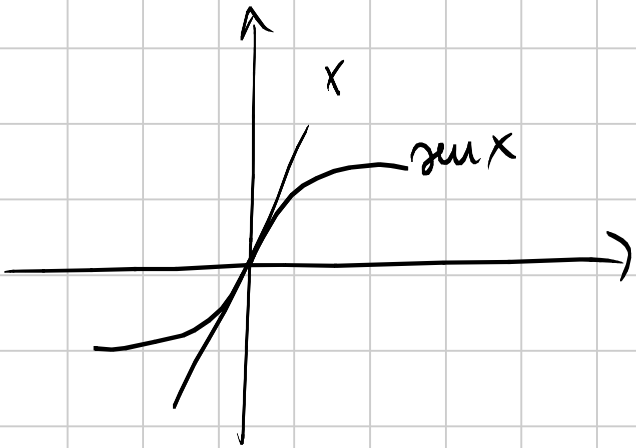
$$\lim_{x\to 0} \frac{\operatorname{sen} x}{x} = 1$$

$\operatorname{sen} x \to 0$

$x \to 0$

$\frac{0}{0}$

dico che $\operatorname{sen} x \sim x$, $x \to 0$



$x \to 0$

Conseguenze

$$\boxed{\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}}$$

$\frac{0}{0}$

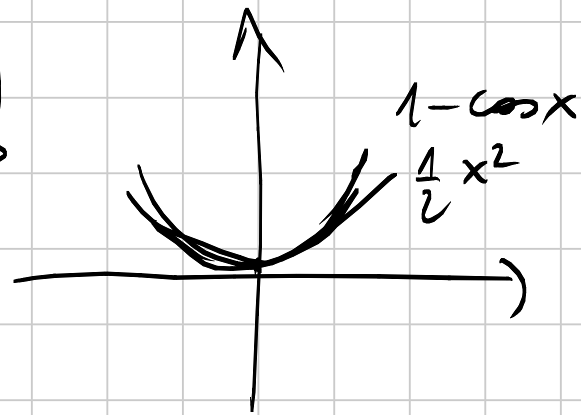
$$\frac{1-\cos x}{x^2} = \frac{(1-\cos x)(1+\cos x)}{x^2\,(1+\cos x)} = \frac{\operatorname{sen}^2 x}{x^2\,(1+\cos x)}$$

$$= \left(\frac{\operatorname{sen} x}{x}\right)^2 \cdot \frac{1}{1+\cos x} \xrightarrow[x\to 0]{} \frac{1}{2}$$

$1-\cos x$ e x^2 sono infinitesime per $x \to 0$

e poiché $\frac{1-\cos x}{x^2} \to \frac{1}{2}$ sono infinitesime dello stesso ordine

$$\boxed{1-\cos x \sim \frac{1}{2}x^2, \quad x \to 0}$$



$$\lim_{x\to 0} \underbrace{\left(\frac{1-\cos x}{x\cdot x}\right)}_{\to 1/2} \cdot x = 0$$

$$\lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

si dice che

$1-\cos x$ è infinitesimo
di ordine superiore
rispetto a x

- $$\lim_{x\to 0} \frac{\operatorname{tg} x}{x} = \lim_{x\to 0} \underbrace{\frac{\operatorname{sen} x}{x}}_{\to 1} \cdot \frac{1}{\cos x} = 1$$

$\operatorname{tg} x \sim x \qquad x \to 0 \qquad e^x - 1 = y$

es. $\displaystyle\lim_{x \to 0} \frac{\operatorname{sen}(e^x - 1)}{(e^x - 1)} = \lim_{y \to 0} \frac{\operatorname{sen} y}{y} = 1$

$\displaystyle\lim_{x \to 0} \frac{\operatorname{sen}(e^x)}{e^x} = \frac{\operatorname{sen} 1}{1} = \operatorname{sen} 1$

no! limite notevole

$\displaystyle\lim_{x \to 0} \frac{\operatorname{sen} 3x}{x} = \lim_{x \to 0} \frac{\operatorname{sen} 3x}{3x} \cdot 3$

$$= \lim_{y \to 0} \frac{\text{sen}\, y}{y} \cdot 3 = 3 \qquad 3x = y$$

$$\cdot \lim_{x \to 0} \frac{\text{sen}\, 3x}{\text{sen}\, 5x} \cdot \frac{5x}{3x} \cdot \frac{3\cancel{x}}{5\cancel{x}} = \frac{3}{5}$$

$\to 1$

$\to 1$

$$\cdot \lim_{x \to \pi^+} \frac{\text{sen}\, x}{\sqrt{x - \pi}} =$$

$x - \pi = y$

$y \to 0^+$

$x = y + \pi$

$$= \lim_{y \to 0^+} \frac{\text{sen}\,(y + \pi)}{\sqrt{y}} =$$

$$= \lim_{y \to 0^+} \frac{\sin y}{\sqrt{y}\,\sqrt{y}} \cdot \sqrt{y} = 0$$

Altro limite notevole

$$\boxed{\lim_{x \to \pm\infty} \left(1 + \frac{1}{x}\right)^x = e}$$

conseguenza di $\lim_{n} \left(1 + \frac{1}{n}\right)^n = e$

$$\lim_{n} \left(1 + \frac{1}{a_n}\right)^{a_n} = e$$

$\forall \{a_n\}$ $a_n \to \pm\infty$

$$\lim_{x \to \pm\infty} \left(1 + \frac{1}{x}\right)^{x} = e$$

$\frac{x}{\alpha} = y$ $\quad x = \alpha y$

$$\lim_{x \to \pm\infty} \left(1 + \frac{\alpha}{x}\right)^{x} = \lim_{y \to \pm\infty} \left(1 + \frac{1}{y}\right)^{y\alpha} = e^{\alpha}$$

$$\boxed{\lim_{x \to \pm\infty} \left(1 + \frac{\alpha}{x}\right)^x = e^\alpha}$$

$$\boxed{\lim_{y \to 0} \frac{\log(1+y)}{y} = 1}$$

$$x = \frac{1}{y}$$

$$y = \frac{1}{x}$$

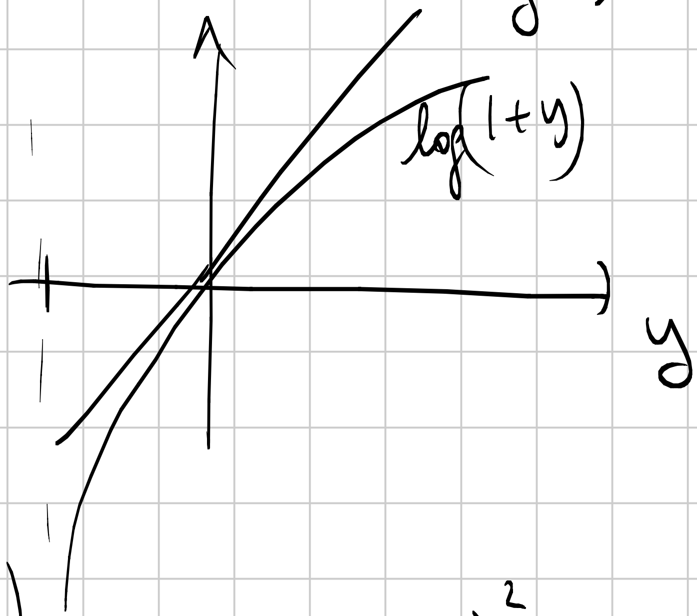
$$x \to \pm\infty$$

$$\frac{\log(1+y)}{y} = x \log\left(1 + \frac{1}{x}\right)$$

$$= \log\left[\left(1 + \frac{1}{x}\right)^x\right] \longrightarrow 1$$

$\searrow e$

$$\lim_{y \to 0} \frac{\log(1+y)}{y} = 1 \iff \log(1+y) \underset{y \to 0}{\sim} y$$



es. $\displaystyle \lim_{x \to 0} \frac{\log(1 - x^2)}{x^2} =$ $\qquad -x^2 = y$

$$= \lim_{y \to 0} \frac{\log(1+y)}{-y} = -1$$

- $$\lim_{x \to 0} \frac{\log x^2}{x^2} = -\infty$$

~~no~~

$$\frac{\lg(1+y)}{y}$$

$x \to 0$

$\log x^2 \to -\infty$

$x^2 \to 0^+$

$$\frac{-\infty}{0^+} = -\infty.$$

non si può applicare il limite notevole!

$$\boxed{\lim_{x \to 0} \frac{e^x - 1}{x} = 1}$$

$y = e^x - 1$ $\qquad x \to 0 \Rightarrow y \to 0$

$e^x = y + 1$ $\qquad x = \log(y+1)$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{y \to 0} \frac{y}{\log(y+1)} = 1$$

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1 \iff e^x - 1 \sim x \,,\ x \to 0$$

$$\lim_{x\to c} f(x) = e \implies \lim_{x\to c} \frac{1}{f(x)} = \frac{1}{e}$$

$$\boxed{\lim_{x\to 0} \frac{a^x - 1}{x} = \log a}$$

$$\lim_{x\to 0} \frac{7^x - 1}{x}$$

$a \in \mathbb{R},\ a > 0$

$$a^x = e^{\log a^x} = e^{x \log a}$$

$$\uparrow$$

$$\frac{a^x - 1}{x} = \frac{e^{x\log a} - 1}{x} =$$

$x \log a = y$

$x = \frac{y}{\log a}$

$x \to 0$

$\Rightarrow y \to 0$

$$= \underbrace{\frac{e^y - 1}{y}}_{\to 1} \cdot \log a$$

es. $\lim_{x \to 0} \frac{5^x - 7^x}{x} = \lim_{x \to 0} \underbrace{7^x}_{\to 1} \underbrace{\frac{\left(\frac{5}{7}\right)^x - 1}{x}}_{x \to 0}$

$= \log \frac{5}{7}$ $\quad a = \frac{5}{7}$

$$\log\frac{5}{7} \leftarrow \frac{a^x-1}{x}$$

$$\bullet \quad \lim_{x\to+\infty} \frac{5^x - 7^x}{x} = \lim_{x\to+\infty} \frac{+7^x\left(\overbrace{\left(\frac{5}{7}\right)^x}^{\to 0} - 1\right)}{x}$$

$$= -\infty$$

fare anche fu $x \to -\infty$

$$\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha, \quad \forall \alpha \in \mathbb{R}$$

es.

$$\lim_{x\to 0} \frac{\sqrt[3]{1+x} - 1}{\sqrt{x}\,\sqrt{x}} \cdot \sqrt{x} = 0$$

$$\lim_{x\to 0} \frac{\sqrt[3]{1+x} - 1}{x} = \frac{1}{3}$$

$1/3$

$$\lim_{x \to 0} \frac{(1+x)^{\alpha} - 1}{x} = \alpha \quad , \forall \alpha \in \mathbb{R}$$

$$y = (1+x)^{\alpha} - 1$$

$x \to 0$
$y \to 0$

$$(1+x)^{\alpha} = 1 + y$$

$$\log (1+x)^{\alpha} = \log (1+y)$$

$$\underbrace{\alpha \log (1+x)} = \underbrace{\log (1+y)}$$

$y \to 0$

$$\underbrace{\frac{\log(1+y)}{y}}_{\substack{\downarrow\, y\to 0 \\ 1}} = \frac{\alpha \log(1+x)}{(1+x)^{\alpha}-1} = \qquad x \to 0$$

$$= \alpha \underbrace{\frac{\log(1+x)}{x}}_{\substack{x\to 0 \\ 1}} \cdot \frac{1 \cdot x}{(1+x)^{\alpha}-1}$$

Quindi

$$\alpha \frac{x}{(1+x)^{\alpha}-1} \underset{x\to 0}{\longrightarrow} 1$$

$$\frac{(1+x)^{\alpha}-1}{x\cdot\alpha} \to 1 \qquad x\to 0$$

$$\frac{(1+x)^{\alpha}-1}{x} \to \alpha \qquad x\to 0$$

$$\lim_{x\to 0} \frac{(1+x)^{\sqrt{2}}-1}{x} = \sqrt{2}$$

$$\lim_{x\to 0} \frac{\sqrt[3]{1+x}-1}{\sqrt{x}}$$

$$\sqrt[3]{1+x} \to (1+x)^{1/3}$$

$$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{\sqrt{x}} = \lim_{x\to 0} \frac{x}{\sqrt{1+x}+1} \cdot \frac{1}{\sqrt{\cancel{x}}}$$

$$\sqrt{1+x}-1 = \frac{\left(\sqrt{1+x}-1\right)\left(\sqrt{1+x}+1\right)}{\sqrt{1+x}+1} =$$

$$= \frac{\cancel{1}+x-\cancel{1}}{\sqrt{1+x}+1} = \frac{x}{\sqrt{1+x}+1}$$

$$= \lim_{x \to 0} \frac{\sqrt{x}}{\sqrt{1+x} + 1} = 0$$

$$\lim_{x \to 0} \frac{\sqrt[3]{1+x} - 1}{\sqrt{x}}$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$\underbrace{\sqrt[3]{1+x}}_{a} - \underbrace{1}_{b}$$

$$\frac{\left(\sqrt[3]{1+x}-1\right)\cdot\left(\sqrt[3]{(1+x)^2}+\sqrt[3]{1+x}+1\right)}{\left(\sqrt[3]{(1+x)^2}+\sqrt[3]{1+x}+1\right)}=$$

$$=\frac{(\cancel{1}+x)-\cancel{1}}{\sqrt[3]{(1+x)^2}+\sqrt[3]{1+x}+1}$$

$$\lim_{x\to 0}\frac{\sqrt[3]{1+x}-1}{\sqrt{x}}=\lim_{x\to 0}\frac{\overset{\sqrt{x}}{\cancel{x}}}{\left(\sqrt[3]{(1+x)^2}+\sqrt[3]{1+x}+1\right)}\frac{1}{\cancel{\sqrt{x}}}$$

$$=0$$

$$\lim_{x \to 0} \frac{\text{sen } x}{x} = 1$$

$$\lim_{x \to \pm\infty} \left(1 + \frac{1}{x}\right)^{x} = e$$

$$\lim_{x \to c} \frac{\text{sen } \varepsilon(x)}{\varepsilon(x)} =$$

$$= \lim_{y \to 0} \frac{\text{sen } y}{y} = 1$$

$\varepsilon(x) \to 0$

$x \to c$

$y = \varepsilon(x)$

limiti di polinomi

$$\lim_{x \to \pm\infty} a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 =$$

$$= \lim_{x \to \pm\infty} a_n x^n \left(1 + \underbrace{\frac{a_{n-1}}{a_n} \frac{1}{x}}_{\to 0} + \dots + \underbrace{\frac{a_0}{a_n x^n}}_{\to 0}\right)$$

$$= \lim_{x \to \pm\infty} a_n x^n$$

es. $$\lim_{x \to +\infty} -x^3 + 8x - 2 = -\infty$$

$$\lim_{x \to -\infty} x^6 - 3x + 8 = +\infty$$

Limiti di funzioni razionali (cioè quozienti di polinomi)

$$\lim_{x \to \pm\infty} \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0} =$$

$$= \lim_{x \to +\infty} \frac{a_n x^n}{b_m x^m} = \begin{cases} \pm\infty \left(\frac{a_n}{b_m}\right) & n > m \\ \frac{a_n}{b_m} & n = m \\ 0 & n < m \end{cases}$$

$$\lim_{x \to x_0} \left(f(x)\right)^{g(x)}$$

$f(x) > 0$ in un intorno di $x = x_0$

$$\left(f(x)\right)^{g(x)} = e^{\log\left(f(x)\right)^{g(x)}} = e^{g(x)\log f(x)}$$

- caso di ieri

$$\lim_{x \to +\infty} \left(\frac{x+3}{x+4}\right)^{\frac{x^2+7}{2x+1}}$$

$\underline{b)}$ $\lim_{x \to +\infty} x^{\frac{1}{\log(x+1)}}$

$$x^{\frac{1}{\log(x+1)}} = e^{\log\left(x^{\frac{1}{\log(x+1)}}\right)} =$$

$$= e^{\frac{1}{\log(x+1)} \log x}$$

intanto facciamo $\lim_{x \to +\infty} \frac{\log x}{\log(x+1)} =$

$$= \lim_{x \to +\infty} \frac{\log x}{\log\left(x\left(1+\frac{1}{x}\right)\right)} = \lim_{x \to +\infty} \frac{\log x}{\log x + \log\left(1+\frac{1}{x}\right)}$$

$$= \lim_{x \to +\infty} \frac{1}{1 + \underbrace{\left(\frac{\log\left(1+\frac{1}{x}\right)}{\log x}\right)}_{\to 0}} = 1$$

$$\lim_{x \to +\infty} x^{\left(\quad\right)} = e^{1} = e$$

PC.
$$\lim_{x \to +\infty} \left(\frac{x+1}{2x + 3^{-1/x}}\right)^{\sqrt{x^2+1} - x}$$

$$\left(\sqrt{x^2+1} - x\right) \log\left(\frac{x+1}{2x+3^{-1/x}}\right)$$