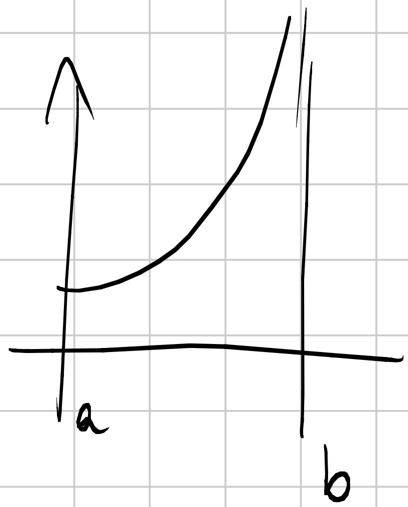
Lezione del 19 Dicembre

1) $f : [a,b) \to \mathbb{R} \quad \lim_{x \to b^-} f(x) = +\infty$

$$\int_a^b f(x)\,dx := \underbrace{\lim_{\omega \to b} \int_a^\omega f(x)\,dx}$$

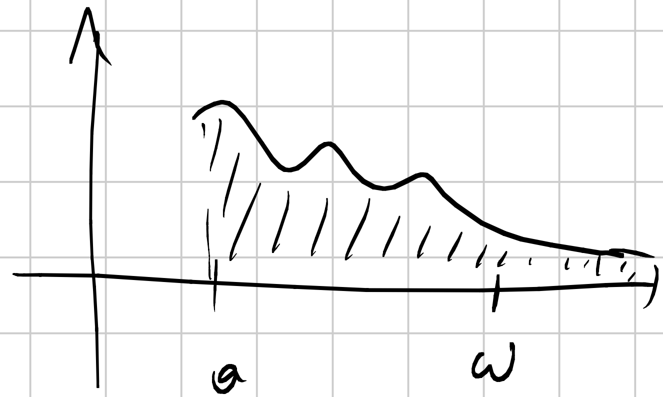


$f, g \geq 0$

- confronto $0 \leq f \leq g$
- confronto asintotico $f \sim g, \ x \to b$

Integrazione su intervalli illimitati

$f : [a, +\infty) \to \mathbb{R}$ continua



$$\int_a^{+\infty} f(x)\,dx \doteq \lim_{\omega \to +\infty} \int_a^{\omega} f(x)\,dx$$

Def. se il limite esiste finito allora f è <u>integrabile</u> in $(a, +\infty)$ o che $\int_a^{+\infty} f(x)\,dx$ è <u>convergente</u>.

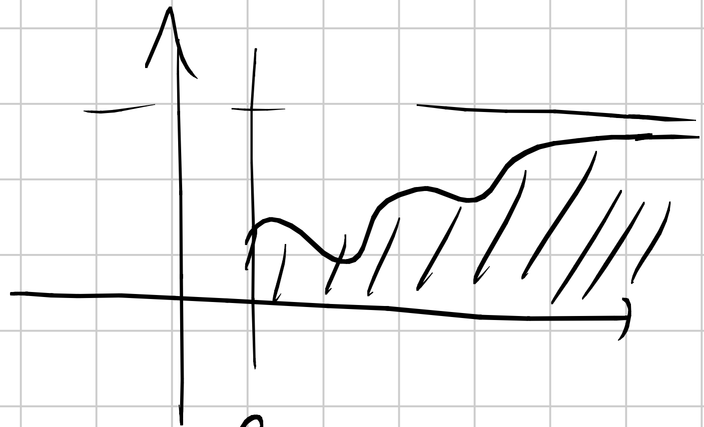
Se il limite è $\pm\infty \Rightarrow \int_a^{+\infty} f(x)\,dx$ è <u>divergente</u>

Se il limite non esiste $\Rightarrow \int$ non esiste

Analog. in $(-\infty, a]$ $\int_{-\infty}^{a} f(x)dx \doteq \lim_{\omega \to -\infty} \int_{\omega}^{a} f(x)dx$

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{+\infty} f(x)dx$$

<u>OSS.</u> Se $\lim_{x \to +\infty} f(x) = L \neq 0$



$$\int_{a}^{+\infty} f(x)dx = +\infty$$

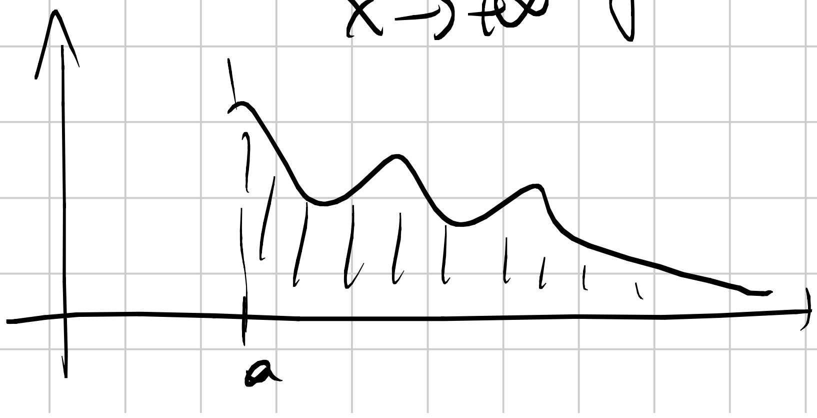
$\Rightarrow$ f <u>non</u> è integrabile es. $f(x) = \dfrac{3x^2+1}{2x^2+5}$ è integrabile in $[1, +\infty)$?

$$\lim_{x \to +\infty} \frac{3x^2+1}{2x^2+5} = \frac{3}{2} \neq 0$$

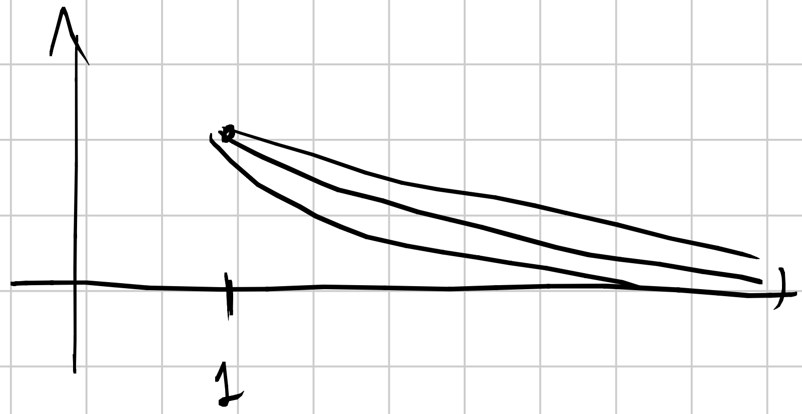
no!

quindi è necessario che $\lim_{x \to +\infty} f(x) = 0$

(ma non sufficiente)



Es. $f(x) = \dfrac{1}{x^\alpha}$ $\quad \alpha > 0$



$$\lim_{x \to +\infty} \frac{1}{\underbrace{x^\alpha}_{+\infty}} = 0$$

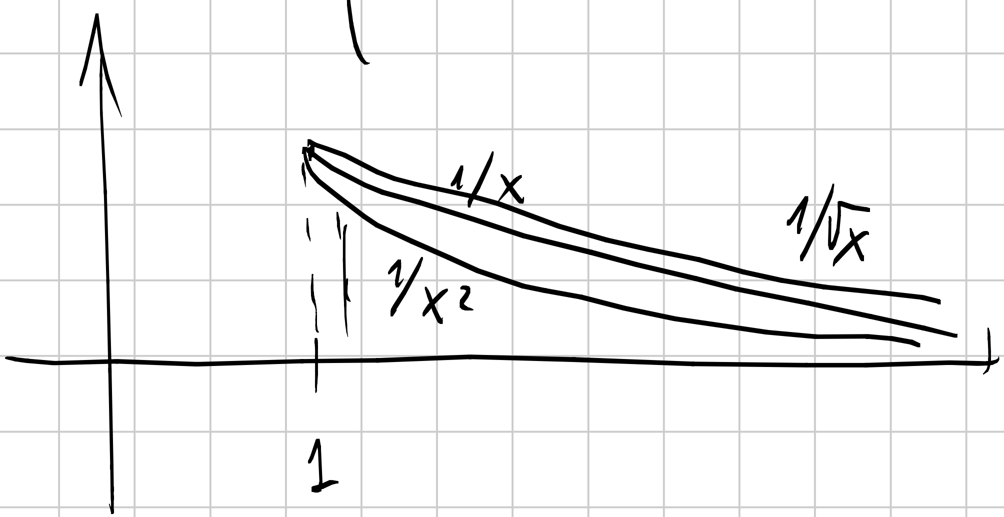
$$\alpha = 1 \qquad \int_1^{+\infty} \frac{1}{x}\,dx \doteq \lim_{\omega \to +\infty} \int_1^{\omega} \frac{1}{x}\,dx = \lim_{\omega \to +\infty} \log \omega = +\infty$$

$\dfrac{1}{x}$ non è integrabile in $[1, +\infty)$

$$\alpha \neq 1 \qquad \int_1^{+\infty} \frac{1}{x^\alpha}\,dx \doteq \lim_{\omega \to +\infty} \int_1^{\omega} x^{-\alpha}\,dx = \lim_{\omega \to +\infty} \frac{x^{1-\alpha}}{1-\alpha}\Bigg/_1^{\omega} =$$

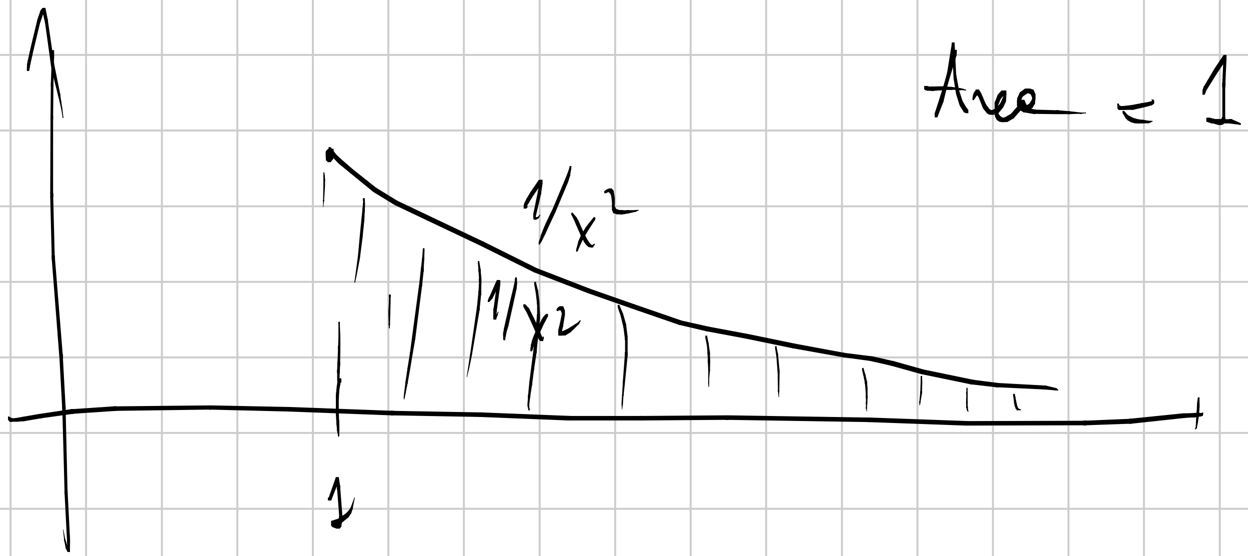
$$= \lim_{\omega \to +\infty} \frac{\omega^{1-\alpha}}{1-\alpha} - \frac{1}{1-\alpha} = \begin{cases} +\infty & \alpha < 1 \\ \dfrac{1}{\alpha - 1} & \alpha > 1 \end{cases}$$

$$\int_1^{+\infty} \frac{1}{x^\alpha}\,dx = \begin{cases} \text{divergente a } +\infty \quad \text{se} \quad \alpha \leq 1 \\ \text{convergente} = \dfrac{1}{\alpha-1} \quad \text{se} \quad \alpha > 1 \end{cases}$$

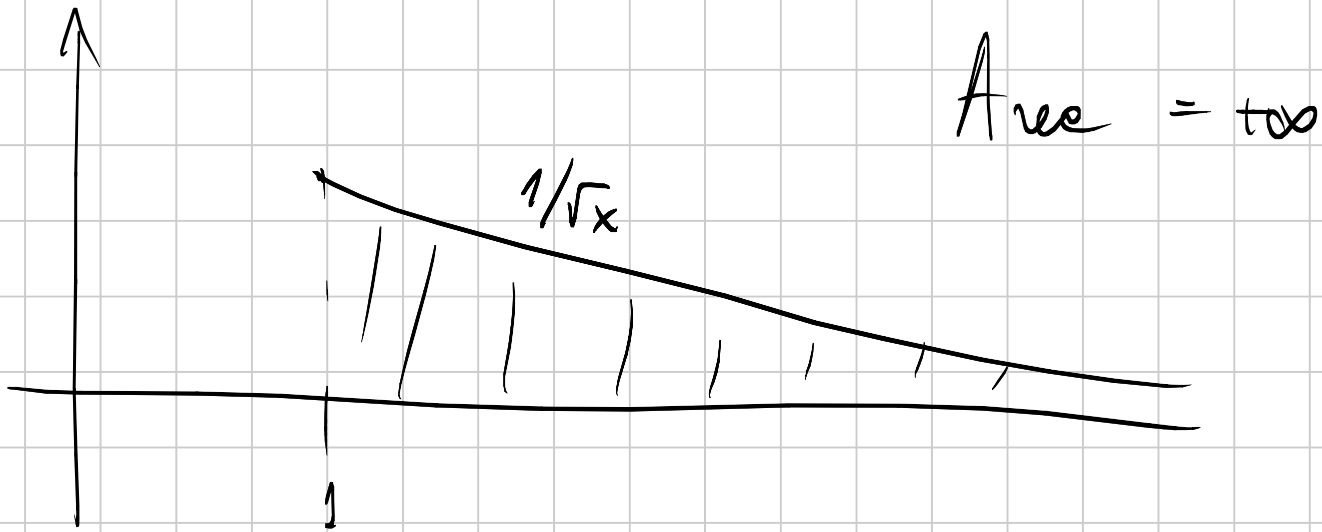


$$\int_1^{+\infty} \frac{1}{x^2}\,dx \quad \text{converg.}$$

$$\int_1^{+\infty} \frac{1}{\sqrt{x}}\,dx \quad \text{diverg.}$$



Aree = 1
1/x^2
1/x2
1



Aree = +∞
1/√x
1

$$\int_0^{+\infty} \frac{1}{x}\,dx = \underbrace{\int_0^1 \frac{1}{x}\,dx}_{+\infty} + \underbrace{\int_1^{+\infty} \frac{1}{x}\,dx}_{+\infty}$$

$= +\infty$ divergente

$$\int_0^{+\infty} \frac{1}{x^\alpha}\,dx = \underbrace{\int_0^1 \frac{1}{x^\alpha}\,dx}_{\text{conv.} \iff \alpha<1} + \underbrace{\int_1^{+\infty} \frac{1}{x^\alpha}\,dx}_{\text{conv.} \iff \alpha>1}$$

sempre divergente $\forall \alpha > 0$.

Altre classe di funzioni

$$f(x) = \frac{1}{x|\log x|} \qquad \begin{matrix} x>0 \\ x \neq 0 \\ x \neq 1 \end{matrix}$$

$$\int_2^{+\infty} \frac{1}{x \log x} \, dx =$$

$$\lim_{x \to +\infty} \frac{1}{x \lg x}$$

$$\lim_{x \to 0^+} \frac{1}{x|\lg x|} = +\infty$$

$$\lim_{x \to 1} \frac{1}{x|\lg x|} = +\infty$$

$$\int_2^{+\infty} \frac{1}{x \log x} \, dx = \lim_{\omega \to +\infty} \int_2^{\omega} \frac{1}{x \lg x} \, dx = \qquad (\lg x = t)$$

$$= \lim_{\omega \to +\infty} \log(|\log x|)\Big/_{2}^{\omega} = \lim_{\omega \to +\infty} \log(|\log \omega|) - \log(\log 2)$$

$$= +\infty$$

$\frac{1}{x \log x}$ non è integrabile in $[2, +\infty)$.

In generale

$$\int_2^{+\infty} \frac{1}{x (\log x)^{\beta}} \, dx = \begin{cases} \text{converge} & \text{se } \beta > 1 \\ \text{diverge} & \text{se } \beta \leq 1 \end{cases}$$

Es.
$$\int_2^{+\infty} \frac{1}{x(\log x)^2}\,dx \quad \text{converge.}$$

$$\int_2^{+\infty} \frac{1}{x\sqrt{\log x}}\,dx \quad \text{diverge}$$

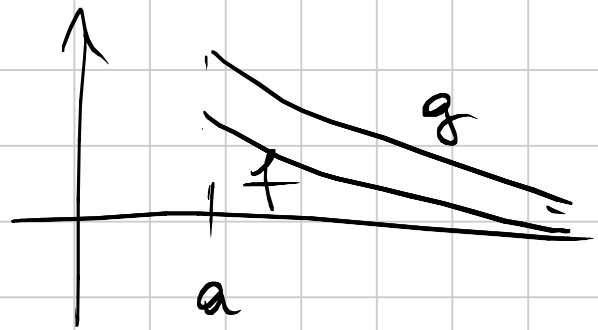
Ancora più in generale

$$\int_2^{+\infty} \frac{1}{x^{\alpha}(\log x)^{\beta}}\,dx \quad \underline{\text{converge}} \iff \begin{cases} \alpha > 1, \ \forall \beta \in \mathbb{R} \\ \alpha = 1, \ \forall \beta > 1 \end{cases}$$

Criteri di integrabilità all'infinito

Confronto $f, g : [a, +\infty)$ continue

$$0 \le f(x) \le g(x) \quad \text{in } [a, +\infty)$$

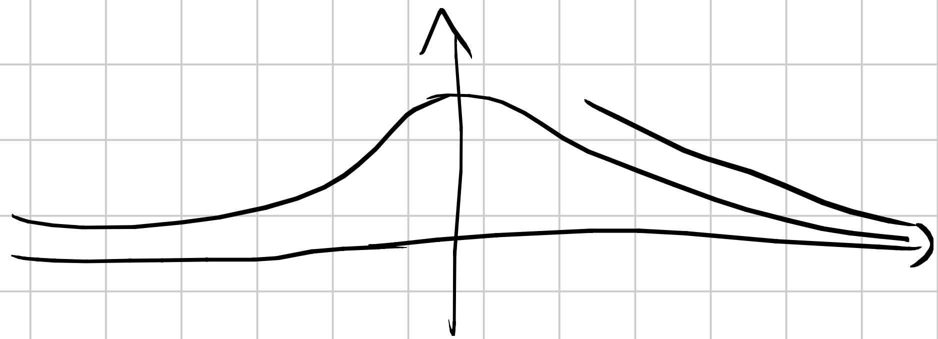


g integrabile in $[a, +\infty)$ $\Rightarrow$ f integrabile in $[a, +\infty)$

f non è integrabile $\Rightarrow$ g non è integrabile

Confronto asintotico $f > 0$, $g > 0$ e $f \underset{x \to +\infty}{\sim} g$

f è integrabile $\Leftrightarrow$ g è integrabile es. $\displaystyle\int_1^{+\infty} e^{-x^2}\,dx$



$x \geq 1 \qquad \underbrace{e^{-x^2}}_{f} \leq \underbrace{e^{-x^2}\cdot x}_{g}$

$-x^2 = t$

$-2x\,dx = dt$

$$\int_1^{+\infty} e^{-x^2}\cdot x\,dx = \lim_{\omega\to+\infty}\int_1^{\omega} e^{-x^2}\, x\,dx = \lim_{\omega\to+\infty} -\frac{1}{2}\left(e^{-x^2}\right)\Big|_1^{\omega}$$

$$= \lim_{\omega \to +\infty} -\frac{1}{2}\left(e^{-\omega^2}\right) + \frac{1}{2} e = \frac{1}{2} e$$

$$\int_1^{+\infty} e^{-x^2} x \, dx \text{ convergente } \Rightarrow \int_1^{+\infty} e^{-x^2} dx \text{ è } \underline{\text{convergente}}$$

ma non allo stesso valore di $\int_1^{+\infty} e^{-x^2} x \, dx$.

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es. $$\int_1^{+\infty} \underbrace{\frac{3x^2+1}{x^5 + e^{1/x} + \operatorname{sen} x}}_{f(x)} \, dx$$

$$\lim_{x \to +\infty} f(x) = 0$$

è integrabile

$$f(x) = \frac{3\cancel{x^2}\left(1 + \frac{1}{3x^2}\right)}{x^{\cancel{5}\,3}\left(1 + \frac{e^{1/x}}{x^5} + \frac{\operatorname{sen} x}{x^5}\right)} \underset{x \to +\infty}{\sim} \frac{3}{x^3}$$

$\frac{3}{x^3}$ integrabile in $[1, +\infty)$

$\Rightarrow$ $f(x)$ è integrabile in $[1, +\infty)$.

es.

$$\int_0^{+\infty} \frac{1}{\sqrt{x^2+1}}\,dx$$

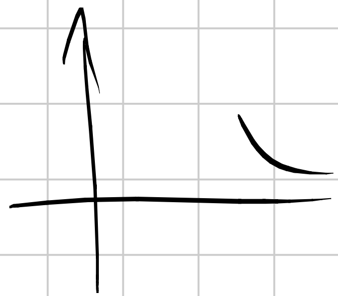
$$f(x) \underset{x\to+\infty}{\sim} \frac{1}{x}$$ non è integrabile in $[1,+\infty)$

$\Rightarrow$ $f(x)$ non è integrabile

es.

$$\int_1^{+\infty} \left(1 - \cos\frac{1}{x}\right) dx$$

$$f(x) \underset{x\to+\infty}{\longrightarrow} 0$$



$$\cos\frac{1}{x} = 1 - \frac{1}{2}\frac{1}{x^2} + o\left(\frac{1}{x^2}\right) \quad x\to+\infty$$

$$1 - \cos\frac{1}{x} = \frac{1}{2}\frac{1}{x^2} + o\left(\frac{1}{x^2}\right) \qquad x \to +\infty$$

$$\underbrace{1 - \cos\frac{1}{x}}_{} \sim \frac{1}{2}\frac{1}{x^2} \qquad x \to +\infty$$

integrabile in $[1, +\infty)$

es: $\displaystyle\int_1^{+\infty} x^{\alpha}\left(\operatorname{sen}\frac{1}{x} - \frac{1}{x}\right)dx$

per quali $\alpha \in \mathbb{R}$ l'$\int$ converge?

$$\operatorname{sen}\frac{1}{x} = \frac{1}{x} - \frac{1}{6}\frac{1}{x^3} + o\left(\frac{1}{x^3}\right)$$

$$\operatorname{sen}\frac{1}{x} - \frac{1}{x} \sim -\frac{1}{6x^3}$$

$$f(x) \sim x^{\alpha}\left(-\frac{1}{6x^3}\right) \qquad x \to +\infty$$

$$f(x) \sim \underbrace{-\frac{1}{6\,x^{3-\alpha}}}$$ è integrabile in $[1, +\infty)$

$$\Updownarrow$$

$$3 - \alpha > 1$$

$$\alpha < 2$$

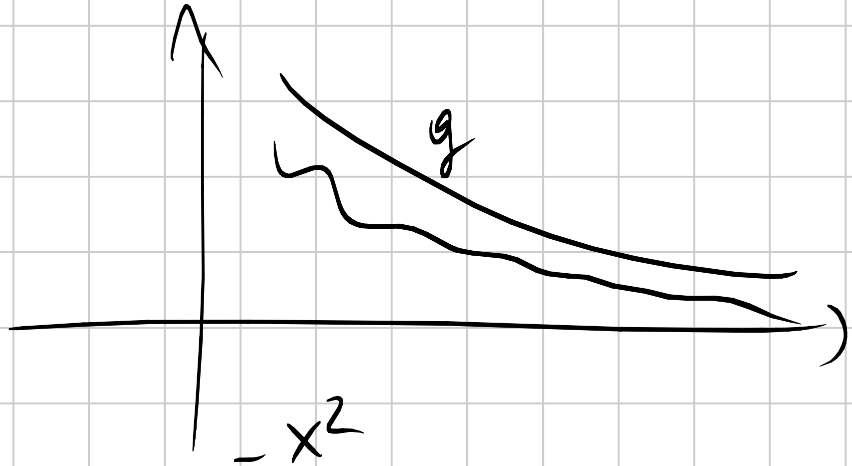
$f(x)$ è integrabile in $[1, +\infty)$ sse $\alpha < 2$.

<u>oss.</u> non importa che $f(x) \sim g(x)$ $x \to +\infty$

ma basta che $f(x)$ vada a zero più velocemente

di una $g(x)$ che è integrabile

cioè $\quad f(x) = o\,(g(x)), \quad x \to +\infty$



$$\int_0^{+\infty} e^{-x^2}\,dx$$

$$e^{-x^2} \sim \frac{1}{x^\alpha} \;?$$

e^{-x^2} non è asintotica a nessuna potenza di $\frac{1}{x^\alpha}$

$$e^{-x^2} = O\left(\frac{1}{x^2}\right)$$

$$= o\left(\frac{1}{x^3}\right)$$

perché $\frac{1}{x^2}$ è integrabile in $[1,+\infty)$

$\Rightarrow e^{-x^2}$ è integrabile in $[0,+\infty)$

$$= o\left(\frac{1}{x^\alpha}\right) \qquad \alpha > 1$$

$$= o\left(\frac{1}{\sqrt{x}}\right) \quad (\Rightarrow \text{non serve}).$$

<u>oss. 2</u> l'osservazione precedente vale anche nel caso f illimitata in $[a, b)$

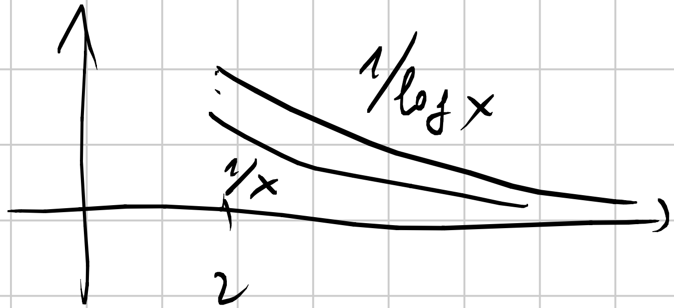
$f \sim g$ ma anche

$f = o(g) \qquad x \to b^-$

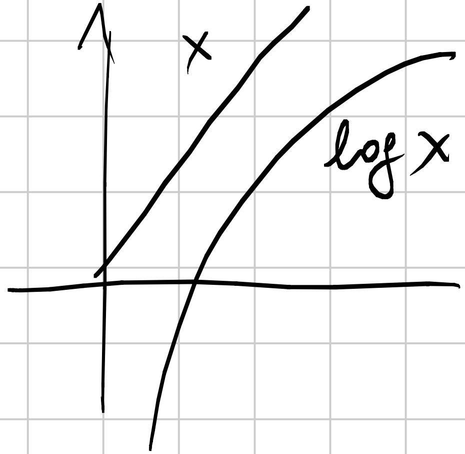
se g è integrabile $\Rightarrow f$ è integrabile.

es.

$$\int_2^{+\infty} \frac{1}{\log x}\, dx$$



$\log x > 0$ $\quad x > 2$



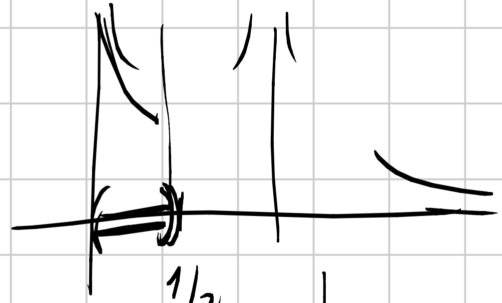
$\log x < x$

$$\frac{1}{\log x} > \frac{1}{x} \quad \forall x \geq 2$$

non integrable in $[2, +\infty)$

$\Rightarrow$ non integrable in $[2, +\infty)$

$$\int_2^{+\infty} \frac{1}{x^\alpha (\log x)^\beta}\, dx$$



$$\int_0^{1/2} \frac{1}{x^\alpha |\log x|^\beta}\, dx$$

converge se $\begin{cases} \alpha < 1, & \forall \beta \in \mathbb{R} \\ \alpha = 1, & \forall \beta > 1 \end{cases}$

es. $$\int_0^{1/2} \frac{1}{x\, |\log x|}\, dx$$ diverge

$$\int_0^{1/2} \frac{1}{\sqrt{x}} |\lg x|^2$$

$\alpha = \frac{1}{2}$

$\beta = -2$

convergente.

es. $f(x) = \dfrac{1}{x^2-1} = \dfrac{1}{(x-1)(x+1)}$

$$\int_0^1 \frac{1}{x^2-1}\,dx = \text{integrale generalizz.}$$

$$\int_2^{+\infty} \frac{1}{x^2-1}\,dx = \text{integrale generalizz.}$$

$$\int_0^{1/2} \frac{1}{x^2-1}\,dx = \text{integrale definito}$$

$$\int_2^{+\infty} \frac{1}{x^2-1}\,dx$$

$$f(x) \sim \frac{1}{x^2} \qquad x \to +\infty$$

è integ. in $(2, +\infty)$

è integrabile

$$\int_0^{1} \frac{1}{x^2-1}\,dx = \int_0^{1} \frac{1}{(x-1)(x+1)}\,dx =$$

$x-1=y \qquad x=y+1 \qquad dx=dy$

$$= \int_{-1}^{0} \frac{1}{y\cdot(y+2)}\,dy$$

$g(y) \sim \frac{1}{2y}$ $y \to 0$

non è integrabile in $(-1,0)$

non è integrabile in $(-1,0)$