Lezione del 18 Gennaio

$$f : A \subseteq \mathbb{R}^n \to \mathbb{R}$$

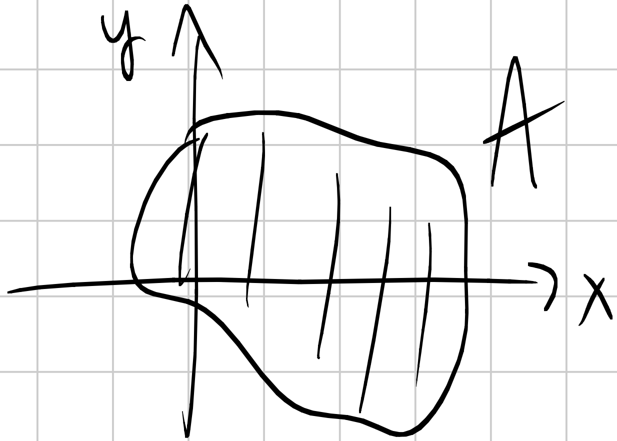
$$\underline{x} = (x_1, x_2, \dots x_n) \longrightarrow f(\underline{x}) \in \mathbb{R}$$

f di due variabili $\quad (x, y) \longleftrightarrow (x_1, x_2)$

tre $\quad$ " $\quad (x, y, z) \longleftrightarrow (x_1, x_2, x_3)$

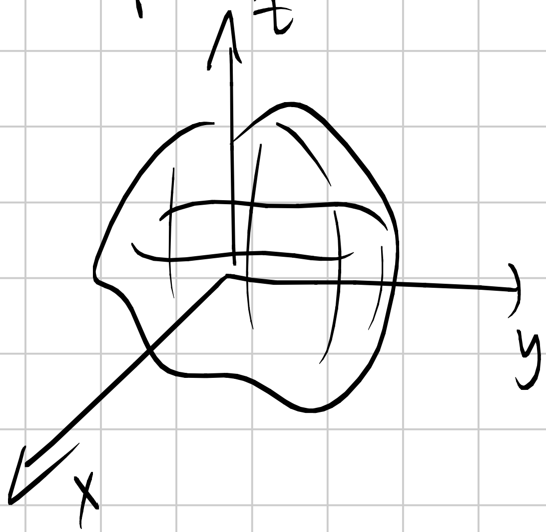
$f(x, y) \qquad A$ dominio $= \{(x, y) \in \mathbb{R}^2 : f$ è definita$\}$

$A \subseteq \mathbb{R}^2$



$f(x,y,z)$ $\quad$ A dominio $= \{(x,y,z) \in \mathbb{R}^3 : f$ è definita$\}$

$A \subseteq \mathbb{R}^3$



<u>es.</u> $f(x,y,z) = \operatorname{sen}(x+y) + \lg z$

$(x, y, z) \longrightarrow f(x, y, z)$

$\text{Dominio} = \{ (x, y, z) \in \mathbb{R}^3 : z > 0 \}$

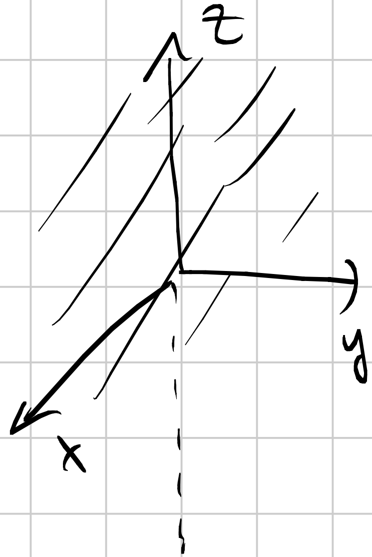
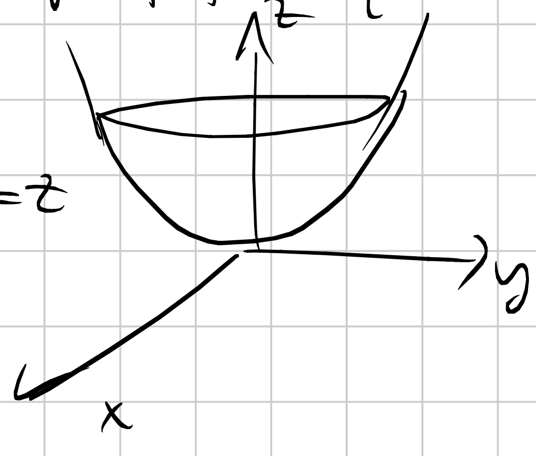


Grafico per f. di due variabili

$f(x, y)$

$\text{graf} f = \{ (x, y, f(x, y)), (x, y) \in A \} \subseteq \mathbb{R}^3$

$f(x, y) = x^2 + y^2 = z$



$$f(x,y)=\sqrt{x^2+y^2} \qquad A=\mathbb{R}^2 \qquad (x,y,\sqrt{x^2+y^2})$$

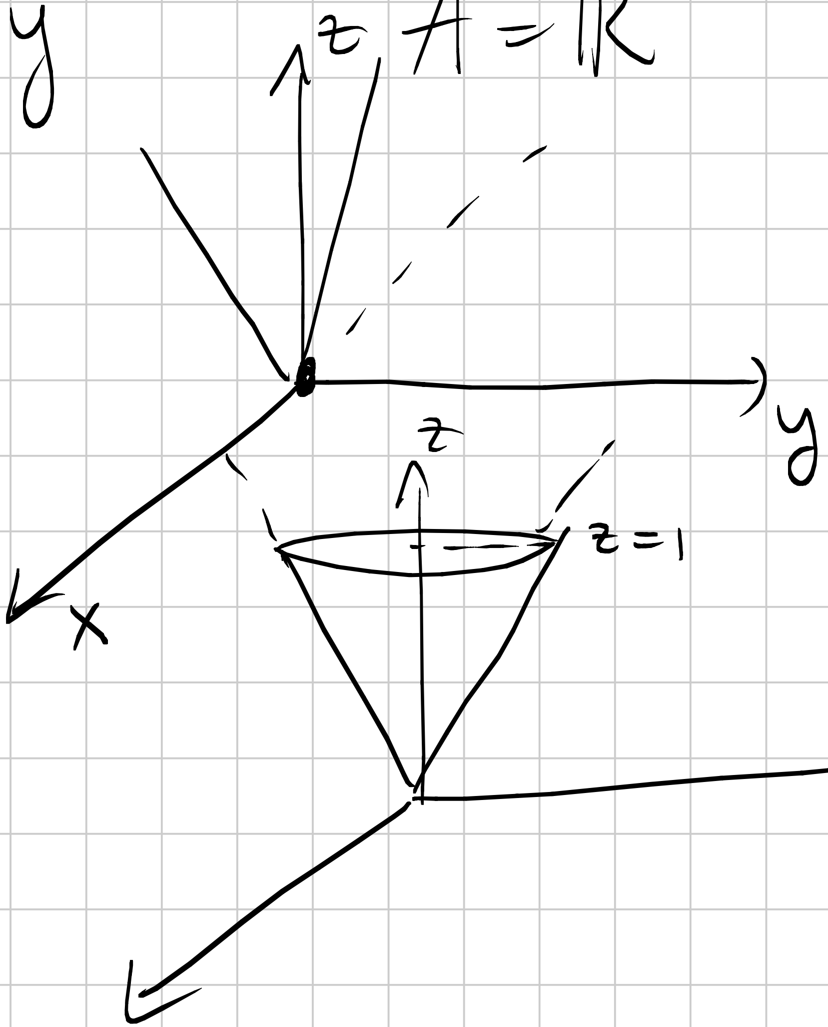
$$z=\sqrt{x^2+y^2}$$

$y=0$ $\Downarrow$

$$z=\sqrt{x^2}=|x|$$

$z=1$ $\quad \sqrt{x^2+y^2}=1$

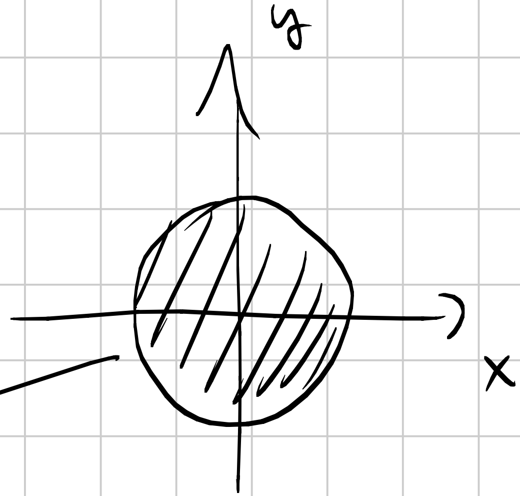
$$x^2+y^2=1$$



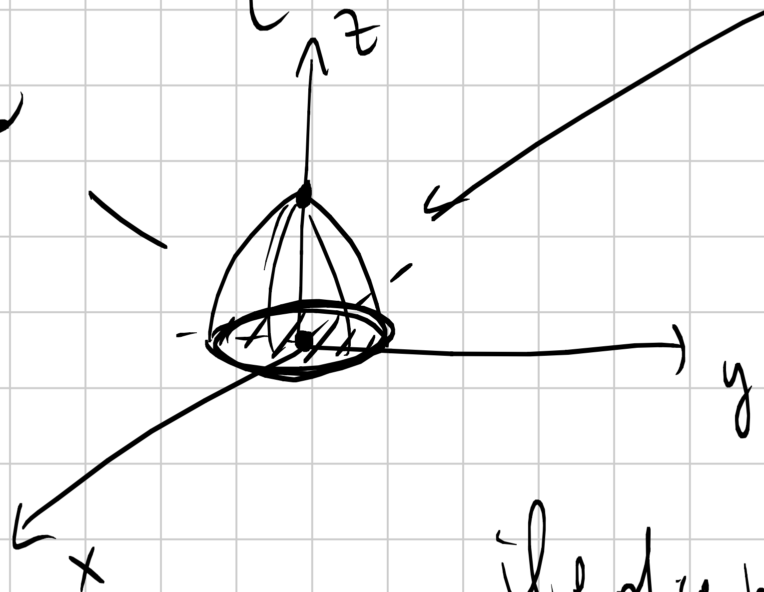
cono

$$f(x,y) = \sqrt{1-(x^2+y^2)}$$

$$D = \{\, x^2+y^2 \le 1 \,\}$$



semi-sfera



$$z = \sqrt{1-(x^2+y^2)}$$

$$x^2+y^2=1 \Rightarrow z=0$$

$x=0$ e $y=0$ $\quad z$

il grafico passa per $(0,0,1)$

$$f(x,y) = x^2 - y^2$$

paraboloide iperbolico

libro!

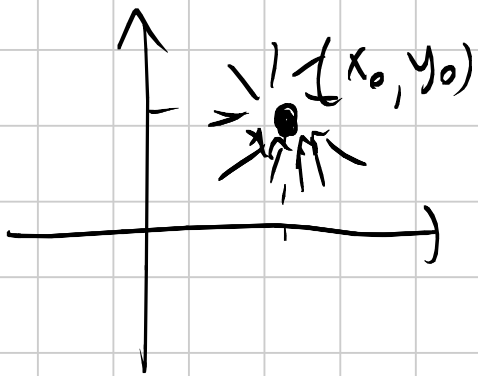
Limiti di funzioni in più variabili

$f : \mathbb{R} \to \mathbb{R}$

$$\lim_{x \to x_0} f(x) = L \iff \forall V \text{ intorno di } L \ \exists U \text{ intorno di } x_0 \text{ t.c. } f(x) \in V \ \forall x \in U$$

x_0

$$f : \mathbb{R}^2 \to \mathbb{R}$$



Intorni sferici

modulo di un vettore (norma)

$$\underline{x} = (x_1, x_2, \dots x_n)$$

$$|\underline{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

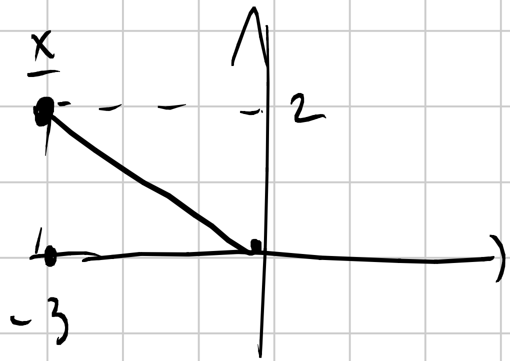
$$(\|x\|)$$

es. $\underline{x} = (1, 2, 3)$

$$|\underline{x}| = \sqrt{1 + 4 + 9} = \sqrt{14}$$

es $\underline{x} = (-3, 2)$

$$|\underline{x}| = \sqrt{9 + 4} = \sqrt{13}$$



= distanza del p.to dall' origine

$|\underline{x}|$ è la sua distanza dall' origine.

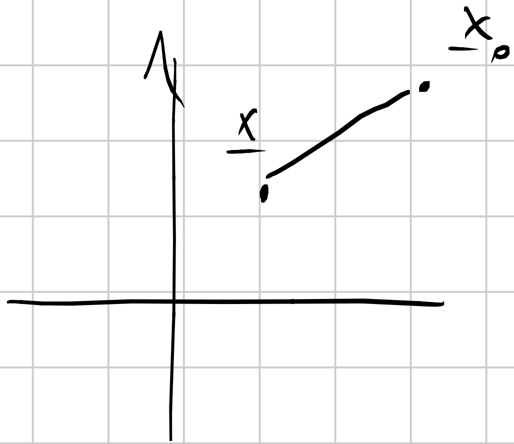
in $\mathbb{R}$ $\quad \underline{x} = x_1$

coincide con il valore assoluto

$$|\underline{x}| = \sqrt{x_1^2} = |x_1|$$

— · —

$\underline{x}, \quad \underline{x}_0$



$$d(\underline{x}, \underline{x}_0) = |\underline{x} - \underline{x}_0|$$

distanza tra due p.ti

es. $\quad \underline{x} = (1, 2) \quad \underline{x}_0 = (0, 5)$

$$d(\underline{x}, x_0) = |\underline{x} - \underline{x}_0| = \sqrt{(1-0)^2 + (2-5)^2} =$$

$$= \sqrt{1 + 9} = \sqrt{10}$$

In generale $\underline{x} = (x_1, x_2, \dots x_n)$

$$\underline{x}_0 = (x_{01}, x_{02}, \dots x_{0n})$$

$$d(\underline{x}, \underline{x}_0) = \sqrt{(x_1 - x_{01})^2 + (x_2 - x_{02})^2 + \dots (x_n - x_{0n})^2}$$

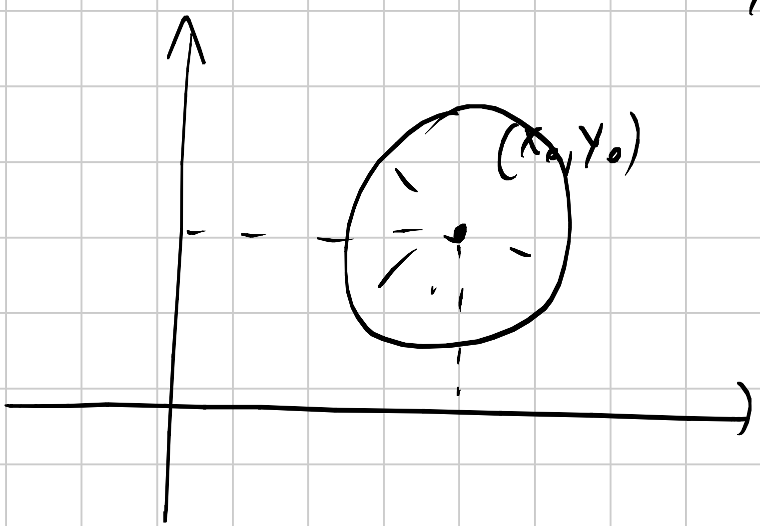
<u>Def.</u> Intorno sferico di $\underline{x}_0 \in \mathbb{R}^n$

$$U_r(\underline{x}_0) = \left\{ \underline{x} \in \mathbb{R}^n : |\underline{x} - \underline{x}_0| < r \right\}$$

r = raggio dell' intorno

x_0 = centro dell'intorno.

in $\mathbb{R}^2$

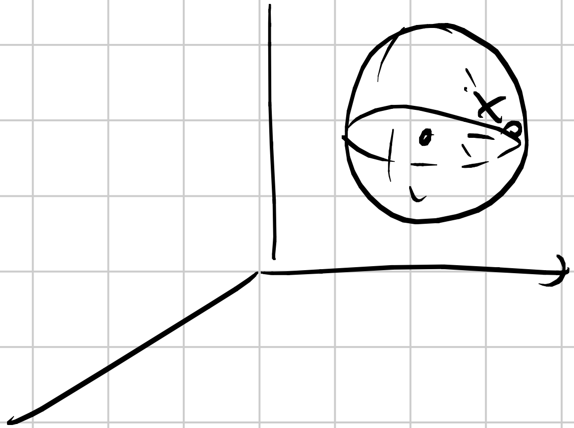


$\underline{x}_0 = (x_0, y_0)$

$U_r(x_0, y_0)$ = i punti che stanno dentro la circonferenza di centro $\underline{x}_0$ e raggio r.

$\underline{x} \in U_r(\underline{x}_0)$

in $\mathbb{R}^3$ gli intorni sferici di raggio r sono le sfere di raggio r



Def. di limite (con gli intorni) Def. 3.3 sul libro

$$f : \mathbb{R}^n \to \mathbb{R}$$

$$\lim_{\underline{x} \to \underline{x}_0} f(\underline{x}) = L \iff \forall \varepsilon > 0 \ \exists \delta : |f(x) - L| < \varepsilon$$

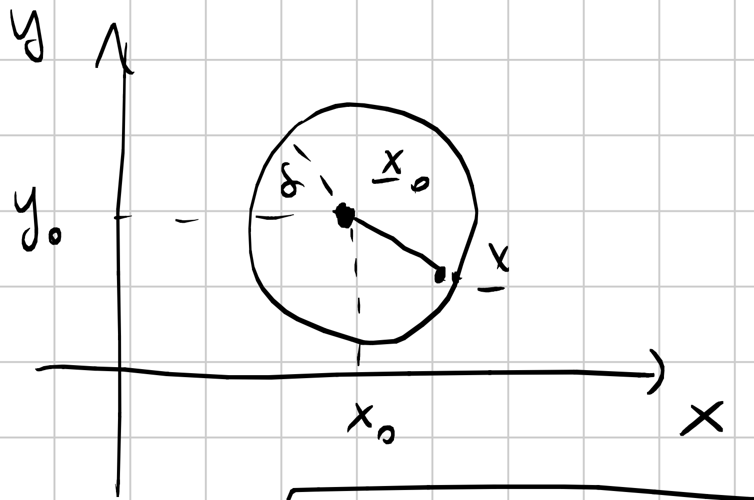
$$\forall x : \quad |\underline{x} - \underline{x}_0| < \delta .$$

$$\underline{x} \neq \underline{x}_0$$

es. in $\mathbb{R}^2$

$\underline{x} = (x,y)$

$\underline{x}_0 = (x_0,y_0)$

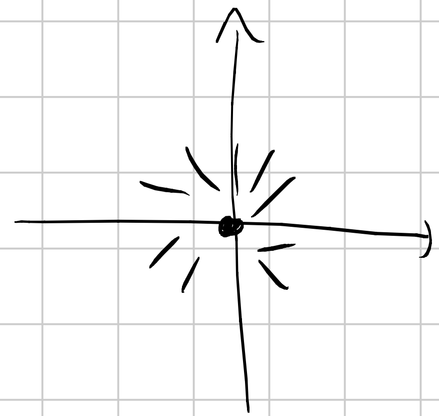


$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

$$|f(x) - L| < \varepsilon \quad \forall \quad |\underline{x} - \underline{x}_0| = \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

cioè (x,y) deve stare dentro la circonferenza di raggio δ e centro (x_0,y_0).

es. $$\lim_{(x,y)\to(0,0)} \underbrace{x^2+y^2}_{f(x,y)} = 0$$



Usiamo la def. di limite

$$\forall \varepsilon > 0 \quad \exists \delta > 0 : \quad |f(x,y) - 0| < \varepsilon \quad \forall (x,y):$$

$$d\big((x,y),(0,0)\big) < \delta$$

$$x^2 + y^2 < \varepsilon$$

$$d\big((x,y),(0,0)\big) = \sqrt{x^2 + y^2} < \delta$$

$$x^2 + y^2 < \varepsilon$$

$$\Rightarrow \sqrt{x^2+y^2} < \sqrt{\varepsilon}$$

Voglio trovare δ quindi $\delta = \sqrt{\varepsilon}$

Def. f continua

$f : \mathbb{R}^n \to \mathbb{R}$ continua in $\underline{x}_0$ se

$$\lim_{\underline{x} \to \underline{x}_0} f(\underline{x}) = f(\underline{x}_0)$$

es. $f(x, y) = x^2 + y^2$

$f(0,0) = 0 + 0 = 0$

$$\lim_{(x,y) \to (0,0)} x^2 + y^2 = 0 = f(0,0)$$

$\Longrightarrow$ $f(x,y) = x^2 + y^2$ è continua in $(0,0)$.

Valgono teoremi analoghi al caso di una variabile: somma, prodotto, composizione... di f. continue sono continue

$\operatorname{sen}(x+y)$, $e^{\left(\log \frac{x}{y}\right)^2} + z$ dove sono definite sono continue

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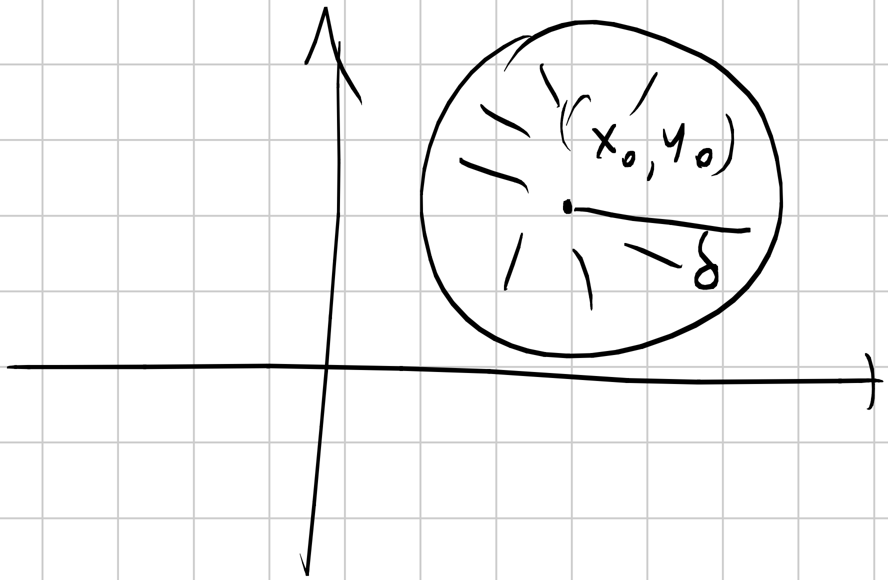
Calcolo dei limiti per funzioni di due variabili

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \iff |f(x,y) - L| < \varepsilon$$

se $(x,y) \to (x_0,y_0)$

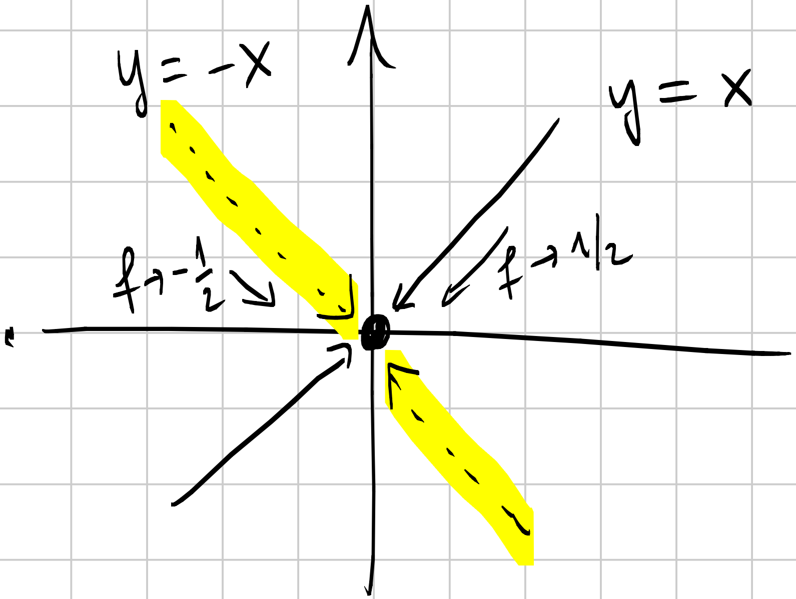
cioè se $\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$

$(x,y) \to (x_0, y_0)$
lungo una
qualsiasi curva
che va verso
(x_0, y_0)



1) se esistono due curve che passano per (x_0, y_0) lungo le quali f tende a due valori diversi $\Rightarrow$ il limite <u>non</u> esiste.

es. $\displaystyle \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ $\left(\frac{0}{0}\right)$



su questa retta

$$f(x,y) = \frac{xy}{x^2+y^2}$$

mi diventa

$$f(x,x) = \frac{\cancel{x^2}}{2\cancel{x^2}} = \frac{1}{2}$$

$f = \frac{1}{2}$ sulla retta $y = x$

$$\lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

ora considero un'altra direzione per arrivare verso l'origine $\quad y = -x$

$$f(x,y) = \frac{xy}{x^2+y^2} \quad \text{se} \quad y = -x$$

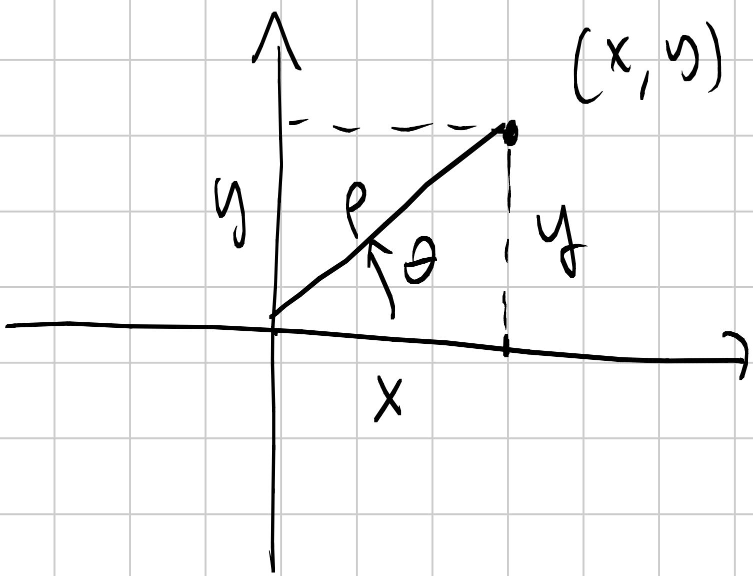
$$f(x,-x) = -\frac{\cancel{x^2}}{2\cancel{x^2}} = -\frac{1}{2} \xrightarrow[x\to 0]{} -\frac{1}{2}$$

quindi su $y = x \quad f \to 1/2$ $\Longrightarrow$ il limite $\nexists$.

su $y = -x \quad f \to -\frac{1}{2}$

Metodo per dimostrare che $\lim\limits_{(x,y)\to(0,0)} f(x,y) = L$

con le coordinate <u>polari</u>

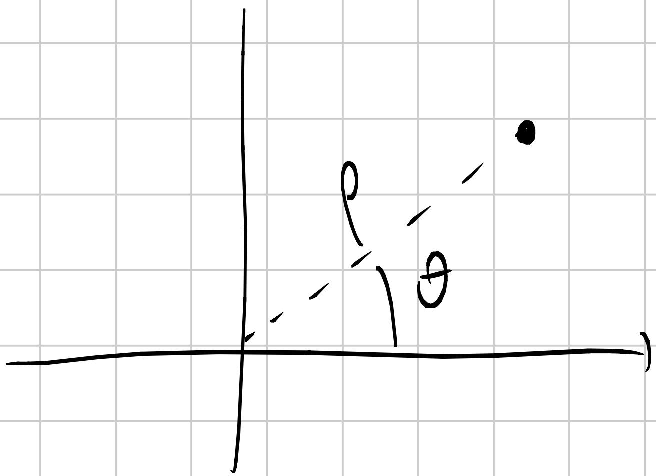


$$\rho = \sqrt{x^2 + y^2}$$

$$\begin{cases} x = \rho \cos\theta \\ y = \rho \operatorname{sen}\theta \end{cases}$$

$\Downarrow$ relazione tra

$$(x, y) \Leftrightarrow (\rho, \theta)$$



se $(x, y) \to (0, 0)$ allora

$$\rho \to 0$$

Quindi dire

$$\lim_{(x,y)\to(0,0)} f(x,y) = l$$ è equivalente a dire

$$\lim_{\rho\to 0} f(\rho\cos\theta, \rho\,\text{sen}\,\theta) = l$$

$\underline{\underline{\forall\theta}}$

<u>es.</u> $$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^2+y^2} =$$ con le coordinate polari

$$x = \rho\cos\theta$$

$$y = \rho\,\text{sen}\,\theta$$

$$= \lim_{\rho \to 0} \frac{2(\rho\cos\theta)^2 \cdot \rho\,\text{sen}\,\theta}{\rho^2\cos^2\theta + \rho^2\text{sen}^2\theta} =$$

$$= \lim_{\rho \to 0} \frac{2\rho^{\cancel{3}}\cos^2\theta\,\text{sen}\,\theta}{\cancel{\rho^2}} =$$

$$= \lim_{\rho \to 0} 2\rho\cos^2\theta\,\text{sen}\,\theta \to 0 \quad \underline{\underline{\forall\theta}}$$

quindi $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^2+y^2} = 0$

es. $$\lim_{(x,y)\to(0,0)} \frac{xy\log(x^2+y^2)}{\sqrt{x^2+y^2}} =$$

$$= \lim_{\rho\to 0} \frac{\cancel{\rho}\cos\theta\,\rho\,\mathrm{sen}\,\theta\,\log(\rho^2)}{\cancel{\rho}} =$$

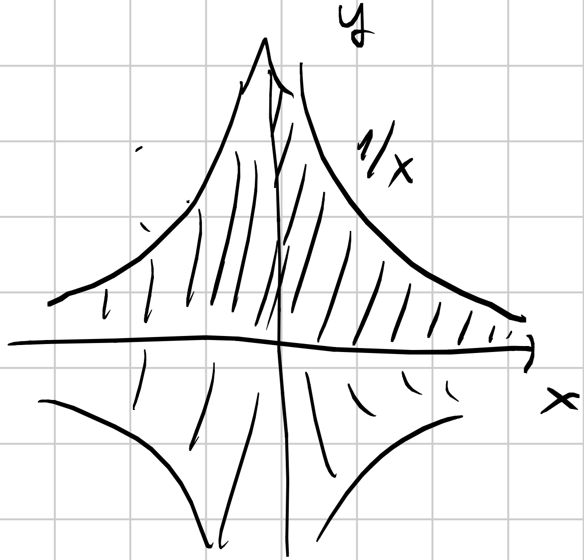
$$= \lim_{\rho\to 0} \underbrace{\rho\log(\rho^2)}_{\to 0}\cos\theta\,\mathrm{sen}\,\theta = 0 \quad \forall\theta$$

es. $f(x,y) = \arcsin(xy)$

Trovare dominio e disegnarlo in $\mathbb{R}^2$

$$|xy| \leq 1$$

$$D = \{ (x,y) : |xy| \leq 1 \}$$



$$|xy| \leq 1$$

1° quadrante $\quad x>0, \ y>0 \quad xy \leq 1 \quad y \leq \frac{1}{x}$

2° " $\quad y>0 \quad x<0 \quad -xy \leq 1 \quad y \leq -\frac{1}{x}$