Lezione del 2 Dicembre

- "o piccolo" $\quad x \to x_0$
- Polinomio di Taylor (o Mc Laurin)

$$f(x) = P_n(x) + \text{resto} \qquad x \to x_0$$

x_0 qualsiasi $\quad$ Polinomio di Taylor

$x_0 = 0$ $\quad$ " $\quad$ di Mc Laurin.

Polinomio di Mc Laurin di $f(x)$, derivabile n volte in $x=0$

$$T_n(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + \dots + f^{(n)}(0)\frac{x^n}{n!}$$

$$= \sum_{k=0}^{n} f^{(k)}(0)\frac{x^k}{k!}$$

$$f^{(0)} = f$$

$$f(x) = e^x \qquad f^{(k)}(x) = e^x \quad \forall k \qquad f^{(k)}(0) = 1$$

$$T_n(x) = 1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!}$$

polinomio di Mc Laurin associato a e^x
di grado n

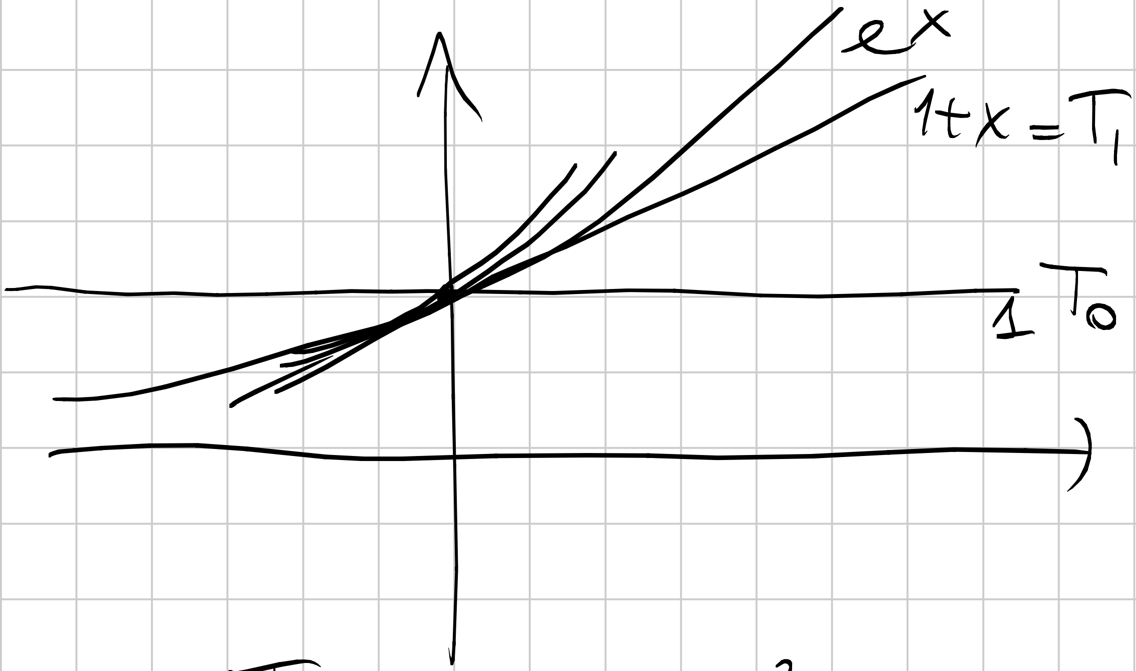
polinomio di grado 0 $\quad T_0(x) = 1$

di grado 1 $\quad T_1(x) = 1 + x$

" 2 $\quad T_2(x) = 1 + x + \frac{x^2}{2}$

$$e^x \longleftrightarrow T_n(x)$$

$x = 0$



$T_0(x) = 1$

ha un p.to in comune con il grafico di e^x

$T_1(x) = 1 + x$

ha un p.to in comune con grafico di e^x e la stessa retta tangente in $x=0$

$T_2 = 1 + x + \frac{x^2}{2}$

ha anche la stessa derivata seconda di e^x in $x=0$

$$T_n(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + \dots + f^{(n)}(0)\frac{x^n}{n!}$$

$$T_n(0) = f(0)$$

$$T_n'(x) = f'(0) + f''(0)x + \dots\dots$$

$$T_n'(0) = f'(0)$$

il polinomio ha un contatto di "ordine n" in $x=0$.

$$T_n''(0) = f''(0)$$

$$\vdots$$

$$T_n^{(n)}(0) = f^{(n)}(0)$$

Teorema Formula di Mc Laurin di ordine n con resto di Peano

f derivabile n volte in $x=0$. Allora

$$f(x) = T_n(x) + o(x^n) \qquad x \to 0$$

es. $f(x) = e^x$

$$x \to 0$$

$$n=1 \qquad e^x = 1 + x + o(x)$$

$n=2 \quad e^x = 1 + x + \frac{x^2}{2} + o(x^2) \quad x \to 0$

<u>Dim.</u> caso $n=1$ e $n=2$

$n=1$ <u>Ts.</u> $f(x) = T_1(x) + o(x)$?

$T_1(x) = f(0) + f'(0)x \quad x \to 0$

? $f(x) = f(0) + f'(0)x + o(x)$?

? $f(x) - f(0) - f'(0)x = o(x)$

$h = o(x)$

per def. di o piccolo $\quad \frac{h}{x} \to 0$

devo verificare $\dfrac{f(x) - f(0) - f'(0)\,x}{x} \underset{x \to 0}{\longrightarrow} 0$

? $\quad \underbrace{\dfrac{f(x) - f(0)}{x}}_{\xrightarrow[x \to 0]{} f'(0)} - f'(0) \underset{x \to 0}{\longrightarrow} 0$

$\underline{n=2}$ $\quad \underline{Ts.}$ $\quad f(x) = T_2(x) + o(x^2)$

$$T_2(x) = f(0) + f'(0)\,x + f''(0)\,\frac{x^2}{2}$$

devo verificare che $\qquad x \to 0$

$$f(x) = f(0) + f'(0)\,x + f''(0)\,\frac{x^2}{2} + o(x^2)$$

$$f(x) - f(0) - f'(0)\,x - f''(0)\,\frac{x^2}{2} = o(x^2) \qquad x \to 0$$

devo verificare che

$$\frac{f(x) - f(0) - f'(0)\,x - f''(0)\,\frac{x^2}{2}}{x^2} \xrightarrow[x \to 0]{} 0 \qquad \frac{0}{0}$$

Hôpital $\quad \dfrac{f'(x) - f'(0) - f''(0)\,x}{2x} \xrightarrow{?} 0$

$\qquad\qquad\qquad\qquad \|$

$$\frac{1}{2}\left(\underbrace{\frac{f'(x)-f'(0)}{x}}_{\to f''(0)} - f''(0)\right) \longrightarrow 0$$

rapporto incrementale di $f'(x)$ in $x=0$

$$f''(0) = \lim_{x\to 0} \frac{f'(x)-f'(0)}{x}$$

#

es. $f(x) = \operatorname{sen} x$

Calcolo il polinomio di Mc Laurin

$f(0) = 0$

$f'(x) = \cos x$ $\qquad$ $f'(0) = 1$

$f''(x) = -\operatorname{sen} x$ $\qquad$ $f''(0) = 0$

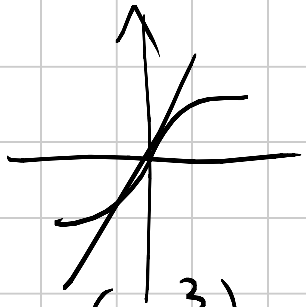
$f'''(x) = -\cos x$ $\qquad$ $f'''(0) = -1$

$f^{IV}(x) = \operatorname{sen} x$ $\qquad$ $f^{IV}(0) = 0$

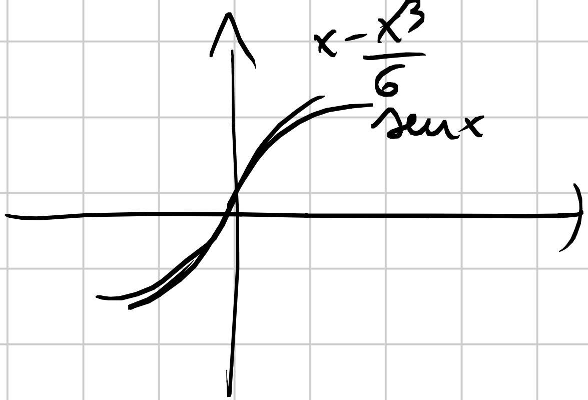
$$\operatorname{sen} x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$x \to 0$ $\qquad$ $+ o\left(x^{2n+1}\right)$

$n=1 \qquad \operatorname{sen} x = x + o(x)$



$n=3 \qquad \operatorname{sen} x = x - \dfrac{x^3}{6} + o(x^3) \qquad x \to 0$



<u>P.C.</u> sviluppo di cos x $\quad x \to 0$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n}) \qquad x \to 0$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots (-1)^{n-1} \frac{x^n}{n} + o(x^n)$$

$x \to 0$

Formula di Taylor di ordine n con resto di Peano

x_0

Polinomio di Taylor

$$T_{n,x_0}(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2} +$$

$$+ \dots \frac{f^{(n)}(x_0)(x-x_0)^n}{n!}$$

$$f(x) = T_{n,x_0}(x) + o\left((x-x_0)^n\right)$$

es. $f(x) = e^x$ $\qquad$ $x = 1$

$f^{(k)}(x) = e^x$ $\qquad$ $f^{(k)}(x) = e$

$$T_{n,1}(x) = e + e \cdot (x-1) + e \frac{(x-1)^2}{2} + \dots$$

sviluppo di Taylor in $x = 1$

Applicazione al calcolo dei limiti

$$\lim_{x \to 0} \frac{x + \operatorname{sen} x + \log(1+x)}{e^x - 1 + x^2} =$$

$\operatorname{sen} x = x + o(x) \qquad x \to 0$

$\log(1+x) = x + o(x)$

$e^x = 1 + x + o(x) \qquad e^x - 1 = x + o(x)$

$$= \lim_{x \to 0} \frac{x + x + o(x) + x + o(x)}{x + o(x) + x^2} =$$

$$= \lim_{x \to 0} \frac{3x + o(x)}{x + o(x)} \stackrel{?}{=}$$

$$x^2 = o(x)_{x \to 0}$$

$$\Downarrow$$

$$\frac{x^{\cancel{2}}}{\cancel{x}} \xrightarrow[x \to 0]{} 0$$

$$= \lim_{x \to 0} \frac{3\cancel{x}}{\cancel{x}} = 3$$

$$\frac{3x + o(x)}{x + o(x)} = \left(\frac{\cancel{x}\left(3 + \frac{o(x)}{x}\right)}{\cancel{x}\left(1 + \frac{o(x)}{x}\right)} \right) \xrightarrow[x \to 0]{} 3$$

$$\frac{o(x)}{x} \to 0$$

In generale $\lim_{x\to 0} \dfrac{x^\alpha + o(x^\alpha)}{x^\beta + o(x^\beta)} =$

$= \lim_{x\to 0} \dfrac{x^\alpha}{x^\beta}$

perché basta $\overline{\text{ordine}}$ $\overline{1}$?

$$\lim_{x\to 0} \frac{x + \operatorname{sen} x + \log(1+x)}{e^x - 1 + x^2}$$

$$\operatorname{sen} x = x - \frac{x^3}{6} + o(x^3)$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$$

N.

$$x + x - \frac{x^3}{6} + o(x^3) + x - \frac{x^2}{2} +$$

$$+ \frac{x^3}{3} + o(x^3) =$$

$$\frac{1}{3} - \frac{1}{6}$$

$$= 3x \underbrace{- \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)}_{o(x)}$$

es. $\lim_{x \to 0} \dfrac{\operatorname{sen} x - x}{x^3} = \lim_{x \to 0} \dfrac{\cancel{x} + o(x) - \cancel{x}}{x^3}$

no informazioni

$\operatorname{sen} x = x + o(x)$

$\operatorname{sen} x = x - \dfrac{x^3}{6} + o(x^3)$

$\lim_{x \to 0} \dfrac{\cancel{x} - \frac{x^3}{6} + o(x^3) - \cancel{x}}{x^3} = \lim_{x \to 0} \dfrac{-\frac{\cancel{x^3}}{6}}{\cancel{x^3}} = \dfrac{-1}{6}$

Proprietà del simbolo "o piccolo"

$x^2 = o(x)$ $\quad x \to 0$

$x^3 = o(x)$ $\quad x \to 0$

$x^3 = o(x^2)$

$f(x) = o(g(x))$ $\quad x \to x_0$

$\dfrac{f(x)}{g(x)} \to 0$ $\quad x \to x_0$

$o(x) + o(x) = o(x)$ $\quad x \to 0$

$o(x) - o(x) = o(x)$

$o(2x^2) = o(x^2)$ $\quad x \to 0$

$\Rightarrow o(kx) = o(x)$ $\quad k \neq 0$

$$o(x) + o(x^2) = o(x)$$

$$x \cdot o(x) = o(x^2)$$

lo verifico

$$\frac{\cancel{x}\; o(x)}{x^{\cancel{2}}} \xrightarrow[x \to 0]{} 0 \quad \leftarrow \text{def.}$$

$\frac{o(x)}{x} \to 0$ def. di "o piccolo".

$$\cdot\; x^2\, o(x^3) = o(x^5) \qquad \underline{x \to 0}$$

$$\cdot \quad o(x)\cdot o(x) = o(x^2)$$

$$x^2 = o(x) \qquad x \to 0$$

$$\boxed{\cancel{x^2 = o(x)} \qquad x \to +\infty}$$

$$\frac{x^2}{x} \to 0 \qquad x \to +\infty$$

$$\boxed{x = o(x^2) \qquad \underline{x \to +\infty}}$$

Esercizi

arcsen y

$|y| \leq 1$

$$f(x) = \arcsin\left(\frac{|x-1|}{x+3}\right)$$

$$D = \left\{ x \neq -3, \quad \left|\frac{x-1}{x+3}\right| \leq 1 \right\}$$

$$\left|\frac{x-1}{x+3}\right| \leq 1 \qquad -1 \leq \frac{x-1}{x+3} \leq 1$$

- $\frac{x-1}{x+3} \leq 1 \Rightarrow \frac{x-1}{x+3} - 1 \leq 0 \Rightarrow \frac{\cancel{x}-1-\cancel{x}-3}{x+3} \leq 0$

$$\frac{-4}{x+3} \leq 0$$

$$\frac{x-1}{x+3} \geq -1 \Rightarrow$$

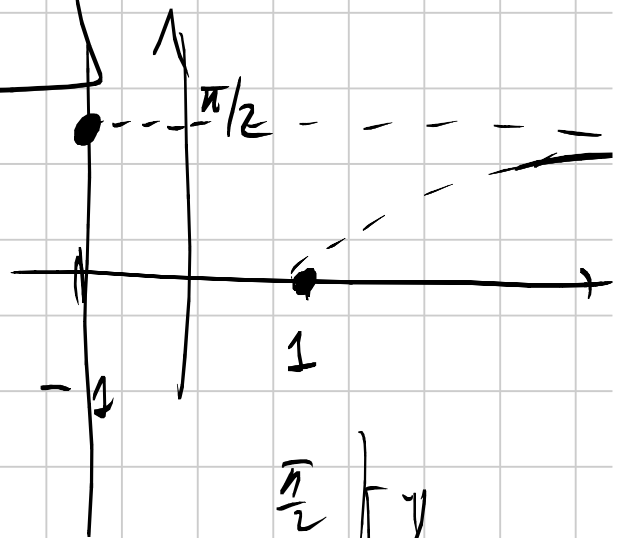
finire

$$\begin{cases} x \geq -1 \\ \text{o} \\ x < -3 \end{cases} \cap \; x > -3$$

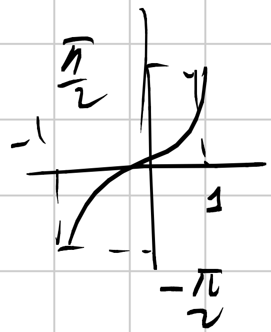
$$\Downarrow$$

$$\boxed{x \geq -1}$$

$$D = \{ x \in \mathbb{R}, \; x \geq -1 \}$$



$$\underline{f(-1)} = \arcsen\left(\frac{2}{2}\right) = \frac{\pi}{2}$$



$$\lim_{x \to +\infty} \arcsin\left(\frac{x-1}{x+3}\right) = \frac{\pi}{2}$$

asintoto orizzontale per $x \to +\infty$

$$f(x) = \begin{cases} \arcsin\left(\dfrac{x-1}{x+3}\right) & x \geq 1 \\ \arcsin\left(\dfrac{1-x}{x+3}\right) \longrightarrow & x < 1 \end{cases}$$

$x \geq 1$

$$f'(x) = \frac{1}{\sqrt{1 - \left(\frac{x-1}{x+3}\right)^2}} \cdot \frac{(x+3) - (x-1)}{(x+3)^2} =$$

$$= \frac{1}{\sqrt{1-\left(\frac{x-1}{x+3}\right)^2}} \frac{4}{(x+3)^2} > 0 \qquad x \geq -1$$

$$= \frac{1}{\sqrt{\frac{(x+3)^2 - (x-1)^2}{(x+3)^2}}} \cdot \frac{4}{(x+3)^2} =$$

$$= \frac{\cancel{(x+3)}}{\sqrt{\cancel{x^2}+9+6x-\cancel{x^2}-1+2x}} \cdot \frac{4}{(x+3)^{\cancel{2}}} =$$

$$= \frac{4}{(x+3)} \frac{1}{\sqrt{8x+8}} = \frac{4}{(x+3)\sqrt{8}\sqrt{x+1}}$$

$$f''(x) = \frac{-2(x+2)}{\sqrt{2}\,(x+3)^2(1+x)^{3/2}}$$

fare

$x \geq 1$

< 0

f strett. concava

se $x < 1$ $\quad f(x) = \arcsen\left(\frac{1-x}{x+3}\right)$

verificare

$$f'(x) = \frac{1}{\sqrt{1-\left(\frac{1-x}{x+3}\right)^2}} \cdot \frac{-4}{(x+3)^2} < 0$$

ii

$x=1$ $\quad f$ è derivabile ?

verificare

verificare

$$\lim_{x\to 1^-} f' = -\frac{1}{4} \qquad \lim_{x\to 1^+} f' = \frac{1}{4}$$

$x=1$ p.to angoloso

attacco in $x = -1$ $\quad (x \leq 1$

$$\lim_{x \to -1^+} f'(x) = -\infty$$

tangente verticale in $x = -1$

ES.

$$f(x) = (x-1)^2 \log\left(1 + \frac{2}{|x-1|}\right)$$

$D = \mathbb{R} \setminus \{1\}$

no f''

$x - 1 \to x$

$$f(x) = x^2 \log\left(1 + \frac{2}{|x|}\right)$$

$$f(-x) = x^2 \log\left(1 + \frac{2}{|x|}\right) = f(x)$$