Lezione del 12 Gennaio 2012

Serie a termini nonnegativi

$\sum a_k$

- non è mai irregolare — Si ha sempre limite (finito o infinito)
- se $a_k \not\to 0$ ⇒ la serie diverge

. se $a_k \to 0$ — s_n esplicitamente
— s_n ? $\Rightarrow$ criteri di convergenza

Criteri per serie a termini nonnegativi

1) Criterio dell'integrale $\int_1^{+\infty} f(x)\,dx \iff \sum_{k=1}^{+\infty} f(k)$

$$\sum_{k=1}^{\infty} \frac{1}{k^\alpha} \text{ converge } \iff \alpha > 1$$

$$\sum_{k=2}^{+\infty} \frac{1}{k^\alpha (\log k)^\beta} \text{ converge se } \begin{cases} \alpha > 1, & \forall \beta \\ \alpha = 1, & \forall \beta > 1 \end{cases}$$

2) Criterio confronto

$0 \le a_k \le b_k$

$\sum b_k$ conv. $\Rightarrow \sum a_k$ conv.

$\sum a_k$ div. $\Rightarrow \sum b_k$ div.

3) Criterio del confronto asintotico

$a_k, b_k \geq 0$ $\quad a_k \underset{k \to +\infty}{\sim} b_k$ $\quad \sum a_k \quad \sum b_k$ hanno lo stesso carattere

$a_k \sim$

<u>oss.</u> $\quad a_k = o(b_k) \quad k \to +\infty$ $\quad \sum b_k$ conv. $\Rightarrow \sum a_k$ conv.

$$\sum \frac{1}{e^k} \qquad a_k \to 0$$

$$\frac{1}{e^k} \sim \frac{1}{k^2} \quad \text{No!}$$

$$\frac{1}{e^k} \longrightarrow \frac{1}{k^2}$$

$$\frac{1}{e^k} = o\left(\frac{1}{k^2}\right)$$

$\sum \frac{1}{k^2}$ conv.

Criterio del rapporto $\quad \sum a_n \,, \quad a_n \geq 0$

$$\lim_n \frac{a_{n+1}}{a_n} = \ell$$

- $\ell < 1$ la serie converge
- $\ell > 1$ la serie diverge

$l = 1$ non si può dire nulla

caso

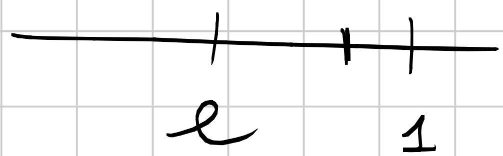
Dim. $l < 1$

$$\lim_n \frac{a_{n+1}}{a_n} = l$$

$$\forall \varepsilon > 0 \quad \exists \nu : \quad \left| \frac{a_{n+1}}{a_n} - l \right| < \varepsilon \quad \forall n > \nu$$

$$\downarrow$$

$$\frac{a_{n+1}}{a_n} < l + \varepsilon$$



$$\boxed{a_{n+1} < (l+\varepsilon)\, a_n} \quad (1)$$

prendo ε t.c. $(\ell+\varepsilon)<1$

$$\boxed{a_n < (\ell+\varepsilon)\, a_{n-1}} \quad 2$$

$$a_{n+1} \underset{1}{<} (\ell+\varepsilon)\, a_n \underset{2}{<} (\ell+\varepsilon)\cdot\underbrace{(\ell+\varepsilon)\, a_{n-1}} =$$

$$= (\ell+\varepsilon)^2\, a_{n-1} <$$

$$a_{n-1} < (\ell+\varepsilon)\, a_{n-2}$$

$$< (\ell+\varepsilon)^2 (\ell+\varepsilon)\, a_{n-2} =$$

$$= (l+\varepsilon)^3 a_{n-2} \leq \dots$$

$$\leq \dots \; (l+\varepsilon)^n a_1$$

$$a_{n+1} \leq \underbrace{(l+\varepsilon)^n a_1}_{b_n}$$

devo dim. che $\sum a_n$ converge

$$\sum b_n = \sum (l+\varepsilon)^n a_1 = a_1 \sum (l+\varepsilon)^n$$

$\sum (l+\varepsilon)^n$ è serie geometrica di ragione $q = l+\varepsilon$

$l + \varepsilon < 1$

e quindi è una serie convergente

$$a_{n+1} \leq \underbrace{(l+\varepsilon)^n}\, a_1$$

$\sum (\ )$ converge $\Rightarrow$

$\sum a_n$ converge

dal principio del confronto. #

es. $\sum \underset{a_n}{\frac{1}{n!}}$ $\quad a_n \to 0$

rapporto $\dfrac{a_{n+1}}{a_n} = \dfrac{1}{(n+1)!}\, n! = \dfrac{\cancel{n!}}{\cancel{n!}\,(n+1)}$

$$\lim_n \frac{a_{n+1}}{a_n} = \lim_n \frac{1}{(n+1)} = 0 < 1$$

$\sum \dfrac{1}{n!}$ converge

__da fare__ $\sum \dfrac{n!}{n^n}$ a_n

rapporto
$\dfrac{a_{n+1}}{a_n} \longrightarrow \dfrac{1}{e} < 1$
(verificare)

prova ad usarlo per la serie armonica

$$\sum \frac{1}{n} \quad a_n \qquad \frac{a_{n+1}}{a_n} = \frac{n}{n+1} \to 1$$

non si può dire nulla.

$$\sum \frac{1}{n^2} \qquad \frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2} \to 1$$

$$\sum (3^n) \qquad 3^n \not\to 0$$

la serie diverge

$$\sum \left(\frac{1}{3}\right)^n \quad a_n$$

$$\frac{a_{n+1}}{a_n} = \left(\frac{1}{3}\right)^{n+1} 3^n = \frac{3^n}{3^{n+1}} = \frac{1}{3} \to \frac{1}{3} < 1$$

$\sum \left(\frac{1}{3}\right)^n$ converge (ma lo sapevamo già!)

<u>Criterio della radice</u> $\sum a_n, \quad a_n \geq 0$

$$\lim_n \sqrt[n]{a_n} = l \begin{cases} l < 1 & \text{la serie converge} \\ l > 1 & \text{la serie diverge} \\ l = 1 & \text{non si può dire nulla} \end{cases}$$

<u>Dim</u> no.

es. $\sum \frac{3^n}{n^n}$ a_n $\left(\frac{3}{n}\right)^n$

$$\sqrt[n]{a_n} = \frac{3}{n} \xrightarrow[n]{} 0 < 1$$

la serie converge.

es. Dire per quali $b \in \mathbb{R}$, $b>0$ $\sum 2^n \frac{b^n}{e^n}$ a_n

converge

$$\sqrt[n]{a_n} = \frac{2b}{e} < 1 \qquad b < \frac{e}{2}$$

la serie converge

$$> 1 \qquad b > \frac{e}{2}$$

la serie diverge

$$\frac{2b}{e} = 1 \qquad b = \frac{e}{2}$$

$$\sum 2^n \frac{b^n}{e^n} = \sum \frac{\cancel{2^n}}{\cancel{e^n}} \frac{\cancel{e^n}}{\cancel{2^n}} = \sum 1$$

diverge.

b.

$$\sum \underbrace{\left(1-\frac{1}{2n}\right)^{5n^2}}_{a_n}$$

$$\sqrt[n]{a_n} = \left(1-\frac{1}{2n}\right)^{\frac{5n^{\cancel{2}}}{\cancel{n}}} =$$

$$= \left(1-\frac{1}{2n}\right)^{\frac{5n\,2}{2}} = \Bigg[\underbrace{\left(1-\frac{1}{2n}\right)^{2n}}_{\to \frac{1}{e}}\Bigg]^{5/2} \to \frac{1}{e^{5/2}} < 1$$

$\sum a_n$ converge.

P.C. $$\sum 3^{n} \left(1 - \frac{1}{n^{3/2}}\right)^{n^{5/2}}$$

$$\sqrt[n]{a_n} = 3 \left(1 - \frac{1}{n^{3/2}}\right)^{\frac{n^{5/2}}{n}} = \ldots \text{finire}$$

$$\left(R. \longrightarrow \frac{3}{e} > 1\right)$$

proprietà

$\sum a_k$, $\sum b_k$ convergenti

$$\sum (a_k + b_k) = \underbrace{\sum a_k} + \underbrace{\sum b_k}$$ è convergente

$$\sum a_k \text{ converge } \Rightarrow \sum c\, a_k = c \sum a_k$$

$c \in \mathbb{R}$ è convergente

es. $$\sum \left(\frac{3}{k^2} - \left(\frac{1}{5}\right)^k \right) = \sum \frac{3}{k^2} - \sum \left(\frac{1}{5}\right)^k$$

$$= 3 \sum_1 \frac{1}{k^2} - \sum_1 \left(\frac{1}{5}\right)^k$$

$\downarrow$ conv.

(serie armonica generalizzata con $\alpha = 2$)

$\downarrow$ conv. (serie geom. di ragione $\frac{1}{5} < 1$)

esercizi

$$\sum \frac{n^2}{2n^2 - 8}$$

$a_n \geq 0$?

$2n^2 - 8 > 0$

$n^2 > 4$

$n > 2$

vale la proprietà $a_n \geq 0$ da un certo n in poi.

$$\sum \frac{n^2}{2n^2 - 25}$$

$$n^2 > \frac{25}{2} \qquad a_n \geq 0$$

$$n > \frac{5}{\sqrt{2}}$$

$$\sum \frac{n^2}{2n^2 - 8} \quad a_n$$

$$a_n \longrightarrow \frac{1}{2} \neq 0$$

la serie <u>non</u> converge

$\Rightarrow$ la serie diverge.

$$\cdot \sum_{k=1}^{\infty} \underbrace{\frac{1}{k^5 (\log^2 k + \log k + 1)}}_{a_k}$$

$a_k \to 0$

$a_k \geq 0$ $\qquad a_k \sim \dfrac{1}{k^5 \log^2 k}$

$\sum \dfrac{1}{k^5 \log^2 k}$ converge $\Rightarrow \sum a_k$ converge.

- $$\sum \frac{k(k-10)}{k^4+1}$$

$a_k \to 0$ $\quad k \to +\infty$

$a_k \geq 0$?

$k \geq 10$

$$a_k \sim \frac{1}{k^2}$$

$\sum \frac{1}{k^2}$ converge

$\Rightarrow \sum a_k$ converge.

es.

$$\sum \frac{n + e^{-n} + \sin n + \log n}{n^5}$$

$a_n \geqslant 0$? si

$$a_n \sim \frac{n}{n^5} = \frac{1}{n^4} \qquad \sum \frac{1}{n^4} \text{ converge}$$

$\Rightarrow \sum a_n$

es. $\displaystyle\sum \frac{e^n + 3^n}{n^3 2^n + 2^{2n}}$

$$a_n \sim \frac{3^n}{4^n} = \left(\frac{3}{4}\right)^n \qquad 4^n\Big(1 + \underbrace{\frac{n^3 2^n}{4^n}}_{\to 0}\Big)$$

$$\sum \left(\frac{3}{4}\right)^n \qquad q = \frac{3}{4} < 1$$

$\sum a_n$ converge.

- Dire per quali $\alpha \in \mathbb{R}$

$$\sum_{n=1}^{\infty} n^{\alpha} \operatorname{arctg}\left(\frac{1}{n^3}\right) \qquad a_n \geq 0$$

$$a_n \sim n^{\alpha} \frac{1}{n^3} = \frac{1}{n^{3-\alpha}} \qquad \operatorname{arctg} x \sim x \quad x \to 0$$

$$x = \frac{1}{n^3}$$

$\sum \frac{1}{n^{3-\alpha}}$ converge sse $3-\alpha > 1$

$$\alpha < 2$$

$\forall \alpha < 2$ la serie converge

$$\sum n^2 \operatorname{arctg} n \to \frac{\pi}{2}$$

$$\sum \frac{\sqrt{n+1} - \sqrt{n}}{n}$$

$$a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n} \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} =$$

$$= \frac{(\cancel{n}+1) - \cancel{n}}{n\left(\sqrt{n+1} + \sqrt{n}\right)} = \frac{1}{n\left(\sqrt{n+1} + \sqrt{n}\right)}$$

$$\sim \frac{1}{n\, 2\sqrt{n}} = \frac{1}{2\, n^{3/2}}$$ converge $\sum \frac{1}{n^{3/2}}$

$\frac{3}{2} > 1$

$\Rightarrow \sum a_n$ converge.

- P.C. $$\sum \frac{\sqrt{n^4 + 2n} - n^2}{\sqrt{n}} \, \underbrace{\operatorname{sen}^2(\sqrt{n})}$$

Serie a termini di segno variabile

es. $$\sum \frac{\operatorname{sen} n}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \dots$$

$$\sum_{n=0}^{+\infty} (-3)^n = 1 - 3 + 9 - 27 + \dots$$

Def. $\sum a_n$ si dice assolutamente convergente se $\sum |a_n|$ converge

es. $\sum \frac{(-1)^n}{n^5} \to a_n$ conv. assolutamente?

$$\sum \left| \frac{(-1)^n}{n^5} \right| = \sum \frac{1}{n^5} \quad \text{converge}$$

$\Rightarrow \sum \frac{(-1)^n}{n^5}$ conv. assolutamente

Es. $\sum \frac{\operatorname{sen} n}{n^2}$ conv. assolutamente?

$$|a_n| = \frac{|\operatorname{sen} n|}{n^2}$$

$\sum \frac{|\operatorname{sen} n|}{n^2}$ converge?

≥ 0

per $\sum |a_n|$ posso usare i criteri delle serie a termini nonnegativi

$$\frac{|\operatorname{sen} n|}{n^2} \leq \frac{1}{n^2} \implies \sum \frac{1}{n^2} \text{ converge}$$

dal principio del confronto

converge $\sum \frac{|\operatorname{sen} n|}{n^2}$

Cioè $\sum \frac{\operatorname{sen} n}{n^2}$ converge assolutamente.

Esercizi vari su <u>integrali</u>

<u>es.</u> $$\int_1^{+\infty} \frac{\cos((2-\alpha)x)+3}{(e^x+1)^\alpha}\,dx$$

per quali $\alpha \in \mathbb{R}$ è convergente?

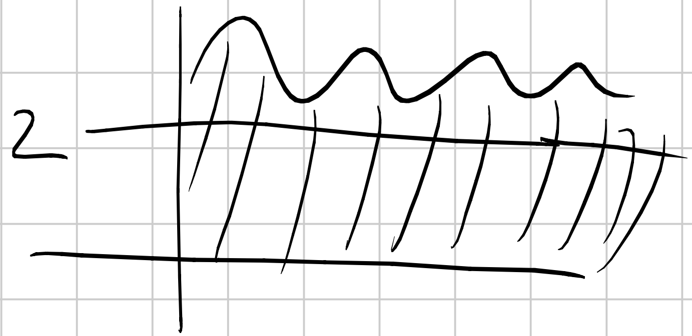
$$\int_1^{+\infty} f(x)\,dx \qquad f(x) \geq 0$$

$\alpha < 0 \longrightarrow \lim_{x \to +\infty} f(x) = +\infty$

non converge

$\alpha = 0 \qquad f(x) = \cos(2x) + 3 \geq 2$

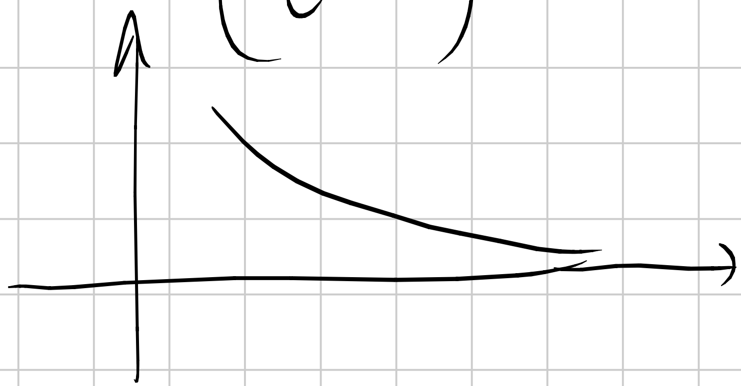
$\lim_{x \to +\infty} f(x) \nexists$



l'integrale diverge

$\alpha > 0 \qquad f(x) = \dfrac{\cos((2-\alpha)x) + 3}{(e^x+1)^\alpha}$

$\lim_{x \to +\infty} f(x) = 0$



$$f(x) \underset{x \to +\infty}{\sim} \frac{\cos((2-\alpha)x) + 3}{(e^x)^\alpha} \underset{x \to +\infty}{=} o\left(\frac{1}{x^2}\right)$$

poiché $\int_1^{+\infty} \frac{1}{x^2}\,dx$ converge $\Rightarrow \int_1^{+\infty} f(x)\,dx$ converge $\forall \alpha > 0$.

es. $\displaystyle\int_0^1 \frac{\operatorname{sen}(x^\alpha)}{x^5 (x+1)\cos x}\, dx$ $\qquad x \to 0$

$\alpha > 0$

$f(x) \geq 0$ $\qquad \displaystyle f(x) \underset{x\to 0}{\sim} \frac{x^\alpha}{x^5} = \frac{1}{x^{5-\alpha}}$

$\displaystyle\int_0^1 \frac{1}{x^\beta}\, dx$ $\qquad \beta < 1$ $\qquad 5 - \alpha < 1$

$$\boxed{\alpha > 4}$$

es.

$$\int_1^{+\infty} \frac{|\operatorname{sen}(x^\alpha)|}{x^5(x+1)}\,dx$$

$x \to +\infty$

$$f(x) = \frac{|\operatorname{sen}(x^\alpha)|}{x^5(x+1)} \leq \frac{1}{x^5(x+1)} \sim \frac{1}{x^6}$$

$x \to +\infty$

poiché $\int_1^{+\infty} \frac{1}{x^6}\,dx$ conv. $\Rightarrow \int_1^{+\infty} f(x)\,dx$ converge. $\forall\alpha$

es.

$$\int_{\alpha}^{1} \frac{1}{x^5\sqrt{x+1}}\,dx$$

Dire per quali $\alpha \geq 0$ converge anche $\alpha = +\infty$

1

$\alpha = 0$

$$\int_{0}^{1} \frac{1}{x^5\sqrt{x+1}}\,dx$$

$$f(x) \underset{x \to 0}{\sim} \frac{1}{x^5}$$

$$\int_{0}^{1} \frac{1}{x^5}\,dx$$ non è convergente

$\Rightarrow f(x)$ non è integrabile quando $\alpha = 0$

2° $\quad \alpha \in (0,1)$

$$\int_\alpha^1 \frac{1}{x^5\sqrt{x+1}}\,dx \Longrightarrow$$ integrale definito è un numero

f definita in $[\alpha, 1]$ e limitata

3° $\quad \alpha = +\infty \quad \displaystyle\int_{+\infty}^{1} \frac{1}{x^5\sqrt{x+1}}\,dx =$

$$= -\int_{1}^{+\infty} \frac{1}{x^5\sqrt{x+1}}\,dx$$

$x \to +\infty$

$$f(x) \sim \frac{1}{x^5\, x^{1/2}} = \frac{1}{x^{5+\frac{1}{2}}}$$

è integrabile in $(1, +\infty)$

$\Rightarrow f(x)$ è integrabile in $(1, +\infty)$.

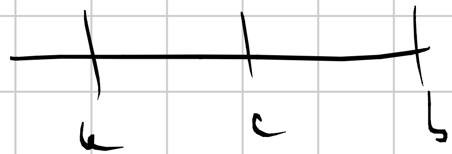
Integrali definiti di funzioni con valore assoluto.

es. $$\int_0^5 |x-1|\,dx \qquad |x-1| = \begin{cases} x-1 & x \geq 1 \\ -x+1 & x < 1 \end{cases}$$

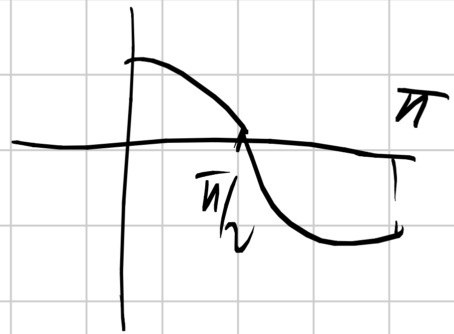
$$= \int_0^1 |x-1|\,dx + \int_1^5 |x-1|\,dx =$$

$$= \int_0^1 (-x+1)\,dx + \int_1^5 (x-1)\,dx$$

$$\underbrace{\int_a^b f} = \underbrace{\int_a^c f} + \int_c^b f$$



$= \ldots\ldots$



es. $\displaystyle\int_0^{\pi} x\,|\cos x|\,dx =$

$$= \int_0^{\pi/2} \underbrace{x\cos x\,dx} + \int_{\pi/2}^{\pi} \underbrace{x(-\cos x)\,dx}$$

es.

$$\int_0^2 \left| x^2+2x-3 \right| dx$$

$$x^2+2x-3>0$$

$$x=-1\pm\sqrt{1+3}=$$

$$=-1\pm 2 \begin{cases} 1 \\ -3 \end{cases}$$

$$x^2+2x-3>0 \qquad x>1 \quad \text{o} \quad x<-3$$

$x^2+2x-3<0$ $\quad$ $\underbrace{x^2+2x-3>0}$

0 $\quad$ 1 $\quad$ 2

$$=\int_0^1\left(-x^2-2x+3\right)dx+\int_1^2\left(x^2+2x-3\right)dx$$

es.

$$\int_{-1}^{1} \frac{|x| + \operatorname{sen} x}{1+x^2}\,dx =$$

$$= \int_{-1}^{1} \underbrace{\frac{|x|}{1+x^2}}_{f(-x)=f(x)}\,dx + \cancel{\int_{-1}^{1} \underbrace{\frac{\operatorname{sen} x}{1+x^2}}_{f \text{ dispari}}\,dx}$$

$f(-x) = f(x)$ è pari

f dispari

$$\int_{-1}^{1} f \text{ dispari} = 0$$

$|x| = x$ se $x > 0$

$$= 2\int_0^1 \frac{x}{1+x^2}\,dx$$

$= \ldots$