Lezione del 18 Novembre

Calcolo differenziale

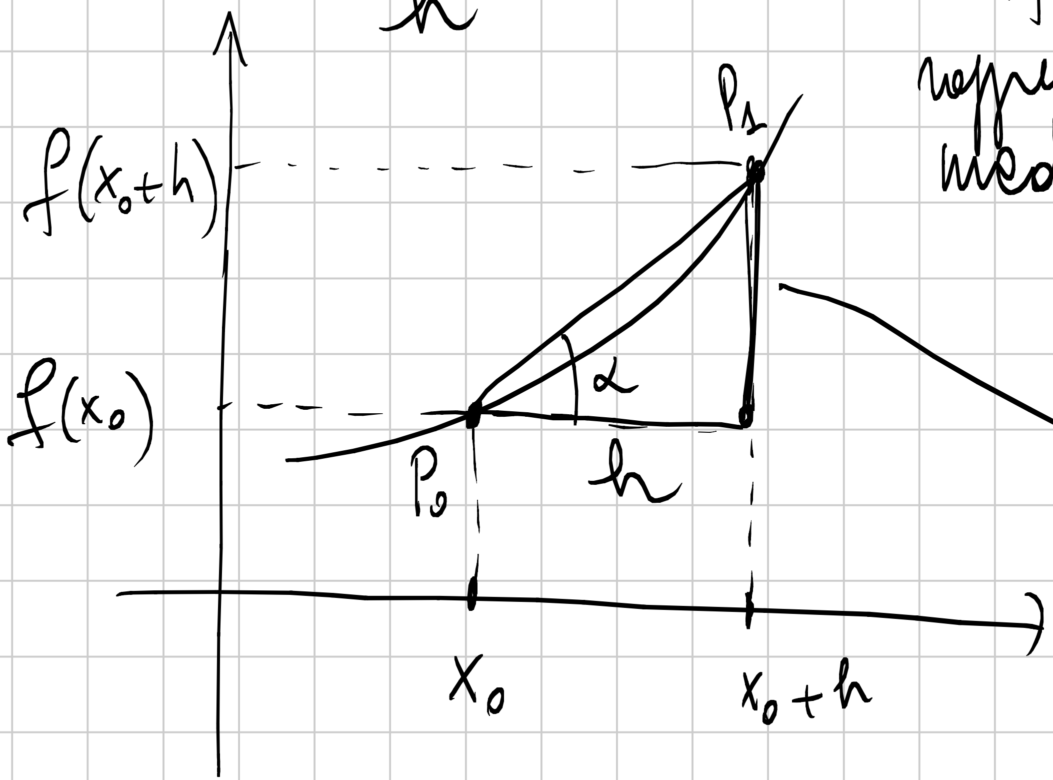
Def. $f:(a,b)\to\mathbb{R}$ si dice derivabile in $x_0\in(a,b)$ se esiste finito

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} =: f'(x_0)$$ è la derivata prima di f in x_0.

$$\dot{f}(x_0)\ ,\quad \frac{df}{dx}(x_0)\ ,\quad Df(x_0)$$

$$\frac{f(x_0+h)-f(x_0)}{h} = \text{rapporto } \underline{\text{incrementale}} \text{ di } f \text{ in } x_0$$

rappresenta il tasso di variazione media tra x_0 e x_0+h



$h>0$

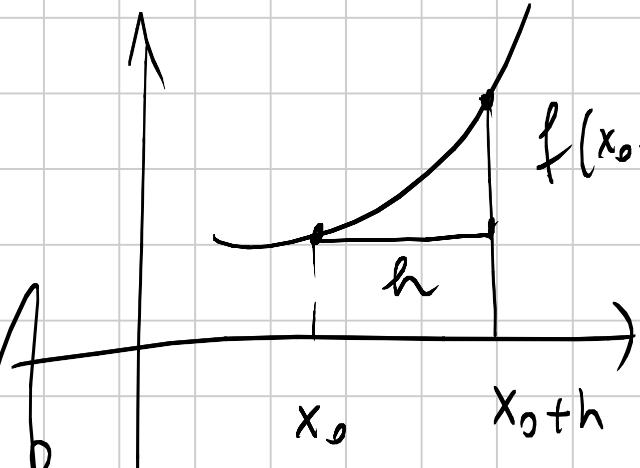
$f(x_0+h)-f(x_0)$

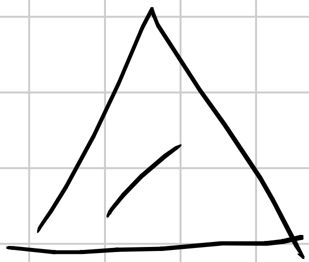
$$\frac{f(x_0+h)-f(x_0)}{h}$$

rappresenta la pendenza della secante tra P_0 e P_1

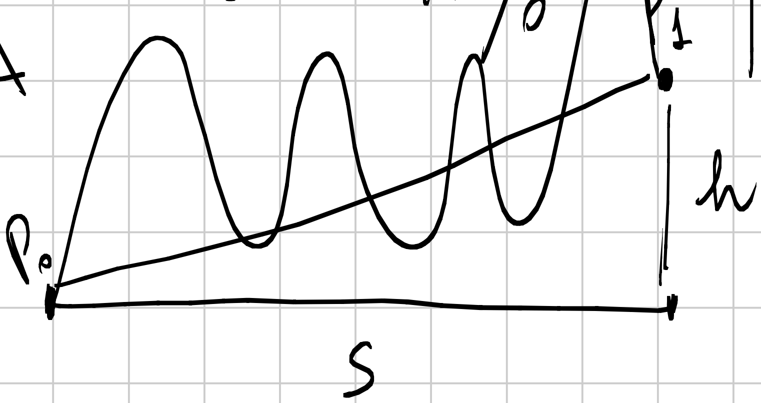
$$\frac{f(x_0+h)-f(x_0)}{h}$$

Tasso di variazione media





Tendenza media 10%



$$\frac{h}{S} = 10\%$$

Variazione media (rapporto incrementale)

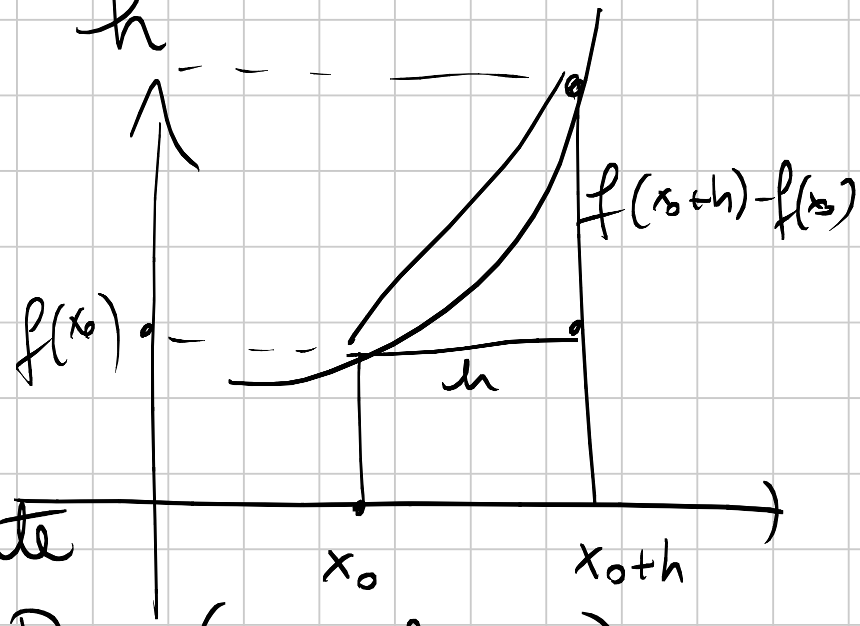
Ci interessa la variazione istantanea cioè $S \to 0$

$$f'(x_0) \doteq \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

si può dim. che

$$f'(x_0)=\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = m =$$

$=$ coeff. angolare della retta tangente al grafico di f in $P_0 = (x_0, f(x_0))$.



eq. retta tangente in $P_0 = (x_0, f(x_0))$

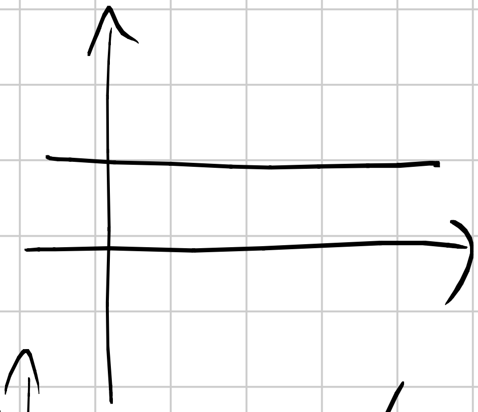
$$y - f(x_0) = f'(x_0)(x - x_0)$$

$$\boxed{y = f(x_0) + f'(x_0)(x - x_0)}$$

eq. retta tg al grafico di f in $(x_0, f(x_0))$

se $f'(x_0) = 0 \Rightarrow y = f(x_0)$

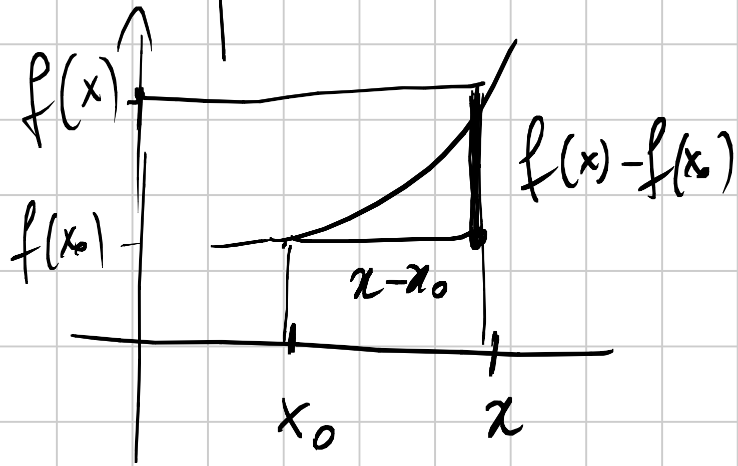
retta orizzontale



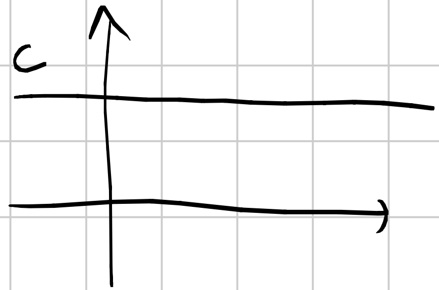
Altro modo di scrivere:

Rapporto incrementale.

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$



Derivate di funzioni <u>elementari</u>



- $f(x) = C$

$$\frac{f(x_0+h)-f(x_0)}{h} = \frac{C-C}{h} = \frac{0}{h} = 0 \underset{h\to 0}{\to} 0$$

$$f'(x_0) = 0 \quad \overset{\Rightarrow}{\forall x_0 \in \mathbb{R}} \quad f'(x) = 0 \quad \forall x \in \mathbb{R}$$

retta tg.

$$y = f(x_0) + \cancel{f'(x_0)}(x-x_0)$$

$$\hookrightarrow y = C$$

- $f(x) = x^\alpha, \quad \alpha \in \mathbb{R}, \quad x > 0$

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^\alpha - x^\alpha}{h} =$$

$$= \frac{x^\alpha \left(\left(\frac{x+h}{x}\right)^\alpha - 1 \right)}{h} = \frac{x^\alpha}{h} \left(\left(1 + \frac{h}{x}\right)^\alpha - 1 \right)$$

$$\lim_{y \to 0} \frac{(1+y)^\alpha - 1}{y} = \alpha$$

$\frac{h}{x} = y \quad$ se $h \to 0$

$\Rightarrow y \to 0$

.

$$\underbrace{\frac{x^\alpha}{h}}\underbrace{\left(\left(1+\frac{h}{x}\right)^\alpha - 1\right)} = \frac{x^\alpha}{\cancel{h}}\left(\frac{(\quad)}{\frac{h}{x}}\right)\cdot\frac{\cancel{h}}{x}$$

$$\frac{(1+y)^\alpha - 1}{y}\cdot y \to \alpha$$

$$\alpha \uparrow$$

$$\frac{x^\alpha}{\cancel{h}}\left(\frac{\left(1+\frac{h}{x}\right)^\alpha - 1}{\frac{h}{x}}\right)\cdot\frac{\cancel{h}}{x}$$

$$h \to 0 \qquad \frac{h}{x} \to 0$$

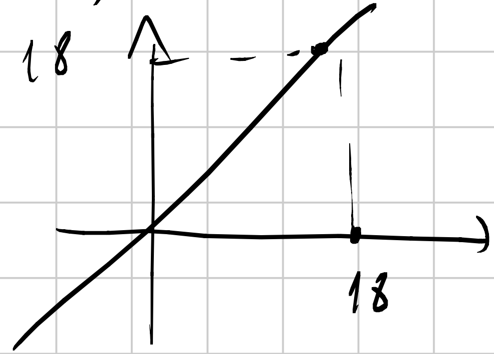
$$\to \alpha \qquad \to \alpha x^{\alpha-1}$$

$f(x)=x^{\alpha} \Rightarrow f'(x) = \alpha x^{\alpha-1}$

es: $f(x) = x$ $(\alpha = 1)$

$f'(x) = 1x^{0} = 1$

$f(18) = 18$



Calcolare $f'(18) = 1$

retta tg. al grafico in 18

$y = f(x_0) + f'(x_0)(x - x_0)$

$y = 18 + 1(x - 18)$

$y = \cancel{18} + x - \cancel{18}$

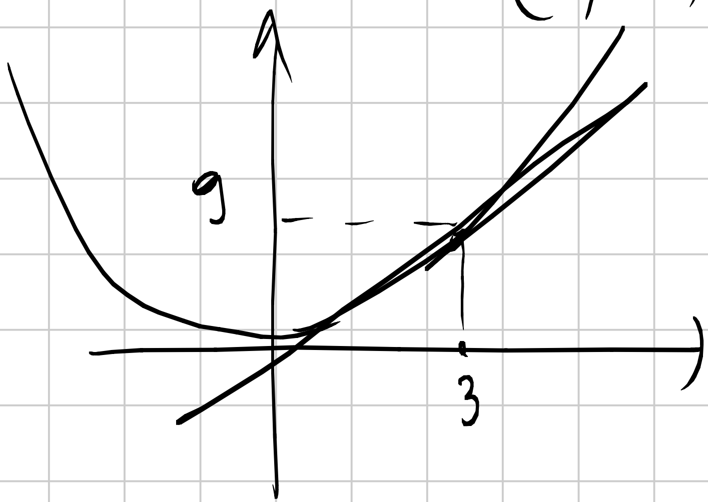
es. $f(x) = x^2$ $\qquad \alpha = 2$ $\qquad f(x) = x^\alpha$

$f'(x) = 2x$ $\qquad\qquad f'(x) = \alpha x^{\alpha-1}$

Calcolare la f' in $x = 3$ $\qquad f'(3) = 2 \cdot 3 = 6$

Calcolare la retta tg al grafico di f nel p.to $(3, 9)$

$x = 3$, $f(3) = 9$



$y = 9 + 6(x - 3)$

$y = 9 + 6x - 18$

$y = 6x - 9$

· **es.** $f(x) = \frac{1}{x} = x^{-1}$ $\quad \alpha = -1$

$f(x) = x^{\alpha}$

$f'(x) = \alpha x^{\alpha - 1}$

$$f'(x) = (-1)\, x^{-2} = \frac{-1}{x^2}$$

es. $f(x) = \sqrt{x} = x^{1/2}$ $\quad f'(x) = \frac{1}{2} x^{-\frac{1}{2}} =$

f è def. $\forall x \geq 0$

$$= \frac{1}{2} \frac{1}{\sqrt{x}}$$

$\forall x > 0$

es. $f(x) = x^{\sqrt{2}} \Rightarrow f'(x) = \sqrt{2}\, x^{\sqrt{2} - 1}$

- $f(x) = \operatorname{sen} x$

$$\frac{f(x+h) - f(x)}{h} = \frac{\operatorname{sen}(x+h) - \operatorname{sen} x}{h} =$$

$$\left(\operatorname{sen}(a+b) = \operatorname{sen} a \cos b + \operatorname{sen} b \cos a \right)$$

$$= \frac{\operatorname{sen} x \cos h + \operatorname{sen} h \cos x - \operatorname{sen} x}{h} =$$

$h \to 0$

$$= \frac{\operatorname{sen} x \left(\cos h - 1 \right)}{h \quad h} h + \frac{\operatorname{sen} h}{h} \cos x \longrightarrow \cos x$$

$\to 0 \qquad \to 1$

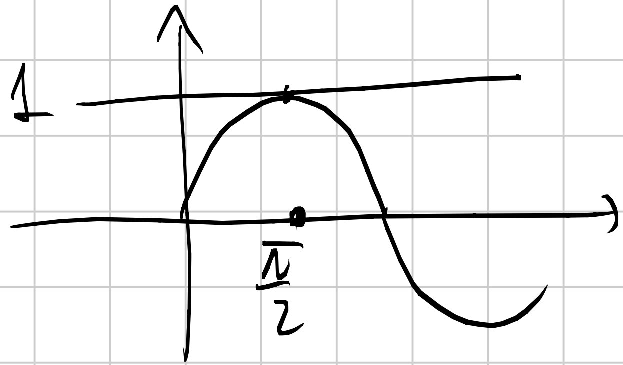
$f(x) = \operatorname{sen} x \Rightarrow f'(x) = \cos x$

Calcolare $f'\left(\frac{\pi}{2}\right)$ cioè la derivata di sen x in $x = \frac{\pi}{2}$

$f'\left(\frac{\pi}{2}\right) = \cos \frac{\pi}{2} = 0$

retta tg. in $\left(\frac{\pi}{2}, 1\right)$ $\quad y = \operatorname{sen}\left(\frac{\pi}{2}\right) + \cancel{f'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right)}$

$$y = 1$$



Analogamente

$\cos(a+b)$

$$f(x) = \cos x \Rightarrow f'(x) = -\operatorname{sen} x$$

- $f(x) = e^x$

$$\frac{f(x+h) - f(x)}{h} = \frac{e^{x+h} - e^x}{h} =$$

$$= e^x \frac{e^h - 1}{h} \xrightarrow[h \to 0]{} e^x$$

$$\frac{e^h - 1}{h} \to 1$$

$$f(x) = e^x \implies f'(x) = e^x$$

$$f'(x) = f(x)$$

Es. Calcolare la derivata di e^x in $x = -8$

$$f'(-8) = e^{-8}$$

• $f(x) = \log x$

$$\frac{f(x+h) - f(x)}{h} = \frac{\log(x+h) - \log x}{h} = \frac{\log\left(\frac{x+h}{x}\right)}{h} =$$

$$= \frac{\log\left(1 + \frac{h}{x}\right)}{\frac{h}{x}\, x} \xrightarrow{h \to 0} \frac{1}{x}$$

$h \to 0$

$\frac{h}{x} \to 0$

$$f(x) = \log x \Rightarrow f'(x) = \frac{1}{x}$$

Se f è derivabile in ogni punto $x \in (a,b)$

è definita $f'(x)$

Se essa è a sua volta derivabile

$$\left(f'(x)\right)' = f''(x) \quad \text{derivata } \underline{\text{seconda}}$$

$$\ddot{f} \qquad \frac{d^2 f}{dx^2}(x) \qquad D^2 f$$

e in generale derivata n-esima

$$f^{(n)}(x)$$

es. $f(x) = x^4$ ē derivabile $\forall x \in \mathbb{R}$

$f'(x) = 4x^3$ // //

$f''(x) = \ldots\ldots$

______________ . ______________

Esercizi su limiti di funzioni

- $$\lim_{x \to 0} \frac{\operatorname{arctg} x}{x} =$$

$\operatorname{arctg} x = y$

$x = \operatorname{tg} y$

$x \to 0$

$y \to 0$

$$= \lim_{y \to 0} \frac{y}{\operatorname{tg} y} = 1 \qquad \underbrace{\frac{y}{\operatorname{sen} y}}_{\to 1} \cdot \cos y$$

$$\operatorname{arctg} x \sim x \quad , \; x \to 0$$

$$\frac{\operatorname{arctg} y}{y} \xrightarrow[y \to 0]{} 1$$

$$\lim_{x \to +\infty} x \left(\frac{\operatorname{arctg}\left(\frac{2}{x+\sin x}\right)}{\frac{2}{x+\sin x}} \right) \qquad \infty \cdot 0 \qquad \frac{2}{x+\sin x}$$

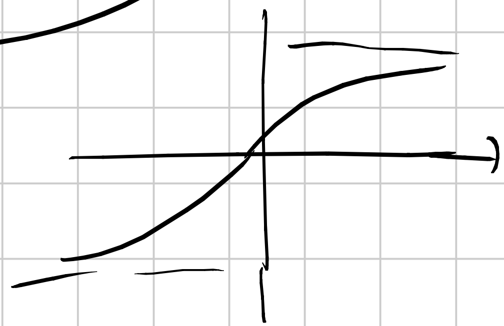
1

$0 \leftarrow y$

$$\lim_{x \to +\infty} \frac{2x}{x+\sin x} \cdot \frac{\operatorname{arctg}\left(\frac{2}{x+\sin x}\right)}{\frac{2}{x+\sin x}} = 2$$

$\to 1$

<u>OSS.</u> se fosse stato



$$\lim_{x \to +\infty} x \underbrace{\operatorname{arctg}(x + \sin x)}_{\to \frac{\pi}{2}} = +\infty$$

---

es. $$\lim_{x \to +\infty} x \left(\frac{\pi}{2} - \operatorname{arctg} x \right)$$

$\frac{\pi}{2} - \operatorname{arctg} x = y$

$\frac{\pi}{2} - y = \operatorname{arctg} x$

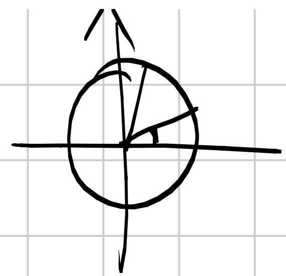
$$= \lim_{y \to 0} \operatorname{tg}\left(\frac{\pi}{2} - y\right) \cdot y$$

$x = \operatorname{tg}\left(\frac{\pi}{2} - y\right)$

$x \to +\infty$

$y \to 0$

$$\operatorname{tg}\left(\frac{\pi}{2}-y\right) = \frac{\operatorname{sen}\left(\frac{\pi}{2}-y\right)}{\cos\left(\frac{\pi}{2}-y\right)} = \frac{\cos y}{\operatorname{sen} y} =$$



$$= \frac{1}{\operatorname{tg} y}$$

$$\lim_{y \to 0} \frac{1}{\operatorname{tg} y} \cdot y = 1$$

$$\frac{\operatorname{sen} y}{y} \to 1 \quad y \to 0$$

$$\frac{\operatorname{arcsen} y}{y} \to 1 \quad y \to 0$$

$$\lim_{x \to 0} \frac{x^2 \,(\operatorname{sen} x)^2 \,(x^2+x^3)}{x^2 \,\arcsin(x^2+x^3)\,(x^2+x^3)} = 1$$

$\to 1$ $\quad$ $\to 1$

<u>Variante</u>:

$$\lim_{x \to 0^+} \frac{(\operatorname{sen} x)^2 - e^{-2/x}}{\arcsin(x^2+x^3) - x^{1/x}}$$

confronto $e^{-2/x}$ con x^2

$\frac{1}{x} = y$

$y \to +\infty$

$$\lim_{x\to 0^+} \frac{e^{-\frac{2}{x}}}{x^2} = \lim_{x\to 0^+} \frac{1}{x^2 e^{2/x}} =$$

$$= \lim_{y\to +\infty} \frac{y^2}{e^{2y}} = 0$$

$$N. = (\operatorname{sen} x)^2 \left(1 - \frac{e^{-2x}}{(\operatorname{sen} x)^2}\right)$$

$\to 0$

$$\lim_{x\to 0^+} x^{\frac{1}{x}} = \lim_{x\to 0^+} e^{\frac{1}{x}\log x} = 0$$

D. $\arcsin(x^2+x^3) - x^{1/x}$

$\arcsin(x^2+x^3) \sim x^2 \quad x \to 0$

$\left(\frac{\arcsin(x^2+x^3)}{x^2} \xrightarrow[x\to 0]{?} 1 \right)$

$$\lim_{x\to 0^+} \frac{x^{1/x}}{x^2} =$$

$\frac{1}{x} = y$ $\quad x = \frac{1}{y}$ $\quad$ PC.

$$= \lim_{y\to +\infty} \frac{1}{y^y} \cdot y^2 = \lim_{y\to+\infty} \frac{1}{y^{y-2}} = 0$$

D. $\arcsin(x^2+x^3)\left(1-\dfrac{x^{1/x}}{\arcsin(x^2+x^3)}\right)$

nel limite iniziale

$$= \lim_{x\to 0^+} \frac{(\operatorname{sen} x)^2 \left(1+ \xrightarrow{} 0\right)}{\arcsin(x^2+x^3)\left(1+ \underset{\searrow 0}{\cdots}\right)}$$

$$= \lim_{x\to 0^+} \frac{(\operatorname{sen} x)^2}{\arcsin(x^2+x^3)}$$

fatto prima.

$$f(x) = \begin{cases} \dfrac{2\sin(x-1) + (x-1)^2}{2(e^{x-1}-1)} & x \neq 1 \\ \alpha - 1 & x = 1 \end{cases}$$

per quali α la $f(x)$ è continua in $\mathbb{R}$.

$$\lim_{x \to 1} \frac{2\sin(x-1) + (x-1)^2}{2(e^{x-1}-1)} = f(1) = \alpha - 1$$

$x-1=t$

$$\Downarrow$$

$$\lim_{t \to 0} \frac{(2\sin t + t^2)}{2(e^t - 1)} \cdot \frac{t}{t} \to 1 = 1$$

$$\frac{2\sin t + t^2}{2t} \cdot \frac{t}{e^t - 1}$$

$\frac{t}{e^t-1} \to 1$

$$\frac{2\sin t}{2t} + \frac{t^2}{2t}$$

$\frac{2\sin t}{2t} \to 1 \qquad \frac{t^2}{2t} \to 0$

la f è continua

$$\alpha - 1 = 1 \Rightarrow \alpha = 2$$

se $\alpha \neq 2$? $\lim_{x \to 1} f(x) = 1 \neq \alpha - 1$

discontinuità eliminabile

es. $\lim_{x \to -\infty} \sqrt{x^2+1} + (x+1)$

$$\left(\sqrt{x^2+1} + (x+1)\right)\frac{\left(\sqrt{x^2+1} - (x+1)\right)}{\sqrt{x^2+1} - (x+1)} = \frac{x^2+1-(x+1)^2}{\sqrt{x^2+1}-(x+1)}$$

$$= \frac{-2x}{\sqrt{x^2+1}-(x+1)} \underset{x\to-\infty}{=}$$

$$\sqrt{x^2+1} = \sqrt{x^2\left(1+\frac{1}{x^2}\right)} \underset{x<0}{=} -x\sqrt{1+\frac{1}{x^2}}$$

$$\sqrt{x^2} = |x| = \begin{cases} x & x > 0 \\ -x & x < 0 \end{cases}$$

$$\frac{-2x}{\sqrt{x^2+1} - (x+1)} =$$

$$= \frac{-2x}{-x\sqrt{1+\frac{1}{x^2}} - x - 1} = \frac{-2\cancel{x}}{\cancel{-x}\underbrace{\left(\sqrt{1+\frac{1}{x^2}} + 1 + \frac{1}{x}\right)}_{\substack{x \to -\infty \\ 2}}}$$

$$\longrightarrow 1$$

$$f(x) = \begin{cases} \dfrac{x-7}{\operatorname{arctg}(x-7)} + \dfrac{e^{x-8}-1}{(x-8)^2} & x \neq 7,\ x \neq 8 \\[2ex] \dfrac{1}{e} & x = 7 \\ e & x = 8 \end{cases}$$

f continua $\forall x \neq 7, 8$

$$? \quad \lim_{x \to 7} f(x) = \frac{1}{e} \longrightarrow$$

$$! \quad \lim_{x \to 8} f(x) = \frac{1}{e}$$

$$\lim_{x\to 7} \underbrace{\frac{x-7}{\operatorname{arctg}(x-7)}}_{1} + \frac{e^{x-8}-1}{(x-8)^2} = \cancel{1} + e^{-1} - \cancel{1}$$

$$\frac{e^{-1}-1}{1}$$

$$= \frac{1}{e}$$

f è continua in $x=7$

$$\lim_{x\to 8} \underbrace{\frac{x-7}{\operatorname{arctg}(x-7)}}_{\frac{1}{\operatorname{arctg} 1}} + \frac{e^{x-8}-1}{(x-8)^2} \to \pm\infty$$

controllo

discontinua (2ª specie) in $x=8$