Lezione del 15/12

G è una primitiva di f

$$\int_a^b f(x)\,dx = G(b) - G(a)$$

$$f(x) =$$

Metodo di integrazione per *sostituzione*

$f(t) \qquad t \in I$

$G(t)$ primitiva di $f \implies G'(t) = f(t)$
$t \in I$

$t = \varphi(x) \qquad \varphi$ funzione derivabile $x \in [a,b]$

$G(t) = G(\varphi(x)) \qquad G'(t) = f(t)$

$$\frac{d}{dx} \underbrace{G(\varphi(x))} = \underbrace{G'(\varphi(x))} \cdot \varphi'(x) =$$

$$= \underbrace{f(\varphi(x))\varphi'(x)}$$

$\Rightarrow$ $G(\varphi(x))$ è una primitiva di $f(\varphi(x))\varphi'(x)$

$$G(\varphi(x)) = \int f(\varphi(x))\varphi'(x)\,dx$$

$$\parallel$$

$$G(t) = \int f(t)\,dt$$

$$\Rightarrow \boxed{\int f(t)\,\underbrace{dt} = \int f(\varphi(x))\,\underbrace{\varphi'(x)\,dx}}$$

$t = \varphi(x)$ solo formalmente

$dt = \varphi'(x)\,dx$

formula di integrazione per sostituzione

$t \in [\alpha, \beta]$

$x \in [a, b]$

$\varphi(a) = \alpha$

$\varphi(b) = \beta$

$$\int_{\alpha}^{\beta} f(t)\,dt = \int_{a}^{b} f(\varphi(x))\varphi'(x)\,dx$$

$$\int \operatorname{sen} x = -\cos x + K$$

es. $\int \operatorname{sen}(3x)\,\underbrace{dx} =$

$3x = t$

$x = \frac{1}{3}t$

$dx = \frac{1}{3}dt$

$= \int \operatorname{sen} t \,\frac{1}{3}dt =$

$= \frac{1}{3}\int \operatorname{sen} t\, dt = \frac{1}{3}\left(-\cos t + K\right)$

$= \frac{1}{3}\left(-\cos 3x + K\right)$

es $\int \frac{1}{2x+1}\,dx = \int \frac{1}{t}\,dt = \log|t| + k$

$2x+1 = t \qquad 2x = t-1$

$x = \frac{t-1}{2}$

$dx = \frac{1}{2}dt$

$= \int \frac{1}{t}\,\frac{1}{2}dt$

$= \frac{1}{2}\int \frac{1}{t}dt = \frac{1}{2}\log|t| + K = \frac{1}{2}\log|2x+1| + k$

$\underline{es.}$ $\displaystyle\int_0^1 \frac{1}{2x+1} = \frac{1}{2}\log|3| - \cancel{\frac{1}{2}\log|1|} =$

$= \frac{1}{2}\lg 3$

$$\int_0^1 \frac{1}{2x+1}dx = \int_1^3 \frac{1}{t}\,\frac{1}{2}dt = \frac{1}{2}\log 3 - \frac{1}{2}\lg 1$$

$2x+1=t$

$x=0 \Rightarrow t=1$

$x=1 \Rightarrow t=3$

$2x=t-1 \qquad x=\frac{t-1}{2}$

es.

$$\int \operatorname{tg} x \, dx = \int \frac{\operatorname{sen} x}{\cos x} \, dx =$$

$$\boxed{\cos x = t} \Rightarrow -\operatorname{sen} x \, dx = dt$$

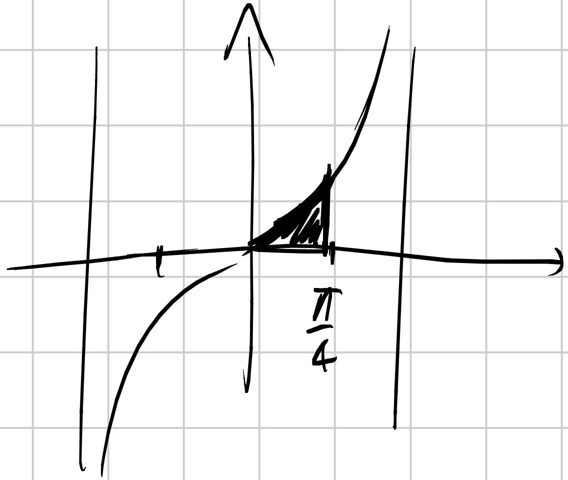
$$\operatorname{sen} x \, dx = -dt$$

$$= \int \frac{1}{t}(-dt) = -\int \frac{1}{t} \, dt =$$

$$= -\log|t| + K = \underline{-\log|\cos x| + K}$$

$$\int_0^{\pi/4} \operatorname{tg} x \, dx =$$

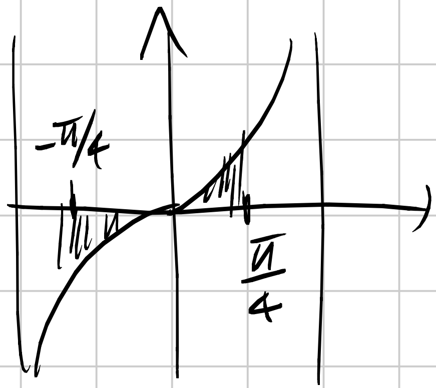
$$\cancel{\left[\int_0^{\pi/2} \operatorname{tg} x \, dx \right]}$$



$$= -\log\left|\cos\frac{\pi}{4}\right| + \cancel{\log|\cos 0|}$$

$$= -\log\frac{\sqrt{2}}{2} \geq 0$$ come ci aspettavamo!

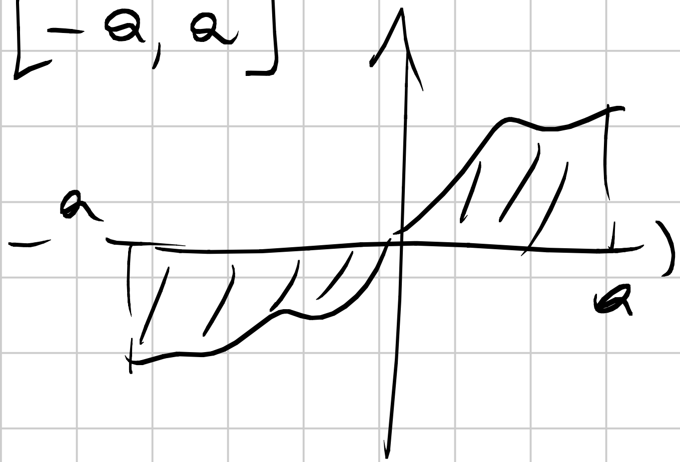
$$\int_{-\pi/4}^{\pi/4} tg\,x = 0$$



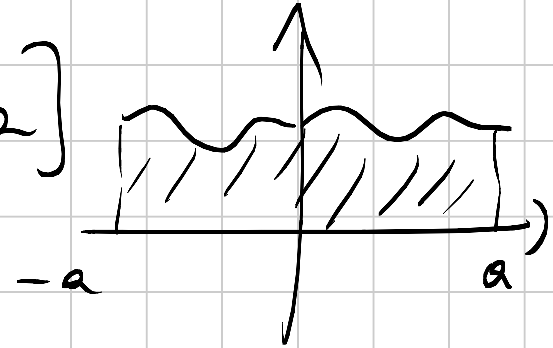
$$= -\log\left|\cos\frac{\pi}{4}\right| + \log\left|\cos\left(-\frac{\pi}{4}\right)\right| = 0$$

<u>Oss.</u> Se f è <u>dispari</u> in $[-a, a]$

$$\int_{-a}^{a} f(x)\,dx = 0$$



Se f è pari in $[-a, a]$



$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

$$\int \operatorname{tg} x \, dx = \int \frac{\overset{f'}{\operatorname{sen} x}}{\underset{f}{\cos x}}\,dx$$

es.

$$\int \frac{x}{\sqrt{x^2-1}}\,dx = \int \frac{2x}{2\sqrt{x^2-1}}\,dx$$

$$\rightarrow \quad x^2-1 = t$$

$$2x\,dx = dt$$

$$= \int \frac{1}{2\sqrt{t}}\,dt$$

$$= \frac{1}{2}\int t^{-1/2}\,dt$$

$$= \frac{1}{2}\left(\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right) + K$$

$$= \frac{1}{\cancel{2}} \cdot \cancel{2}\sqrt{t} + K = \sqrt{x^2-1} + K$$

<u>es.</u> $\displaystyle\int \operatorname{sen}^5 x \, \underbrace{\cos x \, dx}$

$$\operatorname{sen} x = t$$

$$\Downarrow$$

$$\cos x \, dx = dt$$

$$= \int t^5 \, dt =$$

$$= \frac{t^6}{6} + K = \frac{(\operatorname{sen} x)^6}{6} + K$$

$\operatorname{sen}^5 x \cos x$ è dispari $\displaystyle\int_{-3}^{3} \operatorname{sen}^5 x \cos x \, dx = 0$

In generale

$$f(x) = t$$
$$f'(x)\,dx = dt$$

$$\int \frac{f'(x)}{f(x)}\,dx = \int \frac{1}{t}\,dt = \log|t| + K =$$
$$= \log|f(x)| + K$$

es.

$$\int \frac{1}{1+x^2}\,dx = \operatorname{arctg} x + K$$

$$\int \frac{1}{4+x^2}\,dx = \int \frac{1}{4\left(1+\frac{x^2}{4}\right)}\,dx =$$

$$= \frac{1}{4}\int \frac{1}{1+\left(\frac{x}{2}\right)^2}\, dx$$

$$\frac{x}{2} = t$$

$$x = 2t$$

$$dx = 2dt$$

$$= \frac{1}{4}\int \frac{1}{1+t^2}\, 2\,dt$$

$$= \frac{1}{2}\,\text{arctg}\, t + K = \frac{1}{2}\,\text{arctg}\,\frac{x}{2} + K$$

(caso $a=2$)

In generale $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\,\text{arctg}\left(\frac{x}{a}\right) + K$

es. P.C.

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \int \frac{1}{\sqrt{a^2\left(1-\left(\frac{x}{a}\right)^2\right)}}\,dx =$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsen x + K$$

$$= \int \frac{1}{|a|\sqrt{1-\left(\frac{x}{a}\right)^2}}\,dx =$$

$\frac{x}{a} = t$

. . . .

Fare

$$\int \frac{1}{(3+5x)^6}\,dx \quad , \quad \int \sqrt{x+2}\,dx$$

$$\int \frac{x}{\sqrt{2-3x^2}}\,dx \quad , \quad \int \frac{x^3}{\sqrt{1+x^4}}\,dx$$

$$\int \frac{1}{x\log x}\,dx \quad , \quad \int \frac{(\log x)^n}{x}\,dx$$

$$\int 3x\,e^{x^2}\,dx \quad , \quad \int \frac{\operatorname{sen}\sqrt{x}}{\sqrt{x}}\,dx$$

es.

$$\int \frac{1}{1+e^x}\,dx =$$

$e^x = t$

$x = \log t$

$dx = \frac{1}{t}\,dt$

$$= \int \frac{1}{(1+t)}\,\frac{1}{t}\,dt$$

"fratti semplici"

$$\frac{1}{(1+t)\cdot t} = \frac{A}{1+t} + \frac{B}{t}$$

Cerco A e B t.c.

$$\frac{1}{(1+t)\cdot t} = \frac{A\cdot t + B(1+t)}{(1+t)\cdot t}$$

$$\frac{1}{\cancel{(1+t)\cdot t}} = \frac{At + Bt + B}{\cancel{(1+t)\cdot t}} \quad \forall t$$

$$\boxed{1 = (A+B)\cdot t + B}$$

due polinomi sono uguali se i coefficienti davanti ai termini dello stesso grado sono uguali

$$\begin{cases} B = 1 \\ A + B = 0 \end{cases} \qquad A = -B = -1$$

$$\frac{1}{(t+1)\cdot t} = \frac{-1}{1+t} + \frac{1}{t}$$

decomposizione in fratti semplici

$$\int \frac{1}{(t+1)t}\,dt = -\int \frac{1}{1+t}\,dt + \int \frac{1}{t}\,dt$$

$$= -\log|1+t| + \log|t| + K =$$

$$= -\log|1+e^x| + \log|e^x| + K =$$

$$= -\log(1+e^x) + x + K$$

es. $\displaystyle\int \operatorname{sen}^3 x \, dx = \int \operatorname{sen} x \operatorname{sen}^2 x \, dx =$

$\cos x = t$

$$= \int (1-\cos^2 x) \operatorname{sen} x \, dx$$

$= \dots$

es. $\displaystyle\int \frac{1}{\operatorname{sen} x}\,dx = \int \frac{\operatorname{sen} x}{\operatorname{sen}^2 x}\,dx = \int \frac{\operatorname{sen} x}{1-\cos^2 x}\,dx$

$\cos x = t \qquad -\operatorname{sen} x\,dx = dt$

$\operatorname{sen} x\,dx = -dt$

$\displaystyle = \int \frac{-dt}{1-t^2} = -\int \frac{1}{1-t^2}\,dt =$

$\displaystyle = -\int \frac{1}{(1-t)(1+t)}\,dt$

Cerco A, B t.c.

$$\frac{1}{\underbrace{(1-t)}\underbrace{(1+t)}} = \frac{A}{1-t} + \frac{B}{1+t}$$

$$= \frac{A(1+t) + B(1-t)}{(1-t)(1+t)} = \qquad \cancel{1}t$$

$$= \frac{At + A + B - Bt}{(1-t)(1+t)} =$$

$$\frac{1}{\cancel{(1-t)(1+t)}} = \frac{(A-B)\cdot t + (A+B)}{\cancel{(1-t)(1+t)}}$$

$$\left.\begin{aligned} 1 &= A + B \\ 0 &= A - B \end{aligned}\right\} \Rightarrow \begin{aligned} 2A &= 1 \\ A &= B \end{aligned} \qquad \begin{aligned} A &= 1/2 \\ B &= 1/2 \end{aligned}$$

$$\frac{1}{(1-t)(1+t)} = \frac{1}{2}\frac{1}{1-t} + \frac{1}{2}\frac{1}{1+t}$$

decomposizione
in fratti semplici

$$-\int \frac{1}{(1-t)(1+t)}\,dt = -\frac{1}{2}\int \frac{1}{1-t}\,dt - \frac{1}{2}\int \frac{1}{1+t}\,dt$$

$1-t=z$

$$= -\frac{1}{2}\left(-\log|1-t|\right) - \frac{1}{2}\log|1+t| + K$$

$\downarrow$ controllare

$$= \frac{1}{2}\log|1-\cos x| - \frac{1}{2}\log|1+\cos x| + K$$

$$= \frac{1}{2}\log\left(\frac{|1-\cos x|}{|1+\cos x|}\right) + K.$$

Integrale definito con valore assoluto.

$$\int_0^5 |x-1|\,dx =$$

$$|x-1| = \begin{cases} x-1 & x \geq 1 \\ 1-x & x < 1 \end{cases}$$

$$= \int_0^1 \underbrace{|x-1|}_{1-x}\,dx + \int_1^5 \underbrace{|x-1|}_{x-1}\,dx =$$

$$= \int_0^1 (1 - x)\, dx + \int_1^5 (x - 1)\, dx =$$

$$= x\Big|_0^1 - \frac{x^2}{2}\Big|_0^1 + \frac{x^2}{2}\Big|_1^5 - x\Big|_1^5 =$$

$$= 1 - \frac{1}{2} + \frac{25}{2} - \frac{1}{2} - 5 + 1$$

$$= 2 - 1 - 5 + \frac{25}{2} = \ldots$$

es. $\displaystyle\int \frac{\sqrt{x}}{2+\sqrt{x}}\,dx = \dots$ dopo.

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2) Integrazione per parti

f, g derivabili in $[a, b]$

$$(f \cdot g)' = f'g + fg'$$

$$fg' = (f \cdot g)' - f'g$$

$$\int f(x)\,g'(x)\,dx = \underbrace{\int \big(f(x)\,g(x)\big)'\,dx}_{\to\, f(x)\,g(x)} - \int f'(x)\,g(x)\,dx$$

<u>oss.</u> $$\int h'(x)\,dx = h(x) + K$$

$$\boxed{\int \underbrace{f(x)}_{\text{fattore che si deriva}}\,\underbrace{g'(x)}_{\text{fattore che si integra}}\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx}$$

formula di integrazione per parti

$$\int_a^b f(x)\,g'(x)\,dx = f(x)\,g(x)\Big|_a^b - \int_a^b f'(x)\,g(x)\,dx$$

$$\int \underbrace{x}_{f} \cdot \underbrace{\operatorname{sen} x}_{g'} \, dx = \underbrace{x}_{f} \underbrace{(-\cos x)}_{g} - \int 1 \cdot (-\cos x)\,dx =$$

$$g'(x) = \operatorname{sen} x \qquad g(x) = -\cos x$$

$$= -x\cos x + \int \cos x \, dx =$$

$$= -x\cos x + \operatorname{sen} x + K$$

$$\int x \,\text{sen}\, x \, dx = -x\cos x + \text{sen}\, x + K$$

se noi avessimo applicato la formula scambiando f con g si sarebbero complicati i conti

$$\int \underbrace{x}_{g'} \underbrace{\text{sen}\, x}_{f} \, dx = \underbrace{\text{sen}\, x \cdot \frac{x^2}{2} - \int \cos x \, \frac{x^2}{2} \, dx}_{\text{si peggiorano le cose!}}$$

$$g'(x) = x \Rightarrow g(x) = \frac{x^2}{2}$$

es.
$$\int \underbrace{x^2}_{f}\ \underbrace{\operatorname{sen} x}_{g'}\, dx = x^2(-\cos x) - \int 2x\,(-\cos x)\,dx$$

$$= -x^2 \cos x + 2\int x\cos x\, dx$$

$$\int \underbrace{x}_{f}\ \underbrace{\cos x}_{g'}\, dx = x \operatorname{sen} x - \int 1 \cdot \operatorname{sen} x\, dx =$$

$$= x \operatorname{sen} x + \cos x$$

$$\int x^2 \operatorname{sen} x\, dx = -x^2 \cos x + 2\left(x \operatorname{sen} x + \cos x\right) + K$$

$$\int_0^{\pi} \underbrace{x^2 \operatorname{sen} x}_{\geq 0} \, dx = -x^2 \cos x + 2(x \operatorname{sen} x + \cos x) \Big|_0^{\pi}$$

$$= -\pi^2(-1) + \cancel{2(-1)} + \cancel{2}$$

$$= \underbrace{\pi^2}$$

es.

$$\int \underbrace{e^x}_{f} \underbrace{\operatorname{sen} x}_{g'} \, dx = e^x(-\cos x) - \int e^x(-\cos x) \, dx$$

$$= -e^x \cos x + \underbrace{\int e^x \cos x \, dx}$$

$$\int \underbrace{e^x}_{f} \underbrace{\cos x}_{g'} \, dx = e^x \operatorname{sen} x - \int e^x \operatorname{sen} x$$

$$\int e^x \operatorname{sen} x \, dx = -e^x \cos x + e^x \operatorname{sen} x - \int e^x \operatorname{sen} x \, dx$$

$$2 \int e^x \operatorname{sen} x \, dx = e^x (\operatorname{sen} x - \cos x)$$

$$\int e^x \operatorname{sen} x \, dx = \frac{e^x (\operatorname{sen} x - \cos x)}{2} + K$$

es.

$$\int \operatorname{sen}^2 x \, dx = \int \underbrace{\operatorname{sen} x}_{f} \cdot \underbrace{\operatorname{sen} x}_{g'} \, dx =$$

$$= \operatorname{sen} x \, (-\cos x) - \int \cos x \, (-\cos x) =$$

$$= -\operatorname{sen} x \cos x + \int (1 - \operatorname{sen}^2 x) \, dx =$$

$$= -\operatorname{sen} x \cos x + x - \int \operatorname{sen}^2 x \, dx$$

$$2 \int \operatorname{sen}^2 x \, dx = -\operatorname{sen} x \cos x + x$$

$$\int \operatorname{sen}^2 x \, dx = \frac{-\operatorname{sen} x \cos x + x}{2} + K$$

$$\int \cos^2 x \, dx \qquad \text{P.C.}$$

<u>es.</u> $$\int \log x \, dx = \int \underbrace{1}_{g'} \cdot \underbrace{\log x}_{f} \, dx =$$

$$= \log x \cdot x - \int \frac{1}{\cancel{x}} \cdot \cancel{x} \, dx = x \log x - x + K$$

es.

$$\int \underbrace{1}_{g'} \cdot \underbrace{\arcsin x}_{f}\, dx = \underbrace{(\arcsin x) \cdot x}\ -$$

$$- \underbrace{\int \frac{1}{\sqrt{1-x^2}} \cdot x\, dx}$$

$1 - x^2 = t$

$-2x\,dx = dt$

$x\,dx = -\frac{1}{2}dt$

$$\int \frac{x}{\sqrt{1-x^2}}\, dx = \int \frac{1}{\sqrt{t}} \left(-\frac{1}{2}dt\right)$$

$$= -\frac{1}{2}\int t^{-1/2}\,dt = -\frac{1}{\cancel{2}}\, t^{1/2}\,\cancel{2}$$

$$= -\sqrt{1-x^2}$$

$$\int \operatorname{arcsen} x\,dx = x\operatorname{arcsen} x + \sqrt{1-x^2} + K$$

P.C. $\int \operatorname{arctg} x\,dx \qquad \int \arccos x\,dx$

PC.

$$\int x e^{x}\, dx$$

$$\int x^5 \underbrace{e^{x^2}} \to x^2 = t$$

$$\int x^2 \underbrace{\log x}_{f}$$

$$\int e^{x^2}$$

- Funzioni razionali $\dfrac{P(x)}{Q(x)}$

$P(x)$ al massimo di grado 1

$Q(x)$ al massimo di grado 2

- Integrali generalizzati $\displaystyle\int_0^1 \frac{1}{x} \longrightarrow ?$