14 Ottobre

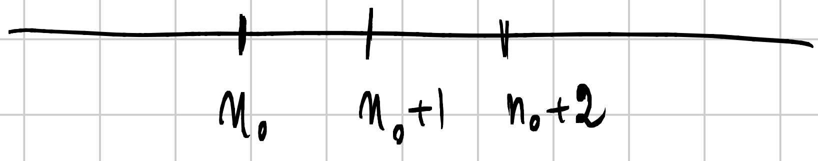
Principio d'induzione

Si applica a teoremi del tipo:

$\forall n \in \mathbb{N}$, $n \geq n_0 \Rightarrow$ vale la proprietà $p(n)$

es. • $\forall n \in \mathbb{N}$ pari $\Rightarrow n^2$ è pari

• $\forall n \in \mathbb{N}$ $\quad 2^n > n$



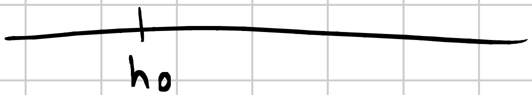
Principio d'induzione Sia $n_0 \in \mathbb{N}$ e $p(n)$ una proprietà.

Si dimostra che

1) $p(n)$ è vera per $n = n_0$

2) suppongo che $p(n)$ sia vera e dimostro che $p(n+1)$ è vera

$\Longrightarrow$ $p(n)$ è vera per ogni $n \geq n_0$



(se $n_0 = 0 \Rightarrow p(n)$ è vera $\forall n \in \mathbb{N}$)

nel punto 2) Hp $p(n)$ vera $\rightarrow$ ipotesi d'induzione

Ts $p(n+1)$ vera

Es. Dimostrare che $2^n > n$, $\forall n \in \mathbb{N}$

1) $n_0 = 0$ è vero per $n=0$? $1 > 0$ sì!

2) Hp. $2^n > n$

Ts. $2^{n+1} > n+1$

ipotesi d'induzione

$$2^{n+1} = 2 \cdot 2^n = \underbrace{2^n}_{>n} + \underbrace{2^n}_{>1} > n+1$$

Es. $1+2+3+ \dots +100 = ? = \frac{\cancel{100}^{50}(101)}{\cancel{2}} = 50 \cdot 101$

$$\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}} \quad \forall n \geq 1$$

1) $n_0 = 1 \qquad 1 = \frac{1 \cdot \cancel{2}}{\cancel{2}}$ è vera!

2) $\underline{Hp.} \quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

$\underline{Ts.} \quad \sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$?

$$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1) = \frac{n(n+1)}{2} + (n+1) =$$

$$= \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

<u>Disuguaglianza di Bernoulli</u>

$(1+h)^n \geq 1 + hn \quad , \forall h > -1, \ h \in \mathbb{R}$
$\forall n \in \mathbb{N}$

1) $n_0 = 0 \qquad 1 \geq 1$ vera!

2) $\underline{Hp}$ $(1+h)^n \geqslant 1+hn$

$\underline{Ts.}$ $(1+h)^{n+1} \geqslant 1+h(n+1)$?

$$(1+h)^{n+1} = (1+h)^n \cdot (1+h) \geqslant (1+hn)(1+h) =$$

ipotesi d'induzione

$$= 1+h+hn+h^2n = 1+h(1+n)+\underbrace{h^2n}_{\geqslant 0}$$

$$\geqslant 1+h(n+1)$$ vero!

es. per casa

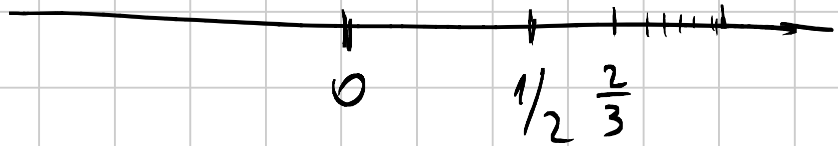
- n^2+n è pari, $\forall n \in \mathbb{N}$
- $2n+1 < n^2$, $\forall n \geqslant 3$ $(n_0 = 3)$
- $\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}$

Esercizio $A = \left\{ x \in \mathbb{R} : \ x = \frac{n}{n+1}, \ n \in \mathbb{N} \right\}$

$n = 0$
$x = 0$

$n = 1 \quad x = \frac{1}{2}$

$n = 2$



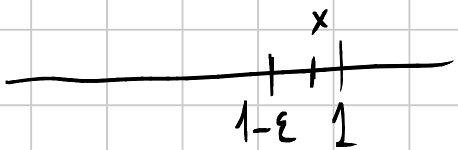
$1 \notin A \qquad 1 \neq \frac{n}{n+1}$

$\cancel{n} + 1 = \cancel{n}$ no!

$0 = \min A \quad \begin{cases} 1) & 0 \in A \quad (\text{si prende } n = 0) \\ 2) & 0 \leq x, \ \forall x \in A \end{cases}$

$$0 \leq \frac{n}{n+1} \quad \forall n \in \mathbb{N}.$$

$\sup A = 1 \quad \begin{cases} 1) & x \leq 1, \ \forall x \in A \\ 2) & \forall \varepsilon > 0 \ \exists \ x \in A : \ x > 1 - \varepsilon \end{cases}$



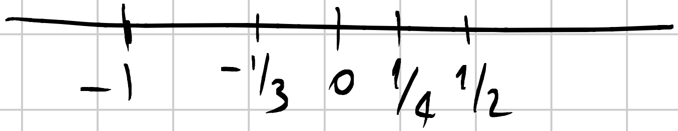
$$x > 1-\varepsilon \Rightarrow \frac{n}{n+1} > 1-\varepsilon$$

$$n > (1-\varepsilon)(n+1) \qquad \cancel{n} > \cancel{n} + 1 - \varepsilon n - \varepsilon$$

$$\varepsilon n > 1-\varepsilon \qquad \bar{n} > \frac{1-\varepsilon}{\varepsilon}$$

$\forall \varepsilon \ \exists \ \bar{n}$ e allora esiste $\bar{x} = \dfrac{\bar{n}}{\bar{n}+1}$

<u>Per casa</u> $A = \left\{ x \in \mathbb{R} : \ x = \dfrac{(-1)^n}{n}, \ \begin{matrix} n \geq 1 \\ n \in \mathbb{N} \end{matrix} \right\}$



$\min A = -1$

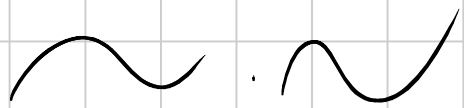
$\max A = 1/2$

• $A = \left\{ x = \dfrac{1}{n} + (-1)^n \right\}$

- se esiste il massimo di un insieme E, questo è unico.

per assurdo $\exists$ due massimi

$$m_1 = \max E$$
$$m_2 = \max E \quad \Longrightarrow \quad (m_1 = m_2)$$



Richiami su potenze, radici n-esime e logaritmi.

$$y = 0 \Longleftrightarrow x = 0$$

$$n \in \mathbb{N}, \quad n \geq 1, \quad y > 0$$

$$\sqrt[n]{y} = x \quad \text{oppure} \quad y^{1/n} = x \quad \text{t.c.} \quad x^n = y$$

$$\Longrightarrow x > 0$$

se n è dispari e $y < 0$

$$\sqrt[n]{y} := -\sqrt[n]{-y}$$

n pari $\sqrt{x^2} = |x|$ $\qquad \underbrace{\sqrt{4}}_{>0} = 2$

$n = 3 \qquad y = -8$

$$\sqrt[3]{-8} := -\sqrt[3]{8} = -2$$

- $y^{1/n}$

Potenze a esponente $\underset{r}{\underline{\text{razionale}}}$

- y^r, $r = \frac{m}{n}$ $\qquad y^r := (y^m)^{1/n}$

$y > 0$

e come sopra si può estendere la definizione se $y<0$ e n è dispari.

<u>Logaritmi</u> $\quad a>0, \quad y>0$

$$\log_a y := x \iff a^x = y$$

proprietà sul libro.

Capitolo 2 FUNZIONI

Def. A, B insiemi. Una funzione f ($f: A \to B$) è una legge che ad ogni elemento di A associa uno e uno solo elemento di B.

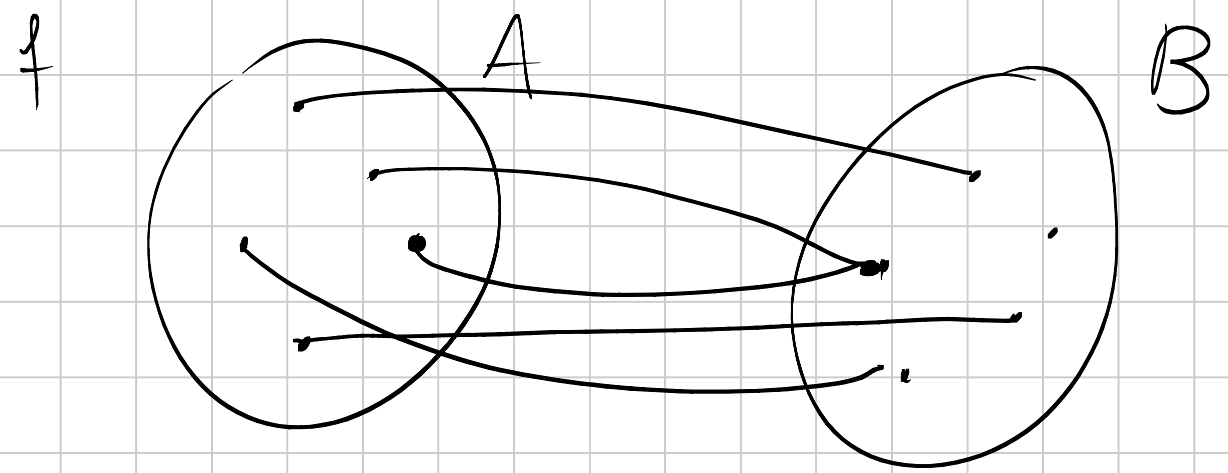
f "definita" da A a B

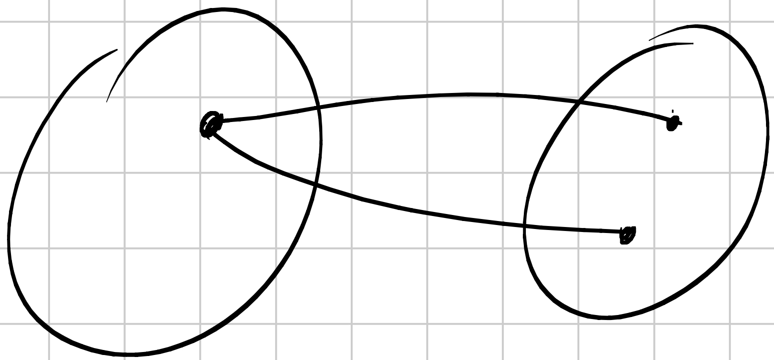
$x \in A$ $\qquad f: x \to y \in B$

$$y = f(x)$$

A = dominio di f

B = codominio di f





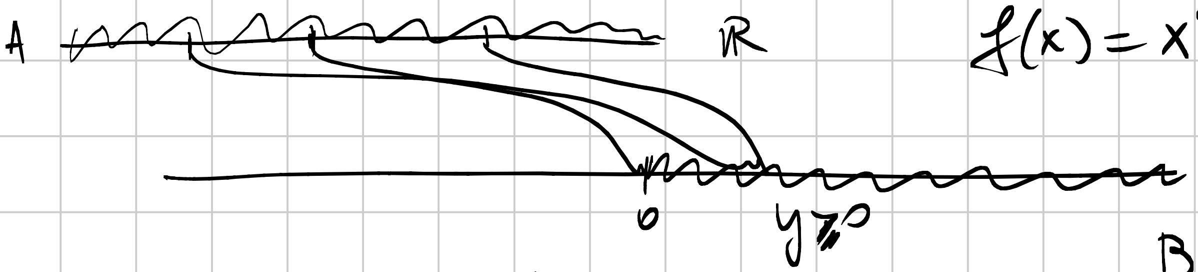
non è una funzione

es. $f(x) = x^4$ $A = \mathbb{R}$

$f: \mathbb{R} \to \mathbb{R}$ $B = \mathbb{R}$

$x \longrightarrow x^4 = y = f(x)$

$\exists\, y \in B = \mathbb{R}$ t.c. y non sono rappresentabili come



$f(x) = x^4$

$B = \mathbb{R}$

$x \longrightarrow x^4$

Immagine di A attraverso f

$$f(A) = \operatorname{im} f = \{ y \in B \text{ t.c. } \exists\, x \in A \text{ t.c. } y = f(x) \}$$

$$x \xrightarrow{f} y = f(x)$$

es. $f(x) = x^4$ $\quad f : \mathbb{R} \to \mathbb{R}$
$\qquad\qquad\qquad\quad A \to B$

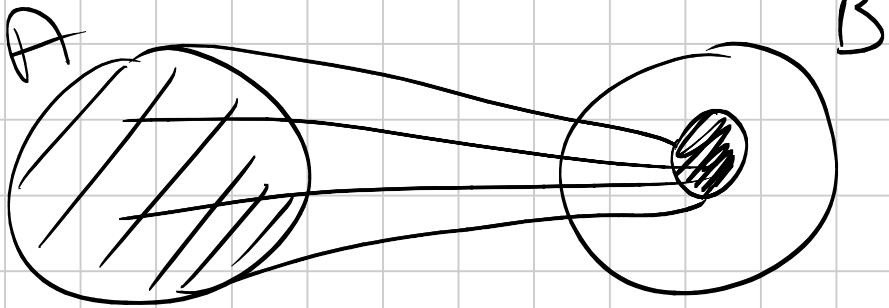
$$f(\mathbb{R}) = [0, +\infty) = \{ y \in \mathbb{R} : y = x^4 \}$$

es. $f(x) = x^3$ $\quad f: \mathbb{R} \to \mathbb{R}$

$x \to x^3$

$f(\mathbb{R}) = \{ y : y = x^3, x \in \mathbb{R} \} = \mathbb{R}$

es.



$f(A) \subseteq B$

$f: A \subseteq \mathbb{R} \longrightarrow B \subseteq \mathbb{R}$

$f: \mathbb{R} \to \mathbb{R}$ funzioni reali di variabile reale

$x \in \mathbb{R}$ $\quad f(x) \in \mathbb{R}$

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$A \qquad \mathbb{R}^2 = \{ (x,y) \,,\; x,y \in \mathbb{R} \}$

$$(x,y) \longrightarrow z \in \mathbb{R} \qquad f(x,y) = z$$

$$f(x,y) = \frac{x+y}{x^2}$$

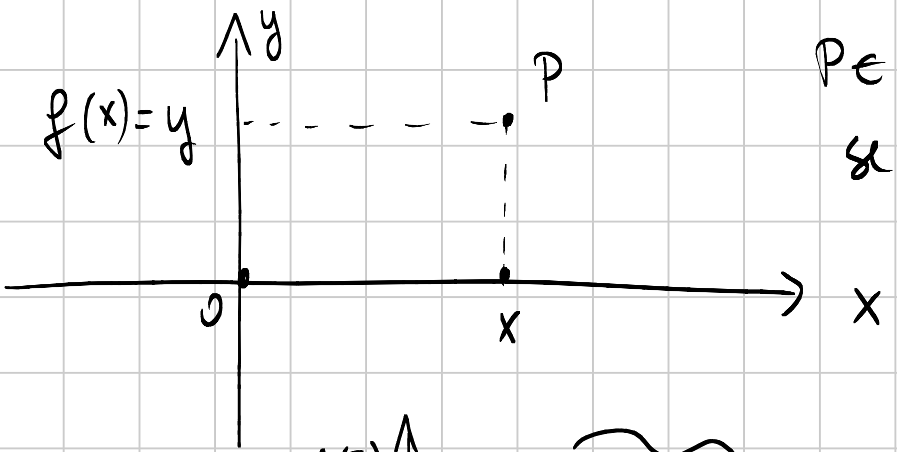
D'ora in poi consideriamo

$$f : D \longrightarrow \mathbb{R}, \qquad D \subseteq \mathbb{R}$$

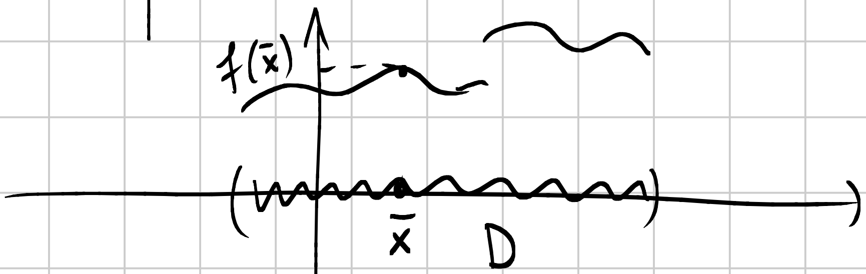
$D =$ dominio di f

$$x \in \mathbb{R} \longrightarrow f(x) \in \mathbb{R}$$

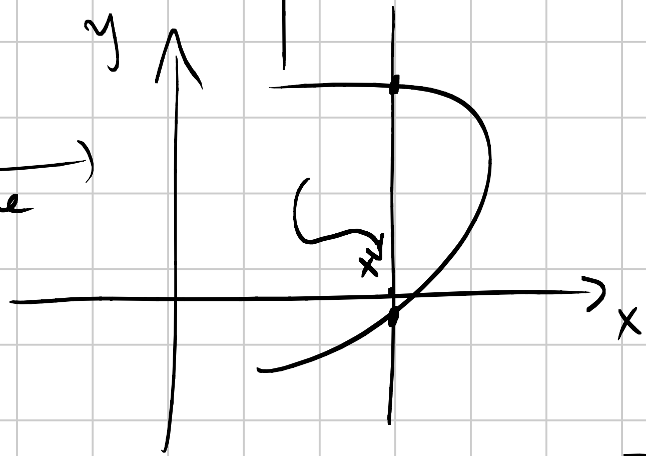
$$\text{graf } f = \{ (x,y) \in \mathbb{R}^2 \text{ t.c. } \underset{x \in D}{y = f(x)} \}$$

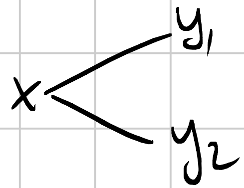


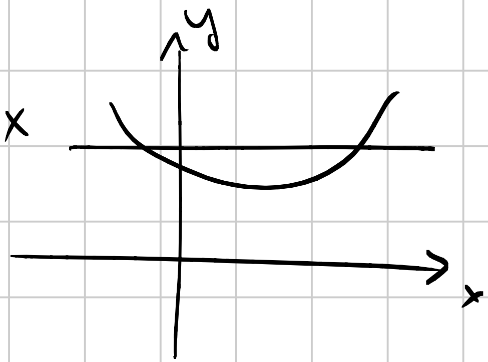
$P \in$ graf f
se $y = f(x)$



No funzione







Dominio di una funzione

$f(x) = x^2 + 1 \qquad D = \mathbb{R}$

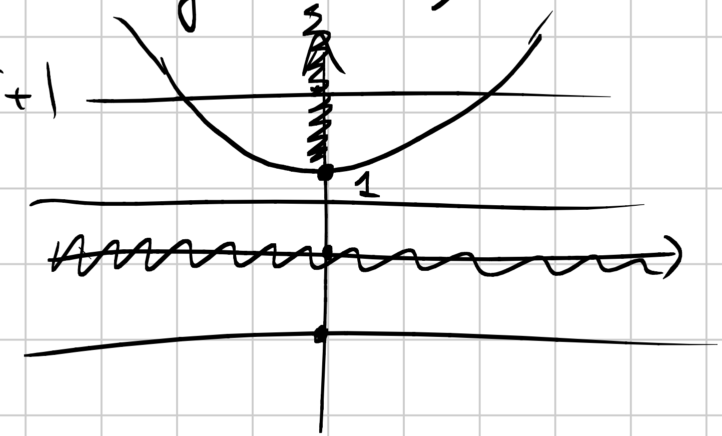
$f(x) = \sqrt{x+1} \qquad D = \{x \geq -1\} = [-1, +\infty)$

$\text{im} f = f(\mathbb{R}) = [1, +\infty)$

$y = x^2 + 1 \geq 1$

$f(x) = x^2 + 1$

$f(0) = 1$



• $f(x,y) = \dfrac{1}{x-y}$ $\qquad f: \mathbb{R}^2 \to \mathbb{R}$

$D = \{ (x,y) \in \mathbb{R}^2 : \quad x \neq y \}$

• $f(x) = \log_2 x$, $\quad x > 0$

$D = (0, +\infty)$

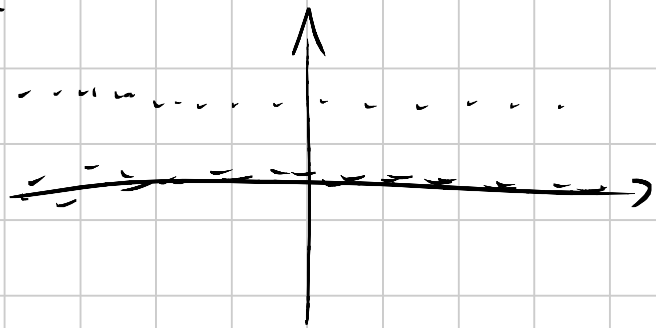
• $f(x) = \dfrac{1}{x} \qquad D = \mathbb{R} \setminus \{0\}$

Dirichlet

• $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

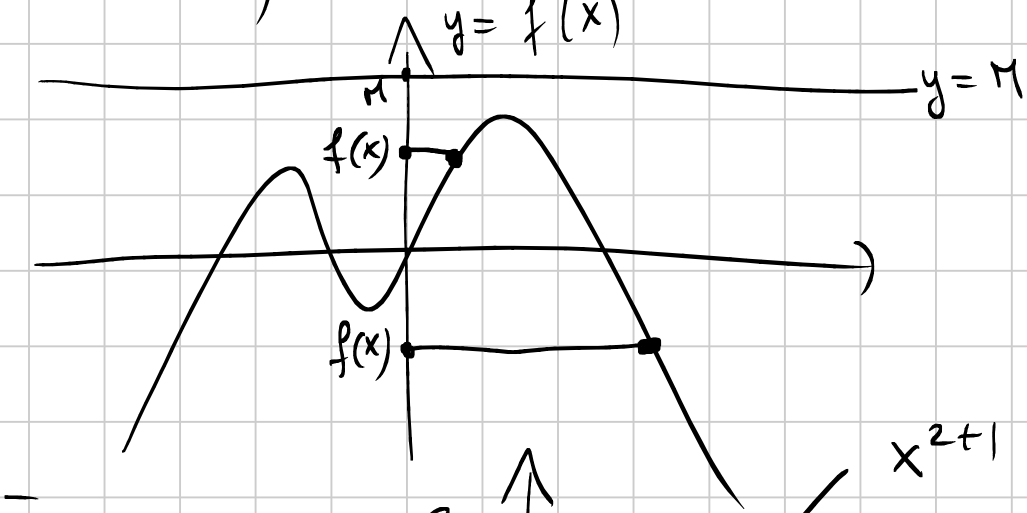
$D = \mathbb{R}$

$\mathrm{Im}\, f = \{0, 1\}$

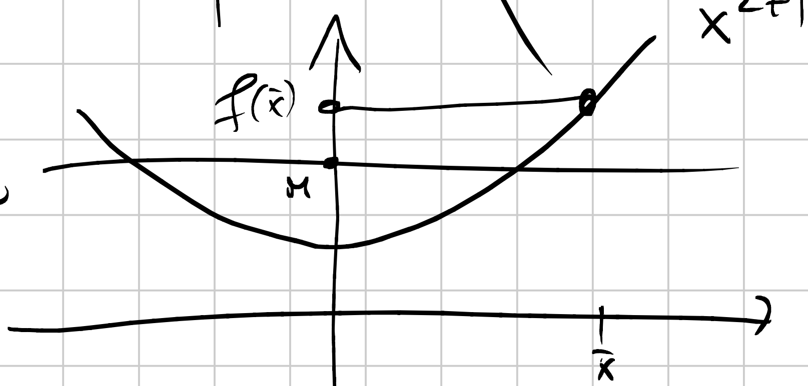


Def. <u>Funzioni limitate</u> $f : D \to \mathbb{R},\ D \subseteq \mathbb{R}$

f è <u>limitata superiormente</u> se $\exists\ M$ t.c.
$f(x) \leq M, \quad \forall\, x \in D$



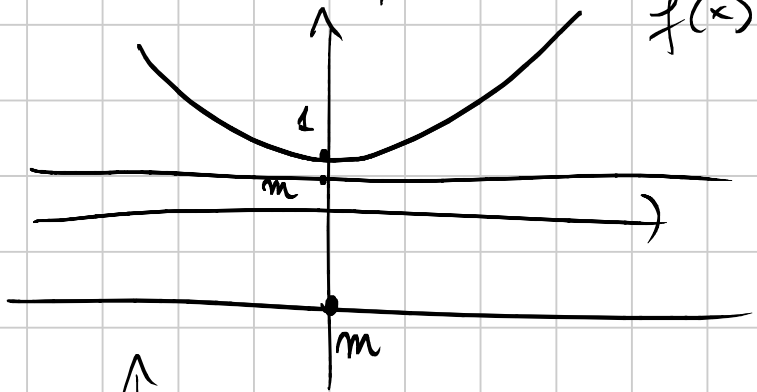
non è limitata superiormente

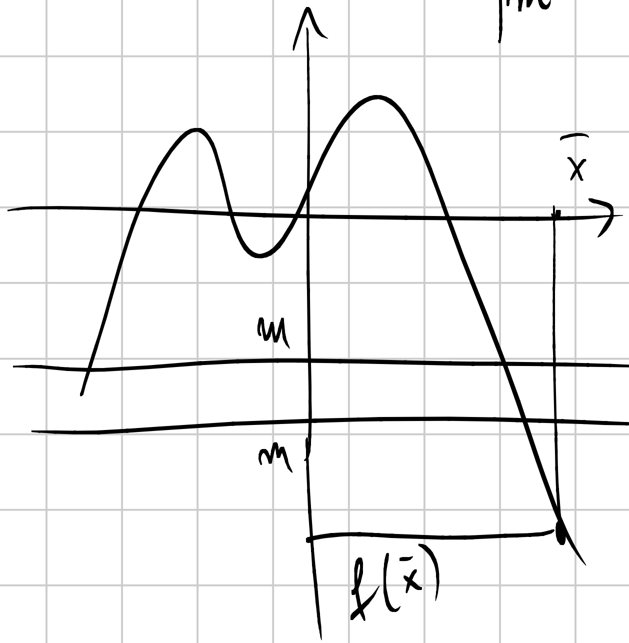


$$\forall M \ \exists\, \bar{x} : f(\bar{x}) > M$$

Def. f è **limitata inferiormente** se $\exists\, m$ t.c. $f(x) \geq m \quad , \forall x \in D$

$f(x) = x^2+1$





$\forall m \ \exists \bar{x}$: $f(\bar{x}) < m$

Def. f è limitata se è limitata superiormente e inferiormente

(cioè esistono m, M t.c. $m \le f(x) \le M$ $\forall x \in D$)

(o equivalentemente $\exists K > 0$: $|f(x)| \le K$ $\forall x \in D$)

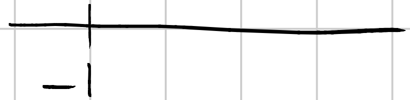
$$\overset{m}{-K} \le f(x) \le \overset{M}{K}$$

es. $f(x) = \dfrac{1}{1+x^2}$ $\qquad D = \mathbb{R}$

$$\underbrace{0}_{m} < \frac{1}{1+x^2} \le \underbrace{1}_{M}$$

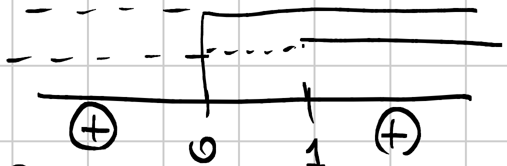
es. $|x+1|x - 2x > 0$

1) $x+1 \geq 0$ $\quad (x \geq -1)$



$(x+1)x - 2x > 0$ $\quad x^2 + x - 2x > 0$ $\quad x^2 - x > 0$

$x(x-1) > 0$



$x > 1$ o $x < 0$ con $x \geq -1$

$x > 1$ o $-1 \leq x < 0$

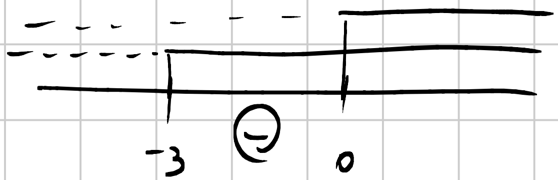
2) $x+1 < 0$ $\quad x<-1$ $\quad (-x-1)x - 2x > 0$

$-x^2 - x - 2x > 0$ $\quad -x^2 - 3x > 0$

$x^2 + 3x < 0$ $\quad x(x+3) < 0$

$-3 < x < 0$

e $x < -1$



$-3 < x \le -1$

Le soluzioni sono $\underbrace{x > 1}$ e $-3 < x < 0$

Es. $$\frac{1}{x-3} \le \frac{1}{2|x|}$$ $x \ne 3$, $x \ne 0$

se $\frac{1}{x-3} < 0$ la disequazione è sempre verificata ($\forall x$)

se $x - 3 < 0 \Rightarrow x < 3$

se $\frac{1}{x-3} > 0$ se $x - 3 > 0$ $x > 3$

$$2|x| \le x-3$$

$$2x \le x-3$$

$$\frac{1}{x-3} \le \frac{1}{2|x|}$$

poiché $x>3 \Rightarrow x>0$

$$x \le -3$$

$x<3$ tutti gli x sono soluzioni

$x>3$ non esistono soluzioni

Soluzioni $x<3,\ x \neq 0$.