Lezione del primo di Dicembre

Teorema di De l'Hôpital

$f, g : (a, b) \to \mathbb{R}$

i) $\lim_{x \to a^+} f = \lim_{x \to a^+} g = 0$

$= +\infty$

$= -\infty$

ii) f, g derivabili in (a, b) e $g'(x) \neq 0 \quad \forall x \in (a, b)$

$g(x) \neq 0 \quad$ "

iii) $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L$

allora $\displaystyle\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$

<u>Dim.</u> (caso facile) $\displaystyle\lim_{x\to a^+} f = \lim_{x\to a^+} g = 0 = f(a)$
$= g(a)$

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f(x)-f(a)}{(x-a)} \cdot \frac{(x-a)}{g(x)-g(a)}$$

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} \cdot \frac{(x-a)}{g(x)-g(a)} =$$

$$= \frac{f'_+(a)}{g'_+(a)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$$

$\Downarrow$ conseguenze del teorema di Lagrange #

<u>Es.</u> $\lim_{x \to +\infty} \frac{\log x}{x}$ $\frac{\infty}{\infty}$

$$\lim_{x \to +\infty} \frac{f'}{g'} = \lim_{x \to +\infty} \frac{\frac{1}{x}}{1} = 0 \implies \lim_{x \to +\infty} \frac{\log x}{x} = 0$$

<u>Es.</u> $\lim_{x \to 0^+} x \log x = \lim_{x \to 0^+} \frac{\log x}{\left(\frac{1}{x}\right)}$

$$\lim_{x \to 0^+} \frac{\frac{1}{x}}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0$$

es. $\lim_{x \to +\infty} \frac{x + \operatorname{sen} x}{x} = 1$ ma

$\lim_{x \to +\infty} \frac{1 + \cos x}{1} \cancel{\nexists}$

se $\cancel{\nexists} \lim \frac{f'}{g'} \cancel{\Rightarrow} \nexists \lim \frac{f}{g}$

semplicemente non si può applicare l'Hôpital

OSS. si applica solo alle forme indeterminate

es. $\lim_{x \to 0} \frac{\cos x}{x^2} = +\infty$

$\frac{\cos x}{x^2}$ non è forma indeterminata

$\frac{1}{0}$

Hopital $\lim_{x \to 0} \frac{-\operatorname{sen} x}{2x} = -\frac{1}{2}$

$\underline{\qquad} \cdot \underline{\qquad}$

es. $\lim_{x \to 0^+} \frac{e^{-1/x^2}}{x}$ f g

$\frac{e^{-1/x^2}}{x}$

$$\lim_{x\to 0^+} \frac{f'}{g'} = \left(\frac{e^{-1/x^2}}{1} \cdot \left(\frac{-2}{x^3} \right) \right)$$

peggio di prima!

$$\frac{1}{x} = y$$

$$\lim_{x\to 0^+} \frac{e^{-\frac{1}{x^2}}}{x} = \lim_{y\to +\infty} y\, e^{-y^2} =$$

$$= \lim_{y\to +\infty} \frac{y}{e^{y^2}} \qquad \lim_{y\to +\infty} \frac{1}{e^{y^2}\cdot 2y} = 0$$

$$\lim_{x \to +\infty} \frac{2^{x^4} - 2x^x}{3^x + e^{3x}}$$

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es.

$$\lim_{x \to +\infty} \frac{3x^3 + 5x + 2}{2x^3 + 1}$$

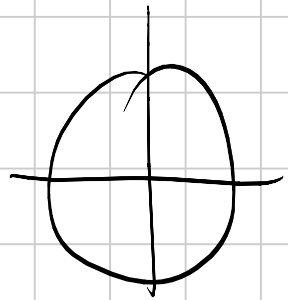
$$\frac{f'}{g'} = \frac{9x^2 + 5}{6x^2} = \frac{\overline{f}}{\overline{g}}$$

di nuovo le derivate

$$\frac{18\cancel{x}}{12\cancel{x}}$$

es.

$$\frac{\operatorname{sen} x}{x} \xrightarrow[x \to 0]{} 1$$



$$\frac{\operatorname{sen} x}{x} = \frac{f}{g} \qquad \frac{f'}{g'} = \frac{\cos x}{1} \xrightarrow[x \to 0]{} 1$$

$$\frac{\operatorname{sen} x}{x} \to 1$$

$$\lim_{x \to 0} \frac{\operatorname{sen} x + \cos x - e^{x}}{\log(1+x) - x} =$$

Hopital

$$\lim_{x \to 0} \frac{\cos x - \operatorname{sen} x - e^{x}}{\frac{1}{1+x} - 1}$$

Hopital

$$\lim_{x \to 0} \frac{-\operatorname{sen} x - \cos x - e^{x}}{-\frac{1}{(1+x)^2}} = 2$$

$\exists = 2$

Definizione di "o piccolo"

Def. $f(x) = o(g(x))$, $x \to x_0$ significa

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$$

es. $1 - \cos x = o(x)$, $x \to 0$

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0 \quad ?$$

$$\frac{1-\cos x}{x^{\cancel{2}}}\ \cancel{x}^{\cancel{2}} \xrightarrow[x\to 0]{} 0$$

$\frac{1}{2}$

<u>oss.</u> se $f, g \to 0$, $x \to x_0$ (cioè sono infinitesimi)

$f = o(g)$ $\quad \frac{f}{g} \to 0$ $\quad$ f è infinitesimo di ordine superiore rispetto a g

<u>es.</u> $\lim_{x\to 0} \frac{x + x^2 + x^3}{\operatorname{sen} x}$

$$= \lim_{x \to 0} \frac{x(1 + x + x^2)}{\operatorname{sen} x}$$

$$= \lim_{x \to 0} \frac{x}{\operatorname{sen} x}$$

$$x + x^2 + x^3 = x + o(x)$$

$$x^2 + x^3 = o(x) \qquad x \to 0$$

$$\Updownarrow$$

$$\frac{x^2+x^3}{x} \to 0 \quad x \to 0$$

$$x + x^2 \to 0 \qquad x \to 0$$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} \qquad \frac{\quad}{\quad}$$

$$f(x) = P(x) + o(x^\alpha)$$

$$g(x) = Q(x) + o(x^\beta)$$

$P(x)$, $Q(x)$ polinomi

Relazione tra "asintotico" e "o piccolo"

$$x \to x_0 \qquad f(x) \sim g(x) \iff f(x) = g(x) + o(g(x))$$

$$\Downarrow$$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$$

?

<u>es.</u> $x \to 0 \qquad \operatorname{sen} x \sim x \iff \boxed{\operatorname{sen} x \underset{x \to 0}{=} x + o(x)}$

lo verifico:

?

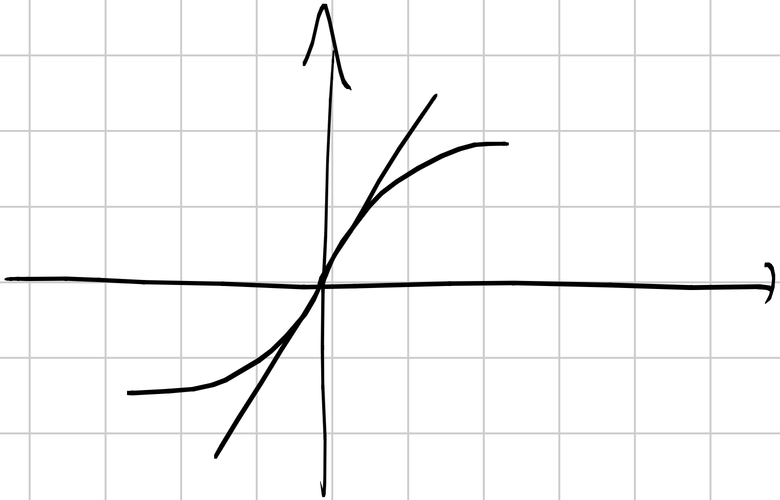
$$\operatorname{sen} x - x = o(x)$$

$$\lim_{x \to 0} \frac{\operatorname{sen} x - x}{x} =$$

$$= \lim_{x \to 0} \left(\frac{\operatorname{sen} x}{x} - 1 \right) = 0$$

$$\boxed{\operatorname{sen} x \underset{x \to 0}{=} x + o(x)}$$

sen x = polinomio di grado 1 + "robe" che va a zero più velocemente di x



es. $$\lim_{x \to 0} \frac{1-\cos x}{\frac{x^2}{2}} = 1$$

$$x \to 0 \qquad 1-\cos x \sim \frac{x^2}{2} \iff 1-\cos x = \frac{x^2}{2} + o\left(\frac{x^2}{2}\right)$$

$$-\cos x = -1 + \frac{x^2}{2} + o\left(\frac{x^2}{2}\right)$$

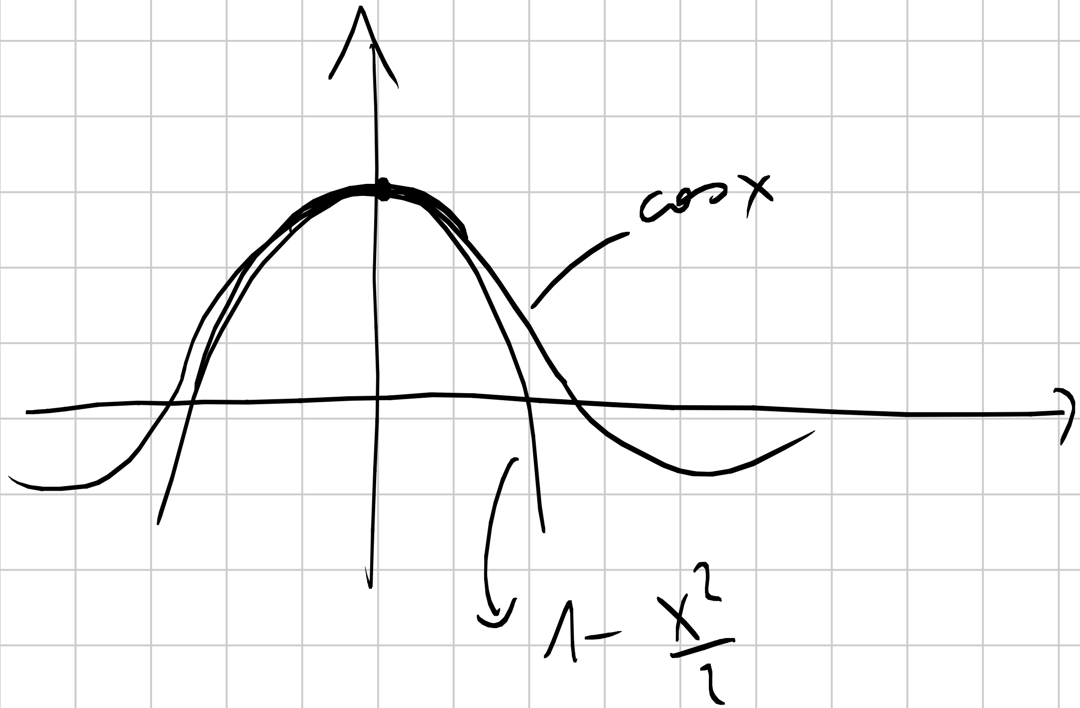
$$\cos x = 1 - \frac{x^2}{2} - o\left(\frac{x^2}{2}\right)$$

"o piccolo" esprime una proprietà e non è una particolare funzione

$$o\left(\frac{x^2}{2}\right) = o\left(x^2\right)$$

$$-\,o\left(x^2\right) = o\left(x^2\right)$$

$$\cos x = 1 - \frac{x^2}{2} + o\left(x^2\right) \qquad x \to 0$$



es. $e^x - 1 \underset{x\to 0}{\sim} x \iff \boxed{e^x = 1 + x + o(x) \quad x \to 0}$

$e^x = P(x) + \text{resto}$

es.

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$\Downarrow$$

$$\lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right) = 0$$

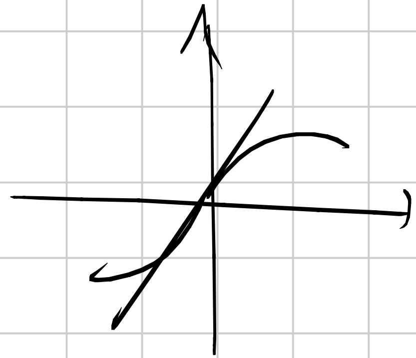
$$\lim_{x \to x_0} \frac{\big(f(x) - f(x_0)\big) - f'(x_0)(x - x_0)}{x - x_0} = 0$$

$\Downarrow$ per def. di "o piccolo"

$$f(x) - f(x_0) - f'(x_0)(x - x_0) = o(x - x_0)_{x \to x_0}$$

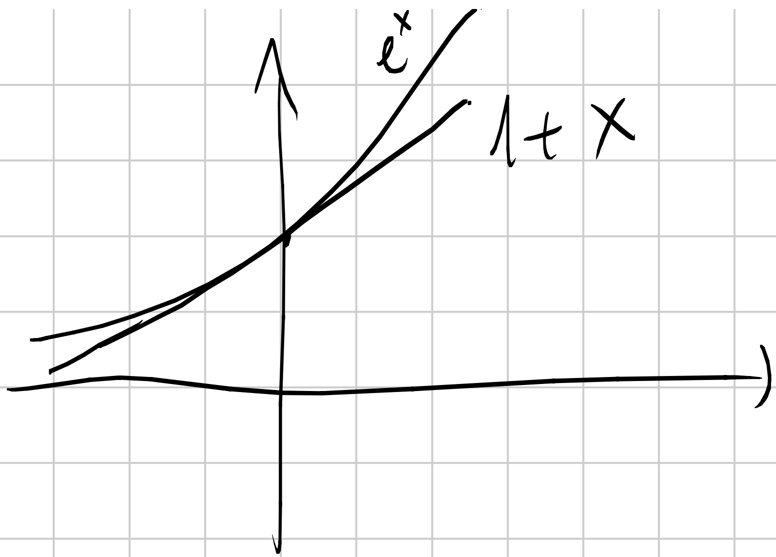
$$f(x) = \underbrace{f(x_0) + f'(x_0)(x-x_0)}_{\text{polinomio di grado 1}} + o(x-x_0)$$

eq. retta tangente al grafico di f in x_0



$$f(x) = r(x) + o(x-x_0)$$

$$e^x = \underbrace{1 + x}_{\text{eq. retta tangente al grafico di } e^x \text{ in } x=0} + \underset{x \to 0}{o}(x)$$



Vogliamo scrivere

$$f(x) = \underbrace{P_n(x)}_{\text{polinomio di grado } n} + o\left(x^n\right)$$

polinomio di Taylor di grado n

$$\boxed{x=0}$$

$f(x)$ derivabile n volte in $x=0$

$f(0)$, $f'(0)$, $f''(0)$, ... $f^{(n)}(0)$

$$T_n(x) = f(0) + f'(0)\,x + f''(0)\,\frac{x^2}{2} + f'''(0)\,\frac{x^3}{3!} + \dots + f^{(n)}(0)\,\frac{x^n}{n!}$$

$x=0$

polinomio di McLaurin

es. $f(x) = e^x$ $\quad f(0) = 1$

$f'(x) = e^x$ $\quad f'(0) = 1$

$$f''(x) = e^x \qquad f''(0) = 1$$

$$T_n(x) = 1 + 1x + 1\frac{x^2}{2} + 1\frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \frac{x^n}{n!}$$

Esercizi su studio di grafici di funzione.

$$f(x) = |\operatorname{arctg}(\log x)|$$

$D = \{x > 0\}$ $\quad f \geq 0 \quad f = 0 \iff x = 1$

$$\lim_{x \to 0^+} f(x) = \frac{\pi}{2}$$

$$\lim_{x \to +\infty} f(x) = \frac{\pi}{2}$$ asintoto orizz. $x \to +\infty$

$$f(x) = \begin{cases} \operatorname{arctg}(\log x) & x \geq 1 \\ -\operatorname{arctg}(\log x) & 0 < x < 1 \end{cases}$$

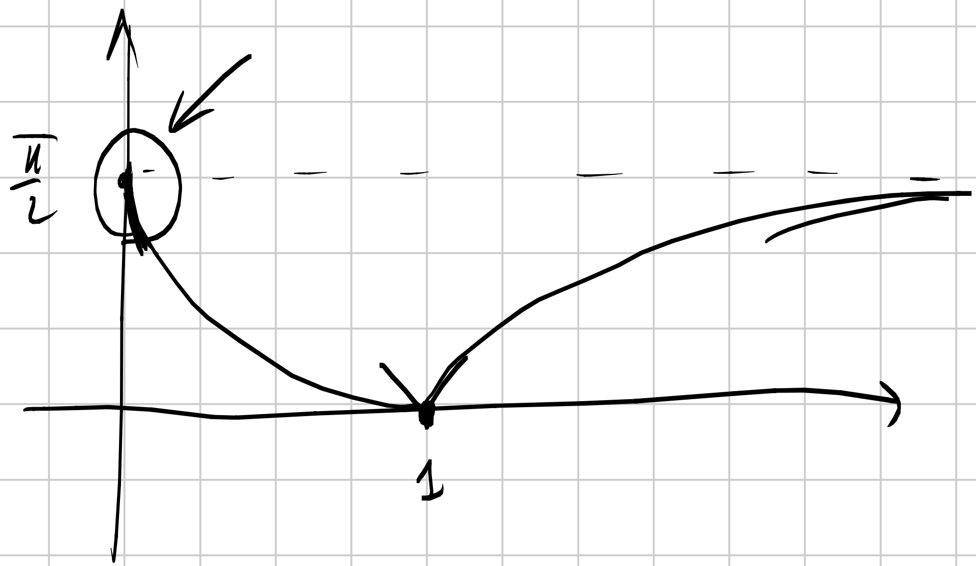
$x = 1$ $\quad$ f è derivabile ?

$$f'(x) = \begin{cases} \dfrac{1}{1+\log^2 x} \cdot \dfrac{1}{x} > 0 & x > 1 \\ -\dfrac{1}{1+\lg^2 x} \cdot \dfrac{1}{x} < 0 & 0 < x < 1 \end{cases}$$

$$\lim_{x \to 1^+} f' = 1 = f'_+(1) \qquad \lim_{x \to 1^-} f' = -1 = f'_-(1)$$

$x = 1$ punto di minimo globale

$f(1) = 0$



f è strett. crescente per $x > 1$

" decresc. per $0 < x < 1$

$x > 1$

$$f'(x) = \frac{1}{(1 + \log^2 x) \cdot x} \qquad x > 1$$

$$f''(x) = -\frac{1}{(1+\log^2 x)^2 x^2} \cdot \left(2\log x \cdot \cancel{\frac{1}{x}} \cdot \cancel{x} + (1+\log^2 x) \right)$$

$$= \frac{-1}{(1+\log^2 x)^2 x^2} \cdot (1+\log x)^2 < 0$$

$x > 1$

f è strett. concava $\forall x > 1$

$x < 1 \qquad f'' = -\left(\qquad \right) > 0$

strett. converse $0 < x < 1$

attacco $\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} -\frac{1}{1+\log^2 x} \cdot \frac{1}{x} = -\infty$

D. $\lim_{x \to 0} x(1+\log^2 x) = \underbrace{x}_{\to 0} + \underbrace{x \log^2 x}_{\to 0} \to 0$

$\underline{\qquad\qquad} \cdot \underline{\qquad\qquad}$

Es. $f(x) = \log\left(e^{x} + e^{-x}\right) + x$

$D = \mathbb{R}$

$\lim_{x \to +\infty} f(x) = +\infty \longrightarrow$ c'è asintoto obliquo? per $x \to +\infty$

$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \log\left(e^{-x}\left(1 + e^{2x}\right)\right) + x$

$= \lim_{x \to -\infty} \underbrace{\log\left(e^{-x}\right)}_{\cancel{-x}} + \log\left(1 + e^{2x}\right) + \cancel{x} =$

$$= \lim_{x \to -\infty} \log\left(1 + e^{2x}\right) = 0$$

asintoto orizzontale
per $x \to -\infty$.

$$\cdot \lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \frac{\log(e^{x} + e^{-x})}{x} + 1$$

Hopital

$$= \lim_{x \to +\infty} \frac{\log\left(e^{x}\left(1 + e^{-2x}\right)\right)}{x} + 1$$

$$= \lim_{x \to +\infty} \frac{x + \log\left(1 + e^{-2x}\right)}{x} + 1$$

$$= \lim_{x \to +\infty} 1 + \frac{\log(1+e^{-2x})}{x} + 1 = 2$$

$\to 0$

$$\frac{\log(e^{x}+e^{-x})}{x}$$

Hôpital

$$\frac{1}{e^{x}+e^{-x}} \cdot e^{x} - e^{-x} = \frac{\cancel{e^{x}}(1-e^{-2x})}{\cancel{e^{x}}(1+e^{-2x})}$$

$\to 1$

1

$$\lim_{x \to +\infty} f(x) - 2x =$$

$$= \lim_{x \to +\infty} \log\left(e^{x} + e^{-x}\right) + x - 2x =$$

$$= \lim_{x \to +\infty} \log\left(e^{x} + e^{-x}\right) - x$$

$$= \lim_{x \to +\infty} \cancel{x} + \log\left(1 + e^{-2x}\right) \cancel{- x} = 0$$

$q = 0$ $\qquad$ $y = 2x$ asintoto obliquo $\text{per } x \to +\infty$

$$f(x) = \log\left(e^{x} + e^{-x}\right) + x = \log(2\cosh x) + x$$

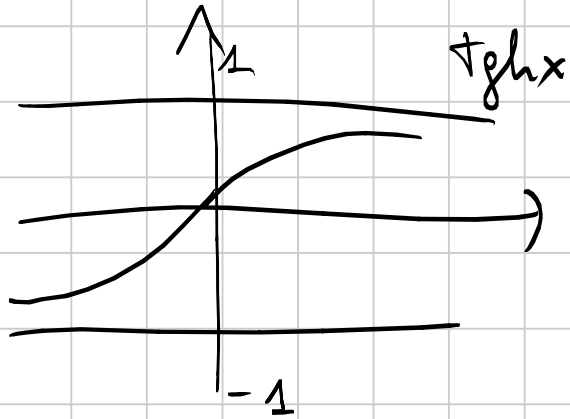
$x > 0 \Rightarrow f(x) > 0$

sempre: no!

$\exists\, f'(x) \quad \forall x \in \mathbb{R}$

$$f'(x) = \frac{1}{e^x + e^{-x}} \cdot e^x - e^{-x} + 1$$

$$f'(x) = 0 \qquad f'(x) = \operatorname{tgh} x + 1$$



$$f'(x) = \operatorname{tgh} x + 1 > 0$$

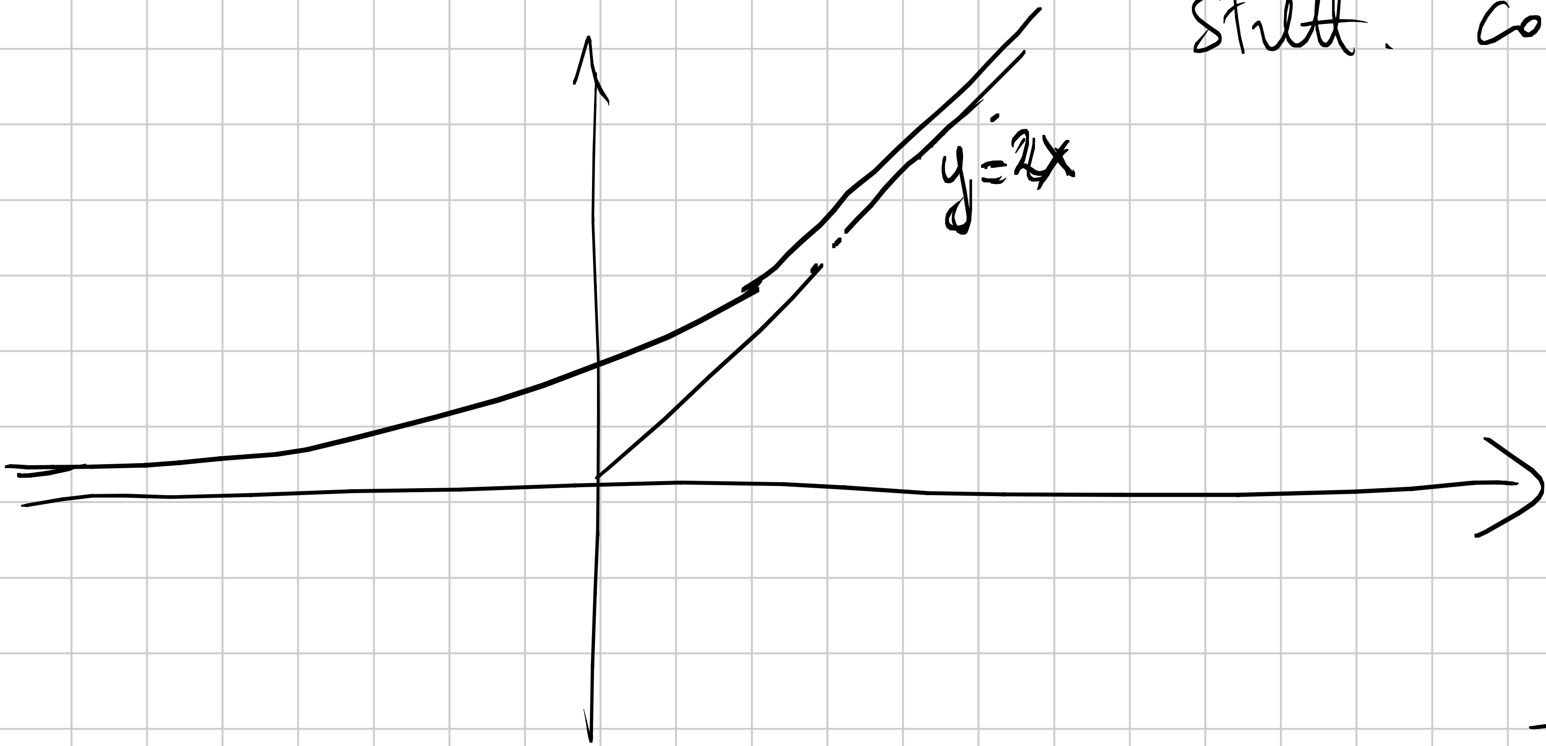
$$\frac{e^x - e^{-x}}{e^x + e^{-x}} + 1 > 0$$

$$f'(x) = \frac{\operatorname{senh} x}{\cosh x} + 1$$

f strett. crescente

$$f''(x) = \frac{\cosh x \cdot \cosh x - \operatorname{senh} x \operatorname{senh} x}{\cosh^2 x} = \frac{1}{\cosh^2 x} > 0$$

strett. convessa



$$\Rightarrow \log\left(e^{x} + e^{-x}\right) + x > 0 \quad \forall x \in \mathbb{R}$$